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Fixed point results via hesitant fuzzy mapping on extended b-metric spaces

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Abstract. In the present research article, the notion of hesitant fuzzy mapping, contraction hesitant fuzzy mapping, generalized contraction of hesitant fuzzy mapping are introduced in setting of extended b-metric structure, which in particular case reduced to b-metric space. The newly introduced concepts are used to prove hesitant fuzzy fixed point theorems. The main results have extended and unified recent results in literature. The results are validated with the help of examples. Finally, as an application, the results are applied to solve integral equation of fredholm type.

1. Introduction and preliminaries

It was M.Frechet [1] who in his doctoral thesis introduced the concept of metric space which has an extension over usual notion of distance. In 1922, Banach [2] represented most important and powerful result that is popularly known as Banach Contraction Principle (BCP) in the theory of metric fixed point. Kannan [3] proved that there exist a mapping which is not continuous in the domain but it have fixed points.

Chatterjea [4] and Zamfirescu [5] proved fixed point theorems under the contractive mapping on metric space. In 1969, Nadler [6] introduced the notion of multivalued mappings and extended BCP. Further, In the year 1981, Heilpern [7] generalized Nadler's results. In 1987, R.K.Bose [8] extended the Heilpern's, fixed point results for non-expansive mapping.

Recently, In 2021, Aliouche and Hamaizia [17] proved fixed point theorems for two multivalued mappings in the setting of complete b-metric spaces and extended the main theorem of Khojasteh [21], and theorem of Demma [22] and Rhoades [23]. In 2022, Savanović [18] introduced the new type of multivalued quasi-contractive mapping with nonlinear comparisons functions in b-metric spaces [24]-[31]. In 2022, Kuber Singh [19] introduced the generalized "α_*-contraction" in b-metric space using multivalued mappings. In 2022, Tassaddiq [20] proved the fixed point theorems for single and multivalued mapping in strong b-metric spaces.

In 1965, L.A.Zadeh [9] firstly introduced the concept of fuzzy set and gave mathematical model to undefined collection. Later this set is extended in form of Hesitant fuzzy set by V.Torra [10].

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It was Wilson [33], who in 1931 generalized the notion of metric space by dropping out the property of symmetry and named it "quasimetric". In literature, different authors have different views about the origin of quasimetric, for more details and the survey the reader is suggested to see [36]. Further in the year 1993, Czerwik [11] (see also [34], [35]) coined the term of quasimetric space as b-metric space. After that quasimetric (b-metric) space was further extended by T.Kamran [12] in the form of extended b-metric space. It was K.E.Osawaru [13] who designed hesitant fuzzy mappings and utilized this new concept to prove fixed point theorems.

In this paper, we first time studied fixed point theorem for hesitant fuzzy mappings in the structure of extended b-metric space. These new type of results are applied is solving some Fredholm type integral equations.

Definition 1.1. *[11] Assume E a non-empty set and $1 \leq s \in \mathbb{R}$. A distance function $d : E \times E \to [0, \infty)$ satisfying the following conditions is called b-metric if $\forall\ \zeta, \eta, \gamma \in E$:*

(i) $d(\zeta, \eta) = 0 \ \Leftrightarrow\ \zeta = \eta$;
(ii) $d(\zeta, \eta) \ = \ d(\eta, \zeta)$;
(iii) $d(\zeta, \gamma) \ \leq \ s[d(\zeta, \eta) + d(\eta, \gamma)]$,

then the pair (E, d) is known as b-metric space.

Definition 1.2. *[12] Assume $E(\neq \phi)$ be a set and $\phi : E \times E \to [1, \infty)$. A distance function $d_\phi : E \times E \to [0, \infty)$ satisfying the following conditions is called extended b-metric, if $\forall\ \zeta, \eta, \gamma \in E$:*

(i) $d_\phi(\zeta, \eta) = 0$ *if and only if* $\zeta = \eta$;
(ii) $d_\phi(\zeta, \eta) = d_\phi(\eta, \zeta)$;
(iii) $d_\phi(\zeta, \gamma) \leq \phi(\zeta, \gamma)[d_\phi(\zeta, \eta) + d_\phi(\eta, \gamma)]$,

then the pair (E, d_ϕ) is known as extended b-metric space. If $\phi(\zeta, \gamma) = s$ and $s \geq 1$, then it is called b-metric space and if $\phi(\zeta, \gamma) = s = 1$ then it is called metric space.

V.Torra [10] generalized Fuzzy sets due to L.A.Zadeh [9] in 1965, by introducing the new notion of the hesitant fuzzy sets and hesitant fuzzy logic.

Definition 1.3. *[10] Suppose E be a non-empty set and S be a family of all subsets of the interval [0,1]. A hesitant fuzzy set on E is a fuzzy set on E is characterized by the map $h : E \to S$ such that $h(\zeta) \in S$. A hesitant fuzzy map reduces to fuzzy map when h is single-valued for all ζ in E. Further, we also denote $H(E)$ by a collection of hesitant fuzzy set on E.*

Xia and Xu [14] defined a comparison on hesitant fuzzy membership by comparing their scores. They defined a score of a hesitant membership value $A_1 \in S$ as

$$s(A_1) = \frac{1}{n(A_1)} \sum_{a \in A_1} a$$

where $n(A_1)$ denotes the cardinality of A_1 and $s(A_1) \in [0, 1]$.

Definition 1.4. *[32] Let us assume that h be a hesitant fuzzy set on E. Then the α-cut of a hesitant fuzzy set A is defined as*

$$h^A_\alpha = \{\zeta \in E : s(h^A(\zeta)) \geq \alpha\} \text{ for any } \alpha \in (0, 1]$$

and

$$h^A_{\{0\}} = C(\{\zeta \in E : s(h^A(\zeta)) > \{0\}\}) = C(B)$$

with $\alpha = \{0\} \in S$ is said to be an α-cut (level set) of a hesitant fuzzy set, where $C(B)$ means the closure of B.

A relation on the set hesitant fuzzy membership values is defined if $s(A_1) > s(A_2)$ then $A_1 > A_2$ and A_1 is similar to A_2 if $s(A_1) = s(A_2)$ for all $A_1, A_2 \in S$.

Liao and Xu [15] said that the relation is not true for some special cases. To resolve this issue, Chen, Xu and Xia [16] defined the deviation degree. The deviation degree of a hesitant fuzzy membership value $A_1 \in S$ is given as

$$d(A_1) = \sqrt{\frac{1}{n(A_1)} \sum_{a \in A_1} (a - s(A_1))^2}$$

and defined a comparisons on sets of a hesitant fuzzy membership value as

(i) $A_1 < A_2$ if $s(A_1) < s(A_2)$ or if $s(A_1) = s(A_2)$ and $d(A_1) > d(A_2)$,
(ii) $A_1 = A_2$ if $s(A_1) = s(A_2)$ and $d(A_1) = d(A_2)$,
(iii) $A_1 > A_2$ if $s(A_1) = s(A_2)$ and $d(A_1) < d(A_2)$.

We illustrate this with the following example.

Example 1.5. *Let the pair* $(\mathbb{Z}, d)$ *represent a b-metric space and the distance function* d *is defined as* $d(\zeta, \eta) = |\zeta - \eta|^2 \; \forall \; \zeta, \eta \in \mathbb{Z}$. *Suppose* $h : E = \{1 \leq \zeta \leq 6\} \to S$ *is a hesitant fuzzy map, where*

$$h(\zeta) = \{\frac{1}{s} \in [0,1], s \text{ is a multiple of } \zeta, s \leq 10\}.$$

Then, we prove the comparison on sets of hesitant fuzzy membership value using deviation degree.
Solution: *First we find the value of hesitant fuzzy map on the interval* $[1, 6]$ *and then get the score of hesitant fuzzy membership values and deviation degree of a hesitant fuzzy membership on the interval* $[1, 6]$. *Finally, we compare the values.*

$$h(1) = \{1, 0.5, 0.33, 0.25, 0.2, 0.17, 0.14, 0.13, 0.11, 0.1\},$$
$$h(2) = \{0.5, 0.25, 0.17, 0.13, 0.1\},$$
$$h(3) = \{0.33, 0.17, 0.11\},$$
$$h(4) = \{0.25, 0.13\},$$
$$h(5) = \{0.2, 0.1\},$$
$$h(6) = \{0.16\}.$$

Then,

$$s(h(1)) = \frac{1}{n(h(1))} \sum_{a \in h(1)} a,$$
$$= \frac{1}{10}(1 + 0.5 + 0.33 + 0.25 + 0.2 + 0.17 + 0.14 + 0.13 + 0.11 + 0.1) = 0.29,$$
$$s(h(2)) = \frac{1}{5}(0.5 + 0.25 + 0.17 + 0.13 + 0.1) = 0.23,$$
$$s(h(3)) = \frac{1}{3}(0.33 + 0.17 + 0.11) = 0.20,$$
$$s(h(4)) = \frac{1}{2}(0.25 + 0.13) = 0.19,$$
$$s(h(5)) = \frac{1}{2}(.2 + 0.1) = 0.15,$$
$$s(h(6)) = 0.16.$$

and

$$d(h(1)) = \sqrt{\frac{1}{h(1)} \sum_{a \in h(1)} (a - s(h(1)))^2}, \quad d(h(2)) = \sqrt{\frac{1}{h(2)} \sum_{a \in h(2)} (a - s(h(2)))^2}$$
$$d(h(3)) = \sqrt{\frac{1}{h(3)} \sum_{a \in h(3)} (a - s(h(3)))^2}, \quad d(h(4)) = \sqrt{\frac{1}{h(4)} \sum_{a \in h(4)} (a - s(h(4)))^2}$$
$$d(h(5)) = \sqrt{\frac{1}{h(5)} \sum_{a \in h(5)} (a - s(h(5)))^2}, \quad d(h(6)) = \sqrt{\frac{1}{h(6)} \sum_{a \in h(6)} (a - s(h(6)))^2}.$$

Let $\alpha = \{0.1, 0.3\}$ *then*

$$s(\alpha) = \frac{1}{n(\alpha)} \sum_{a \in \alpha} a = \frac{1}{2}(0.1 + 0.3) = 0.2$$

If we take $\alpha = \{0.1\}$, *then* $s(0.1) = 0.2$ *and* $d(0.1) = \frac{1}{\sqrt{5}}$. *If we take* $\alpha = \{0.3\}$, *then* $s(0.3) = 0.2$ *and* $d(0.3) = \frac{1}{\sqrt{15}}$. *Therefore, if* $\alpha = 0.1 \leq \alpha = 0.3 \implies d(\alpha = 0.1) \geq d(\alpha = 0.3)$.

Definition 1.6. *[13] A hesitant fuzzy subset h of E is a hesitant fuzzy approximate quantity iff its* α *level set is a convex subset of E* $\forall \alpha \in S$ *and* $sup_{\zeta \in E}\{h(\zeta)^+\} = \{1\}$.

Example 1.7. *Let the pair* $(\mathbb{Z}, d)$ *represent a b-metric space and the distance function d is given as* $d(\zeta, \eta) = |\zeta - \eta|^2 \ \forall \ \zeta, \eta \in \mathbb{Z}$. *Suppose* $h : E = \{1 \leq \zeta \leq 3\} \rightarrow S$ *is a hesitant fuzzy map, where,*

$$h(\zeta) = \{\frac{1}{s} \in [0, 1], s \text{ is a multiple of } \zeta, s \leq 6\}.$$

Then, $sup_{\zeta \in E}\{h(\zeta)^+\} = \{1\}$.
Solution: *Firstly, we find the values of hesitant fuzzy map on the interval* $[1, 3]$.

$$h(1) = \{1, 0.5, 0.33, 0.25, 0.2, 0.17\}, \ \ h(2) = \{0.5, 0.25, 0.17\}, \ \ h(3) = \{0.33, 0.17\},$$

then $sup_{\zeta \in E}\{h(\zeta)^+\} = \{1\}$.

Definition 1.8. *[13] Let* $W(E) \subset H(E)$ *be a collection of hesitant fuzzy approximate quantities of E and* $h, k \in W(E)$ *and* $\alpha \in S$. *Then the* α *set-space of h and k is defined as*

$$p_\alpha(h, k) = inf_{\zeta \in h_\alpha, \eta \in k_\alpha} d(\zeta, \eta),$$

$$p(h, k) = sup_\alpha \ p_\alpha(h, k).$$

If $h, k \in W(E)$ *then the fuzzy approximate quantity h is said to be more precise than k represented by* $h \subset k$ *if and only if* $h(\zeta) \leq k(\zeta)$ *for each* $\zeta \in E$.

Definition 1.9. *[13] Let* $h, k \in W(E)$ *and* $\alpha \in S$. *Then the* α *set-distance of h and k is defined as*

$$D_\alpha(h, k) = HD(h_\alpha, k_\alpha)$$

where, HD denotes the Hausdorff distance.

Let $h, k \in W(E)$ *and* $\alpha \in S$. *Then the distance between h and k is defined as*

$$D(h, k) = sup_\alpha \ D_\alpha(h, k).$$

Definition 1.10. *[13] Assume* $E(\neq \phi)$ *be a set with* (E, d) *a metric space.* $W(E)$ *be a sub collection of hesitant fuzzy approximate quantities of* $H(E)$. *Then the hesitant mapping is defined as* $H_F : E \rightarrow W(E)$ *such that* $H_F(\zeta) \in W(E)$ *for each* $\zeta \in E$.

Definition 1.11. *[13] Assume $E(\neq \phi)$ be a set with (E, d) a metric space. $W(E)$ be a sub collection of hesitant fuzzy approximate quantities of the collection of $H(E)$ of hesitant fuzzy sets of E. Then the pair of hesitant fuzzy maps is defined as $H_{F_1}, H_{F_2} : E \to W(E)$ such that*

$$D(H_{F_1}(\zeta), H_{F_2}(\eta)) \leq a_1 p(\zeta, H_{F_1}(\zeta)) + a_2 p(\eta, H_{F_2}(\eta)) + a_3 p(\eta, H_{F_1}(\zeta)) + a_4 p(\zeta, H_{F_2}(\eta)) + a_5 d(\zeta, \eta),$$

for any $\zeta, \eta \in E$, where $\sum_{i=1}^{5} a_i < 1$, $a_1 = a_2$ or $a_3 = a_4$ $(a_i \in \mathbb{R}^+)$.

Theorem 1.12. [13] Let (E, d) be a metric space and $H_{F_1}, H_{F_2} : E \to W(E)$ hesitant maps such that

$$D(H_{F_1}(\zeta), H_{F_2}(\eta)) \leq a_1 p(\zeta, H_{F_1}(\zeta)) + a_2 p(\eta, H_{F_2}(\eta)) + a_3 p(\eta, H_{F_1}(\zeta)) + a_4 p(\zeta, H_{F_2}(\eta)) + a_5 d(\zeta, \eta),$$

for any $\zeta, \eta \in E$, where $\sum_{i=1}^{5} a_i < 1$, and $a_1 = a_2$ or $a_3 = a_4$ $(a_i \in \mathbb{R}^+)$. Then $\exists\, \zeta^* \in E$ such that $\{\zeta^*\} \in H_{F_1}(\zeta^*)$ and $\{\zeta^*\} \in H_{F_2}(\zeta^*)$ also hold.

Definition 1.13. *[13]Let s be the coefficient of a b-metric space (E, d) and h be a hesitant fuzzy set on E. Then the α cut of a hesitant fuzzy set of b-metric space is defined as*

$$h_\alpha = \{\zeta \in E : s(h(\zeta)) \geq \alpha\},$$

for any $\alpha \in (0, 1]$, and

$$h_{\{0\}} = C(\{\zeta \in E : s(h(\zeta)) > \{0\}\}),$$

with $\alpha = \{0\} \in S$ is said to be an α cut of a hesitant fuzzy set, where $C(B)$ means the closure of B.

Definition 1.14. *[13] Assume $E(\neq \phi)$ be a set and the pair (E, d) represents a b-metric space. $W(E)$ be a sub collection of hesitant fuzzy approximate quantities of $H(E)$. Then the hesitant fuzzy mapping on b-metric space is defined as*

$$H_F : E \to W(E),$$

such that $H_F(\zeta) \in W(E)$ for each $\zeta \in E$.

Definition 1.15. *[13] Let s be the coefficient of a b-metric space (E, d). Then the hesitant fuzzy map $H_F : E \to W(E)$ on b-metric space is said to be a contraction hesitant fuzzy map on b-metric space if*

$$D(H_{F_\zeta}, H_{F_\eta}) \leq a d(\zeta, \eta),$$

for any $\zeta, \eta \in E$, where $a \in (0, \frac{1}{s})$ and $s \geq 1$.

Definition 1.16. *[13] Let E be a non-empty set and s be the coefficient of a b-metric space (E, d). $W(E)$ be a sub collection of hesitant approximate quantities of the collection $H(E)$ of hesitant fuzzy sets of E. Then the generalized contraction of hesitant fuzzy maps on a b-metric space is defined as*

$$H_{F_1}, H_{F_2} : E \to W(E),$$

such that

$$D(H_{F_1}(\zeta), H_{F_2}(\eta)) \leq \frac{1}{s}\Big[a_1 p(\zeta, H_{F_1}(\zeta)) + a_2 p(\eta, H_{F_2}(\eta)) + a_3 p(\eta, H_{F_1}(\zeta)) + a_4 p(\zeta, H_{F_2}(\eta)) + a_5 d_\phi(\zeta, \eta)\Big],$$

for any $\zeta, \eta \in E$ where $a_1 + a_2 + s[a_3 + a_4] + a_5 < 1$, and $a_1 = a_2$ or $a_3 = a_4$ $(a_i \in \mathbb{R}^+)$.

Theorem 1.17. [12] Assume $E(\neq \phi)$ be a set and (E, d_ϕ) be a complete extended b-metric space and d_ϕ is a continuous functional. Assume $T : E \to E$ and $\exists\, \zeta_0 \in E$ such that:

$$d_\phi(T\eta, T^2\eta) \leq k d_\phi(\eta, T\eta) \text{ for each } \eta \in O(\zeta_0) = \text{orbit of } \zeta_0$$

where $k \in [0, 1)$ be such that for $\zeta_0 \in E$, $\lim_{n,m\to\infty} \phi(\zeta_n, \zeta_m) < \frac{1}{k}$, here $\zeta_n = T^n(\zeta_0)$, n=1,2,3.... . Then $T^n\zeta_0 \to \zeta \in E$. Therefore, ζ is a fixed point of T if and only if $G(\zeta) = d(\zeta, T\zeta)$ is T-orbitally semi continuous at ζ.

2. Main Results

In this section, first we introduce the new notion of hesitant and contractive fuzzy mapping on extended b-metric space in the following way.

Definition 2.1. *Assume $E(\neq \phi)$ be a set and the pair (E, d_ϕ) be an extended b-metric space. Also, let the collection $H(E)$ of hesitant fuzzy sets E has a sub collection $W(E)$ of hesitant approximate quantities. Then the hesitant fuzzy mapping on extended b-metric space is defined as $H_F : E \to W(E)$ such that $H_F(\zeta) \in W(E)$ for each $\zeta \in E$.*

Definition 2.2. *Let (E, d_ϕ) be an extended b-metric space with coefficient $\phi(\zeta, \eta)$ and $\phi : E \times E \to [1, \infty)$. Then the hesitant fuzzy map $H_F : E \to W(E)$ on extended b-metric space is called contraction hesitant fuzzy map on extended b-metric space if*

$$D(H_{F_\zeta}, H_{F_\eta}) \leq a d_\phi(\zeta, \eta),$$

for any $\zeta, \eta \in E$, where $a \in (0, \frac{1}{\phi(\zeta,\eta)})$.

Remark 2.3. (i) If $\phi(\zeta, \eta) = s$ and s ≥ 1, then contraction hesitant fuzzy map on extended b-metric space became contraction hesitant fuzzy map on b-metric space.
(ii) If $\phi(\zeta, \eta) = s$ and s $= 1$, then contraction hesitant fuzzy map on extended b-metric space became contraction hesitant fuzzy map on metric space.

Definition 2.4. *Assume $E(\neq \phi)$ be a set and $\phi(\zeta, \eta)$ be coefficient of an extended b-metric space (E, d_ϕ). Also, let the collection $H(E)$ of hesitant fuzzy sets E has a sub collection $W(E)$ of hesitant approximate quantities. Then the generalized contraction of hesitant fuzzy maps on an extended b-metric space is defined as $H_{F_1}, H_{F_2} : E \to W(E)$ such that*

$$\begin{aligned} D(H_{F_1}(\zeta), H_{F_2}(\eta)) \leq \frac{1}{\phi(\zeta, \eta)} \Big[& a_1 p(\zeta, H_{F_1}(\zeta)) + a_2 p(\eta, H_{F_2}(\eta)) + a_3 p(\eta, H_{F_1}(\zeta)) \\ & + a_4 p(\zeta, H_{F_2}(\eta)) + a_5 d_\phi(\zeta, \eta) \Big], \end{aligned}$$

for any $\zeta, \eta \in E$, where $a_1 + a_2 + \phi(\zeta, \eta)[a_3 + a_4] + a_5 < 1$, and $a_1 = a_2$ or $a_3 = a_4$ ($a_i \in \mathbb{R}^+$).

Remark 2.5. (i) If $\phi(\zeta, \eta) = s$ and s ≥ 1, then generalized contraction hesitant fuzzy map on extended b-metric space became generalized contraction hesitant fuzzy maps on b-metric space.
(ii) If $\phi(\zeta, \eta) = s$ and s $= 1$, then generalized contraction hesitant fuzzy map on extended b-metric space became generalized contraction hesitant fuzzy map on metric space.

Lemma 2.6. Let E be an extended b-metric space with $\zeta \in E, h \in W(E)$ and $\{\zeta\}$ is a hesitant fuzzy set whose hesitant membership function is equal to the hesitant characteristic function of the set $\{\zeta\}$. If $\{\zeta\} \subset h$ then $p_\alpha(\zeta, h) = 0$ for each $\alpha \in S$.
Proof: If $\{\zeta\} \subset h$ then $\zeta \in h_\alpha$ for each $\alpha \in S$ and h is an approximate quantity. So, $p_\alpha(\zeta, h) = inf_{\eta \in h_\alpha} d_\phi(\zeta, \eta) = 0$.

Lemma 2.7. Let the pair (E, d_ϕ) be an extended b-metric space having coefficient $\phi(\zeta, \gamma)$ then

$$p_\alpha(\zeta, h) \leq \phi(\zeta, \gamma)[d_\phi(\zeta, \eta) + p_\alpha(\eta, h)],$$

for any $\zeta, \eta \in E$.
Proof: We know that

$$\begin{aligned} p_\alpha(\zeta, h) &= inf_{\gamma \in h_\alpha} d_\phi(\zeta, \gamma) \\ &\leq inf_{\gamma \in h_\alpha} \phi(\zeta, \gamma)[d_\phi(\zeta, \eta) + d_\phi(\eta, \gamma)] \\ &\leq \phi(\zeta, \gamma)[inf_{\gamma \in h_\alpha} d_\phi(\zeta, \eta) + inf_{\gamma \in h_\alpha} d_\phi(\eta, \gamma)] \\ &= \phi(\zeta, \gamma)[d_\phi(\zeta, \eta) + p_\alpha(\eta, h)] \end{aligned}$$

The proof is complete.

Lemma 2.8. Let the pair (E, d_ϕ) be an extended b-metric space having coefficient $\phi(\zeta, \gamma)$. If $\{\zeta_0\} \subset h$ and $h \in W(E)$ then for each $k \in W(E)$ we have that $p_\alpha(\zeta_0, k) \leq D_\alpha(h, k)$.
Proof: We know that

$$\begin{aligned} p_\alpha(\zeta, k) &= inf_{\eta \in k_\alpha}\, d_\phi(\zeta, \eta), \\ &\leq sup_{\zeta \in h_\alpha} inf_{\eta \in k_\alpha}\, d_\phi(\zeta, \eta), \\ &\leq D_\alpha(h, k). \end{aligned}$$

The proof is complete.

Lemma 2.9. Assume (E, d_ϕ) be a complete extended b-metric space having coefficient $\phi(\zeta, \gamma)$ and $h \in W(E)$. Then

$$p_\alpha(\zeta, h) \leq \phi(\zeta, \gamma) d_\phi(\zeta, \eta),$$

if $\{\eta\} \subset h$.
Proof: By lemma 2.7 we can say that, for any $\eta \in E$, we have that

$$p_\alpha(\zeta, h) \leq \phi(\zeta, \gamma)[d_\phi(\zeta, \eta) + p_\alpha(\eta, h)].$$

Since $\eta \in h$, then using lemma 2.6, we can say that $p_\alpha(\eta, h) = 0$, then

$$p_\alpha(\zeta, h) \leq \phi(\zeta, \gamma) d_\phi(\zeta, \eta)$$

The proof is complete.

Theorem 2.10. Assume the pair (E, d) be a complete extended b-metric space having coefficient $\phi(\zeta, \eta)$ and $H_{F_1}, H_{F_2} : E \to W(E)$ hesitant maps such that

$$\begin{aligned} D(H_{F_1}(\zeta), H_{F_2}(\eta)) \leq \frac{1}{\phi(\zeta, \eta)}&[a_1 p(\zeta, H_{F_1}(\zeta)) + a_2 p(\eta, H_{F_2}(\eta)) + a_3 p(\eta, H_{F_1}(\zeta)) \\ &+ a_4 p(\zeta, H_{F_2}(\eta)) + a_5 d_\phi(\zeta, \eta)], \end{aligned}$$

for any $\zeta, \eta \in E$ with $a_1 + a_2 + \phi(\zeta, \eta)[a_3 + a_4] + a_5 < 1$, and $a_1 = a_2$ or $a_3 = a_4$ ($a_i \in \mathbb{R}^+$). Then $\exists\, \zeta^* \in E$ such that $\{\zeta^*\} \subset H_{F_1}(\zeta^*)$ and $\{\zeta^*\} \subset H_{F_2}(\zeta^*)$ hold.
Proof: Let $\zeta_0 \in E$ then $H_{F_1}(\zeta_0) \in W(E)$. Let $\zeta_1 \subset H_{F_1}(\zeta_0)$ then there is $\zeta_2 \in E$ such that $\zeta_2 \subset H_{F_2}(\zeta_1) \in W(E)$. So, $d_\phi(\zeta_1, \zeta_2) \leq D(H_{F_1}(\zeta_0), H_{F_2}(\zeta_1))$. Also there is $\zeta_3 \in E$ such that $\zeta_3 \subset H_{F_3}(\zeta_2) \in W(E)$. So, continuing in this manner, we have that there is $\zeta_n \in E$ such that

$$\{\zeta_{2n+1}\} \subset H_{F_1}(\zeta_{2n}),$$

and

$$\{\zeta_{2n+2}\} \subset H_{F_2}(\zeta_{2n+1}),$$

so that

$$d_\phi(\zeta_{2n+1}, \zeta_{2n+2}) \leq D(H_{F_1}(\zeta_{2n}), H_{F_2}(\zeta_{2n+1})) \tag{1}$$

and

$$d_\phi(\zeta_{2n+2}, \zeta_{2n+3}) \leq D(H_{F_2}(\zeta_{2n+1}), H_{F_1}(\zeta_{2n+2})) \tag{2}$$

If $n = 0$, then from equation (1)

$$\begin{aligned}
d_\phi(\zeta_1,\zeta_2) &\le D(H_{F_1}(\zeta_0),H_{F_2}(\zeta_1))\\
&\le \frac{1}{\phi(\zeta_0,\zeta_1)}[a_1p(\zeta_0,H_{F_1}(\zeta_0)) + a_2p(\zeta_1,H_{F_2}(\zeta_1)) + a_3p(\zeta_1,H_{F_1}(\zeta_0)) + a_4p(\zeta_0,H_{F_2}(\zeta_1)) + a_5d_\phi(\zeta_0,\zeta_1)]\\
&\le a_1p(\zeta_0,H_{F_1}(\zeta_0)) + a_2p(\zeta_1,H_{F_2}(\zeta_1)) + a_3p(\zeta_1,H_{F_1}(\zeta_0)) + a_4p(\zeta_0,H_{F_2}(\zeta_1)) + a_5d_\phi(\zeta_0,\zeta_1)\\
&\le a_1d_\phi(\zeta_0,\zeta_1) + a_2d_\phi(\zeta_1,\zeta_2) + a_3d_\phi(\zeta_1,\zeta_1) + a_4d_\phi(\zeta_0,\zeta_2) + a_5d_\phi(\zeta_0,\zeta_1)\\
&\le a_1d_\phi(\zeta_0,\zeta_1) + a_2d_\phi(\zeta_1,\zeta_2) + a_4\phi(\zeta_0,\zeta_2)[d_\phi(\zeta_0,\zeta_1) + d_\phi(\zeta_1,\zeta_2)] + a_5d_\phi(\zeta_0,\zeta_1)\\
&\le (a_1 + \phi(\zeta_0,\zeta_2)a_4 + a_5)d_\phi(\zeta_0,\zeta_1) + (a_2 + \phi(\zeta_0,\zeta_2)a_4)d_\phi(\zeta_1,\zeta_2)\\
&\le \frac{(a_1 + \phi(\zeta_0,\zeta_2)a_4 + a_5)}{(1 - a_2 - \phi(\zeta_0,\zeta_2)a_4)}d_\phi(\zeta_0,\zeta_1)\\
&\le td_\phi(\zeta_0,\zeta_1)
\end{aligned}$$

where

$$t = \frac{(a_1 + \phi(\zeta_0,\zeta_2)a_4 + a_5)}{(1 - a_2 - \phi(\zeta_0,\zeta_2)a_4)},$$

But $a_1 + a_2 + \phi(\zeta,\eta)[a_3 + a_4] + a_5 < 1$, and $a_1 = a_2$ or $a_3 = a_4$ ($a_i \in \mathbb{R}^+$) gives $a_1 + \phi(\zeta,\eta)a_3 + a_5 < 1 - a_2 - \phi(\zeta,\eta)a_4$ so that if $a_3 \ge a_4$, implies that $0 < t < 1$.

If $n = 0$, then from equation (2)

$$\begin{aligned}
d_\phi(\zeta_2,\zeta_3) &\le D(H_{F_2}(\zeta_1),H_{F_1}(\zeta_2))\\
&\le D(H_{F_1}(\zeta_2),H_{F_2}(\zeta_1))\\
&\le \frac{1}{\phi(\zeta_1,\zeta_2)}[a_1p(\zeta_2,H_{F_1}(\zeta_2)) + a_2p(\zeta_1,H_{F_2}(\zeta_1)) + a_3p(\zeta_1,H_{F_1}(\zeta_2)) + a_4p(\zeta_2,H_{F_2}(\zeta_1)) + a_5d_\phi(\zeta_1,\zeta_2)]\\
&\le a_1p(\zeta_2,H_{F_1}(\zeta_2)) + a_2p(\zeta_1,H_{F_2}(\zeta_1)) + a_3p(\zeta_1,H_{F_1}(\zeta_2)) + a_4p(\zeta_2,H_{F_2}(\zeta_1)) + a_5d_\phi(\zeta_1,\zeta_2)\\
&\le a_1d_\phi(\zeta_2,\zeta_3) + a_2d_\phi(\zeta_1,\zeta_2) + a_3d_\phi(\zeta_1,\zeta_3) + a_4d_\phi(\zeta_2,\zeta_2) + a_5d_\phi(\zeta_1,\zeta_2)\\
&\le a_1d_\phi(\zeta_2,\zeta_3) + a_2d_\phi(\zeta_1,\zeta_2) + a_3\phi(\zeta_1,\zeta_3)[d_\phi(\zeta_1,\zeta_2) + d_\phi(\zeta_2,\zeta_3)] + a_5d_\phi(\zeta_1,\zeta_2)\\
&\le (a_2 + \phi(\zeta_1,\zeta_3)a_3 + a_5)d_\phi(\zeta_1,\zeta_2) + (a_1 + \phi(\zeta_1,\zeta_3)a_3)d_\phi(\zeta_2,\zeta_3)\\
&\le \frac{(a_2 + \phi(\zeta_1,\zeta_3)a_3 + a_5)}{(1 - a_1 - \phi(\zeta_1,\zeta_3)a_3)}d_\phi(\zeta_1,\zeta_2)\\
&\le \frac{(a_2 + \phi(\zeta_1,\zeta_3)a_3 + a_5)}{(1 - a_1 - \phi(\zeta_1,\zeta_3)a_3)}\frac{(a_1 + \phi(\zeta_0,\zeta_2)a_4 + a_5)}{(1 - a_2 - \phi(\zeta_0,\zeta_2)a_4)}d_\phi(\zeta_0,\zeta_1)\\
&\le tfd_\phi(\zeta_0,\zeta_1)
\end{aligned}$$

where

$$f = \frac{(a_2 + \phi(\zeta_1,\zeta_3)a_3 + a_5)}{(1 - a_1 - \phi(\zeta_1,\zeta_3)a_3)},$$

and

$$t = \frac{(a_1 + \phi(\zeta_0,\zeta_2)a_4 + a_5)}{(1 - a_2 - \phi(\zeta_0,\zeta_2)a_4)},$$

But $a_1 + a_2 + \phi(\zeta,\eta)[a_3 + a_4] + a_5 < 1$, and $a_1 = a_2$ or $a_3 = a_4$ ($a_i \in \mathbb{R}^+$) gives $a_1 + \phi(\zeta,\eta)a_4 + a_5 < 1 - a_2 - \phi(\zeta,\eta)a_3$ so that if $a_4 \ge a_3$, implies that $0 < f < 1$. Then $a_3 = a_4$ and $0 < tf < 1$ for both cases.

If $n = 1$, then from equation (1)

$$\begin{aligned}
d_\phi(\zeta_3,\zeta_4) &\le D(H_{F_1}(\zeta_2),H_{F_2}(\zeta_3))\\
&\le \frac{1}{\phi(\zeta_2,\zeta_3)}[a_1p(\zeta_2,H_{F_1}(\zeta_2))+a_2p(\zeta_3,H_{F_2}(\zeta_3))+a_3p(\zeta_3,H_{F_1}(\zeta_2))+a_4p(\zeta_2,H_{F_2}(\zeta_3))+a_5d_\phi(\zeta_2,\zeta_3)]\\
&\le a_1p(\zeta_2,H_{F_1}(\zeta_2))+a_2p(\zeta_3,H_{F_2}(\zeta_3))+a_3p(\zeta_3,H_{F_1}(\zeta_2))+a_4p(\zeta_2,H_{F_2}(\zeta_3))+a_5d_\phi(\zeta_2,\zeta_3)\\
&\le a_1d_\phi(\zeta_2,\zeta_3)+a_2d_\phi(\zeta_3,\zeta_4)+a_3d_\phi(\zeta_3,\zeta_3)+a_4d_\phi(\zeta_2,\zeta_4)+a_5d_\phi(\zeta_2,\zeta_3)\\
&\le a_1d_\phi(\zeta_2,\zeta_3)+a_2d_\phi(\zeta_3,\zeta_4)+a_4\phi(\zeta_2,\zeta_4)[d_\phi(\zeta_2,\zeta_3)+d_\phi(\zeta_3,\zeta_4)]+a_5d_\phi(\zeta_2,\zeta_3)\\
&\le (a_1+\phi(\zeta_2,\zeta_4)a_4+a_5)d_\phi(\zeta_2,\zeta_3)+(a_2+\phi(\zeta_2,\zeta_4)a_4)d_\phi(\zeta_3,\zeta_4)\\
&\le \frac{(a_1+\phi(\zeta_2,\zeta_4)a_4+a_5)}{(1-a_2-\phi(\zeta_2,\zeta_4)a_4)}d_\phi(\zeta_2,\zeta_3)\\
&\le \frac{(a_1+\phi(\zeta_2,\zeta_4)a_4+a_5)}{(1-a_2-\phi(\zeta_2,\zeta_4)a_4)}\frac{(a_2+\phi(\zeta_1,\zeta_3)a_3+a_5)}{(1-a_1-\phi(\zeta_1,\zeta_3)a_3)}\frac{(a_1+\phi(\zeta_0,\zeta_2)a_4+a_5)}{(1-a_2-\phi(\zeta_0,\zeta_2)a_4)}d_\phi(\zeta_0,\zeta_1)\\
&\le (t^2)fd_\phi(\zeta_0,\zeta_1)
\end{aligned}$$

Set

$$\frac{(a_1+\phi(\zeta_i,\zeta_{i+2})a_4+a_5)}{(1-a_2-\phi(\zeta_i,\zeta_{i+2})a_4)}=t,\quad i=0,2,4,......$$

Similarly,

$$\frac{(a_2+\phi(\zeta_i,\zeta_{i+2})a_3+a_5)}{(1-a_1-\phi(\zeta_i,\zeta_{i+2})a_3)}=f,\quad i=1,3,5,....$$

and $\phi(\zeta_i,\zeta_{i+2})=\phi(\zeta,\eta)\ \ \forall\ \ i\in\mathbb{N}$ and we also know that $\phi(\zeta,\eta)\ge 1$.
After adusting t and f, we get $0<t^2f<1$ and

$$d_\phi(\zeta_3,\zeta_4)\le (t^2)fd_\phi(\zeta_0,\zeta_1)$$

If $n = 1$, then from equation (2)

$$\begin{aligned}
d_\phi(\zeta_4,\zeta_5) &\le D(H_{F_2}(\zeta_3),H_{F_1}(\zeta_4))\\
&\le D(H_{F_1}(\zeta_4),H_{F_2}(\zeta_3))\\
&\le \frac{1}{\phi(\zeta_3,\zeta_4)}[a_1p(\zeta_4,H_{F_1}(\zeta_4))+a_2p(\zeta_3,H_{F_2}(\zeta_3))+a_3p(\zeta_3,H_{F_1}(\zeta_4))+a_4p(\zeta_4,H_{F_2}(\zeta_3))+a_5d_\phi(\zeta_3,\zeta_4)]\\
&\le a_1p(\zeta_4,H_{F_1}(\zeta_4))+a_2p(\zeta_3,H_{F_2}(\zeta_3))+a_3p(\zeta_3,H_{F_1}(\zeta_4))+a_4p(\zeta_4,H_{F_2}(\zeta_3))+a_5d_\phi(\zeta_3,\zeta_4)\\
&\le a_1d_\phi(\zeta_4,\zeta_5)+a_2d_\phi(\zeta_3,\zeta_4)+a_3d_\phi(\zeta_3,\zeta_5)+a_4d_\phi(\zeta_4,\zeta_4)+a_5d_\phi(\zeta_3,\zeta_4)\\
&\le a_1d_\phi(\zeta_4,\zeta_5)+a_2d_\phi(\zeta_3,\zeta_4)+a_3\phi(\zeta_3,\zeta_5)[d_\phi(\zeta_3,\zeta_4)+d_\phi(\zeta_4,\zeta_5)]+a_5d_\phi(\zeta_3,\zeta_4)\\
&\le (a_2+\phi(\zeta_3,\zeta_5)a_3+a_5)d_\phi(\zeta_3,\zeta_4)+(a_1+\phi(\zeta_3,\zeta_5)a_3)d_\phi(\zeta_4,\zeta_5)\\
&\le \frac{(a_2+\phi(\zeta_3,\zeta_5)a_3+a_5)}{(1-a_1-\phi(\zeta_3,\zeta_5)a_3)}d_\phi(\zeta_3,\zeta_4)\\
&\le (tf)^2d_\phi(\zeta_0,\zeta_1),\qquad 0<(tf)^2<1
\end{aligned}$$

Similarly,

$$\begin{aligned}
d_\phi(\zeta_5,\zeta_6)&\le (tf)^2td_\phi(\zeta_0,\zeta_1),\\
d_\phi(\zeta_6,\zeta_7)&\le (tf)^3d_\phi(\zeta_0,\zeta_1),
\end{aligned}$$

Continue this process, for n terms with n is even

$$\begin{aligned}
d_\phi(\zeta_{n-1},\zeta_n) &\le \frac{(a_1+\phi(\zeta_{n-2},\zeta_n)a_4+a_5)}{(1-a_2-\phi(\zeta_{n-2},\zeta_n)a_4)}d_\phi(\zeta_{n-2},\zeta_{n-1})\\
&\le (tf)^{n-1}td_\phi(\zeta_0,\zeta_1).
\end{aligned}$$

For n is odd, we have

$$\begin{aligned} d_\phi(\zeta_{n-1},\zeta_n) &\leq \frac{(a_2+\phi(\zeta_{n-2},\zeta_n)a_3+a_5)}{(1-a_1-\phi(\zeta_{n-2},\zeta_n)a_3)} d_\phi(\zeta_{n-2},\zeta_{n-1}) \\ &\leq (tf)^{n-1} d_\phi(\zeta_0,\zeta_1). \end{aligned}$$

Continue this process, we have that for any $n \in \mathbb{N}$ and n even.

$$\begin{aligned} d_\phi(\zeta_1,\zeta_n) &\leq \phi(\zeta_1,\zeta_n)[d_\phi(\zeta_1,\zeta_2)+d_\phi(\zeta_2,\zeta_3)+d_\phi(\zeta_3,\zeta_4)+..........+d_\phi(\zeta_{n-1},\zeta_n)] \\ &\leq \phi(\zeta_1,\zeta_n)\Big[\frac{(a_1+\phi(\zeta_0,\zeta_2)a_4+a_5)}{(1-a_2-\phi(\zeta_0,\zeta_2)a_4)}+\frac{(a_1+\phi(\zeta_0,\zeta_2)a_4+a_5)}{(1-a_2-\phi(\zeta_0,\zeta_2)a_4)}\frac{(a_2+\phi(\zeta_1,\zeta_3)a_3+a_5)}{(1-a_1-\phi(\zeta_1,\zeta_3)a_3)} \\ &+\frac{(a_1+\phi(\zeta_0,\zeta_2)a_4+a_5)}{(1-a_2-\phi(\zeta_0,\zeta_2)a_4)}\frac{(a_2+\phi(\zeta_1,\zeta_3)a_3+a_5)}{(1-a_1-\phi(\zeta_1,\zeta_3)a_3)}\frac{(a_1+\phi(\zeta_2,\zeta_4)a_4+a_5)}{(1-a_2-\phi(\zeta_2,\zeta_4)a_4)} \\ &+................................\Big]d_\phi(\zeta_0,\zeta_1) \\ &\leq \phi(\zeta_1,\zeta_n)[t+tf+t^2f+(tf)^2+(tf)^2t+........+(tf)^{n-1}t]d_\phi(\zeta_0,\zeta_1). \end{aligned}$$

If n is odd then we have that

$$\begin{aligned} d_\phi(\zeta_1,\zeta_n) &\leq \phi(\zeta_1,\zeta_n)[d_\phi(\zeta_1,\zeta_2)+d_\phi(\zeta_2,\zeta_3)+d_\phi(\zeta_3,\zeta_4)+..........+d_\phi(\zeta_{n-1},\zeta_n)] \\ &\leq \phi(\zeta_1,\zeta_n)\Big[\frac{(a_1+\phi(\zeta_0,\zeta_2)a_4+a_5)}{(1-a_2-\phi(\zeta_0,\zeta_2)a_4)}+\frac{(a_1+\phi(\zeta_0,\zeta_2)a_4+a_5)}{(1-a_2-\phi(\zeta_0,\zeta_2)a_4)}\frac{(a_2+\phi(\zeta_1,\zeta_3)a_3+a_5)}{(1-a_1-\phi(\zeta_1,\zeta_3)a_3)} \\ &+\frac{(a_1+\phi(\zeta_0,\zeta_2)a_4+a_5)}{(1-a_2-\phi(\zeta_0,\zeta_2)a_4)}\frac{(a_2+\phi(\zeta_1,\zeta_3)a_3+a_5)}{(1-a_1-\phi(\zeta_1,\zeta_3)a_3)}\frac{(a_1+\phi(\zeta_2,\zeta_4)a_4+a_5)}{(1-a_2-\phi(\zeta_2,\zeta_4)a_4)} \\ &+................................\Big]d_\phi(\zeta_0,\zeta_1) \\ &\leq \phi(\zeta_1,\zeta_n)[t+tf+t^2f+(tf)^2+(tf)^2t+........+(tf)^{n-1}]d_\phi(\zeta_0,\zeta_1). \end{aligned}$$

Next we shall show that any sequence $\{\zeta_n\}$ in E is Cauchy. Let $k,l \in \mathbb{N}$ with $(k>l)$ then if k is even, we have that

$$\begin{aligned} d_\phi(\zeta_l,\zeta_k) &\leq \phi(\zeta_l,\zeta_k)[d_\phi(\zeta_l,\zeta_{l+1})+d_\phi(\zeta_{l+1},\zeta_{l+2})+...........+d_\phi(\zeta_{k-1},\zeta_k)] \\ &\leq \phi(\zeta_l,\zeta_k)[(tf)^{l-1}t+(tf)^lt+(tf)^{l+1}t+............+(tf)^{k-2}t]d_\phi(\zeta_0,\zeta_1) \\ &\leq \phi(\zeta_l,\zeta_k)(tf)^{l-1}t[1+(tf)+(tf)^2+(tf)^3+.........(tf)^{k-l-1}]d_\phi(\zeta_0,\zeta_1) \end{aligned}$$

and if k is odd, we have that

$$\begin{aligned} d_\phi(\zeta_l,\zeta_k) &\leq \phi(\zeta_l,\zeta_k)[d_\phi(\zeta_l,\zeta_{l+1})+d_\phi(\zeta_{l+1},\zeta_{l+2})+..............+d_\phi(\zeta_{k-1},\zeta_k)] \\ &\leq \phi(\zeta_l,\zeta_k)[(tf)^l+(tf)^{l+1}+(tf)^{l+2}+...............+(tf)^{k-1}]d_\phi(\zeta_0,\zeta_1) \\ &\leq \phi(\zeta_l,\zeta_k)(tf)^l[1+(tf)+(tf)^2+(tf)^3+...........(tf)^{k-l-1}]d_\phi(\zeta_0,\zeta_1) \end{aligned}$$

Since,

$$\sum_{i=1}^{5} a_i < \frac{1}{\phi(\zeta_l,\zeta_k)}.$$

For k is even, then we take the coefficients of RHS

$$\phi(\zeta_l,\zeta_k)(tf)^{l-1}t[1+(tf)+(tf)^2+(tf)^3+................(tf)^{k-l-1}] < \frac{1}{\phi(\zeta_l,\zeta_k)},$$

$$(tf)^{l-1}t[1+(tf)+(tf)^2+(tf)^3+................(tf)^{k-l-1}] < \frac{1}{[\phi(\zeta_l,\zeta_k)]^2}.$$

For n^{th} term we put $l-1=n$

$$(tf)^n t[1+(tf)+(tf)^2+(tf)^3+................(tf)^{k-n}] < \frac{1}{[\phi(\zeta_{n-1},\zeta_k)]^2},$$

we know that $0 < tf < 1$ then $(tf)^n \to 0 \; as \; n \to \infty$. Then RHS of inequality became zero. So, $d_\phi(\zeta_l,\zeta_k) < \epsilon$ for k is even. The same is true if k is odd.

Therefore, the sequence $\{\zeta_n\}$ in E is a cauchy sequence. Thus $\exists\, \zeta^* \in E$ such that $\{\zeta_n\} \to \zeta^*$ as $n \to \infty$. Since (E,d) is a complete space. Now,

$$p_0(\zeta^*, H_{F_2}(\zeta^*)) \le \phi(\zeta^*,\eta^*)[d_\phi(\zeta^*,\zeta_{2n+1}) + H_{F_2}(\zeta_{2n+1}, H_{F_1}(\zeta^*))]$$

$$p_0(\zeta^*, H_{F_2}(\zeta^*)) \le \phi(\zeta^*,\eta^*)[d_\phi(\zeta^*,\zeta_{2n+1}) + D(\zeta_{2n}, H_{F_2}(\zeta^*))] \tag{3}$$

But

$$\begin{aligned}
D(\zeta_{2n}, H_{F_2}(\zeta^*)) &\le \frac{1}{\phi(\zeta^*,\eta^*)}[a_1 p(\zeta_{2n}, H_{F_1}(\zeta_{2n})) + a_2 p(\zeta^*, H_{F_2}(\zeta^*)) + a_3 p(\zeta^*, H_{F_1}(\zeta_{2n})) + a_4 p(\zeta_{2n}, H_{F_2}(\zeta^*)) \\
&\quad + a_5 d_\phi(\zeta_{2n},\zeta^*)] \\
&\le a_1 d_\phi(\zeta_{2n},\zeta_{2n+1}) + a_2[d_\phi(\zeta^*,\zeta_{2n+1}) + D(\zeta_{2n}, H_{F_2}(\zeta^*))] + a_3 d_\phi(\zeta^*,\zeta_{2n+1}) \\
&\quad + a_4[d_\phi(\zeta_{2n},\zeta_{2n+1}) + D(\zeta_{2n}, H_{F_2}(\zeta^*))] + a_5 d_\phi(\zeta_{2n},\zeta^*) \\
&\le (a_1+a_4) d_\phi(\zeta_{2n},\zeta_{2n+1}) + (a_2+a_3) d_\phi(\zeta^*,\zeta_{2n+1}) + (a_2+a_4) D(\zeta_{2n}, H_{F_2}(\zeta^*)) + a_5 d_\phi(\zeta_{2n},\zeta^*) \\
&\le \frac{(a_1+a_4)}{(1-a_2-a_4)} d_\phi(\zeta_{2n},\zeta_{2n+1}) + \frac{(a_2+a_3)}{(1-a_2-a_4)} d_\phi(\zeta^*,\zeta_{2n+1}) + \frac{a_5}{(1-a_2-a_4)} d_\phi(\zeta_{2n},\zeta^*)
\end{aligned}$$

Using this value in equation (3)

$$\begin{aligned}
p_0(\zeta^*, H_{F_2}(\zeta^*)) \le \phi(\zeta^*,\eta^*)\Big[& d_\phi(\zeta^*,\zeta_{2n+1}) + \frac{(a_1+a_4)}{(1-a_2-a_4)} d_\phi(\zeta_{2n},\zeta_{2n+1}) + \frac{(a_2+a_3)}{(1-a_2-a_4)} d_\phi(\zeta^*,\zeta_{2n+1}) \\
&+ \frac{a_5}{(1-a_2-a_4)} d_\phi(\zeta_{2n},\zeta^*)\Big],
\end{aligned}$$

$p_0(\zeta^*, H_{F_2}(\zeta^*)) \to 0 \;\; as \; n \to \infty$. Therefore, $\{\zeta^*\} \subset H_{F_2}(\zeta^*)$, Using Lemma 2.6. Similarly, we can prove $\{\zeta^*\} \subset H_{F_1}(\zeta^*)$ and the proof is complete.

Remark 2.11. (i) If we put $\phi(\zeta,\eta) = s$ and s ≥ 1, in theorem 2.10 then this result also hold for hesitant fuzzy map on b-metric space.

(ii) If we put $\phi(\zeta,\eta) = s$ and s $= 1$, in theorem 2.10 then this result also hold for hesitant fuzzy map on metric space.

Corollary 2.12. Let $E(\neq \phi)$ be a set and (E,d) be a complete extended b-metric space. $H_F : E \to W(E)$ hesitant self map such that

$$D(H_F(\zeta), H_F(\eta)) \le a d_\phi(\zeta,\eta) \tag{4}$$

for any $\zeta,\eta \in E$, $a < \frac{1}{\phi(\zeta^*,\eta^*)}$. Then $\exists$ a unique $\zeta^* \in E$ such that $\{\zeta^*\} \subset H_F(\zeta^*)$ holds.

Proof: We have our desired result $\{\zeta^*\} \subset H_F(\zeta^*)$ if we put $H_F = H_{F_1} = H_{F_2}$, $a_i = 0$ for $i = 1,2,3,4$ and $a_5 = a$ in above theorem.

Next, to prove uniqueness of ζ^*. Suppose $\exists\, \zeta^* \neq \eta^*$ $\eta^* \subset H_F(\eta^*)$, then we have

$$\begin{aligned}
d_\phi(\zeta^*,\eta^*) &\le D_1(H_F(\zeta^*), H_F(\eta^*)) \\
&\le D(H_F(\zeta^*), H_F(\eta^*)) \\
&\le a d_\phi(\zeta^*,\eta^*)
\end{aligned}$$

So, $0 < a < \frac{1}{\phi(\zeta^*,\eta^*)}$, a contradiction, therfore, $\zeta^* = \eta^*$ and the proof is complete.

Remark 2.13. (i) If we put $\phi(\zeta^*, \eta^*) = s$ and s ≥ 1, in corollary 2.12 then this result also hold for hesitant fuzzy map on b-metric space.

(ii) If we put $\phi(\zeta^*, \eta^*) = s$ and s $= 1$, in corollary 2.12 then this result also hold for hesitant fuzzy map on metric space.

Example 2.14. *Assume $E = \{0, 1, 2\}$ and (E, d_ϕ) is a complete extended b-metric space. Define $d_\phi : E \times E \to [0, \infty)$ and $\phi : E \times E \to [1, \infty)$ by $d_\phi(\zeta, \eta) = |\zeta - \eta|$ and*

$$D(H_F(\zeta), H_F(\eta)) = \begin{cases} 0 & ,\zeta = \eta \\ \frac{1}{5} & ,\zeta \neq \eta \text{ and } \zeta, \eta \in \{0, 1\} \\ \frac{1}{2} & ,\zeta \neq \eta \text{ and } \zeta, \eta \in \{0, 2\} \\ 1 & ,\zeta \neq \eta \text{ and } \zeta, \eta \in \{1, 2\} \end{cases}$$

Define hesitant fuzzy mapping $H_F : E \to W(E)$ and $W(E)$ is a subcollection of hesitant fuzzy approximate quantities of $H(E)$.

$$H_{F_0}(t) = H_{F_1}(t) = \begin{cases} \frac{1}{2} & , t = 0 \\ 0 & , t = 1, 2 \end{cases}, \quad H_{F_2}(t) = \begin{cases} 0 & , t = 0, 2 \\ \frac{1}{2} & , t = 1 \end{cases}.$$

Define $\alpha : E \to (0, 1]$ by $\alpha(\zeta) = \frac{1}{2}$ for all $\zeta \in E$. Now we obtain that

$$[H_F(\zeta)]_{\frac{1}{2}} = \begin{cases} \{0\} & , \zeta = 0, 1 \\ \{1\} & , \zeta = 2 \end{cases}$$

Then we find the condition for $D(H_F(\zeta, H_F(\eta))) \leq ad_\phi(\zeta, \eta)$.
Solution: *We know α (set-distance) that is*

$$\begin{aligned} D(H_F(\zeta), H_F(\eta)) &= Sup_{\alpha=\frac{1}{2}} D_{\frac{1}{2}}(H_F(\zeta), H_F(\eta)) \\ &= HD([H_F(\zeta)]_{\frac{1}{2}}, [H_F(\eta)]_{\frac{1}{2}}). \end{aligned}$$

If we take $\zeta = 0$ and $\eta = 1$, then we get

$$\begin{aligned} D(H_F(0), H_F(1)) &= HD([H_F(0)]_{\frac{1}{2}}, [H_F(1)]_{\frac{1}{2}}) \\ &= HD(0, 0) \\ &= 0 \end{aligned}$$

from the above corollary 2.12, we say that

$$\begin{aligned} D(H_F(0), H_F(1)) &\leq ad_\phi(\zeta, \eta) \\ 0 &\leq ad_\phi(0, 1) \end{aligned}$$

This condition (4) is hold and similarly, on setting $\zeta = 0$ and $\eta = 2$, then we get

$$\begin{aligned} D(H_F(0), H_F(2)) &= HD([H_F(0)]_{\frac{1}{2}}, [H_F(2)]_{\frac{1}{2}}) \\ &= HD(0, 1) \\ &= \frac{1}{5} \end{aligned}$$

Then,

$$\begin{aligned}
D(H_F(0), H_F(2)) &\leq ad_\phi(\zeta, \eta)\\
\frac{1}{5} &\leq ad_\phi(\zeta, \eta)\\
\frac{1}{5} &\leq 2a\\
\frac{1}{10} &\leq a\\
\frac{1}{10} &\leq \frac{1}{\phi(\zeta, \eta)}\\
\phi(\zeta, \eta) &\leq 10
\end{aligned}$$

Hence, condition (4) holds, if $1 \leq \phi(\zeta, \eta) \leq 10$.
Similarly, we take $\zeta = 1$ *and* $\eta = 2$, *then we get*

$$\begin{aligned}
D(H_F(1), H_F(2)) &= HD([H_F(1)]_{\frac{1}{2}}, [H_F(2)]_{\frac{1}{2}})\\
&= HD(0, 1)\\
&= \frac{1}{5}
\end{aligned}$$

Then,

$$\begin{aligned}
D(H_F(1), H_F(2)) &\leq ad_\phi(\zeta, \eta)\\
\frac{1}{5} &\leq ad_\phi(\zeta, \eta)\\
\frac{1}{5} &\leq a\\
\frac{1}{5} &\leq \frac{1}{\phi(\zeta, \eta)}\\
\phi(\zeta, \eta) &\leq 5
\end{aligned}$$

Hence, condition (4) holds, if $1 \leq \phi(\zeta, \eta) \leq 5$. *Therefore all the conditions of corollary 2.12 and there exist a point* $0 \in E$ *such that* $0 \in [H_F(0)]_{\frac{1}{2}}$ *is a hesitant fuzzy fixed point.*

Theorem 2.15. Assume $E(\neq \phi)$ be a complete extended b-metric space. Suppose H_{F_1} *and* H_{F_2} be hesitant fuzzy mapping from E into $W(E)$. If $\exists$ a constant $a \in [0, 1)$, such that for each $\zeta, \eta \in E$,

$$D(H_{F_1}(\zeta), H_{F_2}(\eta)) \leq a\, max\{d_\phi(\zeta, \eta), p(\zeta, H_{F_1}(\zeta)), p(\eta, H_{F_2}(\eta)), \frac{p(\zeta, H_{F_2}(\eta)) + p(\eta, H_{F_1}(\zeta))}{1 + \phi(\zeta, \eta)}\} \tag{5}$$

then $\exists\, \zeta^* \in E$ such that $\zeta^* \subset H_{F_1}(\zeta^*)$ *and* $\zeta^* \subset H_{F_2}(\zeta^*)$.
Proof: Let $\zeta_0 \in E$ and $\zeta_1 \subset H_{F_1}(\zeta_0)$. Then $\exists\, \zeta_2 \in E \subset H_{F_2}(\zeta_1)$

and

$$\begin{aligned}
d_\phi(\zeta_1,\zeta_2) &\leq D_1(H_{F_1}(\zeta_0),H_{F_2}(\zeta_1))\\
&\leq D(H_{F_1}(\zeta_0),H_{F_2}(\zeta_1))\\
&\leq \frac{a}{\phi(\zeta_0,\zeta_1)}\, max\{d_\phi(\zeta_0,\zeta_1),p(\zeta_0,H_{F_1}(\zeta_0)),p(\zeta_1,H_{F_2}(\zeta_1)),\frac{p(\zeta_0,H_{F_2}(\zeta_1))+p(\zeta_1,H_{F_1}(\zeta_0))}{1+\phi(\zeta_0,\zeta_1)}\}\\
&\leq \frac{a}{\phi(\zeta_0,\zeta_1)}\, max\{d_\phi(\zeta_0,\zeta_1),d_\phi(\zeta_0,\zeta_1),d_\phi(\zeta_1,\zeta_2),\frac{d_\phi(\zeta_0,\zeta_2)+d_\phi(\zeta_1,\zeta_1)}{1+\phi(\zeta_0,\zeta_1)}\}\\
&\leq \frac{a}{\phi(\zeta_0,\zeta_1)}\, max\{d_\phi(\zeta_0,\zeta_1),d_\phi(\zeta_1,\zeta_2),\frac{d_\phi(\zeta_0,\zeta_2)}{1+\phi(\zeta_0,\zeta_1)}\}\\
&\leq \frac{a}{\phi(\zeta_0,\zeta_1)}\, max\{d_\phi(\zeta_0,\zeta_1),d_\phi(\zeta_1,\zeta_2),\frac{\phi(\zeta_0,\zeta_1)[d_\phi(\zeta_0,\zeta_1)+d_\phi(\zeta_1,\zeta_2)]}{1+\phi(\zeta_0,\zeta_1)}\}
\end{aligned}$$

But, we know that

$$\frac{\phi(\zeta_0,\zeta_1)(a+b)}{1+\phi(\zeta_0,\zeta_1)} \leq a+b,\ \forall\, a,b\in\mathbb{R}^+ \text{ and } \phi(\zeta_0,\zeta_1)\geq 1.$$

Set,

$$\phi(\zeta_i,\zeta_{i+1})=\phi(\zeta,\eta)\ \forall\ i=0,1,2,3,...$$

$$\begin{aligned}
d(\zeta_1,\zeta_2) &\leq \frac{a}{\phi(\zeta,\eta)}\, max\{d(\zeta_0,\zeta_1),d(\zeta_1,\zeta_2)\}\\
&\leq \frac{a}{\phi(\zeta,\eta)}d(\zeta_0,\zeta_1).
\end{aligned}$$

Also, since $\zeta_1\in E$ and $\zeta_2\subset H_{F_2}(\zeta_1)$. Then $\exists\,\zeta_3\in E$ such that $\zeta_3\subset H_{F_1}(\zeta_2)$ and

$$\begin{aligned}
d_\phi(\zeta_2,\zeta_3) &\leq D_1(H_{F_2}(\zeta_1),H_{F_1}(\zeta_2))\\
&\leq D(H_{F_2}(\zeta_1),H_{F_1}(\zeta_2))\\
&\leq \frac{a}{\phi(\zeta_1,\zeta_2)}\, max\{d_\phi(\zeta_1,\zeta_2),p(\zeta_1,H_{F_2}(\zeta_1)),p(\zeta_2,H_{F_1}(\zeta_2)),\frac{p(\zeta_1,H_{F_1}(\zeta_2))+p(\zeta_2,H_{F_2}(\zeta_1))}{1+\phi(\zeta_1,\zeta_2)}\}\\
&\leq \frac{a}{\phi(\zeta_1,\zeta_2)}\, max\{d_\phi(\zeta_1,\zeta_2),d_\phi(\zeta_1,\zeta_2),d_\phi(\zeta_2,\zeta_3),\frac{d_\phi(\zeta_1,\zeta_3)}{1+\phi(\zeta_1,\zeta_2)}\}\\
&\leq \frac{a}{\phi(\zeta_1,\zeta_2)}\, max\{d_\phi(\zeta_1,\zeta_2),d_\phi(\zeta_2,\zeta_3),\frac{\phi(\zeta_1,\zeta_2)[d_\phi(\zeta_1,\zeta_2)+d_\phi(\zeta_2,\zeta_3)]}{1+\phi(\zeta_1,\zeta_2)}\}\\
&\leq \frac{a}{\phi(\zeta,\eta)}d_\phi(\zeta_1,\zeta_2)\\
&\leq \frac{a^2}{[\phi(\zeta,\eta)]^2}d_\phi(\zeta_0,\zeta_1).
\end{aligned}$$

Continue this process, we have a sequence $\{\zeta_n\in E\}$ with $n\geq 0$ such that

$$\zeta_{2n+1}\subset H_{F_1}(\zeta_{2n}),$$

and

$$\zeta_{2n+2}\subset H_{F_2}(\zeta_{2n+1}),$$

such that

$$\begin{aligned} d_\phi(\zeta_n, \zeta_{n+1}) &\leq D_1(H_{F_2}(\zeta_{n-1}), H_{F_1}(\zeta_n)) \\ &\leq D(H_{F_2}(\zeta_{n-1}), H_{F_1}(\zeta_n)) \\ &\leq \frac{a}{\phi(\zeta_{n-1}, \zeta_n)}\ max\{d_\phi(\zeta_{n-1}, \zeta_n), p(\zeta_n, H_{F_1}(\zeta_n)), p(\zeta_{n-1}, H_{F_2}(\zeta_{n-1})), \\ &\frac{p(\zeta_{n-1}, H_{F_1}(\zeta_n)) + p(\zeta_n, H_{F_2}(\zeta_{n-1}))}{1+\phi(\zeta_{n-1}, \zeta_n)}\} \\ &\leq \frac{a}{\phi(\zeta_{n-1}, \zeta_n)}\ max\{d_\phi(\zeta_{n-1}, \zeta_n), d_\phi(\zeta_n, \zeta_{n+1}), d_\phi(\zeta_{n-1}, \zeta_n), \frac{d_\phi(\zeta_{n-1}, \zeta_{n+1}) + d_\phi(\zeta_n, \zeta_n)}{1+\phi(\zeta_{n-1}, \zeta_n)}\} \\ &\leq \frac{a}{\phi(\zeta_{n-1}, \zeta_n)}\ max\{d_\phi(\zeta_{n-1}, \zeta_n), d_\phi(\zeta_n, \zeta_{n+1}), \frac{d_\phi(\zeta_{n-1}, \zeta_{n+1})}{1+\phi(\zeta_{n-1}, \zeta_n)}\} \\ &\leq \frac{a}{\phi(\zeta_{n-1}, \zeta_n)}\ max\{d_\phi(\zeta_{n-1}, \zeta_n), d_\phi(\zeta_n, \zeta_{n+1}), \frac{\phi(\zeta_{n-1}, \zeta_n)[d_\phi(\zeta_{n-1}, \zeta_n) + d_\phi(\zeta_n, \zeta_{n+1})]}{1+\phi(\zeta_{n-1}, \zeta_n)}\} \\ &\leq \frac{a}{\phi(\zeta, \eta)}\ max\{d_\phi(\zeta_{n-1}, \zeta_n), d_\phi(\zeta_n, \zeta_{n+1})\} \\ &\leq \frac{a}{\phi(\zeta, \eta)}\ d_\phi(\zeta_{n-1}, \zeta_n) \\ &\leq (\frac{a}{\phi(\zeta, \eta)})^n\ d_\phi(\zeta_0, \zeta_1) \end{aligned}$$

To prove that every sequence in E is Cauchy. Let $p, q \in \mathbb{N}$ with (p is greater than q). Then,

$$\begin{aligned} d(\zeta_p, \zeta_q) &\leq \phi(\zeta, \eta)[d(\zeta_q, \zeta_{q+1}) + d(\zeta_{q+1}, \zeta_{q+2}) + ... + d(\zeta_{p-1}, \zeta_p)] \\ &\leq \phi(\zeta, \eta)\Big[(\frac{a}{\phi(\zeta, \eta)})^q d(\zeta_0, \zeta_1) + (\frac{a}{\phi(\zeta, \eta)})^{q+1} d(\zeta_0, \zeta_1) + ... + (\frac{a}{\phi(\zeta, \eta)})^{p-1} d(\zeta_0, \zeta_1)\Big] \\ &\leq \phi(\zeta, \eta)(\frac{a}{\phi(\zeta, \eta)})^q\Big[1 + \frac{a}{\phi(\zeta, \eta)} + (\frac{a}{\phi(\zeta, \eta)})^2 + ...(\frac{a}{\phi(\zeta, \eta)})^{p-q-1}\Big]\ d(\zeta_0, \zeta_1) \end{aligned}$$

We also know that, $0 < (\frac{a}{\phi(\zeta,\eta)}) < 1$. So, for n^{th} term $(\frac{a}{\phi(\zeta,\eta)})^n \to 0$ as $n \to \infty$. Then

$$d(\zeta_l, \zeta_k) < \epsilon$$

Therefore, the sequence $\{\zeta_n\}$ in E is a cauchy sequence.
Thus $\exists\ \zeta^* \in E$ such that $\{\zeta_n\} \to \zeta^*\ as\ n \to \infty$. Since (E, d) is a complete space. Now,

$$\begin{aligned} p_0(\zeta^*, H_{F_2}(\zeta^*)) &\leq \phi(\zeta^*, \eta^*)[d(\zeta^*, \zeta_{2n+1}) + H_{F_2}(\zeta_{2n+1}, H_{F_1}(\zeta^*))] \\ &\leq \phi(\zeta^*, \eta^*)[d(\zeta^*, \zeta_{2n+1}) + D(\zeta_{2n}, H_{F_2}(\zeta^*))] \end{aligned} \tag{6}$$

$$\begin{aligned} D(\zeta_{2n}, H_{F_2}(\zeta^*)) &\leq a\ max\{d(\zeta_{2n}, \zeta^*), p(\zeta_{2n}, H_{F_1}(\zeta_{2n})), p(\zeta^*, H_{F_2}(\zeta^*)), \frac{p(\zeta_{2n}, H_{F_2}(\zeta^*)) + p(\zeta^*, H_{F_1}(\zeta_{2n}))}{1+\phi(\zeta^*, \eta^*)}\} \\ &\leq a\ max\{d(\zeta_{2n}, \zeta^*), d(\zeta_{2n}, \zeta_{2n+1}), \phi(\zeta^*, \eta^*)[d(\zeta^*, \zeta_{2n+1}) + D(\zeta_{2n}, H_{F_2}(\zeta^*))], \\ &\frac{\phi(\zeta^*, \eta^*)[d(\zeta_{2n}, \zeta_{2n+1}) + D(\zeta_{2n}, H_{F_2}(\zeta^*))] + d(\zeta^*, \zeta_{2n+1})}{1+\phi(\zeta^*, \eta^*)}\} \\ &\leq a\ max\{d(\zeta_{2n}, \zeta^*), d(\zeta^*, \zeta_{2n+1})\} \\ &\leq a\ d(\zeta_{2n}, \zeta^*) \end{aligned}$$

Using this value in equation (6), then

$$p_0(\zeta^*, H_{F_2}(\zeta^*)) \leq \phi(\zeta^*, \eta^*)[d(\zeta^*, \zeta_{2n+1}) + a\ d(\zeta_{2n}, \zeta^*)]$$

$$p_0(\zeta^*, H_{F_2}(\zeta^*)) \to 0 \; as \; n \to \infty.$$

Therefore, $\{\zeta^*\} \subset H_{F_2}(\zeta^*)$ Using lemma 2.6.
Similarly, we can prove $\{\zeta^*\} \subset H_{F_1}(\zeta^*)$ and the proof is complete.

Remark 2.16.
1. If we put $\phi(\zeta, \eta) = s$ and s ≥ 1, in theorem 2.15 then this result also hold for hesitant fuzzy map on b-metric space.
2. If we put $\phi(\zeta, \eta) = s$ and s = 1, in theorem 2.15 then this result also hold for hesitant fuzzy map on metric space.

3. Application

In this section, we establish the application of the fixed point theorem for an integral equation of Fredholm type.

$$\zeta(u) = \int_i^j N(u, v, \zeta(v))dv + h(u), \quad u, v \in [i, j] \tag{7}$$

Assume $E = C([i, j], \mathbb{R})$ be the space of all continuous real valued functions defined on [i,j] and the metric $d_\phi : E \times E \to [0, \infty)$, $\phi : E \times E \to [1, \infty)$ are defined by
$d_\phi(\zeta, \eta) = sup_{u \in [i,j]} |\zeta(u) - \eta(u)|^2$ with $\phi(\zeta, \eta) = |\zeta(u)| + |\eta(u)| + 3$.
We know that (E, d_ϕ) is a complete extended b-metric space. Assume $T : E \to E$ the operator is given by:

$$T\zeta(u) = \int_i^j N(u, v, \zeta(v))dv + h(u) \quad \forall \; u, v \in [i, j]$$

The integral equation (7) has a solution if and only if T has a fixed point.
Statement: Assume the following conditions are satisfied:

(i) $N : [i, j] \times [i, j] \times \mathbb{R} \to \mathbb{R}$ and $h : [i, j] \to \mathbb{R}$ are continuous functions.
(ii) $\exists \; a \in (0, 1)$ such that

$$|N(u, v, \zeta(v)) - N(u, v, T\zeta(v))| \leq a \; |\zeta(v) - T\zeta(v)| \text{ for each } u, v \in [i, j] \text{ and } \zeta \in E.$$

Then the fredholm integral equation (7) has a unique solution.
Proof: We know that (E, d_ϕ) is a complete extended b-metric space. Assume $T : E \to E$ the operator is given by:

$$T\zeta(u) = \int_i^j N(u, v, \zeta(v))dv + h(u) \quad \forall \; u, v \in [i, j]$$

Further, letting that the following condition is satisfied:

$$|N(u, v, \zeta(v)) - N(u, v, T\zeta(v))| \leq \frac{1}{2} |\zeta(v) - T\zeta(v)| \text{ for each } u, v \in [i, j] \text{ and } \zeta \in E$$

Employing that integral equation (7) exihibit a solution. T satisfies all the conditions of theorem 1.17. For any $\zeta \in E$, we have:

$$|T\zeta(u) - T(T\zeta(u))|^2 \leq \Big(\int_i^j |N(u, v, \zeta(v)) - N(u, v, T\zeta(v))| dv \Big)^2$$

$$|T\zeta(u) - T(T\zeta(u))|^2 \leq \frac{1}{4} |\zeta(v) - T\zeta(v)|^2$$

$$|T\zeta(u) - T(T\zeta(u))| \leq \frac{1}{2} |\zeta(v) - T\zeta(v)| \tag{8}$$

Therefore, T has a fixed point.
But from the contraction of hesitant fuzzy map on extended b-metric space. we have:

$$D(H_F(\zeta), H_F(\eta)) \leq a d_\phi(\zeta, \eta) \tag{9}$$

for any $\zeta, \eta \in E$, where $a \in \left(0, \frac{1}{\phi(\zeta,\eta)}\right)$ and we also know that $\phi(\zeta, \eta) \geq 1$.
From the equations (8) and (9), we get similar results. So, T has a fixed point and also say that $\zeta = \eta$ and equation (7) possesses a solution. So, Fredholm integral equation (7) has a solution.

Remark 3.1.

1. If we put $\phi(\zeta, \eta) = s$ and s ≥ 1, in equation (9) then this result also hold for hesitant fuzzy map on b-metric space.
2. If we put $\phi(\zeta, \eta) = s$ and s $= 1$, in equation (9) then this result also hold for hesitant fuzzy map on metric space.

4. Conclusion

In this article, The concept of hesitant fuzzy mapping, contraction hesitant fuzzy mapping and generalized contraction of hesitant fuzzy mappings are introduced in the framework of extended b-metric spaces, which in particular case reduces to b-metric spaces. Through several concluding remarks, it is assured that by setting $\phi(\zeta, \eta) = s$ and $s = 1$, the main results reduce to the results in b-metric spaces and usual metric spaces. In future, the existence of these results can be studied in other types of metric spaces like fuzzy metric spaces, modular metric spaces, neutrosophic metric-like spaces and controlled metric spaces etc by using a different type of contraction.

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