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Rough set paradigms via containment neighborhoods and ideals

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Abstract. Imperfect information causes indistinguishability of objects and inability of making an accurate decision. To deal with this type of vague problem, Pawlak proposed the concept of rough set. Then, this concept has been studied from different points of view like topology and ideals. In this manuscript, we use the system of containment neighborhoods to present new rough set models generated by topology and ideals. We discuss their fundamental characterizations and reveal the relationships among them. Also, we prove that the current approximation spaces produce higher accuracy measures than those given by some previous approximation spaces. Ultimately, we provide a medical example to demonstrate that the current approach is one of the preferable and useful techniques to eliminate the ambiguity of the data in practical problems.

1. Introduction

In 1982, Pawlak [26] put forward an important mathematical approach to deal with vagueness of information systems called "rough set". Its methodology is based on handling subsets of data by a pair of exact sets called lower and upper approximations. These approximations approximate subsets to minimal exact set contained in the subset and maximal exact set containing the subset in terms of equivalence classes. Rough set theory has strong representation ability for incomplete information, so it has been widely applied in many fields, such as computer science, data mining and pattern recognition [24, 32].

An equivalent relation is sometimes difficult to obtain in real-world problems because of the vagueness and incompleteness of human knowledge. As a result, Yao et al [37, 38] defined new approximation operators using a non-equivalent relation. These operators induced from left and right neighborhoods. It can be seen from the published articles that canceling an equivalent condition leads to advent different types of neighborhood systems which automatically produce several sorts of approximation spaces. In this regard, Allam et al [5, 6] studied two types of approximation spaces induced from minimal left and minimal right neighborhoods. Afterward, Abd El-Monsef et al [1] familiarized four types of neighborhood systems, namely, intersection neighborhoods, union neighborhoods, intersection minimal neighborhoods, and union minimal neighborhoods. These systems of neighborhoods have been applied to initiate some kind of rough set models.

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Mareay [23] displayed novel kinds of neighborhood systems inspired by the equality relation between Yao's neighborhoods and applied to set forth some approximation operators. Further types of these sorts of neighborhood systems were established in [13]. Quite recently, Al-shami et al [11] have established the system of E_j-neighborhoods and demonstrated its relationships with previous systems of neighborhoods. Then, two types of neighborhood systems called containment neighborhoods and subset neighborhoods have been discussed by [7] and [10], respectively. They have been employed to rank individuals working in a specific facility in terms of infection of COVID-19. The concept of maximal neighborhoods was introduced and their main features were explored by [9]. It is worthily noting that many authors have generalized equivalence relation and have used many types of relations such as tolerance relation [29], similarity relation [2, 3, 30], dominance relation [40] and arbitrary binary relation [36, 42].

Another interesting approach studying approximation spaces is topology. In fact, there exists a close relationship between rough set theory and topology because it was proven that the pair of lower and upper approximations induced by reflexive and transitive relations is the same as the pair of interior and closure operators [25, 27, 37]. By using a topology, we can get approximations for qualitative concepts (subsets). Recently, researchers and scholars interested in topology and uncertain issues have exploited this similarity to build different types of approximation operators induced by a topology; see, [4, 8, 22, 41]. Kandi et al., in 2013, [21] provided a novel method to construct approximation spaces depending on the structure of ideal. They aimed to improve approximation operators and increase the accuracy measures. Then, Hosny [15, 16] introduced new rough set models induced from topological and ideal structures. Also, Al-shami and Hosny [12] applied the system of maximal neighborhoods with ideal structures to get rid of uncertainty via information systems.

The goal of the current work is to give a new rough paradigm by using the concepts of "containment neighborhoods and ideals". The main motivation for us to introduce and study this paradigm is to provide a new environment to describe incomplete data that cannot be treated by Pawlak's model, and to increase the accuracy measures of subsets by increasing their lower approximations and decreasing their upper approximations.

This article is structured in the following manner. In Section 2, we recall the some types of neighborhood systems via rough set theory and we mention the followed techniques to establish approximation spaces from the structures of topologies and ideals. In Section 3, we scrutinize the main properties and relationships of approximation spaces induced by containment neighborhood. We point out that the accuracy measures obtained from these approximation paces are better than the approaches introduced in [1, 4, 11]. In Section 4, we construct new rough paradigms generated directly by using containment neighborhoods and ideals. We explore their master characterizations and demonstrate that they enlarge the knowledge obtained from the subsets of data compared to the previous methods displayed in [7, 14]. To confirm the ability and importance of the followed technique, we apply this technique in Section 5 to describe an information system of COVID-19 and make a more accurate decision. Finally, in Section 6, we outline the paper's contributions and propose a plan for a future work.

2. Preliminaries

In this part, we recall the concepts and properties that we need to make the manuscript self-contained.

Definition 2.1. ([8]) A subset $R \subseteq U \times U$ is called a binary relation, it is said to be reflexive if $(v, v) \in R \quad \forall\, v \in U$, symmetric if $(u, v) \in R$ whenever $(v, u) \in R$, transitive if $(v, w) \in R$ whenever $(v, u) \in R$ and $(u, w) \in R$ and equivalence if R is reflexive, symmetric and transitive.

At the first, Pawlak [26] associated a subset with two crisp sets called lower and upper approximations. These approximations defined with respect to the equivalent classes as follows.

Definition 2.2. ([26]) If R is an equivalence relation on a universe U and $[x]_R$ is the equivalence class containing $x \in U$. The lower approximation $\underline{F}(A)$ and upper approximation $\overline{F}(A)$ of a set M of U are given by

i. $\underline{F}(M) = \{x \in U : [x]_R \subseteq M\}$.

ii. $\overline{F}(M) = \{x \in U : [x]_R \cap M \neq \phi\}$.

Then, the equivalence relation has been replaced by specific (or arbitrary) relation to expand the scope of applications of rough set theory. This leads to deal with the so-called neighborhoods of an element instead of its equivalence classes. As a result, various sorts of rough-set paradigms have been introduced in the published literature. In what follows, we recall a set of these neighborhoods and paradigms that we need to show the importance and robustness of this work.

Definition 2.3. [j-neighborhoods] ([5, 6, 38]) Let R be an arbitrary binary relation on a universe set U, then the j-neighborhoods of an element $x \in U$ are defined as follows
$j \in \{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$:

i. $N_r(x) = \{y \in U : {}_xR_y\}$

ii. $N_l(x) = \{y \in U : {}_yR_x\}$

iii. $N_i(x) = N_r(x) \cap N_l(x)$

iv. $N_u(x) = N_r(x) \cup N_l(x)$

v. $N_{\langle r\rangle}(x) = \bigcap\limits_{x \in N_r(y)} N_r(y)$

vi. $N_{\langle l\rangle}(x) = \bigcap\limits_{x \in N_l(x)} N_l(y)$

vii. $N_{\langle i\rangle}(x) = N_{\langle r\rangle}(x) \cap N_{\langle l\rangle}(x)$

viii. $N_{\langle u\rangle}(x) = N_{\langle r\rangle}(x) \cup N_{\langle l\rangle}(x)$

Definition 2.4. [j-neighborhood space] ([1]) Let R be an arbitrary binary relation on U and $\psi_j : U \longrightarrow P(U)$ be a mapping which assigns for each z in U its j-neighborhood in $P(U)$. then (U, R, ψ_j) is called a j-neighborhood space (N_jS).

Definition 2.5. ([5, 6, 38]) Let (U, R, ψ_j) be a N_jS, then the N_j-lower approximation $F_{N_j}(X)$, N_j-upper approximation $F^{N_j}(X)$ and N_j-accuracy measure $\mu_{N_j}(X)$ of a set $X \subseteq U$ are defined as follows:

i. $F_{N_j}(X) = \{x \in U : N_j(x) \subseteq X\}$

ii. $F^{N_j}(X) = \{x \in U : N_j(x) \cap X \neq \phi\}$

iii. $\mu_{N_j}(X) = \frac{|F_{N_j}(X) \cap X|}{|F^{N_j}(X) \cup X|}$

Where $|F^{N_j}(X) \cup X| \neq 0$.

Definition 2.6. [j-adhesion neighborhoods] [13, 23] Let R be an arbitrary binary relation on a universe set U, then the j-adhesion neighborhoods of element $x \in U$ are defined as follows $j \in \{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$:

i. $P_r(x) = \{y \in U : N_r(x) = N_r(y)\}$

ii. $P_l(x) = \{y \in U : N_l(x) = N_l(y)\}$

iii. $P_i(x) = P_r(x) \cap P_l(x)$

iv. $P_u(x) = P_r(x) \cup P_l(x)$

v. $P_{\langle r\rangle}(x) = \{y \in U : \bigcap\limits_{x \in N_r(y)} N_r(y) = \bigcap\limits_{y \in N_r(x)} N_r(x)\}$

vi. $P_{\langle l\rangle}(x) = \{y \in U : \bigcap\limits_{x\in N_l(y)} N_l(y) = \bigcap\limits_{y\in N_l(x)} N_l(x)\}$

vii. $P_{\langle i\rangle}(x) = P_{\langle r\rangle}(x) \cap P_{\langle l\rangle}(x)$

viii. $P_{\langle u\rangle}(x) = P_{\langle r\rangle}(x) \cup P_{\langle l\rangle}(x)$

Definition 2.7. [E_j-neighborhoods] ([11]) Let R be an arbitrary binary relation on a universe set U, then the E_j-neighborhoods are defined as follows $j \in \{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$:

i. $E_r(x) = \{y \in U : N_r(y) \cap N_r(x) \neq \phi\}$

ii. $E_l(x) = \{y \in U : N_l(y) \cap N_l(x) \neq \phi\}$

iii. $E_i(x) = E_r(x) \cap E_l(x)$

iv. $E_u(x) = E_r(x) \cup E_l(x)$

v. $E_{\langle r\rangle}(x) = \{y \in U : N_{\langle r\rangle}(y) \cap N_{\langle r\rangle}(x) \neq \phi\}$

vi. $E_{\langle l\rangle}(x) = \{y \in U : N_{\langle l\rangle}(y) \cap N_{\langle l\rangle}(x) \neq \phi\}$

vii. $E_{\langle i\rangle}(x) = E_{\langle r\rangle}(x) \cap E_{\langle l\rangle}(x)$

viii. $E_{\langle u\rangle}(x) = E_{\langle r\rangle}(x) \cup E_{\langle l\rangle}(x)$

Definition 2.8. ([11]) Let (U, R, ψ_j) be a N_jS, then the E_j-lower approximation $F_{E_j}(X)$, E_j-upper approximation $F^{E_j}(X)$ and μ_{Ej}-accuracy measure $M_{E_j}(X)$ of a set $X \subseteq U$ are defined as follows:

i. $F_{E_j}(X) = \{x \in U : E_j(x) \subseteq X\}$

ii. $F^{E_j}(X) = \{x \in U : E_j(x) \cap X \neq \phi\}$

iii. $\mu_{E_j}(X) = \frac{|F_{E_j}(X)\cap X|}{|F^{E_j}(X)\cup X|}$,

where $|F^{Ej}(X) \cup X| \neq 0$.

Definition 2.9. [C_j-neighborhoods] ([7]) Let R be an arbitrary binary relation on a universe set U, then the C_j-neighborhoods are defined as follows $j \in \{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$:

i. $C_r(x) = \{y \in U : N_r(y) \subseteq N_r(x)\}$

ii. $C_l(x) = \{y \in U : N_l(y) \subseteq N_l(x)\}$

iii. $C_i(x) = C_r(x) \cap C_l(x)$

iv. $C_u(x) = C_r(x) \cup C_l(x)$

v. $C_{\langle r\rangle}(x) = \{y \in U : N_{\langle r\rangle}(y) \subseteq N_{\langle r\rangle}(x)\}$

vi. $C_{\langle l\rangle}(x) = \{y \in U : N_{\langle l\rangle}(y) \subseteq N_{\langle l\rangle}(x)\}$

vii. $C_{\langle i\rangle}(x) = C_{\langle r\rangle}(x) \cap C_{\langle l\rangle}(x)$

viii. $C_{\langle u\rangle}(x) = C_{\langle r\rangle}(x) \cup C_{\langle l\rangle}(x)$

Definition 2.10. ([7]) Let (U, R, ψ_j) be a N_jS, then the C_j-lower approximation $F_{C_j}(X)$, C_j-upper approximation $F^{C_j}(X)$ and C_j-accuracy measure $\mu_{C_j}(X)$ of a set $X \subseteq U$ are defined as follows:

i. $F_{C_j}(X) = \{x \in U : C_j(x) \subseteq X\}$

ii. $F^{C_j}(X) = \{x \in U : C_j(x) \cap X \neq \phi\}$

iii. $\mu_{C_j}(X) = \frac{|F_{C_j}(X)|}{|F^{C_j}(X)|}$,

where $|F^{Cj}(X)| \neq 0$. ==

Definition 2.11. ([8]) A subset $T \subseteq P(U)$ is called a topology on U if $\phi, U \in T$ and T is closed under arbitrary union and finite intersection. We call the order pair (T, U) a topological space. A set M is called an open set if it is a member of T, and it is called a closed set if its complement a member of T. For any subset M of U, the interior points of M, denoted by $int(M)$ is the union of all open sets that are contained in M, and the closure points of M, denoted by $cl(M)$ is the intersection of all closed sets containing M.

The rough set paradigms have been studied topologically in several published literature. The followed methods to link neighborhoods systems and topological structures are proved in the following results.

Theorem 2.1. ([1, 11, 14, 23]) *Let (U, R, ψ_j) be a N_jS. Then each one of the following collections is a topology on U for each j.*

i. $T_{Nj} = \{M \in U : \forall\, y \in M, N_j(y) \subseteq M\}$ *[1].*

ii. $T_{Pj} = \{M \in U : \forall\, y \in M, P_j(y) \subseteq M\}$ *[4, 23].*

iii. $T_{Ej} = \{M \in U : \forall\, y \in M, E_j(y) \subseteq M\}$ *[11].*

iv. $T_{Cj} = \{M \in U : \forall\, y \in M, C_j(y) \subseteq M\}$ *[14]*

Definition 2.12. ([1, 11, 14, 23]) Let (U, R, ψ_j) be a N_jS. Then, the following are some types of lower and upper approximations and accuracy measures of a subset $M \subseteq U$ induced from topological spaces T_{Nj}, T_{Pj}, T_{Ej} and T_{Cj}, are respectively defined as follows.

$$\begin{array}{lll} \underline{N}_j(M) = int_{N_j}(M) & \overline{N}_j(M) = cl_{N_j}(M) & \rho_{N_j}(M) = \frac{|\underline{N}_j(M)|}{|\overline{N}_j(M)|} \\ \underline{P}_j(M) = int_{P_j}(M) & \overline{P}_j(M) = cl_{P_j}(M) & \rho_{P_j}(M) = \frac{|\underline{P}_j(M)|}{|\overline{P}_j(M)|} \\ \underline{E}_j(M) = int_{E_j}(M) & \overline{E}_j(M) = cl_{E_j}(M) & \rho_{E_j}(M) = \frac{|\underline{E}_j(M)|}{|\overline{E}_j(M)|} \\ \underline{C}_j(M) = int_{C_j}(M) & \overline{C}_j(M) = cl_{C_j}(M) & \rho_{C_j}(M) = \frac{|\underline{C}_j(M)|}{|\overline{C}_j(M)|} \end{array}$$

Definition 2.13. ([20]) A non-empty collection I of subsets of a set X is called an ideal on X if it satisfies the following conditions:

i. If $A \in I$ and $B \in I$, then $A \cup B \in I$.

ii. If $A \in I$ and $B \subseteq A$, then $B \in I$.

To improve the approximation operators and increase the accuracy measure of a set, the topological structures given in Theorem 2.1 were enlarged by inserting ideals as illustrated in the next theorem.

Theorem 2.2. ([14]) *Let (U, R, ψ_j) be N_jS and I be an ideal on U. Then each one of the following collections is a topology on U for each j.*

i. $T^I_{Nj} = \{M \subseteq U : \forall\, z \in M, (N_j(z) - M) \in I\}$ *[21].*

ii. $T^I_{Pj} = \{M \subseteq U : \forall\, z \in M, (P_j(z) - M) \in I\}$ *[19].*

iii. $T^I_{Ej} = \{M \subseteq U : \forall\, z \in M, (E_j(z) - M) \in I\}$ *[18].*

iv. $T^I_{Cj} = \{M \subseteq U : \forall z \in M, (C_j(z) - M) \in I\}$ *[14].*

Definition 2.14. ([14]) Let (U, R, ψ_j) be N_jS, I be an ideal on U and $M \subseteq U$, then for each j, the lower approximation $\underline{C}^I_j$, upper approximation $\overline{C}^I_j$, boundary region B^I_{Cj}, positive region POS^I_j, negative region NEG^I_j and accuracy measure ρ^I_{Cj} of M are defined by:

i. $\underline{C}^I_j(M) = int^I_{Cj}(M)$.

ii. $\overline{C}^I_j(M) = cl^I_{Cj}(M)$.

iii. $B^I_{C_j}(M) = \overline{C}^I_j(M) - \underline{C}^I_j(M)$

iv. $POS^I_{C_j}(M) = \underline{C}^I_j(M)$.

v. $NEG^I_{C_j}(M) = U - \overline{C}^I_j(M)$.

vi. $\rho^I_{Cj}(M) = \frac{|\underline{C}^I_j(M)|}{|\overline{C}^I_j(M)|}$,

where $|\overline{C}^I_j(M)| \neq 0$

3. Topologies generated by containment neighborhoods

In this section, we study the relation between the 8 topologies initiated in 4 of Theorem 2.1 as well as the relation between these topologies and those studied in [1, 11, 23]. Then, we construct new rough approximations from these topologies and research the properties of these approximations. Also, we elucidate the relation between these approximations and those introduced in [7].

Definition 3.1. Let (U, R, ψ_j) be N_jS, then for each j, a subset X of U is called C_j-open set if $X \in T_{C_j}$ and the complement of a C_j-open set is said to be C_j-closed set. The collection F_{Cj} of all C_j-closed sets is given by:

$$F_{Cj} = \{F \subseteq U : F^c \in T_{C_j}\}$$

The following theorem states the relations between the new topologies generated by 4 of Theorem 2.1 as well as the relations between these new topologies and those generated in[1, 11, 23].

Theorem 3.1. *The following properties hold for the topologies generated by C_j-neighborhoods:*

i. $T_{C_u} \subseteq T_{C_r} \subseteq T_{C_i}$.

ii. $T_{C_u} \subseteq T_{C_l} \subseteq T_{C_i}$.

iii. $T_{C\langle u\rangle} \subseteq T_{C\langle r\rangle} \subseteq T_{C\langle i\rangle}$.

iv. $T_{C\langle u\rangle} \subseteq T_{C\langle l\rangle} \subseteq T_{C\langle i\rangle}$.

v. If R is symmetric, then
$T_{C_r} = T_{C_l} = T_{C_i} = T_{C_u}$ *and*
$T_{C\langle r\rangle} = T_{C\langle l\rangle} = T_{C\langle i\rangle} = T_{C\langle u\rangle}$

vi. $T_{C_j} \subseteq T_{P_j}$ *for each j.*

vii. If R is equivalence relation, then $T_{P_j} = T_{C_j}$ *for each j.*

viii. If R is equivalence relation, then
$T_{C_r} = T_{C_l} = T_{C_i} = T_{C_u} = T_{C\langle r\rangle} = T_{C\langle l\rangle} = T_{C\langle i\rangle} = T_{C\langle u\rangle}$

ix. If R is reflexive relation, then $T_{E_j} \subseteq T_{N_j} \subseteq T_{C_j}$ *for each j*

Proof. :

i. Since $C_i(x) \subseteq C_r(x) \subseteq C_u(x) \quad \forall\, x \in U$, we obtain $T_{C_u} \subseteq T_{C_r} \subseteq T_{C_i}$.

ii. Since $C_i(x) \subseteq C_l(x) \subseteq C_u(x) \quad \forall\, x \in U$, we obtain $T_{C_u} \subseteq T_{C_l} \subseteq T_{C_i}$.

iii. Since $C_{\langle i\rangle}(x) \subseteq C_{\langle r\rangle}(x) \subseteq C_{\langle u\rangle}(x)$, we obtain $T_{C\langle u\rangle} \subseteq T_{C\langle r\rangle} \subseteq T_{C\langle i\rangle}$.

iv. Since $C_{\langle i\rangle}(x) \subseteq C_{\langle l\rangle}(x) \subseteq C_{\langle u\rangle}(x)$, we obtain $T_{C\langle u\rangle} \subseteq T_{C\langle l\rangle} \subseteq T_{C\langle i\rangle}$.

v. Since R is symmetric, then $C_r(x) = C_l(x) = C_i(x) = C_u(x)$ and $C_{\langle r\rangle}(x) = C_{\langle l\rangle}(x) = C_{\langle i\rangle}(x) = C_{\langle u\rangle}(x) \forall\, x \in U$.

vi. $P_j(x) \subseteq C_j(x)\ \forall\, x \in U$ for each j.

vii. Since R is equivalence relation, then $C_j(x) = P_j(x)\ \forall\, x \in U$ for each j.

viii. Since R is equivalence relation, then
$C_r(x) = C_l(x) = C_i(x) = C_u(x) = C_{\langle r\rangle}(x) = C_{\langle l\rangle}(x) = C_{\langle i\rangle}(x) = C_{\langle u\rangle}(x)\ \forall\, x \in U$.

ix. Since R is reflexive, then $C_j(x) \subseteq N_j(x) \subseteq E_j(x)\ \forall\, x \in U$ for each j.

□

Definition 3.2. Let (U, R, ψ_j) be N_jS and $M \subseteq U$, then for each j the C_j-interior and the C_j-closure of M are defined by:

$$int_{C_j}(M) = \cup\{G \in T_{C_j} : G \subseteq M\}$$

$$cl_{C_j}(M) = \cap\{H \in F_{C_j} : M \subseteq H\}$$

Definition 3.3. Let (U, R, ψ_j) be N_jS and $M \subseteq U$, then for each j, the lower approximation $\underline{C}_j$, upper approximation $\overline{C}_j$, boundary region B_{C_j}, positive region POS_{C_j}, negative region NEG_{C_j} and accuracy measure ρ_{C_j} of M are defined by:

i. $\underline{C}_j(M) = int_{Cj}(M)$.

ii. $\overline{C}_j(M) = cl_{Cj}(M)$.

iii. $B_{C_j}(M) = \overline{C}_j(M) - \underline{C}_j(M)$

iv. $POS_{C_j}(M) = \underline{C}_j(M)$.

v. $NEG_{C_j}(M) = U - \overline{C}_j(M)$.

vi. $\rho_{C_j}(M) = \frac{|\underline{C}_j(M)|}{|\overline{C}_j(M)|}$.

Where $|\overline{C}_j(M)| \neq 0$

The following theorem states the properties of the lower approximation $\underline{C}_j$ and upper approximation $\overline{C}_j$.

Theorem 3.2. *Let* (U, R, ψ_j) *be* N_jS *and M and N be subsets of U, then for each j:*

L1 $\underline{C}_j(M) \subseteq M$.

L2 $\underline{C}_j(\phi) = \phi$.

L3 $\underline{C}_j(U) = U$.

L4 $\underline{C}_j(M \cap N) = \underline{C}_j(M) \cap \underline{C}_j(N)$.

L5 *If* $M \subseteq N$, *then* $\underline{C}_j(M) \subseteq \underline{C}_j(N)$.

L6 $\underline{C}_j(M) \cup \underline{C}_j(N) \subseteq \underline{C}_j(M \cup N)$.

L7 $\underline{C}_j(M^c) = [\overline{C}_j(M)]^c$.

L8 $\underline{C}_j[\underline{C}_j(M)] = \underline{C}_j(M)$.

U1 $M \subseteq \overline{C}_j(M)$.

U2 $\overline{C}_j(\phi) = \phi$.

U3 $\overline{C}_j(U) = U$.

U4 $\overline{C}_j(M \cup N) = \overline{C}_j(M) \cup \overline{C}_j(N)$.

U5 *If* $M \subseteq N$, *then* $\overline{C}_j(M) \subseteq \overline{C}_j(N)$.

U6 $\overline{C}_j(M) \cap \overline{C}_j(N) \supseteq \overline{C}_j(M \cap N)$.

U7 $\overline{C}_j(M^c) = [\overline{C}_j(M)]^c$.

U8 $\overline{C}_j[\overline{C}_j(M)] = \overline{C}_j(M)$.

Proof. These follow from the properties of the interior and closure operators. □

Definition 3.4. Let (U, R, ψ_j) be N_jS and $M \subseteq U$, then for each j, M is called:

i. Totally C_j-definable or C_j-exact if $\underline{C}_j(M) = M = \overline{C}_j(M)$.

ii. Internally C_j-definable if $\underline{C}_j(M) = M$ and $\overline{C}_j(M) \neq M$.

iii. Externally C_j-definable if $\underline{C}_j(M) \neq M$ and $\overline{C}_j(M) = M$.

iv. C_j-Rough if $\underline{C}_j(M) \neq M \neq \overline{C}_j(M)$.

We elaborate, in the next two results, the relationships between the approximation operators and accuracy measure produced by containment neighborhood as given in [7] and their counterparts that we construct in this section.

Theorem 3.3. *Let M be a subset of U, then $\underline{C}_j(M) \subseteq F_{C_j}(M)$ and $F^{C_j}(M) \subseteq \overline{C}_j(M)$.*

Proof. First, let $z \in \underline{C}_j(M)$, then $z \in int_{C_j}(M)$. Thus there exists an open set G such that $z \in G \subseteq M$. Therefore $C_j(z) \subseteq G \subseteq M$. Hence $\underline{C}_j(M) \subseteq F_{C_j}(M)$.

Second, let $z \notin \overline{C}_j(M)$, then $z \notin cl_{C_j}(M)$. So there exists a closed set $F \supseteq M$ such that $z \notin F$. Therefore $z \in F^c$ which is open set and $F^c \cap M = \phi$. Since $z \in F^c$ which is open, thus $C_j(z) \subseteq F^c$ and $C_j(z) \cap M = \phi$. Therefore $z \notin F^{C_j}(M)$. Hence $F^{C_j}(M) \subseteq \overline{C}_j(M)$. □

Corollary 3.1. *Let M be a subset of U, then $\rho_{C_j}(M) \leq \mu_{C_j}(M)$.*

Example 3.1. Let $U = \{m_1, m_2, m_3\}$ be a universe set and $R = \{(m_1, m_1), (m_1, m_3), (m_2, m_3), (m_3, m_3)\}$ be a binary relation on U. Then the power set of U is given by: $P(U) = \{\{m_1\}, \{m_1, m_2\}, \{m_1, m_3\}, \{m_2\}, \{m_2, m_3\}, \{m_3\}, U, \phi\}$.
The j-neighborhoods and C_j-neighborhoods are given in Tables (1,2) respectively.
The 8 topologies generated from containment neighborhoods are as follows:
$T_{C_r} = \{U, \phi, \{m_2, m_3\}\}$
$T_{C_l} = \{U, \phi, \{m_1, m_2\}, \{m_2\}\}$
$T_{C_i} = \{U, \phi, \{m_1, m_2\}, \{m_2\}, \{m_2, m_3\}\}$
$T_{C_u} = \{U, \phi\}$
$T_{C_{\langle r\rangle}} = \{U, \phi, \{m_2\}, \{m_2, m_3\}\}$
$T_{C_{\langle l\rangle}} = \{U, \phi, \{m_1\}\}$
$T_{C_{\langle i\rangle}} = \{U, \phi, \{m_1\}, \{m_2\}, \{m_1, m_2\}, \{m_2, m_3\}\}$
$T_{C_{\langle u\rangle}} = \{U, \phi\}$
F_{C_j}, F^{C_j}, μ_{C_j}, $\underline{C}_j$, $\overline{C}_j$ and ρ_{C_j} are given in Tables (3, 4, 5, 6, 7, 8) respectively.

Table 1: j-Neighborhoods

z	$N_r(z)$	$N_l(z)$	$N_i(z)$	$N_u(z)$	$N_{\langle r\rangle}(z)$	$N_{\langle l\rangle}(z)$	$N_{\langle i\rangle}(z)$	$N_{\langle u\rangle}(z)$
m_1	$\{m_1, m_3\}$	$\{m_1\}$	$\{m_1\}$	$\{m_1, m_3\}$	$\{m_1, m_3\}$	$\{m_1\}$	$\{m_1\}$	$\{m_1, m_3\}$
m_2	$\{m_3\}$	ϕ	ϕ	$\{m_3\}$	ϕ	U	ϕ	U
m_3	$\{m_3\}$	U	$\{m_3\}$	U	$\{m_3\}$	U	$\{m_3\}$	U

Table 2: C_j-Neighborhoods

z	$C_r(z)$	$C_l(z)$	$C_i(z)$	$C_u(z)$	$C_{\langle r\rangle}(z)$	$C_{\langle l\rangle}(z)$	$C_{\langle i\rangle}(z)$	$C_{\langle u\rangle}(z)$
m_1	U	$\{m_1, m_2\}$	$\{m_1, m_2\}$	U	U	$\{m_1\}$	$\{m_1\}$	U
m_2	$\{m_2, m_3\}$	$\{m_2\}$	$\{m_2\}$	$\{m_2, m_3\}$	$\{m_2\}$	U	$\{m_2\}$	U
m_3	$\{m_2, m_3\}$	U	$\{m_2, m_3\}$	U	$\{m_2, m_3\}$	U	$\{m_2, m_3\}$	U

Table 3: F_{C_j}

Set (M)	$F_{C_r}(M)$	$F_{C_l}(M)$	$F_{C_i}(M)$	$F_{C_u}(M)$	$F_{C_{\langle r\rangle}}(M)$	$F_{C_{\langle l\rangle}}(M)$	$F_{C_{\langle i\rangle}}(M)$	$F_{C_{\langle u\rangle}}(M)$
$\{m_1\}$	ϕ	ϕ	ϕ	ϕ	ϕ	$\{m_1\}$	$\{m_1\}$	ϕ
$\{m_2\}$	ϕ	$\{m_2\}$	$\{m_2\}$	ϕ	$\{m_2\}$	ϕ	$\{m_2\}$	ϕ
$\{m_3\}$	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
$\{m_1, m_2\}$	ϕ	$\{m_1, m_2\}$	$\{m_1, m_2\}$	ϕ	$\{m_2\}$	$\{m_1\}$	$\{m_1, m_2\}$	ϕ
$\{m_1, m_3\}$	ϕ	ϕ	ϕ	ϕ	ϕ	$\{m_1\}$	$\{m_1\}$	ϕ
$\{m_2, m_3\}$	$\{m_2, m_3\}$	$\{m_2\}$	$\{m_2, m_3\}$	$\{m_2\}$	$\{m_2, m_3\}$	ϕ	$\{m_2, m_3\}$	ϕ

Table 4: F^{C_j}

Set (M)	$F^{C_r}(M)$	$F^{C_l}(M)$	$F^{C_i}(M)$	$F^{C_u}(M)$	$F^{C_{\langle r\rangle}}(M)$	$F^{C_{\langle l\rangle}}(M)$	$F^{C_{\langle i\rangle}}(M)$	$F^{C_{\langle u\rangle}}(M)$
$\{m_1\}$	$\{m_1\}$	$\{m_1, m_3\}$	$\{m_1\}$	$\{m_1, m_3\}$	$\{m_1\}$	U	$\{m_1\}$	U
$\{m_2\}$	U	U	U	U	U	$\{m_2, m_3\}$	$\{m_2, m_3\}$	U
$\{m_3\}$	U	$\{m_3\}$	$\{m_3\}$	U	$\{m_1, m_3\}$	$\{m_2, m_3\}$	$\{m_3\}$	U
$\{m_1, m_2\}$	U	U	U	U	U	U	U	U
$\{m_1, m_3\}$	U	$\{m_1, m_3\}$	$\{m_1, m_3\}$	U	$\{m_1, m_3\}$	U	$\{m_1, m_3\}$	U
$\{m_2, m_3\}$	U	U	U	U	U	$\{m_2, m_3\}$	$\{m_2, m_3\}$	U

Table 5: μ_{C_j}

Set (M)	$\mu_{C_r}(M)$	$\mu_{C_l}(M)$	$\mu_{C_i}(M)$	$\mu_{C_u}(M)$	$\mu_{C_{\langle r\rangle}}(M)$	$\mu_{C_{\langle l\rangle}}(M)$	$\mu_{C_{\langle i\rangle}}(M)$	$\mu_{C_{\langle u\rangle}}(M)$
$\{m_1\}$	0	0	0	0	0	$\frac{1}{3}$	1	0
$\{m_2\}$	0	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	0	$\frac{1}{2}$	0
$\{m_3\}$	0	0	0	0	0	0	0	0
$\{m_1, m_2\}$	0	$\frac{2}{3}$	$\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	0
$\{m_1, m_3\}$	0	0	0	0	0	$\frac{1}{3}$	$\frac{1}{2}$	0
$\{m_2, m_3\}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	0	1	0

Table 6: $\underline{C}_j$

Set (M)	$\underline{C}_r(M)$	$\underline{C}_l(M)$	$\underline{C}_i(M)$	$\underline{C}_u(M)$	$\underline{C}_{\langle r\rangle}(M)$	$\underline{C}_{\langle l\rangle}(M)$	$\underline{C}_{\langle i\rangle}(M)$	$\underline{C}_{\langle u\rangle}(M)$
$\{m_1\}$	ϕ	ϕ	ϕ	ϕ	ϕ	$\{m_1\}$	$\{m_1\}$	ϕ
$\{m_2\}$	ϕ	$\{m_2\}$	$\{m_2\}$	ϕ	$\{m_2\}$	ϕ	$\{m_2\}$	ϕ
$\{m_3\}$	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
$\{m_1, m_2\}$	ϕ	$\{m_1, m_2\}$	$\{m_1, m_2\}$	ϕ	$\{m_2\}$	$\{m_1\}$	$\{m_1, m_2\}$	ϕ
$\{m_1, m_3\}$	ϕ	ϕ	ϕ	ϕ	ϕ	$\{m_1\}$	$\{m_1\}$	ϕ
$\{m_2, m_3\}$	$\{m_2, m_3\}$	$\{m_2\}$	$\{m_2, m_3\}$	ϕ	$\{m_2, m_3\}$	ϕ	$\{m_2, m_3\}$	ϕ

Table 7: $\overline{C}_j$

Set (M)	$\overline{C}_r(M)$	$\overline{C}_l(M)$	$\overline{C}_i(M)$	$\overline{C}_u(M)$	$\overline{C}_{\langle r\rangle}(M)$	$\overline{C}_{\langle l\rangle}(M)$	$\overline{C}_{\langle i\rangle}(M)$	$\overline{C}_{\langle u\rangle}(M)$
$\{m_1\}$	$\{m_1\}$	$\{m_1, m_3\}$	$\{m_1\}$	U	$\{m_1\}$	U	$\{m_1\}$	U
$\{m_2\}$	U	U	U	U	U	$\{m_2, m_3\}$	$\{m_2, m_3\}$	U
$\{m_3\}$	U	$\{m_3\}$	$\{m_3\}$	U	$\{m_1, m_3\}$	$\{m_2, m_3\}$	$\{m_3\}$	U
$\{m_1, m_2\}$	U	U	U	U	U	U	U	U
$\{m_1, m_3\}$	U	$\{m_1, m_3\}$	$\{m_1, m_3\}$	U	$\{m_1, m_3\}$	U	$\{m_1, m_3\}$	U
$\{m_2, m_3\}$	U	U	U	U	U	$\{m_2, m_3\}$	$\{m_2, m_3\}$	U

Table 8: ρ_{C_j}

Set (M)	$\rho_{C_r}(M)$	$\rho_{C_l}(M)$	$\rho_{C_i}(M)$	$\rho_{C_u}(M)$	$\rho_{C_{\langle r\rangle}}(M)$	$\rho_{C_{\langle l\rangle}}(M)$	$\rho_{C_{\langle i\rangle}}(M)$	$\rho_{C_{\langle u\rangle}}(M)$
$\{m_1\}$	0	0	0	0	0	$\frac{1}{3}$	1	0
$\{m_2\}$	0	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	0	$\frac{1}{2}$	0
$\{m_3\}$	0	0	0	0	0	0	0	0
$\{m_1, m_2\}$	0	$\frac{2}{3}$	$\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	0
$\{m_1, m_3\}$	0	0	0	0	0	$\frac{1}{3}$	$\frac{1}{2}$	0
$\{m_2, m_3\}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{2}{3}$	0	1	0

Note that the converse of the following relations doesn't hold in general.

I. $T_{C_u} \subseteq T_{C_r} \subseteq T_{C_i}$.

  i. $T_{C_u} = \{U, \phi\}$.

  ii. $T_{C_r} = \{U, \phi, \{m_2, m_3\}\}$.

  iii. $T_{C_i} = \{U, \phi, \{m_1, m_2\}, \{m_2\}, \{m_2, m_3\}\}$.

II. $T_{C_u} \subseteq T_{C_l} \subseteq T_{C_i}$.

i. $T_{C_u} = \{U, \phi\}$.
ii. $T_{C_l} = \{U, \phi, \{m_1, m_2\}, \{m_2\}\}$.
iii. $T_{C_i} = \{U, \phi, \{m_1, m_2\}, \{m_2\}, \{m_2, m_3\}\}$.

III. $T_{C_{\langle u\rangle}} \subseteq T_{C_{\langle r\rangle}} \subseteq T_{C_{\langle i\rangle}}$.

i. $T_{C_{\langle u\rangle}} = \{U, \phi\}$.
ii. $T_{C_{\langle r\rangle}} = \{U, \phi, \{m_2\}, \{m_2, m_3\}\}$.
iii. $T_{C_{\langle i\rangle}} = \{U, \phi, \{m_1\}, \{m_2\}, \{m_1, m_2\}, \{m_2, m_3\}\}$.

IV. $T_{C_{\langle u\rangle}} \subseteq T_{C_{\langle l\rangle}} \subseteq T_{C_{\langle i\rangle}}$.

i. $T_{C_{\langle u\rangle}} = \{U, \phi\}$.
ii. $T_{C_{\langle l\rangle}} = \{U, \phi, \{m_1\}\}$.
iii. $T_{C_{\langle i\rangle}} = \{U, \phi, \{m_1\}, \{m_2\}, \{m_1, m_2\}, \{m_2, m_3\}\}$.

V. $\underline{C}_j(M) \subseteq M$.

i. $\underline{C}_{\langle r\rangle}(\{m_1, m_2\}) = \{m_2\}$.

VI. If $M \subseteq N$, then $\underline{C}_j(M) \subseteq \underline{C}_j(N)$.

i. $\underline{C}_r(\{m_1\}) = \phi$.
ii. $\underline{C}_r(\{m_2, m_3\}) = \{m_2, m_3\}$.

VII. $[\underline{C}_j(M) \cup \underline{C}_j(N)] \subseteq \underline{C}_j(M \cup N)$.

i. $[\underline{C}_l(\{m_1\}) \cup \underline{C}_l(\{m_2\})] = \{m_2\}$.
ii. $\underline{C}_l(\{m_1\} \cup \{m_2\}) = \{m_1, m_2\}$

VIII. $M \subseteq \overline{C}_j(M)$.

i. $\overline{C}_l(\{m_1\}) = \{m_1, m_3\}$.

IX. If $M \subseteq N$, then $\overline{C}_j(M) \subseteq \overline{C}_j(N)$.

i. $\overline{C}_l(\{m_1\}) = \{m_1, m_3\}$.
ii. $\overline{C}_l(\{m_2\}) = U$.

X. $[\overline{C}_j(M) \cap \overline{C}_j(N)] \supseteq \overline{C}_j(M \cap N)$.

i. $\overline{C}_l(\{m_1, m_3\}) \cap \overline{C}_l(\{m_2, m_3\}) = \{m_1, m_3\}$.
ii. $\overline{C}_l(\{m_1, m_3\} \cap \{m_2, m_3\}) = \{m_3\}$.

XI. $\underline{C}_j(M) \subseteq F_{Cj}(M)$

i. $\underline{C}_u(\{m_2, m_3\}) = \phi$.
ii. $F_{C_u}(\{m_2, m_3\}) = \{m_2\}$.

XII. $F^{C_j}(M) \subseteq \overline{C}_j(M)$

i. $F_{C_u}(\{m_1\}) = \{m_1, m_3\}$.
ii. $\underline{C}_u(\{m_1\}) = U$.

XIII. $\rho_{C_j}(M) \leq \mu_{C_j}(M)$

i. $\rho_{C_u}(\{m_2, m_3\}) = 0$.
ii. $\mu_{C_u}(\{m_2, m_3\}) = \frac{1}{3}$.

4. New types of approximations in terms of containment neighborhoods and ideals

In this section, we define new rough approximation spaces directly generated from containment neighborhoods and ideals. We explore their basic characterizations and illustrate the relationships between them. Furthermore, we demonstrate the advantages of them compared to the approximation spaces displayed in Section 3 and those studied in [7, 14].

Definition 4.1. Let (U, R, ψ_j) be N_jS, I be an ideal on U and $M \subseteq U$, then the lower approximation ${}^{I}F^{-}_{C_j}$, upper approximation ${}^{I}F^{+}_{C_j}$ and accuracy measure ${}^{I}\rho_{C_j}$ of M are defined by:

i. ${}^{I}F^{-}_{C_j}(M) = \{z \in U : [C_j(z) - M] \in I\}$.

ii. ${}^{I}F^{+}_{C_j}(M) = \{z \in U : [C_j(z) \cap M] \notin I\}$.

iii. $\mu^{I}_{C_j}(M) = \frac{|{}^{I}F^{-}_{C_j} \cap M|}{|{}^{I}F^{+}_{C_j} \cup M|}$

Where $|{}^{I}F^{+}_{C_j} \cup M| \neq 0$

First, we compare the proposed approximation spaces in terms of lower approximations ${}^{I}F^{-}_{C_j}$, upper approximations ${}^{I}F^{+}_{C_j}$ and accuracy measures $\mu^{I}_{C_j}$.

Theorem 4.1. *Let (U, R, ψ_j) be N_jS, I be an ideal on U and $M \subseteq U$, then the following properties hold:*

i. ${}^{I}F^{-}_{C_u}(M) \subseteq {}^{I}F^{-}_{C_r}(M) \subseteq {}^{I}F^{-}_{C_i}$ *and* ${}^{I}F^{-}_{C_u}(M) \subseteq {}^{I}F^{-}_{C_l}(M) \subseteq {}^{I}F^{-}_{C_i}$.

ii. ${}^{I}F^{+}_{C_i}(M) \subseteq {}^{I}F^{+}_{C_r}(M) \subseteq {}^{I}F^{+}_{C_u}$ *and* ${}^{I}F^{+}_{C_i}(M) \subseteq {}^{I}F^{+}_{C_l}(M) \subseteq {}^{I}F^{+}_{C_u}$.

iii. ${}^{I}\rho_{C_u}(M) \subseteq {}^{I}\rho_{C_r}(M) \subseteq {}^{I}\rho_{C_i}$ *and* ${}^{I}\rho_{C_u}(M) \subseteq {}^{I}\rho_{C_l}(M) \subseteq {}^{I}\rho_{C_i}$.

iv. ${}^{I}F^{-}_{C_{<u>}}(M) \subseteq {}^{I}F^{-}_{C_{<r>}}(M) \subseteq {}^{I}F^{-}_{C_{<i>}}$ *and* ${}^{I}F^{-}_{C_{<u>}}(M) \subseteq {}^{I}F^{-}_{C_{<l>}}(M) \subseteq {}^{I}F^{-}_{C_{<i>}}$.

v. ${}^{I}F^{+}_{C_{<i>}}(M) \subseteq {}^{I}F^{+}_{C_{<r>}}(M) \subseteq {}^{I}F^{+}_{C_{<u>}}$ *and* ${}^{I}F^{+}_{C_{<i>}}(M) \subseteq {}^{I}F^{+}_{C_{<l>}}(M) \subseteq {}^{I}F^{+}_{C_{<u>}}$.

vi. $\mu^{I}_{C_{<u>}}(M) \subseteq \mu^{I}_{C_{<r>}}(M) \subseteq \mu^{I}_{C_{<i>}}$ *and* $\mu^{I}_{C_{<u>}}(M) \subseteq \mu^{I}_{C_{<l>}}(M) \subseteq \mu^{I}_{C_{<i>}}$.

Proof. Straightforward. □

The basic properties of lower approximations ${}^{I}F^{-}_{C_j}$ and upper approximations ${}^{I}F^{+}_{C_j}$ are listed in the next result.

Theorem 4.2. *Let (U, R, ψ_j) be N_jS, I be an ideal on U and M, M_1 and M_2 be subsets of U. Then for each j, the following properties hold:*

i. ${}^{I}F^{-}_{C_j}(U) = U$;

ii. If $M_1 \subseteq M_2$, *then* ${}^{I}F^{-}_{C_j}(M_1) \subseteq {}^{I}F^{-}_{C_j}(M_2)$;

iii. ${}^{I}F^{-}_{C_j}(M_1 \cap M_2) = {}^{I}F^{-}_{C_j}(M_1) \cap {}^{I}F^{-}_{C_j}(M_2)$;

iv. ${}^{I}F^{-}_{C_j}(M^c) = [{}^{I}F^{+}_{C_j}(M)]^c$;

v. If $M^c \in I$, *then* ${}^{I}F^{-}_{C_j}(M) = U$;

vi. ${}^{I}F^{+}_{C_j}(\phi) = \phi$;

vii. If $M_1 \subseteq M_2$*, then* ${}^I F^+_{C_j}(M_1) \subseteq {}^I F^+_{C_j}(M_2)$*;*

viii. ${}^I F^+_{C_j}(M_1 \cup M_2) = {}^I F^+_{C_j}(M_1) \cup {}^I F^+_{C_j}(M_2)$*;*

ix. ${}^I F^+_{C_j}(M^c) = [{}^I F^-_{C_j}(M)]^c$*;*

x. If $M \in I$*, then* ${}^I F^+_{C_j}(M) = \phi$*;*

Proof. i. Since $[C_j(z) - U] = \phi \ \forall z \in U$, hence ${}^I F^-_{C_j}(U) = U$.

ii. Let $z \in {}^I F^-_{C_j}(M_1)$, then $[C_j(z) - M_1] \in I$. Since $M_1 \subseteq M_2$, then $[C_j(z) - M_2] \in I$. Thus $z \in {}^I F^-_{C_j}(M_2)$. Hence ${}^I F^-_{C_j}(M_1) \subseteq^I F^-_{C_j}(M_2)$.

iii. Since $(M_1 \cap M_2) \subseteq M_1$ and $(M_1 \cap M_2) \subseteq M_2$, then ${}^I F^-_{C_j}(M_1 \cap M_2) \subseteq {}^I F^-_{C_j}(M_1)$ and ${}^I F^-_{C_j}(M_1 \cap M_2) \subseteq {}^I F^-_{C_j}(M_2)$. Hence ${}^I F^-_{C_j}(M_1 \cap M_2) \subseteq {}^I F^-_{C_j}(M_1) \cap {}^I F^-_{C_j}(M_2)$.

Conversely, let $z \in {}^I F^-_{C_j}(M_1) \cap {}^I F^-_{C_j}(M_2)$, then $z \in {}^I F^-_{C_j}(M_1)$ and $z \in {}^I F^-_{C_j}(M_2)$. Thus $[C_j(z) - M_1] \in I$ and $[C_j(z) - M_2] \in I$. Therefore $[C_j(z) - (M_1 \cap M_2)] \in I$. So $z \in {}^I F^-_{C_j}(M_1 \cap M_2)$. Hence ${}^I F^-_{C_j}(M_1) \cap {}^I F^-_{C_j}(M_2) \subseteq {}^I F^-_{C_j}(M_1 \cap M_2)$.

iv. $z \in {}^I F^-_{C_j}(M^c) \Longleftrightarrow [C_j(z) - M^c] \in I \Longleftrightarrow [C_j(z) \cap M] \in I \Longleftrightarrow z \notin {}^I F^+_{C_j}(M) \Longleftrightarrow z \in [{}^I F^+_{C_j}(M)]^c$. Hence ${}^I F^-_{C_j}(M^c) = [{}^I F^+_{C_j}(M)]^c$.

v. Let $M^c \in I$, then $[C_j(z) \cap M^c] \in I \ \forall z \in U$. Thus $[C_j(z) - M] \in I \ \forall z \in U$. Hence ${}^I F^-(M) = U$.

vi. Let $M_1 \subseteq M_2$ and $z \in {}^I F^+_{C_j}(M_1)$, then $[C_j(z) \cap M_1] \notin I$. Since $M_1 \subseteq M_2$, then $[C_j(z) \cap M_2] \notin I$. Thus $z \in {}^I F^+_{C_j}(M_2)$. Hence ${}^I F^+_{C_j}(M_1) \subseteq {}^I F^+_{C_j}(M_2)$.

vii. Since $M_1 \subseteq (M_1 \cup M_2)$ and $M_2 \subseteq (M_1 \cup M_2)$, then ${}^I F^+_{C_j}(M_1) \subseteq {}^I F^+_{C_j}(M_1 \cup M_2)$ and ${}^I F^+_{C_j}(M_2) \subseteq {}^I F^+_{C_j}(M_1 \cup M_2)$. Hence ${}^I F^+_{C_j}(M_1) \cup {}^I F^+_{C_j}(M_2) \subseteq {}^I F^+_{C_j}(M_1 \cup M_2)$. Conversely, Let $z \in {}^I F^+_{C_j}(M_1 \cup M_2)$, then $[C_j(z) \cap (M_1 \cup M_2)] \notin I$. Thus $[C_j(z) \cap M_1] \notin I$ or $[C_j(z) \cap M_2] \notin I$. Therefore $z \in {}^I F^+_{C_j}(M_1)$ or $z \in {}^I F^+_{C_j}(M_2)$. So $z \in [{}^I F^+_{C_j}(M_1) \cup {}^I F^+_{C_j}(M_2)]$. Hence ${}^I F^+_{C_j}(M_1 \cup M_2) \subseteq {}^I F^+_{C_j}(M_1) \cup {}^I F^+_{C_j}(M_2)$.

viii. $z \in {}^I F^+_{C_j}(M^c) \Longleftrightarrow [C_j(z) \cap M^c] \notin I \Longleftrightarrow [C_j(z) - M] \notin I \Longleftrightarrow z \notin {}^I F^-_{C_j}(M) \Longleftrightarrow z \in [{}^I F^-_{C_j}(M)]^c$.

ix. Let $M \in I$, then $[C_j(z) \cap M] \in I \ \forall z \in U$. Thus $z \notin {}^I F^+_{C_j}(M) \ \forall z \in U$. Hence ${}^I F^+_{C_j}(M) = \phi$. □

We demonstrate, in the next four results, the relationships between the approximation spaces that we construct herein and their counterparts induced from containment neighborhoods [7] and topologies generated by containment neighborhoods and ideals [14].

Theorem 4.3. *Let* M *be a subset of* U *and* I *be an Ideal on* U*, then* $F_{C_j}(M) \subseteq {}^I F^-_{C_j}(M)$ *and* ${}^I F^+_{C_j}(M) \subseteq F^{C_j}(M)$

Proof. First, let $z \in F_{C_j}(M)$, then $C_j(z) \subseteq M$. Thus $[C_j(z) - M] = \phi \in I$. So $z \in {}^I F^-_{C_j}(M)$. Hence $F_{C_j}(M) \subseteq {}^I F^-_{C_j}(M)$. Second, let $z \in {}^I F^+_{C_j}(M)$, then $[C_j(M) \cap M] \notin I$. So $[C_j(M) \cap M] \neq \phi$. Therefore $z \in F^{Cj}(M)$. Hence ${}^I F^+_{C_j}(M) \subseteq F^{C_j}(M)$. □

Corollary 4.1. *Let* M *be a subset of* U *and* I *be an ideal on* U*. Then* $\mu_{C_j}(M) \leq \mu^I_{C_j}(M)$*.*

Theorem 4.4. *Let* M *be a subset of* U *and* I *be an ideal on* U*, then* $\underline{C}^I_j(M) \subseteq {}^I F^-_{C_j}(M)$ *and* $\overline{C}^I_j(M) \subseteq {}^I F^+_{C_j}(M)$

Proof. First, let $z \in \underline{C}^I_{C_j}(M)$, then $z \in int^I_{C_j}(M)$. Therefore there exists an open set $G \subseteq M$ such that $z \in G$. Thus $[C^I_j(z) - G] \in I$. Since $G \subseteq M$, then $[C_j(z) - M] \subseteq [C_j(z) - G]$. So by hereditary property of the ideal $[C_j(z) - M] \in I$. Therefore $z \in {}^I F^-_{C_j}(M)$. So $\underline{C}^I_j(M) \subseteq {}^I F^-_{C_j}(M)$. Second, let $z \notin \overline{C}^I_j(M)$, then $z \notin cl^I_j(M)$. Therefore there exists a closed set $F \supseteq M$ such that $z \notin F$. Thus $z \in F^C$ which is open set. So $[C_j(z) - F^c] \in I$. Thus $[C_j(z) \cap F] \in I$. Since $M \subseteq F$, so $[C_j(z) \cap M] \subseteq [C_j(z) \cap F]$. Then by hereditary property of the ideal $[C_j(z) \cap M] \in I$. Therefore $z \notin {}^I F^+_{C_j}(M)$ and so ${}^I F^+_{C_j} \subseteq \overline{C}^I_j$. □

Corollary 4.2. *Let M be a subset of U and I be an ideal on U. Then, $\rho^I_{C_j}(M) \leq \mu^I_{C_j}(M)$.*

In the next example, we elaborate that the converse of the results given in this section fails.

Example 4.1. Continued from Example 3.1. Let $I = \{\phi, \{m_2\}\}$ be an ideal on U. The 8 topologies generated by using containment neighborhoods and ideals are:
$T^I_{C_r} = \{U, \phi, \{m_1, m_3\}, \{m_2, m_3\}, \{m_3\}\}$
$T^I_{C_l} = \{U, \phi, \{m_1\}, \{m_1, m_2\}, \{m_1, m_3\}, \{m_2\}\}$
$T^I_{C_i} = \{U, \phi, \{m_1\}, \{m_2\}, \{m_1, m_3\}, \{m_2, m_3\}, \{m_3\}, \{m_1, m_2\}\}$
$T^I_{C_u} = \{U, \phi, \{m_1, m_3\}\}$
$T^I_{C_{\langle r\rangle}} = \{U, \phi, \{m_1, m_3\}, \{m_2\}, \{m_2, m_3\}, \{m_3\}\}$
$T^I_{C_{\langle l\rangle}} = \{U, \phi, \{m_1\}, \{m_1, m_3\}\}$
$T^I_{C_{\langle i\rangle}} = \{U, \phi, \{m_1\}, \{m_2\}, \{m_3\}, \{m_1, m_2\}, \{m_1, m_3\}, \{m_2, m_3\}\}$
$T^I_{C_{\langle u\rangle}} = \{U, \phi, \{m_1, m_3\}\}$
$\underline{C}^I_j$, $\overline{C}^I_j$, $\rho^I_{C_j}$, ${}^I F^-_{C_j}$, ${}^I F^+_{C_j}$ and ${}^I \rho_{C_j}$ are given in Tables (9, 10, 11, 12, 13, 14) respectively.

Table 9: $\underline{C}^I_j$

Set (M)	$\underline{C}^I_r(M)$	$\underline{C}^I_l(M)$	$\underline{C}^I_i(M)$	$\underline{C}^I_u(M)$	$\underline{C}^I_{\langle r\rangle}(M)$	$\underline{C}^I_{\langle l\rangle}(M)$	$\underline{C}^I_{\langle i\rangle}(M)$	$\underline{C}^I_{\langle u\rangle}(M)$
$\{m_1\}$	ϕ	$\{m_1\}$	$\{m_1\}$	ϕ	ϕ	$\{m_1\}$	$\{m_1\}$	ϕ
$\{m_2\}$	ϕ	$\{m_2\}$	$\{m_2\}$	ϕ	$\{m_2\}$	ϕ	$\{m_2\}$	ϕ
$\{m_3\}$	$\{m_3\}$	ϕ	$\{m_3\}$	ϕ	$\{m_3\}$	ϕ	$\{m_3\}$	ϕ
$\{m_1, m_2\}$	ϕ	$\{m_1, m_2\}$	$\{m_1, m_2\}$	ϕ	$\{m_2\}$	$\{m_1\}$	$\{m_1, m_2\}$	ϕ
$\{m_1, m_3\}$	$\{m_1, m_3\}$	$\{m_1, m_3\}$	$\{m_1, m_3\}$	$\{m_1, m_3\}$	$\{m_1, m_3\}$	$\{m_1, m_3\}$	$\{m_1, m_3\}$	$\{m_1, m_3\}$
$\{m_2, m_3\}$	$\{m_2, m_3\}$	$\{m_2\}$	$\{m_2, m_3\}$	ϕ	$\{m_2, m_3\}$	ϕ	$\{m_2, m_3\}$	ϕ

Table 10: $\overline{C}^I_j$

Set (M)	$\overline{C}^I_r(M)$	$\overline{C}^I_l(M)$	$\overline{C}^I_i(M)$	$\overline{C}^I_u(M)$	$\overline{C}^I_{\langle r\rangle}(M)$	$\overline{C}^I_{\langle l\rangle}(M)$	$\overline{C}^I_{\langle i\rangle}(M)$	$\overline{C}^I_{\langle u\rangle}(M)$
$\{m_1\}$	$\{m_1\}$	$\{m_1, m_3\}$	$\{m_1\}$	U	$\{m_1\}$	U	$\{m_1\}$	U
$\{m_2\}$	$\{m_2\}$	$\{m_2\}$	$\{m_2\}$	$\{m_2\}$	$\{m_2\}$	$\{m_2\}$	$\{m_2\}$	$\{m_2\}$
$\{m_3\}$	U	$\{m_3\}$	$\{m_3\}$	U	$\{m_1, m_3\}$	$\{m_2, m_3\}$	$\{m_3\}$	U
$\{m_1, m_2\}$	$\{m_1, m_2\}$	U	$\{m_1, m_2\}$	U	$\{m_1, m_2\}$	U	$\{m_1, m_2\}$	U
$\{m_1, m_3\}$	U	$\{m_1, m_3\}$	$\{m_1, m_3\}$	U	$\{m_1, m_3\}$	U	$\{m_1, m_3\}$	U
$\{m_2, m_3\}$	U	$\{m_2, m_3\}$	$\{m_2, m_3\}$	U	U	$\{m_2, m_3\}$	$\{m_2, m_3\}$	U

Table 11: $\rho^I_{C_j}$

Set (M)	$\rho^I_{C_r}(M)$	$\rho^I_{C_l}(M)$	$\rho^I_{C_i}(M)$	$\rho^I_{C_u}(M)$	$\rho^I_{C_{\langle r\rangle}}(M)$	$\rho^I_{C_{\langle l\rangle}}(M)$	$\rho^I_{C_{\langle i\rangle}}(M)$	$\rho^I_{C_{\langle u\rangle}}(M)$
$\{m_1\}$	0	$\frac{1}{2}$	1	0	0	$\frac{1}{3}$	1	0
$\{m_2\}$	0	1	1	0	1	0	1	0
$\{m_3\}$	$\frac{1}{3}$	0	1	0	$\frac{1}{2}$	0	1	0
$\{m_1, m_2\}$	0	$\frac{2}{3}$	1	0	$\frac{1}{2}$	$\frac{1}{3}$	1	0
$\{m_1, m_3\}$	$\frac{2}{3}$	1	1	$\frac{2}{3}$	1	$\frac{2}{3}$	1	$\frac{2}{3}$
$\{m_2, m_3\}$	$\frac{2}{3}$	$\frac{1}{2}$	1	0	$\frac{2}{3}$	0	1	0

Table 12: ${}^IF^-_{C_j}$

Set (M)	${}^IF^-_{C_r}(M)$	${}^IF^-_{C_l}(M)$	${}^IF^-_{C_i}(M)$	${}^IF^-_{C_u}(M)$	${}^IF^-_{C_{\langle r\rangle}}(M)$	${}^IF^-_{C_{\langle l\rangle}}(M)$	${}^IF^-_{C_{\langle i\rangle}}(M)$	${}^IF^-_{C_{\langle u\rangle}}(M)$
$\{m_1\}$	ϕ	$\{m_1, m_2\}$	$\{m_1, m_2\}$	ϕ	ϕ	$\{m_1\}$	$\{m_1, m_2\}$	ϕ
$\{m_2\}$	ϕ	$\{m_2\}$	$\{m_2\}$	ϕ	$\{m_2\}$	ϕ	$\{m_2\}$	ϕ
$\{m_3\}$	$\{m_2, m_3\}$	$\{m_2\}$	$\{m_2, m_3\}$	$\{m_2\}$	$\{m_2, m_3\}$	ϕ	$\{m_3\}$	ϕ
$\{m_1, m_2\}$	ϕ	$\{m_1, m_2\}$	$\{m_1, m_2\}$	ϕ	$\{m_2\}$	$\{m_1\}$	$\{m_1, m_2\}$	ϕ
$\{m_1, m_3\}$	U	U	U	U	U	U	U	U
$\{m_2, m_3\}$	$\{m_2, m_3\}$	$\{m_2\}$	$\{m_2, m_3\}$	$\{m_2\}$	$\{m_2, m_3\}$	ϕ	$\{m_2, m_3\}$	ϕ

Table 13: ${}^IF^+_{C_j}$

Set (M)	${}^IF^+_{C_r}(M)$	${}^IF^+_{C_l}(M)$	${}^IF^+_{C_i}(M)$	${}^IF^+_{C_u}(M)$	${}^IF^+_{C_{\langle r\rangle}}(M)$	${}^IF^+_{C_{\langle l\rangle}}(M)$	${}^IF^+_{C_{\langle i\rangle}}(M)$	${}^IF^+_{C_{\langle u\rangle}}(M)$
$\{m_1\}$	$\{m_1\}$	$\{m_1, m_3\}$	$\{m_1\}$	$\{m_1, m_3\}$	$\{m_1\}$	U	$\{m_1\}$	U
$\{m_2\}$	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
$\{m_3\}$	U	$\{m_3\}$	$\{m_3\}$	U	$\{m_1, m_3\}$	$\{m_2, m_3\}$	$\{m_3\}$	U
$\{m_1, m_2\}$	$\{m_1\}$	$\{m_1, m_3\}$	$\{m_1\}$	$\{m_1, m_3\}$	$\{m_1\}$	U	$\{m_1\}$	U
$\{m_1, m_3\}$	U	$\{m_1, m_3\}$	$\{m_1, m_3\}$	U	$\{m_1, m_3\}$	U	$\{m_1, m_3\}$	U
$\{m_2, m_3\}$	U	$\{m_3\}$	$\{m_3\}$	U	$\{m_1, m_3\}$	$\{m_2, m_3\}$	$\{m_3\}$	U

Table 14: $\mu^I_{C_j}$

Set(M)	$\mu^I_{C_r}(M)$	$\mu^I_{C_l}(M)$	$\mu^I_{C_i}(M)$	$\mu^I_{C_u}(M)$	$\mu^I_{C_{\langle r\rangle}}(M)$	$\mu^I_{C_{\langle l\rangle}}(M)$	$\mu^I_{C_{\langle i\rangle}}(M)$	$\mu^I_{C_{\langle u\rangle}}(M)$
$\{m_1\}$	0	$\frac{1}{2}$	1	0	0	$\frac{1}{3}$	1	0
$\{m_2\}$	0	1	1	0	1	0	1	0
$\{m_3\}$	$\frac{1}{3}$	0	1	0	$\frac{1}{2}$	0	1	0
$\{m_1, m_2\}$	0	$\frac{2}{3}$	1	0	$\frac{1}{2}$	$\frac{1}{3}$	1	0
$\{m_1, m_3\}$	$\frac{2}{3}$	1	1	$\frac{2}{3}$	1	$\frac{2}{3}$	1	$\frac{2}{3}$
$\{m_2, m_3\}$	$\frac{2}{3}$	$\frac{1}{2}$	1	$\frac{1}{3}$	$\frac{2}{3}$	0	1	0

Note that the converse of the following relations doesn't hold in general

I. If $M_1 \subseteq M_2$, then ${}^IF^-_{C_j}(M_1) \subseteq {}^IF^-_{C_j}(M_2)$

i. ${}^IF^-_{C_l}(\{m_1\}) = \phi$.

ii. ${}^IF^-_{C_l}(\{m_3\}) = \{m_2, m_3\}$.

II. If $M_1 \subseteq M_2$, then ${}^I F^+_{C_j}(M_1) \subseteq {}^I F^+_{C_j}(M_2)$.

i. ${}^I F^+_{C_i}(\{m_2\}) = \phi$.

ii. ${}^I F^+_{C_i}(\{m_3\}) = \{m_3\}$.

III. $F_{C_j}(M) \subseteq {}^I F^-_{C_j}(M)$

i. $F_{C_r}(\{m_3\}) = \phi$

ii. ${}^I F^-_{C_j}(\{m_3\}) = \{m_2, m_3\}$

IV. ${}^I F^+_{C_j}(M) \subseteq F^{C_j}(M)$

i. ${}^I F^+_{C_r}(\{m_2\}) = \phi$

ii. $F^{C_r}(\{m_2\}) = U$

V. $\mu_{C_j}(M) \leq \mu^I_{C_j}(M)$

i. $\mu_{C_r}(\{m_3\}) = 0$

ii. $\mu^I_{C_r}(\{m_3\}) = \frac{1}{3}$

VI. $\underline{C}^I_j(M) \subseteq {}^I F^-_{C_j}(M)$

i. $\underline{C}^I_l(\{m_1\}) = \{m_1\}$

ii. ${}^I F^-_{C_l}(\{m_1\}) = \{m_1, m_2\}$

VII. ${}^I F^+_{C_j}(M) \subseteq \overline{C}^I_j(M)$

i. ${}^I F^+_{C_l}(\{m_2\}) = \phi$

ii. $\overline{C}^I_l(\{m_2\}) = \{m_2\}$

VIII. $\rho^I_{C_j}(M) \leq \mu^I_{C_j}(M)$

i. $\rho^I_{C_u}(\{m_2, m_3\}) = 0$

ii. $\mu^I_{C_u}(\{m_2, m_3\}) = \frac{1}{3}$

5. Medical example

One of the global diseases that disturb humanity is COVID-19. It affects the systems of health, economy, politics, and society. According to the data from World Health Organization, the major way of its spread is through physical contact or nearness between individuals, which means the different types of neighborhood systems and their rough set models are useful instruments to represent the information system of individuals under suspicion with respect to infection of COVID-19.

In this section, we analyze the pandemic of COVID-19 via the approximation spaces generated by containment neighborhoods and ideals. Then, we demonstrate the good performance of the current approach in terms of accuracy values and approximation operators compared to some approaches introduced in the literature [18, 21].

To do this, we deal with the data of six individuals suspected with infection of COVID-19 $U = \{m_1, m_2, m_3, m_4, m_5, m_6\}$ as displayed in Table 15. The data of these individuals are given according to the most common symptoms of COVID-19: fever F, cough C, tiredness T, and loss of taste or smell L.

The attributes have two values: ✓ refers to the patient has symptoms and × refers to the patient has no symptoms. The made decision also has the same two values which refer to possessing COVID-19 disease or not.

Table 15: Covid 19 information system

P	F	C	T	L	Covid 19
x_1	✓	✓	×	×	×
x_2	✓	×	×	✓	×
x_3	×	✓	✓	✓	✓
x_4	✓	✓	✓	×	✓
x_5	✓	✓	×	✓	✓
x_6	✓	×	×	×	×

To begin analysis, we define a map h, of a set of parameters into the power set of U, which associates each patient with his/her symptoms as follows.

$h(m_1)$	$h(m_2)$	$h(m_3)$	$h(m_4)$	$h(m_5)$	$h(m_6)$
$\{F, C\}$	$\{F, L\}$	$\{C, T, L\}$	$\{F, C, T\}$	$\{F, C, L\}$	$\{T, L\}$

Let us consider the experts of the system proposing that the two individuals are related if they have two similar symptoms at least. This assumption can be described by the following binary relation.

$$m_i R m_j \Longleftrightarrow \mid h(m_i) \cap h(m_j) \mid \geq 2. \tag{1}$$

We draw attention to that this relation is changed according to the viewpoint of system experts. According to relation in 1, we obtain $(m_1, m_5), (m_5, m_2) \in R$ whereas $(m_1, m_2) \notin R$. So, R is not transitive. For this reason, Pawlak's model fails to deal with this information system. In contrast, a relation R is symmetry, which means that the $N_i(m) = N_r(m) = N_l(m) = N_u(m)$ for each $m \in U$. Moreover, R is reflexive.

Now, we list the relation in 1 to construct the ideal approximation space: $R = \{(m_1, m_1), (m_2, m_2), (m_3, m_3), (m_4, m_4), (m_5, m_5), (m_6, m_6), (m_1, m_4), (m_4, m_1), (m_1, m_5), (m_5, m_1), (m_2, m_5), (m_5, m_2), (m_3, m_4), (m_4, m_3), (m_3, m_5), (m_5, m_3), (m_3, m_6), (m_6, m_3), (m_4, m_5), (m_5, m_4)\}$.

Then we compute the neighborhood systems N_j in cases of $j \in \{r, l, i, u\}$ for each member in U.
$N_j(m_1) = N_j(m_4) = N_j(m_5) = \{m_1, m_4, m_5\}$
$N_j(m_2) = \{m_2, m_5\}$
$N_j(m_3) = \{m_3, m_4, m_5, m_6\}$
$N_j(m_6) = \{m_3, m_6\}$

After that, we compute the E_j-neighborhood and C_j-neighborhood for each member in U.
$E_j(m_1) = E_j(m_2) = E_j(m_4) = E_j(m_5) = U \setminus \{m_6\}$
$E_j(m_3) = U$
$E_j(m_6) = \{m_3, m_6\}$
$C_j(m_1) = C_j(m_4) = C_j(m_5) = \{m_1, m_4, m_5\}$
$C_j(m_2) = \{m_2\}$
$C_j(m_3) = \{m_3, m_6\}$
$C_j(m_6) = \{m_6\}$

Without loss of generality, we consider the ideal is $I = \{\phi\}$.

For a set of patients without infection with COVID-19 $M = \{m_1, m_2, m_6\}$, we calculate their lower and upper approximations, boundary regions and the accuracy measures induced from approaches displayed in [18, 21] and our approach given in the previous section.

Hosny et al.'s approach [18] and Kandil et al.'s approach [21]: The lower and upper approximations are ${}^{I}F^{-}_{E_j}(M) ={}^{I} F^{-}_{N_j}(M) = \phi$ and ${}^{I}F^{+}_{E_j}(M) ={}^{I} F^{+}_{N_j}(M) = U$, respectively. Therefore, the boundary region is ${}^{I}B_{E_j}(M) ={}^{I} B_{N_j}(M) = U$ and the accuracy measure is $\mu^{I}_{E_j}(M) = \mu^{I}_{N_j}(M) = 0$.

Our approach given in the previous section: The lower and upper approximations are ${}^{I}F^{-}_{C_j}(M) = \{m_2, m_6\}$ and ${}^{I}F^{+}_{C_j}(M) = U$, respectively. Therefore, the boundary region is ${}^{I}B_{C_j}(M) = \{m_1, m_3, m_4, m_5\}$ and the accuracy measure is $\mu^{I}_{C_j}(M) = \frac{1}{3}$.

From the above computations, it is obtained that the boundary region of a subset of patients without infection with COVID-19 inspired by the approach given in [18, 21] is the universal set U. In this case, we are unable to decide whether these individuals are infected with COVID-19 or not, which enlarges the area of uncertainty/vagueness and affects the precision of made decision. In contrast, the boundary region inspired by our approach is the subset $\{m_1, m_3, m_4, m_5\}$, which means we minimize the uncertainty in the data and raise up enhance the accuracy measure.

6. Conclusion and future work

Rough set was introduced to deal with intelligent systems characterized by insufficient and incomplete information. This theory has been developed and extended by several ways; one of them is the abstract concepts "neighbourhoods and ideals".

In this manuscript, first, we have studied the main properties of topological approximation spaces defined using the system of containment neighborhoods. The approximation operators of these spaces have been defined by interior and closure operators of a topology. Then, we have produced new types of approximation spaces directly inspired by containment neighborhoods and ideals. In general, we have investigated the main properties of these models and revealed the relationships between them. To point out the importance of the proposed approaches, we have showed their major advantages to increase the accuracy measure of a subset by increasing lower approximation and decreasing upper approximation.

On the one hand, we have elucidated that the accuracy measures obtained from the topological approaches given herein are greater than those in [1, 4, 11]. Moreover, we have proved the effectiveness of the methods proposed in Section 4 to improve the approximation operators and high accuracy values compared to the previous ones presented in [7, 14]. On the other hand, the efficiency of topological approximation operators investigated in Section 3 to remove uncertainty of data are less than those presented in [7] as explained in Theorem 3.3.

Finally, we have provided a medical example to show the efficiency of the proposed technique compared to some previous ones [18, 21] in terms of minimizing the boundary region and increasing the accuracy measures. The obtained computations have illustrated that our approach is one of the preferable techniques to eliminate the ambiguity of the data and obtain an accurate decision in practical problems.

In future work, we will study the containment neighborhoods in the content of soft rough graph, soft rough set and fuzzy rough set [33–35]. Also, we will generate new topologies from other types neighborhoods and ideals.

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