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Hamiltonicity and pancyclicity of superclasses of claw-free graphs

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Abstract. A graph G is called to be fully cycle extendable graph [3] if each vertex of G belongs to a triangle and for any cycle C with $|V(C)| < |V(G)|$ there exists a cycle C' in G such that $V(C) \subset V(C')$ and $|V(C')| = |V(C)|+1$. In this paper, we show that every graph G that is triangularly connected, partly claw-free and $\{K_{1,4}, K_4\}$-free is fully cycle extendable graph if its claw centers set is P_4-free. This paper generalizes the concept of Hendry fully cycle extendable graph [3] for the largest superclass of partly claw-free graphs defined by Abbas and Benmeziane [1].

1. Introduction

Throughout this paper, we will use terms, notations and definitions of [2]. Only undirected simple finite graphs $G = (V, E)$ are considered, with vertices set $V(G)$ and edges set $E(G)$. A graph on n vertices is a complete graph, denoted by K_n, if its vertices are two by two adjacent to each other. A graph G, of at least two vertices, is called stable or independent if its vertices are two by two not adjacent. A graph $G = (V, E)$ is called bipartite if V can be partitioned into two stables V_1 and V_2 and its edges have exactly one end in V_1. If each vertex of V_1 is adjacent to all the vertices of V_2, the graph G is called a complete bipartite graph, denoted $K_{n,m}$ with $|V_1| = n$ and $|V_2| = m$. The graph G is isomorph to the graph H if there exists a bijection f from $V(G)$ to $V(H)$ such that for all pair of vertices $(u, v) \in V(G)^2$, $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$, we denote $G \cong H$.

The path P_k connecting the two vertices u and v or the (u, v)-path is the sequence of vertices edges $P_k = v_1, v_2, \ldots, v_k$ verifying $v_1 = u$, $v_k = v$ and for all i such that $1 \leq i \leq k-1$, $v_i v_{i+1} \in E(G)$. The vertices u and v are called the initial and the final end of the path P_k. A path is called to be elementary if it does not pass twice through the same vertex. A cycle passing through the vertex u is a (u, u)-path. The length of cycle C is the number of its edges $|E(C)|$ (or its vertices $|V(C)|$). We denote by C_n a cycle of length n. C_3 is also called triangle.

The distance between two vertices u and v, denoted by $d(u, v)$, is the number of edges of a shortest (u, v)-path. The eccentricity of a vertex v of a graph G, denoted by $e(v)$, is the maximum distance from the vertex v to all the other vertices of G.

For $S \subset V$, $\langle S \rangle$ will denote the sub-graph inducted by S of vertices set S and edges set those of G witch have its two ends in S and $G - S$ is $\langle V \setminus S \rangle$. A graph G is called without S if it does not contain any sub-graph

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isomorphic to S. The vertices set $S = \{u, v_1, v_2, \ldots, v_k\}$ is a star of the graph G if $\langle S \rangle$ is isomorphic to the bipartite graph $K_{1,k}$ with $d(u) = k$ which called star center. $K_{1,3}$ is also called claw. We denote by A the set of claw centers of the graph G. A subset $D \subseteq V(G)$ is a dominating set of G if every vertex in $V(G) - D$ has a neighbor in D, while D is a 2-dominating set of G if every vertex belonging to $V(G) - D$ is joined by at least two edges with a vertex or vertices in D.

For $v \in V$, the open (respectively closed) neighbourhood of vertex v in G, denoted by $N_G(v)$ and $N[v]$ respectively, is the set of all vertices u adjacent to v. So $N_G(v) = \{u \in V; (v, u) \in E\}$ (respectively $N[v] = N_G(v) \cup \{v\}$). We denote by $N_k(v) = \{u \in V;\ d(v, u) = k\}$ the set of all the vertices at distance k from the vertex v, in particular $N_G(v) = N_1(v)$. The degree of the vertex v, denoted by $d(v)$, is the number of all the edges incident to v. The cardinality of $N_G(v)$ is the degree $d(v)$ of the vertex v. Respectively, $\delta(G)$ and $\Delta(G)$ represent the minimum degree and the maximum degree of G. We denote by $\sigma_k(G)$ the minimal value of the sum of the degrees of k vertices of G two by two non-adjacent.

A graph G is called connected if every pair of vertices is joined by a path. A connected graph G is called 2-connected, if for every vertex $x \in V(G)$, $G - x$ is connected. We called that a vertex v is locally connected if and only if the sub-graph induced by its open neighbourhood $\langle N_G(v) \rangle$, is connected. A graph G is locally connected if and only if for every vertex v, v induces a connected sub-graph in G. A graph G is said to be Hamiltonian if it has a cycle that passes through all the vertices of G one and only once. In [6], a graph G is triangularly connected if for every two edges $e_1; e_2 \in E(G)$, G there exists a sequence of triangles $C_1, C_2, \ldots, C_l$ such that $e_1 \in C_1$, $e_2 \in C_l$ and $E(C_i) \cap E(C_{i+1}) \neq \emptyset$ for $1 \leq i \leq l - 1$. A cycle C in a graph G is extendable if there exists a cycle C' of G, such that $V(C) \subset V(C')$ and $|V(C')| = |V(C)| + 1$. A graph G on n vertices is said to be cycle extendable graph if every non-Hamiltonian cycle C on k vertices, $(k < n)$, is cycle extendable. G is said to be a fully cycle extendable graph if G is a cycle extendable graph and every vertex of G lies on a triangle of G.

In 1990, Hendry [3] proved the following result.

Theorem 1.1 *(G. R. T. Hendry, [3]) If G is a connected graph, locally connected and claw-free graph on at least three vertices, then G is fully cycle extendable graph.*

Ryjáčěk introduced the class of almost claw-free graphs, in [5], as follows.

Definition 1.2 *A graph G is almost claw-free if for all vertices v of G, $< N_G(v) >$ is 2-dominated and the set A of claw centers of G is a stable set.*

In [7], Zhan studied triangular and almost claw-free graphs and proved in the following theorem that

Theorem 1.3 *(M. Zhan, [7]) Every triangularly connected, $K_{1,4}$-free, almost claw-free graph on at least three vertices is fully cycle extendable.*

Az a generalization of claw-free graphs, the class of partly claw-free graphs was introduced by Abbas and Benmeziane in [1].

Definition 1.4 *A graph $G = (V, E)$ is said to be partly claw-free graph if for all vertex $v \in A$, the set of claw centers of G, there exist two vertices $x, y \in V - A$ such that $N_G(v) \subseteq N[x] \cup N[y]$. We say that $< N_G(v) >$ is 2-dominated in $V \setminus A$.*

Abbas and Benmeziane in [1] proved the following results on partly claw-free 2-connected graphs.

Theorem 1.5 *(M. Abbas and Z. Benmeziane, [1])*
If G is a partly claw-free 2-connected graph with $\delta(G) \geq \frac{(n-2)}{3}$, then G is Hamiltonian.

Theorem 1.6 *(M. Abbas and Z. Benmeziane, [1])*
If G is a partly claw-free 2-connected graph with $\sigma_3(G) \geq n$, then G is Hamiltonian.

In this paper we prove that the partly claw-fee graphs defined by Abbas and Benmeziane [1] are fully cycle extendable graphs the property studied in Hendry's theorem 1 [3] and in Zhan's theorem 2 [7].

2. Main results

The following proposition is important in the sense that it locates the two vertices x and y in a smaller domain compared to the definition [1].

Proposition 2.1 *Let G be a partly claw-free graph, $v \in A$ and $x, y \in V - A$ such that $N_G(v) \in N[x] \subset N[y]$. Then $x, y \in [(N_1(v) \cup N_2(v)) - A]$.*

Proof. Let G be a partly claw-free graph, $v \in A$ a claw center of G and $x,\ y \in V - A$ such that $N_G(v) \in N[x] \subset N[y]$. The set $\{\{v\}, N_1(v), N_2(v), \dots, N_{e(v)}(v)\}$ is a partition of V. Without losing generality, suppose $x \in N_k(v)$ with $k \geq 3$. So

$$\begin{cases} N_G(x) \subset N_{k-1}(v) \cup N_k(v) \cup N_{k+1}(v) \\ \text{and} & (\text{ for all } k \geq 3) \\ N[x] \subset N_{k-1}(v) \cup N_k(v) \cup N_{k+1}(v) \end{cases}$$

As

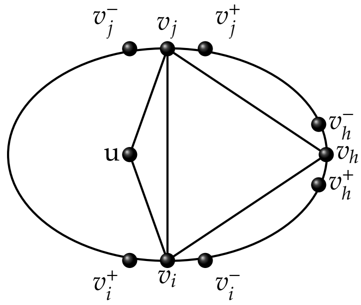
$$\begin{cases} N_G(v) \cap N_{k-1}(v) = \emptyset \\ N_G(v) \cap N_k(v) = \emptyset & (\text{ for all } k \geq 3) \\ N_G(v) \cap N_{k+1}(v) = \emptyset \end{cases}$$

Then $N_G(v) \not\subseteq N[x]$. Similarly, $N_G(v) \not\subseteq N[y]$. Hence, $N_G(v) \not\subseteq N[x] \cup N[y]$, a contradiction. □

Now we can give a theorem by which we show that any triangularly connected partly claw-free graph is fully cycle extendable if it is $\{K_{1,4}, K_4\}$-free and its claw center set is P_4-free.

Theorem 2.2 *Every triangularly connected, partly claw-free and $\{K_{1,4}, K_4\}$-free graph is a fully cycle extendable graph if the sub-graph $\langle A \rangle$ induced by A is P_4-free.*

Proof. This proof is inspired from the work done by M. Zhan [7]. Each vertex of the graph G belongs to a triangle, so it is sufficient to show that for every cycle C of length $r \leq V(G) - 1$ there exists a cycle C' of length $r + 1$ such that $V(C) \in V(C')$. Assume that G contains a non extendable cycle C of length $r \leq V(G) - 1$. An orientation is chosen on the cycle C as following. For all $u \in V(C)$, u^+ and u^- denotes, respectively, the successor and the predecessor of the vertex u on C. For two vertices $u, v \in V(C)$, $C[u, v]$ and $\overleftarrow{C}[u, v]$, denotes the two $(u; v)$-paths in the same direction and in the opposite direction with the orientation of the cycle C. For $u \in V(C)$, $C[u, u]$ and $\overleftarrow{C}[u, u]$ denote the vertex u. When the vertices of $K_{1,3}$ or $K_{1,4}$ are cited, the center is always cited first in the list. A denote the set of all the claw centers in the graph G. Let's be C a cycle of the graph G and the set $\mathfrak{B}(C) = \{\mathcal{B};\ \mathcal{B}$ is a triange and $E(\mathcal{B}) \cap E(C) \neq \emptyset\}$. Clearly, $E(C) \subset \bigcup_{\mathcal{B} \in \mathfrak{B}(C)} E(\mathcal{B})$. If a triangle $\mathcal{B}$ is such that $V(C) \cap V(\mathcal{B}) = 2$, then the sub-graph induced by the set of edges $E(C) \cup E(\mathcal{B}) - (E(C) \cap E(\mathcal{B}))$ extend C, a contradiction. So we suppose that for every triangle $\mathcal{B} \in \mathfrak{B}(C)$; $V(\mathcal{B}) \subseteq V(C)$. Consider the edge e such that e is incident at a single vertex of the cycle C and $\mathcal{B}_e$ is the triangle such that $e \in \mathcal{B}_e$. Clearly, $E(\mathcal{B}_e) \cap E(C) = \emptyset$ so $\mathcal{B}_e \notin \mathfrak{B}(C)$. From G triangularly connected, there is a sequence of triangles $Z_0,\ Z_1,\ \dots,\ Z_k$ such that $Z_0 = \mathcal{B}_e$ and $Z_k \in \mathfrak{B}(C)$. The cycle C, the edge e, and $\mathcal{B}_e$ are chosen such that among all the sets of vertices $V(C)$, the number k of triangles in this sequence is the smallest possible. Therefore, from the definition of the edge e, $k \geq 1$. We also have $|V(Z_0) \cap V(C)| = 2$ and $V(Z_i) \subseteq V(C)$ for all $i \geq 1$. Let $Z_0 = uv_iv_ju$ and $Z_1 = v_iv_jv_hv_i$ be such that $v_h \in C[v_j^+,\ v_i^-]$. The cycle C, the edge e and the triangle $\mathcal{B}_e$ are chosen such that $|\{v_i^+v_i^-,\ v_j^+v_j^-\} \cap E(G)|$ is as large as possible.



□

The following lemma is the first consequence between the vertices of two triangles Z_0, Z_1 and the number of triangles of the sequence k.

Lemma 2.3 **(i)** *$v_i, v_j \in A$, $v_i^+ \notin N_1(v_j^+)$ and $v_i^- \notin N_1(v_j^-)$;*

(ii) *For $k \geq 2$,*

- *$v_h^\circ \notin N_1(v_l^\circ)$ with $l \in \{i, j\}$ and $\circ \in \{-, +\}$;*
- *If $v_i^+ \in N_1(v_i^-)$, then $v_i \notin N_1(v_h^+) \cup N_1(v_h^-)$;*
- *If $v_h \notin N_1(v_i^+) \cup N_1(v_i^-)$, then $v_h^+ \notin N_1(v_h^-)$;*
- *If $v_h \notin A$, then $v_h \notin N_1(v_i^+) \cup N_1(v_j^-)$.*

Proof. (i)

- $1 \leq |\{v_h v_i^-, v_h v_i^+, v_i^- v_i^+\}| \cap E(G) \leq 2$ otherwise either $\langle v_i, v_i^-, v_i^+, u, v_h\rangle \cong K_{1,4}$ or $\langle v_h, v_i^-, v_i, v_i^+\rangle \cong K_4$, a contradiction.
- If $v_i^+ \in N_1(v_j^+)$, then the cycle $C^{'} = v_j u \overleftarrow{C}[v_i, v_j^+]C[v_i^+, v_j]$ extends the cycle C, contradiction. Similarly, $v_i^- \notin N_1(v_j^-)$).

(ii) For $k \geq 2$.

- If $v_h^- \in N_1(v_i^-)$, then v_i and v_h will be adjacent on the cycle $C^{'} = v_i C[v_h, v_i^-]\overleftarrow{C}[v_h^-, v_i]$. Hence, $P^{'} = Z_0 Z_1$, with $Z_1 \in \mathfrak{B}(C^{'})$, is a path of $k = 1$, a contradiction. Similarly $v_h^+ \notin N_1(v_i^+)$, $v_h^- \notin N_1(v_j^-)$ and $v_h^+ \notin N_1(v_j^+)$.
- For $v_i^+ \in N_1(v_i^-)$ and $v_i \in N_1(v_h^-)$, v_i and v_h will be adjacent on the cycle $C^{'} = v_i C[v_h, v_i^-]C[v_i^+, v_h^-]v_i$ and $P^{'} = Z_0 Z_1$ is a path of $k = 1$, contradiction. Similarly $v_i \notin N_1(v_h^+)$.
- It suffices to show that if $v_h^- \in N_1(v_h^+)$, then $v_h \notin N_1(v_i^+) \cup N_1(v_i^-)$. So, for $v_h \in N_1(v_i^+)$, v_i and v_h will be adjacent on the cycle $C^{'} = v_h C[v_i^+, v_h^-]C[v_h^+, v_i]v_h$ and $P^{'} = Z_0 Z_1$ is a path of $k = 1$, a contradiction. Similarly $v_h \notin N_1(v_i^-)$.
- If $v_j^- \in N_1(v_h)$, then $v_i^+ \neq v_j^-$ and $\langle v_h, v_h^-, v_j^-, v_i\rangle \cong K_{1,3}$ so $v_h \in A$ from $v_h^- \notin N_1(v_i)$ otherwise $\langle v_i, v_i^+, v_i^-, v_h^-, u\rangle \cong K_{1,4}$ because $v_h^- \notin N_1(u)$ otherwise $v_i^+ \notin N_1(v_h^-)$. Let the cycle $C^{'} = C[v_j, v_h^-]C[v_i^+, v_j^-]C[v_h, v_i]uv_j$ and $P^{'} = Z_0^{'} Z_1^{'}$, with $Z_0^{'} = \langle u, v_h^-, v_i, u\rangle$, $Z_1^{'} = \langle v_i, v_h^-, v_h, v_i\rangle \in \mathfrak{B}(C)$. So $P^{'}$ is a path of $k = 1$, contradiction. Similarly for $v_i^+ \in N_1(v_h)$, $\langle v_h, v_h^+, v_i^+, v_j\rangle \cong K_{1,3}$ and $v_h \in A$ from $\langle v_j, v_j^+, v_j^-, v_h^+, u\rangle \cong K_{1,4}$ if $v_h^+ \in N_1(v_j)$.

□

We can also confirm that

Lemma 2.4 *For $d \in N_G(v_j) \cap N_G(v_j^+)$,*

(i) *$d \in V(C)$ and $d \notin N_1(u)$;*

(ii) *If* $w \in [V(C) \cap N_1(u) \cap N_1(v_j^+)] - \{v_i, v_j, v_h, N_1(v_l^{--}), N_1(v_l^{++})\}$ *with* $l \in \{i, j\}$, *then* $u \notin N_1(w^{--}) \cup N_1(w^{++})$;

(iii) $v_j^- \in N_1(d)$. *So* $d \neq v_h$;

(iv) *If* $k \geq 2$, *then* $v_h \notin N_1(d)$ *and* $v_h \notin N_1(v_j^-) \cup N_1(v_j^+)$.

Proof. **(i)** By absurdity,

- If $d \notin V(C)$, then $C' = v_j dC[v_j^+, v_j]$ extends the cycle C, a contradiction.
- If $d \in N_1(u)$, then $d \notin \{v_i^-, v_i^+, v_j^-, v_j^+\}$. $v_j^+ \notin N_1(d^+)$ otherwise $C' = v_j u\overleftarrow{C}[d, v_j^+]C[d^+, v_j]$ extends C and $\langle d, d^-, d^+, u, v_j^+\rangle \cong K_{1,4}$ from $v_j^+ \notin N_1(d^-)$ otherwise on the one hand, $\langle d, d^-, d^+, u\rangle \cong K_{1,3}$ from $d^+ \notin N_1(d^-)$ if not $C' = v_j udC[v_j^+, d^-]C[d^+, v_j]$ extends the cycle C and the other hand, according to proposition 1, $d \notin A$. Without losing generality, suppose $d \in C[v_j^{++}, v_h^-]$.
  - $v_j^- \notin N_1(d^-)$ otherwise $C' = v_j uC[d, v_j^-]\overleftarrow{C}[d^-, v_j]$ extends the cycle C.
  - $d^+ \notin N_1(v_j^{++}) \cup N_1(d^{--})$ otherwise $C' = v_j udv_j^+\overleftarrow{C}[d^-, v_j^{++}]C[d^+, v_j]$ or $C' = v_j udd^- C[v_j^+, d^{--}]C[d^+, v_j]$ if $d^+ \in N_1(v_j^{++}) \cup N_1(d^{--})$ extends the cycle C.
  - $v_h \notin N_1(d^-) \cap N_1(v_j^+)$ otherwise
    - For $v_j^+ = d^-$, $v_h \notin N_1(d)$ and $v_h \in A$ de $C' = v_j uC[d, v_h^-]v_j^+ C[v_h, v_j]$ extends the cycle C if $v_j^+ \in N_1(v_h^-)$ and $\langle v_i, v_i^-, v_i^+, u, v_h^-\rangle \cong K_{1,4}$ if $v_h^- \in N_1(v_i)$ from $C' = v_j uv_i\overleftarrow{C}[v_h^-, v_j^+]C[v_h, v_i^-]C[v_i^+, v_j]$ or $C' = v_j u\overleftarrow{C}[v_i, v_h]C[v_j^+, v_h^-]C[v_i^+, v_j]$ extends C if $v_i^+ \in N_1(v_i^-) \cup N_1(v_h^-)$ and $\langle v_i, v_i^-, v_i^+, u, v_h\rangle \cong K_{1,4}$ if $v_h^- \in N_1(v_i^-)$ from $v_h^+ \notin N_1(v_i^+)$ otherwise $C' = v_j uv_i v_h C[v_j^+, v_h^-]\overleftarrow{C}[v_i^-, v_h^+]C[v_i^+, v_j]$ extends the cycle C.
    - For $v_j^+ \neq d^-$, $v_h \in A$ from $C' = v_j uC[d, v_h^-]\overleftarrow{C}[d^-, v_j^+]C[v_h, v_j]$ extends the cycle C if $d^- \in N_1(v_h^-)$ and $\langle v_j, v_j^-, v_j^+, u, d^-\rangle \cong K_{1,4}$ or $\langle v_h, v_h^-, v_j^+, v_i\rangle \cong K_{1,3}$ if $v_j \notin N_1(d^-) \cap N_1(v_h^-)$.
  - $d^{++} \notin N_1(d^-) \cup N_1(v_j^+)$ otherwise $d^{++} \notin N_1(d) \cup N_1(v_j)$ and $\langle v_j, v_j^-, v_j^+, u, v_h\rangle \cong K_{1,4}$ from
    - If $v_j^+ \in N_1(v_h)$, then $d^{++} \in N_1(v_h) - \{v_h^-\}$ otherwise $C' = v_j uC[d, d^{++}]\overleftarrow{C}[d^-, v_j^+]C[v_h, v_j]$ extends C and $v_j^+ \in A$. So $v_h \in A$ because firstly, $v_h^- \notin N_1(v_j)$ otherwise $v_h^- \notin N_1(v_i)$ and $\langle v_h, v_h^+, v_j^+, v_i\rangle \cong K_{1,3}$. Secondly, $d^{++} \notin N_1(v_h^+)$ otherwise $\langle d^+, d^+, d^-, v_j, v_h^+, \rangle \cong K_{1,4}$ from $\langle v_j, v_j^-, v_j^+, u, d^+\rangle \cong K_{1,4}$ or $\langle d, d^+, d^-, u, v_j^+, \rangle \cong K_{1,4}$ if $v_j \in N_1(d^+) \cup N_1(d^-)$ respectively because $C' = v_j udd^+\overleftarrow{C}[v_j^-, d^{++}]\overleftarrow{C}[d^-, v_j]$ extends C or $\langle v_j^+, v_j^-, v_h, d\rangle \cong K_{1,3}$ if $v_j^- \in N_1(d^+) \cup N_1(v_j^+)$ respectively.
    - If $v_i^+ \in N_1(v_j^-)$, then $d^{++} \neq v_h^-$ and $v_i^+ \neq v_j^-$ otherwise $C' = v_j uC[d, d^{++}]\overleftarrow{C}[d^-, v_j^+]\overleftarrow{C}[v_j^-, v_h]v_j$ or $C' = v_j u\overleftarrow{C}[v_i, v_j^+]v_j^- v_j$ extends C. Hence, $\langle d^{++}, d^+, d^-, d^{+++}\rangle \cong K_{1,3}$ from $C' = v_j udC[v_j^+, d^-]d^{++}d^+ C[d^{+++}, v_j]$ or $C' = v_j uC[d, d^{++}]C[v_j^+, d^-]C[d^{+++}, v_j]$ extends C if $d^{+++} \in N_1(d^+) \cup N_1(d^-)$. So $d^{++} \notin N_1(d)$ and $\langle v_j^+, v_j^-, d, d^{++}\rangle \cong K_{1,3}$ from $\langle d^{++}, d^-, d^+, v_j^-, d^{+++}\rangle \cong K_{1,4}$ if $d^{++} \in N_1(v_j^-)$ from $C' = v_j uC[d, v_j^-]\overleftarrow{C}[d^-, v_j]$ or $C' = v_j udd^+\overleftarrow{C}[v_i^-, d^{++}]\overleftarrow{C}[d^-, v_j]$ extends C if $v_j^- \in N_1(d^-) \cup N_1(d^+)$ and $\langle v_j^-, v_j^+, v_j^{--}, d^{+++}\rangle \cong K_{1,3}$ if $d^{+++} \in N_1(v_j^-)$ because $C' = v_j uC[d, d^{++}]\overleftarrow{C}[d^-, v_j^+]\overleftarrow{C}[v_j^-, d^{+++}]v_j^- v_j$ extends C or $\langle v_j^-, v_j, v_j^{--}, d^{++}\rangle \cong K_{1,3}$ if $v_j^{--} \in N_1(v_j^+) \cup N_1(d^{+++})$ from $v_j^{--} \notin N_1(v_j)$ otherwise $C' = v_j uC[d, d^{++}]\overleftarrow{C}[d^-, v_j^+]v_j^- C[d^{+++}, v_j^{--}]v_j$ extends C.

. $v_j^- \notin N_1(v_h)$ otherwise $v_h^+ \neq N_1(v_i)$ and $v_h \in A$ from $v_j \notin N_1(v_h^-) \cup N_1(v_h^+)$ otherwise $v_h \in A$ and $v_h^+ \notin N_1(v_h^-)$ otherwise $v_h^+ \neq N_1(v_i^-)$ and $\langle v_i, v_i^-, v_i^+, u, v_h\rangle \cong K_{1,4}$ because $C' = v_juC[v_i, v_j^-]v_h\overleftarrow{C}[v_i^-, v_h^+]\overleftarrow{C}[v_h^-, v_j]$ or $C' = v_juv_iv_h\overleftarrow{C}[v_j^-, v_i^+]\overleftarrow{C}[v_i^-, v_h^+]\overleftarrow{C}[v_h^-, v_j]$ extends the cycle C if $v_i^- \in N_1(v_h) \cup N_1(v_i^+)$ and $v_h \in A$ if $v_i^+ \in N_1(v_h)$ from $C' = v_ju\overleftarrow{C}[v_i, v_h^+]C[v_i^+, v_j^-]\overleftarrow{C}[v_h, v_j]$ extends the cycle C if $v_h^+ \in N_1(v_i^+)$.

(ii) Without losing generality assume that, $w \in C[v_j, v_h]$ and $w^{--} \in N_1(u)$. Firstly, $\langle w, w^-, w^+, u\rangle \cong K_{1,3}$, $\langle w^{--}, w^-, w^{---}, u\rangle \cong K_{1,3}$ from $C' = v_juwC[v_j^+, w^-]C[w^+, v_j]$ and $C' = wuw^{--}w^-\overleftarrow{C}[w^{---}, v_j^+]C[w, v_j]$ extend the cycle C if $w^- \in N_1(w^+) \cup N_1(w^{---})$ respectively. Secondly, $\langle u, v_k, w, w^{--}\rangle \cong K_{1,3}$ for $k \in \{i, j\}$ from $v_i \notin N_1(w) \cup N_1(w^{--})$ otherwise $\langle u, v_j, w^{--}, w\rangle \cong K_{1,3}$ and $w^{--} \notin N_1(w)$ otherwise $\langle w, w^-, w^+, u, v_j^+\rangle \cong K_{1,4}$ because $C' = v_ju\overleftarrow{C}[w^{--}, v_j^+]C[w^-, v_j]$ or $C' = v_ju\overleftarrow{C}[w, v_j^+]C[w^+, v_j]$ extends the cycle C if $v_j^+ \in N_1(w^-) \cup N_1(w^+)$ respectively.

(iii) For $v_j^- \notin N_1(d)$ and without losing generality, assume $d \in C[v_i, v_j]$, $d \notin \{v_i^+, v_j^{--}\}$ from Lemma 1,(i) and $v_h \notin \{v_i^-, v_j^+\}$ otherwise

- If $v_h = v_i^-$, then $\langle v_j, v_j^-, v_j^+, u, v_h\rangle \cong K_{1,4}$ if $v_j^+ \notin N_1(v_h)$ or $\langle v_j, v_j^-, d, u, v_h\rangle \cong K_{1,4}$ if $v_j^+ \in N_1(v_h)$, a contradiction.
- If $v_h = v_j^+$, then $\langle v_i, v_i^-, v_i^+, u, v_h\rangle \cong K_{1,4}$ if $v_i^- \notin N_1(v_h)$ from $v_i^+ \notin N_1(v_i^-)$ otherwise $C' = v_juv_iC[v_h, v_i^-]C[v_i^+, v_j]$ extends the cycle C, a contradiction.
  - According to proposition 1 $v_i \notin A$ from
    - $v_i^{--} \notin N_1(v_i^+)$ otherwise $C' = v_iu\overleftarrow{C}[v_j, v_i^+]\overleftarrow{C}[v_i^{--}, v_h]v_i^-v_i$ extends the cycle C. $d \notin N_1(v_i^-) \cap N_1(v_i^+)$ otherwise $\langle d, d^-, d^+, v_i^-\rangle \cong K_{1,3}$ and $\langle v_h, v_i^+, d, v_i\rangle \cong K_{1,3}$ from $v_i^- \notin N_1(d^-) \cap N_1(d^+)$ otherwise

$$\begin{cases} C' = v_juC[v_i, d^-]\overleftarrow{C}[v_i^-, v_h]C[d, v_j] & \text{if } v_i^- \in N_1(d^-) \\ C' = v_juC[v_i, d]C[v_h, v_i^-]C[d^+, v_j] & \text{if } v_i^- \in N_1(d^+) \end{cases}$$

extends the cycle C, $C' = v_iu\overleftarrow{C}[v_j, d^+]\overleftarrow{C}[d^-, v_i^+]dC[v_h, v_i]$ extends the cycle C if $d^+ \in N_1(d^-)$ and $v_h^+ \notin N_1(v_i) \cap N_1(d)$ otherwise $v_h^+ \in N_1(v_i^-)$ and $\langle v_h, v_h^+, v_i^-, v_j\rangle \cong K_{1,3}$.
    - Also, $v_i^{++} \notin N_1(v_i^-) \cap N_1(v_i)$ otherwise $v_i^{++} \notin N_1(v_i)$ and $v_i^{++} \neq d^-$ or else $C' = v_juC[v_i, v_i^{++}]\overleftarrow{C}[v_i^-, v_h]C[d, v_j]$ extends the cycle C. Hence, $\langle v_i^{++}, v_i^+, v_i^-, v_i^{+++}\rangle \cong K_{1,3}$ and $\langle v_h, v_h^+, v_i, v_i^{++}\rangle \cong K_{1,3}$ from $v_i^{+++} \notin N_1(v_i^+) \cup N_1(v_i^-)$ otherwise

$$\begin{cases} C' = v_ju\overleftarrow{C}[v_i, v_h]v_i^{++}v_i^+C[v_i^{+++}, v_j] & \text{if } v_i^{+++} \in N_1(v_i^+) \\ C' = v_juC[v_i, v_i^{++}]C[v_h, v_i^-]C[v_i^{+++}, v_j] & \text{if } v_i^{+++} \in N_1(v_i^-) \end{cases}$$

extends the cycle C and $\langle v_h, v_h^+, v_i^-, v_j\rangle \cong K_{1,3}$ if $v_h^+ \in N_1(v_i) \cup N_1(v_i^{++})$ from $v_h^+ \in N_1(v_j)$ otherwise $\langle v_h, v_h^+, v_i, v_j\rangle \cong K_4$ or $\langle v_h, v_h^+, v_i, d\rangle \cong K_{1,3}$.
- If $v_h \notin \{v_j^+, v_i^-\}$, $\langle v_j, v_j^-, d, u, v_h\rangle \cong K_{1,4}$ from $v_h \notin N_1(v_j^-) \cup N_1(d)$ otherwise
  - For $v_h^+ = v_i^-$, $v_i^+ \notin N_1(v_i^-)$ otherwise $C' = v_juv_iv_i^-C[v_i^+, v_j^-]\overleftarrow{C}[v_h, v_j]$ extends C. Consequently, $v_i \notin A$ according to proposition 1.
  - For $v_h^+ \neq v_i^-$,
    - If $v_h \in N_1(d)$, then $v_j^- \notin N_1(v_h)$, $\langle v_h, v_j^-, d, v_i\rangle \cong K_{1,3}$ and $\langle v_h, v_h^-, v_j^-, v_i, d\rangle \cong K_{1,4}$ from $\langle v_i, v_i^-, v_i^+, u, v_h^-\rangle \cong K_{1,4}$ if $v_h^- \in N_1(v_i)$ because $v_h^- \notin N_1(v_i^-) \cup N_1(v_i^+)$

otherwise $C' = v_j u C[v_i, v_j^-]C[v_h, v_i^-]\overleftarrow{C}[v_h^-, v_j]$ or $C' = v_j u \overleftarrow{C}[v_i, v_h]\overleftarrow{C}[v_j^-, v_i^+]\overleftarrow{C}[v_h^-, v_j]$ extends C and $v_i^+ \notin N_1(v_i^-)$ otherwise $\langle d, d^-, d^+, v_j^+\rangle \cong K_{1,3}$ because,
if $d^+ \in N_1(d^-)$, then $C' = v_j u v_i \overleftarrow{C}[v_h^-, v_j^+]C[d^+, v_j^-]C[v_h, v_i^-]C[v_i^-, d]v_j$ extends C and

$$\begin{cases} C' = v_j u v_i \overleftarrow{C}[v_h^-, v_j^+]dC[v_h, v_i^-]C[v_i^+, d^-]C[d^+, v_j] & \text{if } v_j^+ \in N_1(d^-) \\ \text{or} \\ C' = v_j u v_i \overleftarrow{C}[v_h^-, v_j^+]\overleftarrow{C}[d^-, v_i^+]\overleftarrow{C}[v_i^-, v_h]C[d, v_j] & \text{if } v_j^+ \in N_1(d^+) \end{cases}$$

extends C. For $v_h^- \in N_1(v_j^-)$, $v_i^+ \notin N_1(v_i^-) \cup N_1(v_h)$ otherwise
$C' = v_j u v_i C[v_h, v_i^-]C[v_i^+, v_j^-]\overleftarrow{C}[v_h^-, v_j]$
or $C' = v_j u \overleftarrow{C}[v_i, v_h]C[v_i^+, v_j^-]\overleftarrow{C}[v_i^-, v_j]$ extends C. So $v_i^- \in N_1(v_h)$ and $v_h^+ \in N_1(v_i^-)$ otherwise $\langle v_h, v_h^-, v_h^+, v_j, v_i^-\rangle \cong K_{1,4}$ from $C' = v_j u C[v_i, v_j^-]v_h \overleftarrow{C}[v_i^-, v_h^+]\overleftarrow{C}[v_h^-, v_j]$
or $C' = v_j u C[v_i, v_j^-]C[v_h, v_i^-]\overleftarrow{C}[v_h^-, v_j]$ extends C if $v_h^- \in N_1(v_h^+) \cup N_1(v_i^-)$ and if $v_h^+ \in N_1(v_j)$, then $\langle v_j, v_j^-, d, u, v_h^+\rangle \cong K_{1,4}$. Hence, $\langle v_h, v_h^+, v_i, d, v_j^-\rangle \cong K_{1,4}$ from
$C' = v_j u C[v_i, v_j^-]C[v_h^+, v_i^-]\overleftarrow{C}[v_h, v_j]$ extends C or $\langle d, d^-, d^+, v_j, v_h^+\rangle \cong K_{1,4}$
if $v_h^+ \in N_1(v_j^-) \cup N_1(d)$ because $C' = v_j u C[v_i, d^-]C[d^+, v_j^-]\overleftarrow{C}[v_h^-, v_j^+]dC[v_h^+ v_i^-]v_h v_j$ extends C if $d^+ \in N_1(d^-)$ and $C' = v_j u C[v_i, d^-]C[v_h^+, v_i^-]\overleftarrow{C}[v_h, v_j^+]C[d, v_j]$
or $C' = v_j u C[v_i, d]C[v_j^+, v_h]\overleftarrow{C}[v_i^-, v_h^+]C[d^+, v_j]$ extends C if $v_h^+ \in N_1(d^-) \cup N_1(d^+)$.
For $v_h^+ \in N_1(v_j^-)$, $\langle v_i, v_i^-, v_i^+, u, v_h\rangle \cong K_{1,4}$ from $C' = v_j u C[v_i, [v_j^-]C[v_h^+, v_i^-]\overleftarrow{C}[v_h, v_j]$ or
$C' = v_j u \overleftarrow{C}[v_i, v_h^+]\overleftarrow{C}[v_j^-, v_i^+]\overleftarrow{C}[v_h, v_j]$ extends C if $v_h \in N_1(v_i^-) \cup N_1(v_i^+)$ and $v_i^+ \notin N_1(v_i^-)$ otherwise $\langle d, , d^+, v_h, v_j^+\rangle \cong K_{1,3}$ from $C' = v_j u v_i \overleftarrow{C}[v_h, v_j^+]C[d^+, v_j^-]C[v_h^+, v_i^-]C[v_i^+, d]v_j$ extends C if $v_j^+ \notin N_1(d^+)$ and $\langle v_h, v_h^-, v_h^+, v_j, d^+\rangle \cong K_{1,4}$ if $v_h \notin N_1(d^+)$ from
$C' = v_j u v_i v_h \overleftarrow{C}[v_j^-, v_i^+]\overleftarrow{C}[v_i^-, v_h^+]\overleftarrow{C}[v_h^-, v_j]$ extends C if $v_h^+ \notin N_1(v_h^-)$
and if $d^+ \in N_1(v_h^-) \cup N_1(v_h^+)$, then $C' = v_j u v_i \overleftarrow{C}[v_h, v_j^+]\overleftarrow{C}[d, v_i^+]\overleftarrow{C}[v_i^-, v_h^+]C[d^+, v_j]$
or $C' = v_j u v_i C[v_h, v_i^-]C[v_i^+, d]C[v_j^+, v_h^-]C[d^+, v_j]$ extends C. Deduce at the end that $v_j^+ \in N_1(v_j^-)$, $v_j \notin N_1(d^-) \cup N_1(d^+)$ otherwise $\langle v_j, v_j^-, v_h, u, x\rangle \cong K_{1,4}$ if $x \notin N_1(v_j)$ for $x \in \{d^-, d^+\}$ and $v_j \notin N_1(v_h^+) \cup N_1(v_i^-)$ otherwise $\langle v_j, v_j^-, d, u, y\rangle \cong K_{1,4}$ if $y \notin N_1(v_j)$ for $y \in \{v_h^-, v_h^+\}$ from $v_j^- \in N_1(v_h^-) \cup N_1(v_h^+)$ otherwise $\langle v_i, v_i^-, v_i^+, u, v_h\rangle \cong K_{1,4}$ from
$C' = v_j u v_i \overleftarrow{C}[v_h, v_j^+]\overleftarrow{C}[v_j^-, v_i^+]\overleftarrow{C}[v_i^-, v_h^+]dv_j$ extends C if $v_i^+ \in N_1(v_i^-)$
and $C' = v_j u C[v_i, v_j^-]C[v_h^+, v_i^-]\overleftarrow{C}[v_h, v_j]$ or $C' = v_j u \overleftarrow{C}[v_i, v_h^+]\overleftarrow{C}[v_j^-, v_i^+]\overleftarrow{C}[v_h, v_j]$ extends C if $v_h \in N_1(v_i^-) \cup N_1(v_i^+)$. Also, $C' = v_j u v_i C[v_h, v_i^-]C[v_i^+, v_j^-]\overleftarrow{C}[v_h^-, v_j]$
or $C' = v_j u \overleftarrow{C}[v_i, v_h]C[v_i^+, v_j^-]\overleftarrow{C}[v_h^-, v_j]$ extends C if $v_i^+ \in N_1(v_i^-) \cup N_1(v_h)$
and $\langle v_h, v_h^-, v_h^+, v_i, d\rangle \cong K_{1,4}$ if $v_i^- \in N_1(v_h)$ from $C' = v_j u C[v_i, v_j^-]C[v_j^+, v_h^-]C[v_h^+, v_i^-]v_h v_j$ extends C if $v_h^+ \in N_1(v_h^-)$ and $\langle d, d^-, d^+, v_j, v_h^+\rangle \cong K_{1,4}$ if $v_h^+ \in N_1(d)$
from $C' = v_j u C[v_i, d^-]C[d^+, v_j^-]C[v_j^+, v_h]\overleftarrow{C}[v_i^-, v_h^+]dv_j$ extends C if $d^+ \in N_1(d^-)$ and

$$\begin{cases} C' = v_j u C[v_i, d^-]C[v_h^+, v_i^-]\overleftarrow{C}[v_h, v_j^+]\overleftarrow{C}[v_j^-, d]v_j & \text{if } v_h \in N_1(d^-) \\ \text{or} \\ C' = v_j u C[v_i, d]C[v_j^+, v_h]\overleftarrow{C}[v_i^-, v_h^+]C[d^+, v_j] & \text{if } v_h \in N_1(d^+) \end{cases}$$

- If $v_j^- \in N_1(v_h)$, then $\langle v_h, v_h^-, v_h^+, v_i, v_j^-\rangle \cong K_{1,4}$ from
  - $v_h^- \notin N_1(v_i)$ otherwise $v_h^- \notin N_1(v_i^-) \cup N_1(v_i^+)$, $v_i^+ \in N_1(v_i^-)$ and $v_h^+ \notin N_1(v_h^-)$ if not $C' = v_j u v_i v_h \overleftarrow{C}[v_j^-, v_i^+]\overleftarrow{C}[v_i^-, v_h^+]\overleftarrow{C}[v_h^-, v_j]$ extends C. Hence, $\langle v_h, v_h^-, v_h^+, v_j\rangle \cong K_{1,3}$ because $v_j \notin N_1(v_h^+)$ otherwise $\langle v_j, v_j^-, d, u, v_h^+\rangle \cong K_{1,4}$ from $v_h^+ \notin N_1(d)$ otherwise $\langle d, d^-, d^+, v_j^+\rangle \cong K_{1,3}$ and $\langle v_h, v_h^-, v_h^+, v_j^-\rangle \cong K_{1,3}$ from $C' = v_j u v_i \overleftarrow{C}[v_h^-, v_j^+] d C[v_h, v_i^-] C[v_i^+, d^-] C[d^+, v_j]$ extends C if $d^+ \in d^-$, $v_j^+ \notin N_1(d^-) \cap N_1(d^+)$ otherwise $C' = v_j u v_i \overleftarrow{C}[v_h^-, v_j^+]\overleftarrow{C}[d^-, v_i^+]\overleftarrow{C}[v_i^-, v_h]\overleftarrow{C}[v_j^-, d] v_j$ or $C' = v_j u v_i \overleftarrow{C}[v_h^-, v_j^+] C[d^+, v_j^-] C[v_h, v_i^-] C[v_i^+, d] v_j$ extends C and $v_h^- \notin N_1(v_j^-)$ otherwise $C' = v_j u v_i C[v_h, v_i^-] C[v_i^+, v_j^-]\overleftarrow{C}[v_h^-, v_j]$. So $v_j^+ \notin N_1(v_j^-)$ and $v_j \notin A$, according to proposition 1, otherwise $C' = v_j u v_i \overleftarrow{C}[v_h^-, v_j^+]\overleftarrow{C}[v_j^-, v_i^+]\overleftarrow{C}[v_i^-, v_h] v_j$ extends C.
  - $v_h^- \notin N_1(v_j^-)$ from $v_i^+ \notin N_1(v_i^-)$ otherwise $C' = v_j u v_i C[v_h, v_i^-] C[v_i^+, v_j^-]\overleftarrow{C}[v_h^-, v_j]$ extends C. Hence, $v_i \notin A$ according to proposition 1.
  - $v_h^+ \notin N_1(v_j^-)$ otherwise $v_h \notin N_1(v_i^-) \cup N_1(v_i^+)$ or else $C' = v_j u C[v_i, v_j^-] C[v_h^+, v_i^-]\overleftarrow{C}[v_h, v_j]$ or $C' = v_j u \overleftarrow{C}[v_i, v_h^+]\overleftarrow{C}[v_j^-, v_i^+]\overleftarrow{C}[v_h, v_j]$ extends C if $v_h \in N_1(v_i^-) \cup N_1(v_i^+)$. Hence, $v_i^+ \in N_1(v_i^-)$ and $\langle v_h, v_h^-, v_j^-, v_i\rangle \cong K_{1,3}$. Moreover, $v_j^+ \notin N_1(v_j^-)$ otherwise $\langle v_j^-, v_j^+, v_h, v_j^{--}\rangle \cong K_{1,3}$ from $C' = v_j u v_i \overleftarrow{C}[v_h, v_j^+]\overleftarrow{C}[v_j^{--}, v_i^+]\overleftarrow{C}[v_i^-, v_h^+] v_j^- v_j$ or $\langle v_j^-, v_j, v_j^{--}, v_h^+\rangle \cong K_{1,3}$ if $v_j^{--} \in N_1(v_j^+) \cup N_1(v_h)$.

- If $v_h = d$, then $v_h \in A$, $v_h^+ \neq v_i^-$ otherwise $v_i^+ \notin N_1(v_i^-)$ if not $C' = v_j u v_i v_h^- C[v_i^+, v_j^-]\overleftarrow{C}[v_h, v_j]$ and according to proposition 1 $v_i \notin A$, a contradiction.
  - If $v_j^+ = v_h^-$, then $v_j \notin A$ because $v_h^+ \notin N_1(v_j^+)$ and $v_j^{--} \notin N_1(v_j^+) \cap N_1(v_h)$ otherwise $\langle v_i, v_i^-, v_i^+, u, v_h\rangle \cong K_{1,4}$. $v_j^{--} \notin N_1(u)$ otherwise $\langle v_h, v_j^-, v_j^+, v_h^+, v_i\rangle \cong K_{1,4}$.
  - If $v_j^+ \neq v_h^-$, then $v_j \notin N_1(v_h^-) \cup N_1(v_h^+)$ otherwise $\langle v_j, v_j^-, v_j^+, u, x\rangle \cong K_{1,4}$ for $x \in \{v_h^-, v_h^+\}$ and $\langle v_i, v_i^-, v_i^+, u, v_h\rangle \cong K_{1,4}$ from $\langle v_h, v_h^-, v_h^+, y, v_j\rangle \cong K_{1,4}$ for $y \in \{v_h^-, v_h^+\}$ if $v_h \in N_1(v_i^-) \cup N_1(v_i^+)$ and $\langle v_h, v_h^+, v_j^-, v_j^+, v_i\rangle \cong K_{1,4}$ if $v_i^+ \in N_1(v_i^-)$ because $v_h^+ \notin N_1(v_j^-)$ otherwise $v_j \notin A$ according to proposition 1. Indeed, if $v_j^{--} \in N_1(v_j^+)$, then $C' = v_j u v_i \overleftarrow{C}[v_h, v_j^+]\overleftarrow{C}[v_j^{--}, v_i^+]\overleftarrow{C}[v_i^-, v_h^+] v_j^- v_j$ and if $v_j^{++} \in N_1(v_j^-)$, then $v_j^{++} \notin N_1(v_h)$ otherwise $\langle v_j^-, v_j, v_j^{++}, v_i^+\rangle \cong K_{1,3}$.

(iv) Suppose $v_h \in N_1(d)$. Then $v_h \notin N_1(v_j^-) \cup N_1(v_j^+)$. So, $\langle v_j, v_j^-, v_j^+, u, v_h\rangle \cong K_{1,4}$, a contradiction.
□

The following result assures us the lower bound to the number of triangles k.

Lemma 2.5 $k \geq 2$. *Hence,* $v_i \notin N_1(v_j^-) \cup N_1(v_j^+)$ *and* $v_j \notin N_1(v_i^-) \cup N_1(v_i^+)$.

Proof. Indeed, Suppose that $k = 1$, then $v_h \in \{v_i^-, v_j^+\}$. Without losing generality, suppose that $v_h = v_i^-$. So, $v_j^+ \notin N_1(v_j^-)$ and $v_j^+ \in N_1(v_h)$ or else $\langle v_j, v_j^-, v_j^+, v_h, u\rangle \cong K_{1,4}$.

- Hence, according to proposition 1, $v_j \notin A$ from:
  - **Case 01:** If $v_j^+ = v_h^-$, then $v_j^{--} \notin N_1(u)$ from $v_j^{--} \neq v_i^+$ otherwise $C' = v_i u C[v_j^{--}, v_i]$ extends C. So firstly, $\langle v_j^{--}, v_j^-, u, v_j^{---}\rangle \cong K_{1,3}$ and $v_j^{--} \in A$ secondly, $v_j^{--} \notin A$ according to proposition 1, contradiction. $v_j^{--} \notin N_1(v_j^+) \cap N_1(v_h)$ otherwise $\langle v_j^{--}, v_j^-, v_j^+, v_j^{---}, u\rangle \cong K_{1,4}$ from $C' = v_i u \overleftarrow{C}[v_j, v_j^{--}] C[v_j^+, v_i]$ extends C if $v_j^{--} = v_i^+$ and $C' = v_j u C[v_i, v_j^{---}] v_j^- v_j^{--} \overleftarrow{C}[v_h, v_j]$ or $C' = v_j u C[v_i, v_j^{---}] C[v_j^+, v_h] C[v_j^{--}, v_j]$ extends C if $v_j^{---} \in N_1(v_j^-) \cap N_1(v_j^+)$.

· **Case 02:** If $v_j^+ \neq v_h^-$, then more $v_j^{++} \notin N_1(v_j^-)$ otherwise $C' = v_j u C[v_i, v_j^-]C[v_j^{++}, v_h]v_j^+ v_j$ extends C.

- Clearly, $v_j^+ \notin N_1(v_i)$. If $v_j^- \in N_1(v_i)$, then $v_i^+ \in N_1(v_j^-)$ and $v_i^+ \notin N_1(v_h)$. So, $v_i \notin A$ from $v_j^{--} \notin N_1(u) \cup N_1(v_h)$ otherwise $C' = v_i u \overleftarrow{C}[v_j^{--}, v_i^+]C[v_j^-, v_i]$ or $C' = v_i u C[v_j, v_h]\overleftarrow{C}[v_j^{--}, v_i^+]v_j^- v_i$ extends C and $C' = v_j u C[v_i^{++}, v_j^-]\overleftarrow{C}[v_i^+, v_j]$ or $C' = v_i u C[v_j, v_h]C[v_i^{++}, v_j^-]v_i^+ v_i$ extends C if $v_i^{++} \in N_1(u) \cup N_1(v_h)$.

□

The number k also verifies

Lemma 2.6 $k = 2$.

Proof. Indeed, suppose $k \geq 3$. So $v_i^- \neq v_h^+$, $v_h^- \neq v_j^+$, and

$$\begin{cases} v_h \notin N_1(v_k^-) \cup N_1(v_k^+) & \textit{and} \quad v_k \notin N_1(v_h^-) \cup N_1(v_h^+) \quad \textit{for} \quad k \in \{i, j\} \\ v_k \notin N_1(v_m^-) \cup N_1(v_m^+) & \textit{for} \quad k, m \in \{i, j\} \quad \textit{and} \quad k \neq m \end{cases}$$

Also, $v_k^+ \in N_1(v_k^-)$ otherwise $\langle v_k, v_k^-, v_k^+, u, v_h\rangle \cong K_{1,4}$ for $k \in \{i, j\}$. Morever, $v_h^+ \in N_1(v_h^-)$ otherwise $v_h \in A$ and $v_j \notin A$ from

- $v_j^{--} \neq v_h$ and $v_j^{--} \notin N_1(v_h) \cap N_1(v_j^+)$ otherwise $C' = v_j u v_i v_i^+ \overleftarrow{C}[v_i^-, v_j^+]C[v_j^{--}, v_j]$ extends the cycle C and $\langle v_h, v_h^-, v_h^+, v_j, v_j^{--}\rangle \cong K_{1,4}$ from $v_j^{--} \notin N_1(v_h^-) \cap N_1(v_h^+)$ otherwise $C' = v_j u v_i C[v_h, v_i^-]C[v_i^+, v_j^{--}]\overleftarrow{C}[v_h^-, v_j^+]v_j^- v_j$ extends the cycle C or $C' = v_j u v_i v_i^+ \overleftarrow{C}[v_i^-, v_j^+]C[v_j^{--}, v_j]$ extends the cycle C if $v_j^{---} = v_i^+$. So $\langle v_j^{--}, v_j^+, v_h^+, v_j^{---}\rangle \cong K_{1,3}$ from $v_j^{---} \notin N_1(v_j^+) \cup N_1(v_h^+)$ otherwise $C' = v_j u v_i v_i^+ \overleftarrow{C}[v_i^-, v_h^+]v_j^- \overleftarrow{C}[v_h, v_j^+]v_j^- v_j$ extends the cycle C or $\langle v_j^{--}, v_j^+, v_h, v_j^{---}\rangle \cong K_{1,3}$.
- $v_j^{++} \neq v_h$ and $v_j^{++} \notin N_1(v_j^-) \cup N_1(v_h)$otherwise $C' = v_j u v_i C[v_h, v_i^-]C[v_i^+, v_j^-]v_j^+ v_j$ extends the cycle C and $\langle v_h, v_h^-, v_h^+, v_j, v_j^{++}\rangle \cong K_{1,4}$ from $v_j^{++} \neq v_h^-$ otherwise $C' = v_j u v_i C[v_h, v_i^-]C[v_i^+, v_j^-]\overleftarrow{C}[v_j^{++}, v_j]$ extends the cycle C and $v_j^{++} \notin N_1(v_h^+) \cap N_1(v_h^-)$ otherwise $C' = v_j u v_i \overleftarrow{C}[v_h, v_j^{++}]C[v_h^+, v_i^-]C[v_i^+, v_j^-]v_j^+ v_j$ extends C or $\langle v_j^{++}, v_j^-, v_h^-, v_j^{+++}\rangle \cong K_{1,3}$ because $v_j^{+++} \notin N_1(v_j^-) \cup N_1(v_h^-)$ otherwise $C' = v_j u v_i C[v_h, v_i^-]C[v_i^+, v_j^-]C[v_j^{+++}, v_h^-]]\overleftarrow{C}[v_j^{++}, v_j]$ extends the cycle C or $\langle v_j^{++}, v_j^-, v_h, v_j^{+++}\rangle \cong K_{1,3}$.

Finally, $v_i \notin A$ from $v_h \notin N_1(v_i^{++}) \cup N_1(v_i^{--})$ otherwise $C' = v_i u v_j v_h C[v_i^{++}, v_j^-]C[v_j^+, v_h^-]C[v_h^+, v_i^-]v_i^+ v_i$ or $C' = v_i u v_j v_h \overleftarrow{C}[v_i^{--}, v_h^+]\overleftarrow{C}[v_h^-, v_j^+]\overleftarrow{C}[v_j^-, v_i^+]v_i^- v_i$ extends the cycle C. □

The successor and the predecessor of the vertices v_i and v_j is such that

Lemma 2.7 $|\{v_i^- v_i^+, v_j^- v_j^+\} \cap E(G)| = 1$.

Proof. Suppose $|\{v_j^- v_j^+, v_i^- v_i^+\} \cap E(G)| \neq 1$. Then

- **Case 01:** If $|\{v_j^- v_j^+, v_i^- v_i^+\} \cap E(G)| = 0$, then $v_h \in [N_1(v_i^-) \cup N_1(v_i^+)] \cap [N_1(v_j^-) \cup N_1(v_j^+)]$.

  · **Sub-case 01:** For $v_h \in N_1(v_i^-) \cap N_1(v_j^-)$, $\langle v_h, v_h^-, v_i^-, v_j^-\rangle \cong K_{1,3}$ and $v_h^+ \neq v_i^-$ otherwise $v_i \notin A$ from $v_i^{++} \notin N_1(v_i^-) \cap N_1(v_h)$ otherwise $v_i^{++} \neq v_j^-$ and $\langle v_h, v_h^-, v_j^-, v_i, v_i^{++}\rangle \cong K_{1,4}$ from $\langle v_i^{++}, v_i^-, v_i^+, y\rangle \cong K_{1,3}$ for $y \in \{v_j^-, v_h^-\}$ and $v_i^{++} \in N_1(y)$ because $C' = v_i u C[v_j, v_i^-]C[v_i^{++}, v_j^-]v_i^+ v_i$ or $C' = v_i u C[v_j, v_h^-]C[v_i^{++}, v_j^-]v_h v_i$ extends the cycle C if $v_i^+ \in N_1(y)$. So $\langle v_h, v_h^-, v_h^+, v_i, v_j^-\rangle \cong K_{1,4}$ from $\langle v_i, v_i^-, v_i^+, u, z\rangle \cong K_{1,4}$ for $z \in \{v_h^-, v_h^+\}$ and $z \in N_1(v_i)$. Also $v_h^+ \notin N_1(v_h^-)$ otherwise $C' = v_j u \overleftarrow{C}[v_i, v_j^-]\overleftarrow{C}[v_i^-, v_h^+]\overleftarrow{C}[v_h^-, v_j]$ extends the cycle C.

· **Sub-case 02:** For $v_h \in N_1(v_i^+) \cap N_1(v_j^+)$, so $\langle v_h, v_h^+, v_i^+, v_j^+ \rangle \cong K_{1,3}$ and $v_j^+ \neq v_h^-$ otherwise, according to proposition 1, $v_j \notin A$ from $\langle v_h, v_h^+, v_i^+, v_j, v_j^{--} \rangle \cong K_{1,4}$ if $v_j^{--} \notin N_1(v_j^+) \cap N_1(v_h)$ and $\langle v_h, v_h^-, v_h^+, v_i^+, v_j^+ \rangle \cong K_{1,4}$ if $v_j^{++} \notin N_1(v_j^-)$ from $C' = v_i u C[v_j, v_h] C[v_i^+, v_j^+] C[v_h^+, v_i]$ or $C' = v_j u \overleftarrow{C}[v_i, v_j^+] \overleftarrow{C}[v_j^{--}, v_i^+] v_j^- v_j$ extends C if $y \notin N_1(v_j^{--})$ for $y \in \{v_h^+, v_i^+\}$ and $C' = v_j u \overleftarrow{C}[v_i, v_h^+] \overleftarrow{C}[v_h^-, v_j^+] \overleftarrow{C}[v_j^-, v_i^+] v_h v_j^+ v_j$ or $C' = v_j u \overleftarrow{C}[v_i, v_h] C[v_i^+, v_j^-] \overleftarrow{C}[v_j^{++}, v_h^-] v_j^+ v_j$ or $C' = v_j u \overleftarrow{C}[v_i, v_h] \overleftarrow{C}[v_j^-, v_i^+] \overleftarrow{C}[v_h^-, v_j]$ extends C if $v_h^- \notin N_1(v_h^+) \cup N_1(v_j^+) \cup N_1(v_i^+)$.

· **Sub-case 03:** For $v_h \in N_1(v_i^-) \cap N_1(v_j^+)$, $v_j^+ \neq v_h^-$ and $v_h^+ \neq v_i^-$ otherwise $v_i \notin A$ or $v_j \notin A$. Hence, $\langle v_h, v_h^-, v_h^+, v_j, v_i^- \rangle \cong K_{1,4}$ from $x \notin N_1(v_j)$ otherwise $\langle v_j, v_j^-, v_j^+, u, x \rangle \cong K_{1,4}$ for $x \in \{v_h^-, v_h^+\}$ and $v_h^+ \notin N_1(v_h^-)$ otherwise v_h and v_j are adjacents on $C' = v_j v_h C[v_h^+, v_j]$ and $k = 1$, a contradiction. Finally, $v_h^+ \notin N_1(v_i^-)$ otherwise $v_i \notin A$ from $v_i^{--} \neq v_h^+$ and $v_i^{++} \neq v_j^-$, $v_i^{--} \notin N_1(v_i^+) \cap N_1(v_h)$ and $v_i^{++} \notin N_1(v_i^-) \cup N_1(v_h)$ otherwise $\langle v_j, v_j^-, v_j^+, u, y \rangle \cong K_{1,4}$ for $y \in \{v_i^{--}, v_i^{++}\}$ from $y \notin N_1(v_h^-)$ otherwise $\langle y, v_i^-, v_i^+, v_h^- \rangle \cong K_{1,3}$ because $C' = v_i u \overleftarrow{C}[v_j, v_i^+] \overleftarrow{C}[v_h^-, v_j^+] C[v_h, v_i]$ extends the cycle C if $v_i^+ \notin N_1(v_h^-)$.

· **Sub-case 04:** Fir $v_h \in N_1(v_i^+) \cap N_1(v_j^-)$, $v_h \in A$, $v_j^+ \neq v_h^-$ and $v_h^+ \neq v_i^-$ otherwise $v_i \notin A$ or $v_j \notin A$. $v_h^+ \notin N_1(v_h^-)$ otherwise v_h and v_j are adjacents on $C' = v_j v_h \overleftarrow{C}[v_j^-, v_h^+] \overleftarrow{C}[v_h^-, v_j]$ and $k = 1$. And therefore, $\langle v_h, v_h^-, v_h^+, v_i, v_j^- \rangle \cong K_{1,4}$ from $C' = v_i u C[v_j, v_h] C[v_i^+, v_j^-] C[v_h^+, v_i]$ extends C if $v_h^+ \in N_1(v_j^-)$. So, according to proposition 1, $v_i \notin A$ from $\langle v_h, v_h^-, v_h^+, v_i^+, v_j^- \rangle \cong K_{1,4}$ if $v_i^{++} \notin N_1(v_i^-)$ from $v_i^+ \notin N_1(v_j^-)$ otherwise $C' = v_j u v_i C[v_h^+, v_i^-] C[v_i^{++}, v_j^-] v_i^+ \overleftarrow{C}[v_h, v_j]$ extends the cycle C. Also, $v_i^{--} \notin N_1(v_i^+)$ otherwise $\langle v_i, v_i^-, v_i^+, u, v_h^+ \rangle \cong K_{1,4}$ from $v_h^+ \notin N_1(v_i^-)$ otherwise $C' = v_j u v_i v_i^-] C[v_h^+, v_i^{--}] C[v_i^+, v_j^-] \overleftarrow{C}[v_h, v_j]$ extends C.

- **Case 02:** If $|\{v_j^- v_j^+, v_i^- v_i^+\} \cap E(G)| = 2$, then $v_j^+ \neq v_h^-$, $v_h^+ \neq v_i^-$, $v_i^+ \neq v_j^-$ and $v_k \notin N_1(v_h^-) \cup N_1(v_h^+)$ for $k \in \{i, j\}$. Hence, $v_h \notin N_1(v_k^-) \cup N_1(v_k^+)$ for $k \in \{i, j\}$. Without losing generality, assume that $v_h \notin N_1(v_i^-)$, so $\langle v_h, v_h^-, v_h^-, v_j, v_i^- \rangle \cong K_{1,4}$ from $C' = v_j u v_i v_h \overleftarrow{C}[v_i^-, v_h^+] \overleftarrow{C}[v_h^-, v_j^+] \overleftarrow{C}[v_j^-, v_i]$ extends the cycle C if $v_h^+ \in N_1(v_h^-)$ and $v_h^+ \notin N_1(v_i^-)$ otherwise $v_i \notin A$ from $v_i^{++} \notin N_1(v_i^-) \cap N_1(v_h)$ otherwise $\langle v_i^-, v_i, v_i^{++}, v_h^+ \rangle \cong K_{1,3}$ and $\langle v_i^-, v_i, v_i^{--}, v_h^+ \rangle \cong K_{1,3}$ if $v_i^{--} \in N_1(v_i^-) \cap N_1(v_h)$ because $v_i^{--} \neq v_h^+$ otherwise $C' = v_j u \overleftarrow{C}[v_i, v_i^{--}] C[v_i^+, v_j^-] C[v_j^+, v_h] v_j$ extends C.

□

Either then

Lemma 2.8 $v_i^+ \in N_1(v_i^-)$.

Proof. Assume that $v_i^+ \notin N_1(v_i^-)$. Firstly, $v_j^+ \in N_1(v_j^-)$ and $v_h \in N_1(v_i^-) \cup N_1(v_i^+)$. So, $v_h^+ \notin N_1(v_h^-)$ and $v_h \in A$ otherwise $C' = v_i u v_j v_h \overleftarrow{C}[v_i^-, v_h^+] \overleftarrow{C}[v_h^-, v_j^+] \overleftarrow{C}[v_j^-, v_i]$ or $C' = v_i u v_j v_h C[v_i^+, v_j^-] C[v_j^+, v_h^-] C[v_h^+, v_i]$ extends the cycle C if $v_h \in N_1(v_i^+) \cup N_1(v_i^-)$. Secondly, according to proposition 1 $v_i \notin A$ from

- **Case 01:** Assume that $v_h \in N_1(v_i^-) - N_1(v_i^+)$. Then

  – **Sub-case 01:** $v_i^{--} \notin N_1(v_i^+)$, $v_i^{--} \neq v_h^+$ otherwise $C' = v_j u \overleftarrow{C}[v_i, v_i^{--}] C[v_i^+, v_j^-] C[v_j^+, v_h] v_j$ extends the cycle C. Hence, $v_i^{--} \notin N_1(v_i^+) \cap N_1(v_h)$ otherwise $\langle v_h, v_h^-, v_h^+, v_j, v_i^- \rangle \cong K_{1,4}$ from $v_h^+ \notin N_1(v_i^-)$ otherwise $C' = v_j u v_i v_i^- C[v_h^+, v_i^{--}] C[v_i^+, v_j^-] C[v_j^+, v_h] v_j$ extends the cycle C.

– **Sub-case 02:** $v_i^{++} \notin N_1(v_i^+) \cap N_1(v_h)$ otherwise $\langle v_h, v_h^-, v_h^+, v_i, v_i^{++}\rangle \cong K_{1,4}$ from $y \notin N_1(v_i^{++})$ for $y \in \{v_h^+, v_h^-\}$ otherwise $\langle v_i^{++}, v_i^-, v_i^+, y\rangle \cong K_{1,3}$ because $C' = v_i u v_j C[v_h, v_i^-]C[v_i^{++}, v_j^-]C[v_j^+, v_h^-]v_i^+ v_i$ extends the cycle C if $v_i^+ \in N_1(v_h^-)$.

- **Case 02:** Assume that $v_h \in N_1(v_i^+) - N_1(v_i^-)$. Then $v_j^+ \neq v_h^-$ otherwise $C' = v_j u \overleftarrow{C}[v_i, v_h]C[v_i^+, v_j^-]v_j^+ v_j$ extends the cycle C. Also
  - **Sub-case 01:** $v_i^{--} \notin N_1(v_i^+) \cup N_1(v_h)$ otherwise $v_i^{--} \neq v_h^+$ and $\langle v_h, v_h^-, v_h^+, v_j, v_i^+\rangle \cong K_{1,4}$ from $v_i^+ \notin N_1(v_h^-)$ otherwise $\langle v_i^+, v_i, v_i^{--}, v_h^-\rangle \cong K_{1,3}$.
  - **Sub-case 02:** $v_i^{++} \notin N_1(v_i^-)$ otherwise $C' = v_j u v_i C[v_i, v_i^{++}]\overleftarrow{C}[v_i^-, v_j^+]v_j^- v_j$ extends the cycle C if $v_i^{++} = v_j^-$. So $v_h^+ \neq v_i^-$ and $\langle v_h, v_h^-, v_i^-, v_i^+, v_j\rangle \cong K_{1,4}$ from $C' = v_j u v_i v_i^- C[v_i^{++}, v_j^-]C[v_j^+, v_h^-]v_i^+ v_h v_i$ extends the cycle C if $v_i^+ \in N_1(v_h^-)$.
- **Case 03:** Assume that $v_h \in N_1(v_i^+) \cap N_1(v_i^-)$. Then $\langle v_h, v_i^-, v_i^+, v_j\rangle \cong K_{1,3}$ and
  - **Sub-case 01:** $v_i^{--} \notin N_1(v_i^+)$ otherwise $\langle v_h, v_h^+, v_i^-, v_i^+, v_j\rangle \cong K_{1,4}$ from $v_h^+ \notin N_1(v_i^-)$ otherwise $C' = v_i u v_j \overleftarrow{C}[v_h, v_j^+]\overleftarrow{C}[v_j^-, v_i^+]\overleftarrow{C}[v_i^{--}, v_h^+]v_i^- v_i$ extends the cycle C.
  - **Sub-case 02:** $v_i^{++} \notin N_1(v_i^-)$ otherwise $\langle v_h, v_h^-, v_i^-, v_i^+, v_j\rangle \cong K_{1,4}$ from $v_i^+ \notin N_1(v_h^-)$ otherwise $C' = v_i u v_j C[v_h, v_i^-]C[v_i^{++}, v_j^-]C[v_j^+, v_h^-]v_i^+ v_i$ extends the cycle C.

□

The vertex v_h is incident to v_j^- or v_j^+ and belongs to the set A.

Lemma 2.9 $v_h \in N_1(v_j^+) \cup N_1(v_j^-)$, *$v_h^+ v_h^- \notin E(G)$ and $v_h \in A$.*

Proof. If $v_h \notin N_1(v_j^-) \cup N_1(v_j^+)$, then $\langle v_j, v_j^-, v_j^+, u, v_h\rangle \cong K_{1,4}$. $v_h^+ \notin N_1(v_h^-)$ otherwise the cycle

$$\begin{cases} C' = v_j u v_i v_h C[v_j^+, v_h^-]C[v_h^+, v_i^-]C[v_i^+, v_j] & \text{if } v_h \in v_j^+ \\ \text{or} & \\ C' = v_j u v_i v_h \overleftarrow{C}[v_j^-, v_i^+]\overleftarrow{C}[v_i^-, v_h^+]\overleftarrow{C}[v_h^-, v_j] & \text{if } v_h \in v_j^- \end{cases}$$

extends the cycle C. $v_h^+ \neq v_i^-$ otherwise $v_j \notin A$ from

- $v_j^{--} \notin N_1(v_j^+) \cap N_1(v_h)$ or else, $C' = v_j u v_i v_i^- C[v_i^+, v_j^{---}]C[v_j^+, v_h]C[v_j^{--}, v_j]$ if $v_j^{---} \in N_1(v_j^+)$ or $C' = v_j u v_i v_i^- C[v_i^+, v_j^{---}]v_j^- v_j^{--} \overleftarrow{C}[v_h, v_j]$ extends C if $v_j^{---} \in N_1(v_j^-)$.
  Therefore, $\langle v_j^{--}, v_h^-, v_j^+, v_j^{---}\rangle \cong K_{1,3}$ and $\langle v_h, v_h^-, v_j^{--}, v_j\rangle \cong K_{1,3}$, a contradiction.
- $v_j^{++} \notin N_1(v_j^-)$ otherwise $C' = v_j u v_i v_i^- C[v_i^+, v_j^-]C[v_j^{++}, v_h]v_j^+, v_j$ extends C.

So, $\langle v_h, v_h^-, v_h^+, v_i\rangle \cong K_{1,3}$ and $v_h \in A$. □

Also,

Lemma 2.10 $\langle v_j, v_j^-, v_j^+, u, v_h\rangle \cong K_{1,4}$.

Proof. Suppose that $v_h \in N_1(v_j^+) \cup N_1(v_j^-)$. Then

- **Case 01:** For $v_h \in N_1(v_j^+) - N_1(v_j^-)$, $v_i^+ \notin N_1(v_j^-)$ otherwise $C' = v_j u v_i v_i^+ \overleftarrow{C}[v_i^-, v_j]$ extends the cycle C. So $v_j \notin A$ from

· $v_j^{++} \notin N_1(v_j^-)$ otherwise $C' = v_j u v_i C[v_h, v_i^-]C[v_i^+, v_j^-]C[v_j^{++}, v_j]$ extends C if $v_j^{++} = v_h^-$. In this case, $v_j^{++} \notin N_1(v_j^-) \cup N_1(v_h)$ otherwise $C' = v_j u v_i v_i^- \overleftarrow{C}[v_i^-, v_j^-]\overleftarrow{C}[v_j^{++}, v_h]v_j^+ v_j$ extends C if $v_h^+ = v_i^-$. Therefore, $C' = v_j u v_i C[v_h, v_i^-]C[v_i^+, v_j^-]C[v_j^{++}, v_h^-]v_j^+ v_j$ or $C' = v_j u v_i v_h C[v_j^+, v_h^-]C[v_h^+, v_i^-]C[v_i^+, v_j]$ extends C if $v_h^- \in N_1(v_j^+) \cup N_1(v_h^+)$. So, $\langle v_h, v_h^-, v_h^+, v_i, v_j^+\rangle \cong K_{1,4}$.

· $v_j^{--} \notin N_1(v_h) \cup N_1(v_j^+)$ otherwise $C' = v_j u v_i v_i^+ \overleftarrow{C}[v_i^-, v_j^+]C[v_j^{--}, v_j]$ extends C if $v_j^{--} = v_i^{++}$. Therefore,

$$\begin{cases} \langle v_j^{--}, v_j^-, v_j^+, v_j^{---}\rangle \cong K_{1,3} \text{ and } \langle v_h, v_i^-, v_j, v_j^{--}\rangle \cong K_{1,3} \text{ if } v_i^- = v_h^+ \\ or \\ \langle\langle v_h, v_h^+, v_j^{--}, v_j\rangle \cong K_{1,3} \text{ and } v_j^{--}, v_j^-, v_j^+, v_j^{---}\rangle \cong K_{1,3} \text{ if } v_i^- \neq v_h^+ \end{cases}$$

from

* $C' = v_j u C[v_i, v_j^{---}]v_j^- v_j^{--} C[v_j^{--}, v_j]$ or $C' = v_j u C[v_i, v_j^{---}]C[v_j^+, v_i^-]C[v_j^{--}, v_j]$ extends C if $v_j^{---} \in N_1(v_j^-) \cup N_1(v_j^+)$ and $v_j^{--} \notin N_1(v_i^-)$ otherwise $C' = v_j u C[v_i, v_j^{---}]\overleftarrow{C}[v_i^-, v_j^+]C[v_j^{--}, v_j]$ extends C if $v_j^{---} \in N_1(v_i^-)$. Therefore $\langle v_j^{--}, v_j^-, v_j^+, v_i^-, v_j^{--}\rangle \cong K_{1,4}$;
* $v_h^+ \notin N_1(v_j) \cup N_1(v_j^{--})$ otherwise $\langle v_j, v_j^-, v_j^+, u, v_h^+\rangle \cong K_{1,4}$ or $\langle v_j^{--}, v_j^-, v_j^+, v_h^+, v_j^{--}\rangle \cong K_{1,4}$. $v_j^{--} \notin N_1(v_j^-) \cup N_1(v_j^+)$ otherwise $C' = v_j u v_i C[v_h, v_i^-]C[v_i^+, v_j^{---}]v_j^- v_j^{--} v_j^+ v_j$ or $C' = v_j u v_i C[v_h, v_i^-]C[v_i^+, v_j^{---}]v_j^+ C[v_j^{--}, v_j]$ extends C if $v_j^+ = v_h^-$ and therefore $\langle v_h, v_h^-, v_h^+, v_j, v_j^{--}\rangle \cong K_{1,4}$ because for $w \in \{v_h^-, v_h^+\}$, $w \notin N_1(v_j) \cup N_1(v_j^{--})$ otherwise $\langle v_j, v_j^-, v_j^+, u, w\rangle \cong K_{1,4}$ or $\langle v_j^{--}, v_j^-, v_j^+, w\rangle \cong K_{1,3}$.

- **Case 02:** For $v_h \in N_1(v_j^-) - N_1(v_j^+)$, $v_i^+ \neq v_i^-$ otherwise $C' = v_j u v_i v_i^- C[v_i^+, v_j^-]\overleftarrow{C}[v_h, v_j]$. Therefore, $\langle v_h, v_h^-, v_h^+, v_j, v_j^-\rangle \cong K_{1,4}$ because $v_h^+ \notin N_1(v_j^-)$ otherwise $v_j \notin A$ from

  – $v_j^{--} \notin N_1(v_j^+)$ otherwise $C' = v_j u v_i \overleftarrow{C}[v_h, v_j^+]\overleftarrow{C}[v_j^{--}, v_i^+]\overleftarrow{C}[v_i^-, v_h^+]v_j^- v_j$ extends the cycle C.

  – $v_j^{++} \neq v_h^-$ and $v_j^{++} \notin N_1(v_j^-) \cup N_1(v_h)$ otherwise $\langle v_j^-, v_j^{++}, v_j, v_h^+\rangle \cong K_{1,3}$.

- **Case 03:** For $v_h \in N_1(v_j^-) \cap N_1(v_j^+)$, $v_h^+ \neq v_i^-$ otherwise $C' = v_j u v_i v_i^- C[v_i^+, v_j^-]\overleftarrow{C}[v_h, v_j]$ extends the cycle C. So $\langle v_h, v_h^+, v_i, v_j^-, v_j^+\rangle \cong K_{1,4}$ because $v_h^+ \notin N_1(v_j^-)$ otherwise $v_j \notin A$ from

  · $C' = v_j u v_i v_i^+ \overleftarrow{C}[v_i^-, v_h^+]v_j^- \overleftarrow{C}[v_h, v_j]$ if $v_j^- = v_i^+$ and $C' = v_j u v_i \overleftarrow{C}[v_h, v_j^+]\overleftarrow{C}[v_j^{--}, v_i^+]\overleftarrow{C}[v_i^-, v_h^+]v_j^- v_j$ extends C if $v_j^{--} \in N_1(v_j^+)$.

  · If $v_j^+ \neq v_h^-$, then $v_j^{++} \notin N_1(v_j^-)$ otherwise $C' = v_j u v_i C[v_h, v_i^-]C[v_i^+, v_j^i]\overleftarrow{C}[v_j^{++}, v_j]$ extends C if $v_j^{++} = v_h^-$ and therefore $\langle v_h, v_h^-, v_i, v_j^-, v_j^+\rangle \cong K_{1,4}$ from $C' = v_j u v_i C[v_h, v_i^-]C[v_i^+, v_j^-]C[v_j^{++}, v_h^-]v_j^+ v_j$ extends C if $v_j^+ \in N_1(v_h^-)$.

□

As every independent set is also P_4-free, the following corollary is a direct consequence of our theorem.

Corollary 2.11 *Every connected, locally connected and partly claw-free graph of at least three vertices is a fully cycle extendable graph if it is $\{K_4, K_{1,4}\}$-free and its set of claws is independent.*

3. Conclusion

At the end of this work, I will conclude by recalling that a triangularly connected and partly claw-free graph is a fully cycle extendable graph if it is $\{K_{1,4}, K_4\}$-free and if its set of claw centers A is P_4-free. We have proved this result by using a new result of partly claw-free graphs proved in this paper. The partly claw-fee graphs were required to be K_4-free but the set of its claw centers was expanded to be P_4-free instead of an independent set.
Questions that we can ask ourselves and which will be our next perspective are:

· Should we conclude that the triangularly connected, partly claw-free and $K_{1,4}$-free graphs remain fully cycle extendable if the K_4-free condition is removed?

· Should we conclude that the triangularly connected, partly claw-free and $K_{1,4}$-free graphs remain fully cycle extendable if its set of claw centers is K_3-free?

Other distant objectives will be to verify the property of full cycle extendability in the class of $[\mu, \eta]$-regular cycle graphs defined by M.Mollard [4] and the K-regular graphs.

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