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On some inequalities of Fejér's type using the notion of harmonic convexity

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Abstract. In this study, we introduce some new mappings in connection with Hermite-Hadamard and Fejér type integral inequalities which have been proved using the harmonic convex functions. As a consequence, we discover certain new inequalities of the Fejér type that provide refinements of the Hermite-Hadamard and Fejér type integral inequalities that have already been obtained.

1. Introduction

For convex functions the following double inequality has great significance in literature and is known as Hermite-Hadamard's inequality:

Let $f : I \longrightarrow \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, $u, v \in I$ with $u < v$, be a convex function then

$$f\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u}\int_u^v f(x)dx \leq \frac{f(u)+f(v)}{2}, \tag{1}$$

the inequality holds in reversed direction if f is concave.
Fejér [15], established the following double inequality as a weighted generalization of (1):

$$f\left(\frac{u+v}{2}\right)\int_u^v p(x)dx \leq \int_u^v f(x)p(x)dx \leq \frac{f(u)+f(v)}{2}\int_u^v p(x)dx, \tag{2}$$

where $f : I \longrightarrow \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, $u, v \in I$ with $u < v$ is any convex function and $p : [u, v] \rightarrow \mathbb{R}$ is non-negative integrable and symmetric about $x = \frac{u+v}{2}$.
These inequalities have many extensions and generalizations, see [2]-[16] and [19]-[36].
Consider the following mappings on $[0, 1]$:

$$G(t) = \frac{1}{2}\left[f\left(tu + (1-t)\frac{u+v}{2}\right) + f\left(tv + (1-t)\frac{u+v}{2}\right)\right],$$

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$$Q(t)=\frac{1}{2}\left[f\left(tu+(1-t)v\right)+f\left(tv+(1-t)u\right)\right],$$

$$H(t)=\frac{1}{v-u}\int_u^v f\left(tx+(1-t)\frac{u+v}{2}\right)dx,$$

$$H_p(t)=\int_u^v f\left(tx+(1-t)\frac{u+v}{2}\right)p(x)\,dx,$$

$$I(t)=\frac{1}{2}\int_u^v\left[f\left(t\frac{u+x}{2}+(1-t)\frac{u+v}{2}\right)+f\left(t\frac{v+x}{2}+(1-t)\frac{u+v}{2}\right)\right]p(x)\,dx$$

$$P(t)=\frac{1}{2}\int_u^v\left[f\left(\left(\frac{1+t}{2}\right)u+\left(\frac{1-t}{2}\right)x\right)+f\left(\left(\frac{1+t}{2}\right)v+\left(\frac{1-t}{2}\right)x\right)\right]dx,$$

$$P_p(t)=\frac{1}{2(v-u)}\int_u^v\left[f\left(\left(\frac{1+t}{2}\right)u+\left(\frac{1-t}{2}\right)x\right)p\left(\frac{u+x}{2}\right)+f\left(\left(\frac{1+t}{2}\right)v+\left(\frac{1-t}{2}\right)x\right)p\left(\frac{x+v}{2}\right)\right]dx,$$

$$N(t)=\frac{1}{2}\int_u^v\left[f\left(tu+(1-t)\frac{u+x}{2}\right)+f\left(tv+(1-t)\frac{x+v}{2}\right)\right]p(x)\,dx,$$

$$L(t)=\frac{1}{2(v-u)}\int_u^v\left[f\left(tu+(1-t)x\right)+f\left(tv+(1-t)x\right)\right]dx,$$

$$L_p(t)=\frac{1}{2}\int_u^v\left[f\left(tu+(1-t)x\right)+f\left(tv+(1-t)x\right)\right]p(x)\,dx$$

and

$$S_p(t)=\frac{1}{2}\int_u^v\left[f\left(tu+(1-t)\frac{u+x}{2}\right)+f\left(tu+(1-t)\frac{x+v}{2}\right)\right.$$
$$\left.+f\left(tv+(1-t)\frac{u+x}{2}\right)+f\left(tv+(1-t)\frac{x+v}{2}\right)\right]p(x)\,dx,$$

where $f:[u,v]\to\mathbb{R}$ is a convex function and $p:[u,v]\to\mathbb{R}$ is non-negative integrable and symmetric about $x=\frac{u+v}{2}$.

Remark 1.1. *It should be noted that* $H=H_p=I$, $P=P_p=N$ *and* $L=L_p=S_p$ *on* $[0,1]$ *as* $p(x)=\frac{1}{v-u}$, $x\in[u,v]$.

The important results that characterize the properties of the mappings G,H,Q,I,P,N,L and refinement inequalities for (1) are discussed by a number of mathematicians, see for instance the studies conducted in Tseng et al. [27], Dragomir et al. [2], Yang and Hong [32]. Yang and Tseng [33] and Tseng et al. [27] established the following Fejér-type inequalities by using the mappings H_p, P_p which are weighted generalizations of proved in [2], [27], [32] and [33].

Dragomir et al. [7], obtained the Hermite-Hadamard-type inequality given as:

Theorem 1.2. *[7] Let f, H, G and L be defined as above. Then G is convex, increasing on $[0,1]$, L is convex on $[0,1]$ and for all $t \in [0,1]$, we have*

$$H(t) \leq G(t) \leq L(t) \leq \frac{1-t}{v-u}\int_u^v f(x)\,dx + t\cdot\frac{f(u)+f(v)}{2} \leq \frac{f(u)+f(v)}{2}. \tag{3}$$

Tseng et al. [28, 29] provided a weighted generalizations of the inequalities (3) which is given in the theorems below:

Theorem 1.3. *[28] Let f, p, G, H_p and L_p be defined as above. Then L_p is convex, increasing on $[0,1]$, and for all $t \in [0,1]$, we have*

$$H_p(t) \leq G(t)\int_u^v p(x)\,dx \leq L_p(t) \leq (1-t)\int_u^v f(x)p(x)\,dx + t\cdot\frac{f(u)+f(v)}{2}\int_u^v p(x)\,dx \leq \frac{f(u)+f(v)}{2}\int_u^v p(x)\,dx. \tag{4}$$

Theorem 1.4. *[28] Let f, p, G, I and S_p be defined as above. Then S_p is convex, increasing on $[0,1]$, and for all $t \in [0,1]$, we have*

$$I(t) \leq G(t)\int_u^v p(x)\,dx \leq S_p(t) \leq \frac{(1-t)}{2}\int_u^v \left[f\left(\frac{u+x}{2}\right)+f\left(\frac{x+v}{2}\right)\right]p(x)\,dx + t\cdot\frac{f(u)+f(v)}{2}\int_u^v p(x)\,dx \leq \frac{f(u)+f(v)}{2}\int_u^v p(x)\,dx. \tag{5}$$

Teseng et al. [27] used the following result to prove his results.

Lemma 1.5. *[27] Let $f:[u,v]\to\mathbb{R}$ be a convex function and let $u \leq y_1 \leq x_1 \leq x_2 \leq y_2 \leq v$ with $x_1+x_2 = y_1+y_2$. Then*

$$f(x_1)+f(x_2) \leq f(y_1)+f(y_2).$$

The assumptions in Lemma 1.5 can be weakened as in the following lemma:

Lemma 1.6. *[31] Let $f:[u,v]\to\mathbb{R}$ be a convex function and let $u \leq y_1 \leq x_1 \leq y_2 \leq v$ and $u \leq y_1 \leq x_2 \leq y_2 \leq v$ with $x_1+x_2 = y_1+y_2$. Then*

$$f(x_1)+f(x_2) \leq f(y_1)+f(y_2).$$

Lemma 1.7. *[31] Let f, G, Q be defined as above. Then Q is symmetric about $\frac{1}{2}$, Q is decreasing on $\left[0,\frac{1}{2}\right]$ and and increasing on $\left[\frac{1}{2},1\right]$,*

$$G(2t) \leq Q(t),\quad t\in\left[0,\frac{1}{4}\right],$$

$$G(2t) \geq Q(t),\quad t\in\left[\frac{1}{4},\frac{1}{2}\right],$$

$$G(2(1-t)) \geq Q(t),\quad t\in\left[\frac{1}{2},\frac{3}{4}\right],$$

and

$$G(2(1-t)) \leq Q(t),\quad t\in\left[\frac{3}{4},1\right].$$

Tseng et al. [30], proved new Fejér-type inequalities related to the mappings $G, Q, H_p, P_p, I, N, L_p$ and S_p defined above. These results generalize known results obtained in connection to the Hermite-Hadamard inequality (1) and therefore are useful in obtaining various results of means for a given convex function f and particular weight function p. Here we cite two important results form Tseng et al. [30].

Theorem 1.8. *[30] Let f, p, H, P_p, L_p and S_p be defined as above. Then*

(i) The inequalities

$$\int_u^v f(x)p(x)\,dx \le 2\left[\int_u^{\frac{3u+v}{4}} f(x)p(2x-u)\,dx + \int_{\frac{u+3v}{4}}^v f(x)p(v-2x)\,dx\right] \le \int_0^1 P_p(t)\,dt \le \frac{1}{2}\left[\int_u^v f(x)p(x)\,dx + \frac{f(u)+f(v)}{2}\int_u^v p(x)\,dx\right] \quad (6)$$

holds.

(ii) The inequalities

$$L_p(t) \le P_p(t) \le (1-t)\int_u^v f(x)p(x)\,dx + t\cdot\frac{f(u)+f(v)}{2}\int_u^v p(x)\,dx \le \frac{f(u)+f(v)}{2}\int_u^v p(x)\,dx \quad (7)$$

and

$$0 \le N(t) - G(t)\int_u^v p(x)\,dx \le \frac{f(u)+f(v)}{2}\int_u^v p(x)\,dx - N(t) \quad (8)$$

hold for all $t \in [0,1]$.

(iii) If f is differentiable on $[u,v]$, then we have the inequalities

$$0 \le t\left[\frac{1}{v-u}\int_u^v f(x)\,dx - f\left(\frac{u+v}{2}\right)\right]\cdot \inf_{x\in[u,v]} p(x) \le P_p(t) - \int_u^v f(x)p(x)\,dx, \quad (9)$$

$$0 \le P_p(t) - f\left(\frac{u+v}{2}\right)\int_u^v p(x)\,dx \le \frac{(v-u)\left(f'(v)-f'(u)\right)}{4}\int_u^v p(x)\,dx, \quad (10)$$

$$0 \le L_p(t) - H_p(t) \le \frac{(v-u)\left(f'(v)-f'(u)\right)}{4}\int_u^v p(x)\,dx, \quad (11)$$

$$0 \le P_p(t) - L_p(t) \le \frac{(v-u)\left(f'(v)-f'(u)\right)}{4}\int_u^v p(x)\,dx, \quad (12)$$

$$0 \le P_p(t) - H_p(t) \le \frac{(v-u)\left(f'(v)-f'(u)\right)}{4}\int_u^v p(x)\,dx, \quad (13)$$

$$0 \le N(t) - I(t) \le \frac{(v-u)\left(f'(v)-f'(u)\right)}{4}\int_u^v p(x)\,dx, \quad (14)$$

and

$$0 \le S_p(t) - I(t) \le \frac{(v-u)\left(f'(v)-f'(u)\right)}{4}\int_u^v p(x)\,dx \quad (15)$$

hold for all $t \in [0,1]$.

Theorem 1.9. *[30] Let f, p, G, Q, H_p, P_p and S_p be defined as above. Then*

(i) The inequalities

$$H_p(t) \leq Q(t) \int_u^v p(x)\,dx \leq \frac{f(u)+f(v)}{2} \int_u^v p(x)\,dx,\ t \in \left[0, \frac{1}{3}\right], \tag{16}$$

and

$$f\left(\frac{u+v}{2}\right) \int_u^v p(x)\,dx \leq Q(t) \int_u^v p(x)\,dx \leq P_p(t),\ t \in \left[\frac{1}{3}, 1\right] \tag{17}$$

hold for all $t \in [0,1]$.

(ii) The inequality

$$0 \leq S_p(t) - G(t) \int_u^v p(x)\,dx \leq \frac{1}{2}\left[\frac{f(u)+f(v)}{2} + Q(t)\right] \int_u^v p(x)\,dx - S_p(t) \tag{18}$$

hold for all $t \in [0,1]$.

One of the generalizations of the convex functions is harmonic functions:

Definition 1.10. *[18] Define $I \subseteq \mathbb{R}\setminus\{0\}$ as an interval of real numbers. A function f from I to the real numbers is considered to be harmonically convex, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq t f(y) + (1-t) f(x) \tag{19}$$

for all $x, y \in I$ and $t \in [0,1]$. Harmonically concave f is defined as the inequality in (19) reversed.

İşcan used harmonic-convexity to develop the Hermite-Hadamard type.

Theorem 1.11. *[18] Let $f : I \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a harmonically convex function and $u, v \in I$ with $u < v$. If $f \in L([u,v])$ then the following inequalities hold:*

$$f\left(\frac{2uv}{u+v}\right) \leq \frac{uv}{v-u} \int_v^u \frac{f(x)}{x^2}dx \leq \frac{f(u)+f(v)}{2}. \tag{20}$$

Harmonic symmetricity of a function is given in the definition below.

Definition 1.12. *[25] A function $p : [u,v] \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is harmonically symmetric with respect to $\frac{2uv}{u+v}$ if*

$$p(x) = p\left(\frac{1}{\frac{1}{u}+\frac{1}{v}-\frac{1}{x}}\right)$$

holds for all $x \in [u,v]$.

Fejér type inequalities using harmonic convexity and the notion of harmonic symmetricity were presented in Chan and Wu [1].

Theorem 1.13. *[1] Let $f : I \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a harmonically convex function and $u, v \in I$ with $u < v$. If $f \in L([u,v])$ and $p : [u,v] \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is nonnegative, integrable and harmonically symmetric with respect to $\frac{2uv}{u+v}$, then*

$$f\left(\frac{2uv}{u+v}\right) \int_v^u \frac{p(x)}{x^2}dx \leq \int_v^u \frac{f(x)p(x)}{x^2}dx \leq \frac{f(u)+f(v)}{2} \int_v^u \frac{p(x)}{x^2}dx. \tag{21}$$

Some important facts which relate harmonic convex and convex functions are given in the results below.

Theorem 1.14. *[8, 9] If* $[u,v]\subset I\subset(0,\infty)$ *and if we consider the function* $g:\left[\frac{1}{v},\frac{1}{u}\right]\to\mathbb{R}$ *defined by* $g(t)=f\left(\frac{1}{t}\right)$*, then* f *is harmonically convex on* $[u,v]$ *if and only if* g *is convex in the usual sense on* $\left[\frac{1}{v},\frac{1}{u}\right]$*.*

Theorem 1.15. *[8, 9] If* $I\subset(0,\infty)$ *and* f *is convex and nondecreasing function then* f *is HA-convex and if* f *is HA-convex and nonincreasing function then* f *is convex.*

The interested readers are referred to [27] for the results that can be deduced from Theorems 1.8 and 1.9.

The main objective is to define some mappings on $[0,1]$ for harmonic convex function $f:[u,v]\subset I\subset(0,\infty)\to\mathbb{R}$ and a non-negative integrable function $p:[u,v]\to\mathbb{R}$ symmetric about $x=\frac{2uv}{u+v}$ related to the inequalities (20), (21) and to prove some new Hermite-Hadamard type inequalities which refine and extend (20) and (21). We also discuss some properties of these mappings in the next section and establish variants of the Lemmas 1.5-1.7 for harmonic convex functions.

2. Main Results

Let $f:[u,v]\subset(0,\infty)\to\mathbb{R}$ be a harmonic convex mapping and let $G_1,Q,U,U_p,K,V,V_p,\mathcal{L},\mathcal{L}_p,\mathcal{S}_p:[0,1]\to\mathbb{R}$ be defined by

$$G_1(t)=\frac{1}{2}\left[f\left(\frac{2uv}{2vt+(1-t)(u+v)}\right)+f\left(\frac{2uv}{2ut+(1-t)(u+v)}\right)\right],$$

$$Q(t)=\frac{1}{2}\left[f\left(\frac{uv}{tv+(1-t)u}\right)+f\left(\frac{uv}{tu+(1-t)v}\right)\right],$$

$$U(t)=\frac{uv}{v-u}\int_u^v\frac{1}{x^2}f\left(\frac{2uvx}{2uvt+(1-t)x(u+v)}\right)dx \tag{22}$$

$$U_p(t)=\int_u^v f\left(\frac{2uvx}{2uvt+(1-t)x(u+v)}\right)\frac{p(x)}{x^2}dx,$$

$$K(t)=\frac{1}{2}\int_u^v\left[f\left(\frac{2uvx}{t(u+x)v+(1-t)(u+v)x}\right)+f\left(\frac{2uvx}{t(u+x)v+(1-t)(u+v)x}\right)\right]\frac{p(x)}{x^2}dx, \tag{23}$$

$$V(t)=\frac{uv}{2(v-u)}\int_u^v\left[f\left(\frac{2vx}{(1+t)x+(1-t)v}\right)+f\left(\frac{2ux}{(1+t)x+(1-t)u}\right)\right]\frac{dx}{x^2}, \tag{24}$$

$$V_p(t)=\frac{1}{2}\int_u^v\left[f\left(\frac{2vx}{(1+t)x+(1-t)v}\right)\frac{p\left(\frac{2ux}{u+x}\right)}{x^2}+\left(\frac{2ux}{(1+t)x+(1-t)u}\right)\frac{p\left(\frac{2vx}{x+v}\right)}{x^2}\right]dx, \tag{25}$$

$$\mathcal{N}(t)=\frac{1}{2}\int_u^v\left[f\left(\frac{2ux}{2tx+(1-t)(u+x)}\right)+f\left(\frac{2vx}{2tx+(1-t)(v+x)}\right)\right]\frac{p(x)}{x^2}dx,$$

$$\mathcal{L}(t)=\frac{uv}{2(v-u)}\int_u^v\left[f\left(\frac{ux}{tx+(1-t)u}\right)+f\left(\frac{vx}{tx+(1-t)v}\right)\right]\frac{dx}{x^2},$$

$$\mathcal{L}_p(t)=\frac{1}{2}\int_u^v\left[f\left(\frac{ux}{tx+(1-t)u}\right)+f\left(\frac{vx}{tx+(1-t)v}\right)\right]\frac{p(x)}{x^2}dx$$

and

$$\mathcal{S}_p(t) = \frac{1}{2}\int_u^v \left[f\left(\frac{2ux}{2xt+(1-t)(u+x)}\right) + f\left(\frac{2uvx}{2vxt+(1-t)(x+v)}\right) + f\left(\frac{2uvx}{2uxt+(1-t)(u+x)v}\right) + f\left(\frac{2vx}{2xt+(1-t)(x+v)}\right)\right]\frac{p(x)}{x^2}dx. \quad (26)$$

Remark 2.1. *It should be noted that $U = U_p = K$, $V = V_p = \mathcal{N}$ and $\mathcal{L} = \mathcal{L}_p = \mathcal{S}_p$ on $[0,1]$ as $p(x) = \frac{uv}{v-u}$, $x \in [u,v]$.*

The author obtained the refinement inequalities for (20) related to the above mappings:

Theorem 2.2. *[21] Let $f : [u,v] \subset (0,\infty) \to \mathbb{R}$ be a harmonic convex function on $[u,v]$. Then*

(i) S is harmonic convex $(0,1]$ and increases monotonically on $[0,1]$.
(ii) The following inequalities hold:

$$f\left(\frac{2uv}{u+v}\right) = U(0) \le U(t) \le U(1) = \frac{uv}{v-u}\int_u^v \frac{f(x)}{x^2}dx. \quad (27)$$

Theorem 2.3. *[21] Let $f : [u,v] \subset (0,\infty) \to \mathbb{R}$ be a harmonic convex function on $[u,v]$. Then*

(i) V is harmonic convex $(0,1]$ and increases monotonically on $[0,1]$.
(ii) The following inequalities hold:

$$\frac{uv}{v-u}\int_u^v \frac{f(x)}{x^2}dx = V(0) \le V(t) \le V(1) = \frac{f(u)+f(v)}{2}. \quad (28)$$

Lemma 2.4. *[22] Let $f : [u,v] \subset (0,\infty) \to \mathbb{R}$ be a harmonic convex function and let $u \le y_1 \le x_1 \le x_2 \le y_2 \le v$ with $\frac{x_1x_2}{x_1+x_2} = \frac{y_1y_2}{y_1+y_2}$. Then*

$$f(x_1) + f(x_2) \le f(y_1) + f(y_2).$$

The assumptions in Lemma 2.4 can be weakened as in the following lemma:

Lemma 2.5. *Let $f : [u,v] \subset (0,\infty) \to \mathbb{R}$ be a harmonic convex function and let $u \le y_1 \le x_1 \le y_2 \le v$ and $u \le y_1 \le x_2 \le y_2 \le v$ with $\frac{x_1x_2}{x_1+x_2} = \frac{y_1y_2}{y_1+y_2}$. Then*

$$f(x_1) + f(x_2) \le f(y_1) + f(y_2).$$

We also need the following lemma to prove our results.

Lemma 2.6. *Let f, G, Q be defined as above. Then Q is symmetric about $\frac{1}{2}$, Q is decreasing on $\left[0,\frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2},1\right]$,*

$$G(2t) \le Q(t), \quad t \in \left[0,\frac{1}{4}\right], \quad (29)$$

$$G(2t) \ge Q(t), \quad t \in \left[\frac{1}{4},\frac{1}{2}\right], \quad (30)$$

$$G(2(1-t)) \ge Q(t), \quad t \in \left[\frac{1}{2},\frac{3}{4}\right], \quad (31)$$

and

$$G(2(1-t)) \le Q(t), \quad t \in \left[\frac{3}{4},1\right]. \quad (32)$$

Proof. The harmonic convexity of $Q(t)$ on $(0,1]$ follows from the harmonic convexity of f on $[u,v]$. It is clear that $Q(t)$ is symmetric about $\frac{1}{2}$. Let $0<t_1<t_2\le\frac{1}{2}\le t_3<t_4\le 1$, then according to Lemma 2.4, the following inequalities hold:
The inequality

$$f\left(\frac{uv}{t_2v+(1-t_2)u}\right)+f\left(\frac{uv}{t_2u+(1-t_2)v}\right)\le f\left(\frac{uv}{t_1v+(1-t_1)u}\right)+f\left(\frac{uv}{t_1u+(1-t_1)v}\right)$$

holds for $x_1=\frac{uv}{t_2v+(1-t_2)u}$, $x_2=\frac{uv}{t_2u+(1-t_2)v}$, $y_1=\frac{uv}{t_1v+(1-t_1)u}$, $y_2=\frac{uv}{t_1u+(1-t_1)v}$.
The inequality

$$f\left(\frac{uv}{t_3v+(1-t_3)u}\right)+f\left(\frac{uv}{t_3u+(1-t_3)v}\right)\le f\left(\frac{uv}{t_4v+(1-t_4)u}\right)+f\left(\frac{uv}{t_4u+(1-t_4)v}\right).$$

Now, we consider the two cases:
Case 1. $t\in\left[0,\frac{1}{4}\right]$
By choosing $x_1=\frac{2uv}{4vt+(1-2t)(u+v)}$, $x_2=\frac{2uv}{4ut+(1-2t)(u+v)}$, $y_1=\frac{uv}{tv+(1-t)u}$, $y_2=\frac{uv}{tu+(1-t)v}$ in Lemma 2.4, we get

$$f\left(\frac{2uv}{4vt+(1-2t)(u+v)}\right)+f\left(\frac{2uv}{4ut+(1-2t)(u+v)}\right)\le f\left(\frac{uv}{tv+(1-t)u}\right)+f\left(\frac{uv}{tu+(1-t)v}\right)$$

for all $t\in\left[0,\frac{1}{4}\right]$.
Case 2. $t\in\left[\frac{1}{4},\frac{1}{2}\right]$
By choosing $x_1=\frac{uv}{tv+(1-t)u}$, $x_2=\frac{uv}{tu+(1-t)v}$, $y_1=\frac{2uv}{4vt+(1-2t)(u+v)}$, $y_2=\frac{2uv}{4ut+(1-2t)(u+v)}$ in Lemma 2.4, we get

$$f\left(\frac{uv}{tv+(1-t)u}\right)+f\left(\frac{uv}{tu+(1-t)v}\right)\le f\left(\frac{2uv}{4vt+(1-2t)(u+v)}\right)+f\left(\frac{2uv}{4ut+(1-2t)(u+v)}\right)$$

for all $t\in\left[\frac{1}{4},\frac{1}{2}\right]$.
Thus (29) and (30) are established. Using the symmetricity of Q, (31) and (32) follow from (29) and (30), respectively. □

Chan and Wu [1] also defined some mappings related to (21) and discussed important properties of these mappings.

The author proved Fejér type inequalities which extend the inequalities given in Theorem 2.2 and Theorem 2.3 for the mappings related to (21) which in turn provide refinements of the inequalities (21). The author used the Lemma 2.4 to obtain those refinements for (21).

Theorem 2.7. *[24] Let f, U_p, V_p and p be as defined above, then*

$$\begin{aligned}f\left(\frac{2uv}{u+v}\right)\int_u^v\frac{p(x)}{x^2}dx&=U_p(0)\le U_p(t)\\&\le U_p(1)=\int_u^v\frac{f(x)p(x)}{x^2}dx=V_p(0)\le V_p(t)\le V_p(1)=\frac{f(u)+f(v)}{2}\int_u^v\frac{p(x)}{x^2}dx.\end{aligned}\tag{33}$$

Theorem 2.8. *[22]Let f, K, $\mathcal{N}$ and p be as defined above, then*

$$\begin{aligned}f\left(\frac{2uv}{u+v}\right)\int_u^v\frac{p(x)}{x^2}dx&\le K(0)\le K(t)\le K(1)\\&=\frac{1}{2}\int_u^v\left[f\left(\frac{2ux}{u+x}\right)+f\left(\frac{2vx}{x+v}\right)\right]\frac{p(x)}{x^2}dx=\mathcal{N}(0)\le\mathcal{N}(t)\le\mathcal{N}(1)=\frac{f(u)+f(v)}{2}\int_u^v\frac{p(x)}{x^2}dx.\end{aligned}\tag{34}$$

Corollary 2.9. *[22] Let f, p be defined as above. Then we have*

$$f\left(\frac{2uv}{u+v}\right)\int_u^v \frac{p(x)}{x^2}dx \le \frac{f\left(\frac{4uv}{u+3v}\right)+f\left(\frac{4uv}{3u+v}\right)}{2}\int_u^v \frac{p(x)}{x^2}dx \le \frac{1}{2}\int_u^v \left[f\left(\frac{2ux}{u+x}\right)+f\left(\frac{2vx}{x+v}\right)\right]\frac{p(x)}{x^2}dx$$
$$\le \frac{1}{2}\left[f\left(\frac{2uv}{u+v}\right)+\frac{f(u)+f(v)}{2}\right]\int_u^v \frac{p(x)}{x^2}dx \le \frac{f(u)+f(v)}{2}\int_u^v \frac{p(x)}{x^2}dx. \quad (35)$$

Theorem 2.10. *[23] Let $f, p, G_1, \mathcal{S}_p, \mathcal{L}_p$ be defined as above. Then we have the following results:*

(i) $\mathcal{L}_p$ is harmonic convex on $(0,1]$.

(ii) The following inequalities hold for all $t \in [0,1]$:

$$U_p(t) \le G_1(t)\int_u^v \frac{p(x)}{x^2}dx \le \mathcal{L}_p(t) \le (1-t)\int_u^v \frac{f(x)p(x)}{x^2}dx$$
$$+t\cdot\frac{f(u)+f(v)}{2}\int_u^v \frac{p(x)}{x^2}dx \le \frac{f(u)+f(v)}{2}\int_u^v \frac{p(x)}{x^2}dx, \quad (36)$$

$$\mathcal{S}_p(1-t) \le \mathcal{L}_p(t) \quad (37)$$

and

$$\frac{\mathcal{S}_p(t)+\mathcal{S}_p(1-t)}{2} \le \mathcal{L}_p(t). \quad (38)$$

(iii) The following bound is true:

$$\sup_{t\in[0,1]} \mathcal{L}_p(t) = \frac{f(u)+f(v)}{2}\int_u^v \frac{p(x)}{x^2}dx. \quad (39)$$

Theorem 2.11. *[23] Let $f, p, G_1, K, \mathcal{S}_p$ be defined as above. Then we have the following results:*

(i) $\mathcal{S}_p$ is harmonically convex on $[0,1]$.

(ii) The following inequalities hold for all $t \in [0,1]$:

$$K(t) \le G_1(t)\int_u^v \frac{p(x)}{x^2}dx \le \mathcal{S}_p(t)$$
$$\le (1-t)\cdot\frac{1}{2}\int_u^v \left[f\left(\frac{2xv}{x+v}\right)+f\left(\frac{2ux}{u+x}\right)\right]\frac{p(x)}{x^2}dx$$
$$+t\cdot\frac{f(u)+f(v)}{2}\int_u^v \frac{p(x)}{x^2}dx \le \frac{f(u)+f(v)}{2}\int_u^v \frac{p(x)}{x^2}dx, \quad (40)$$

$$K(1-t) \le \mathcal{S}_p(t) \quad (41)$$

and

$$\frac{K(t)+K(1-t)}{2} \le \mathcal{S}_p(t). \quad (42)$$

(iii) The following identity holds:

$$\sup_{t\in[0,1]} \mathcal{S}_p(t) = \frac{f(u)+f(v)}{2}\int_u^v \frac{p(x)}{x^2}dx. \quad (43)$$

We can now prove a variant of Theorem 1.8 for harmonic convex functions.

Theorem 2.12. *Let f, p, U, V_p, $\mathcal{L}_p$ and $\mathcal{S}_p$ be defined as above. Then*

(i) The inequalities

$$\int_u^v \frac{f(x)p(x)}{x^2}dx \le 2\left[\int_u^{\frac{4uv}{u+3v}} f(x)\frac{p\left(\frac{vx}{2v-x}\right)}{x^2}dx + \int_{\frac{4uv}{3u+v}}^v f(x)\frac{p\left(\frac{ux}{2u-x}\right)}{x^2}dx\right] \le \int_0^1 V_p(t)\,dt \le \frac{1}{2}\left[\int_u^v \frac{f(x)p(x)}{x^2}dx + \frac{f(u)+f(v)}{2}\int_u^v \frac{p(x)}{x^2}dx\right] \quad (44)$$

hold.

(ii) The inequalities

$$\mathcal{L}_p(t) \le V_p(t) \le (1-t)\int_u^v \frac{f(x)p(x)}{x^2}dx + t\cdot\frac{f(u)+f(v)}{2}\int_u^v \frac{p(x)}{x^2}dx \le \frac{f(u)+f(v)}{2}\int_u^v \frac{p(x)}{x^2}dx \quad (45)$$

and

$$0 \le \mathcal{N}(t) - G_1(t)\int_u^v \frac{p(x)}{x^2}dx \le \frac{f(u)+f(v)}{2}\int_u^v \frac{p(x)}{x^2}dx - \mathcal{N}(t) \quad (46)$$

hold for all $t \in [0,1]$.

(iii) If f is differentiable on $[u,v]$, then we have the inequalities

$$0 \le t\frac{uv}{v-u}\left[\frac{1}{v-u}\int_u^v \frac{f(x)}{x^2}dx - f\left(\frac{2uv}{u+v}\right)\right] \le V(t) - \frac{uv}{v-u}\int_u^v \frac{f(x)}{x^2}dx, \quad (47)$$

$$0 \le V_p(t) - f\left(\frac{2uv}{u+v}\right)\int_u^v \frac{p(x)}{x^2}dx \le \frac{(v-u)\left(v^2 f'(v) - u^2 f'(u)\right)}{4uv}\int_u^v \frac{p(x)}{x^2}dx, \quad (48)$$

$$0 \le \mathcal{L}_p(t) - U_p(t) \le \frac{(v-u)\left(v^2 f'(v) - u^2 f'(u)\right)}{4uv}\int_u^v \frac{p(x)}{x^2}dx, \quad (49)$$

$$0 \le V_p(t) - \mathcal{L}_p(t) \le \frac{(v-u)\left(v^2 f'(v) - u^2 f'(u)\right)}{4uv}\int_u^v \frac{p(x)}{x^2}dx, \quad (50)$$

$$0 \le V_p(t) - U_p(t) \le \frac{(v-u)\left(v^2 f'(v) - u^2 f'(u)\right)}{4uv}\int_u^v \frac{p(x)}{x^2}dx, \quad (51)$$

$$0 \le \mathcal{N}(t) - K(t) \le \frac{(v-u)\left(v^2 f'(v) - u^2 f'(u)\right)}{4uv}\int_u^v \frac{p(x)}{x^2}dx, \quad (52)$$

and

$$0 \le \mathcal{S}_p(t) - K(t) \le \frac{(v-u)\left(v^2 f'(v) - u^2 f'(u)\right)}{4uv}\int_u^v \frac{p(x)}{x^2}dx \quad (53)$$

hold for all $t \in [0,1]$.

Proof. (i) By using integration techniques and the assumptions on p, we get the following identities:

$$\int_u^v \frac{f(x)\,p(x)}{x^2}dx = \int_u^{\frac{2uv}{u+v}} \int_0^{\frac{1}{2}} \left[f(x) + f\left(\frac{1}{\frac{1}{u}+\frac{1}{v}-\frac{1}{x}}\right)\right] \frac{p(x)}{x^2}dtdx, \tag{54}$$

$$\begin{aligned}
&2\left[\int_u^{\frac{4uv}{u+3v}} f(x)\frac{p\left(\frac{vx}{2v-x}\right)}{x^2}dx + \int_{\frac{4uv}{3u+v}}^{v} f(x)\frac{p\left(\frac{ux}{2u-x}\right)}{x^2}dx\right] \\
&= 2\int_u^{\frac{4uv}{u+3v}} \left[f(x) + f\left(\frac{1}{\frac{1}{u}+\frac{1}{v}-\frac{1}{x}}\right)\right]\frac{p\left(\frac{vx}{2v-x}\right)}{x^2}dx \\
&\qquad = 2\int_u^{\frac{2uv}{u+v}} \int_0^{\frac{1}{2}} \left[f\left(\frac{vx}{v+x}\right) + f\left(\frac{uv}{(u+2v)x-uv}\right)\right]\frac{p(x)}{x^2}dtdx,
\end{aligned} \tag{55}$$

$$\begin{aligned}
\int_0^1 V_p(t)\,dt &= \int_u^{\frac{2uv}{u+v}} \int_0^1 f\left(\frac{vx}{tx+(1-t)v}\right)\frac{p(x)}{x^2}dtdx + \int_{\frac{2uv}{u+v}}^{v} \int_0^1 f\left(\frac{ux}{tx+(1-t)u}\right)\frac{p(x)}{x^2}dtdx \\
&= \int_u^{\frac{2uv}{u+v}} \int_0^1 f\left(\frac{vx}{tx+(1-t)v}\right)\frac{p(x)}{x^2}dtdx + \int_u^{\frac{2uv}{u+v}} \int_0^1 f\left(\frac{uvx}{tvx+(1-t)((u+v)x-uv)}\right)\frac{p(x)}{x^2}dtdx \\
&= \int_u^{\frac{2uv}{u+v}} \int_0^{\frac{1}{2}} \left[f\left(\frac{vx}{tx+(1-t)v}\right) + f\left(\frac{ux}{tx+(1-t)u}\right)\right]\frac{p(x)}{x^2}dtdx \\
&\quad + \int_u^{\frac{2uv}{u+v}} \int_0^1 \left[f\left(\frac{uvx}{(1-t)vx+t((u+v)x-uv)}\right) + f\left(\frac{uvx}{tvx+(1-t)((u+v)x-uv)}\right)\right]\frac{p(x)}{x^2}dtdx
\end{aligned} \tag{56}$$

and

$$\begin{aligned}
&\frac{1}{2}\left[\int_u^v \frac{f(x)\,p(x)}{x^2}dx + \frac{f(u)+f(v)}{2}\int_u^v \frac{p(x)}{x^2}dx\right] \\
&\qquad = \int_u^{\frac{2uv}{u+v}} \int_0^{\frac{1}{2}} [f(u)+f(x)]\frac{p(x)}{x^2}dtdx + \int_u^{\frac{2uv}{u+v}} \int_0^{\frac{1}{2}} \left[f(v) + f\left(\frac{1}{\frac{1}{u}+\frac{1}{v}-\frac{1}{x}}\right)\right]\frac{p(x)}{x^2}dtdx.
\end{aligned} \tag{57}$$

According to Lemma 2.4, the following inequalities hold for all $t \in \left[0, \frac{1}{2}\right]$ and $x \in \left[u, \frac{2uv}{u+v}\right]$:
The inequality

$$f(x) + f\left(\frac{1}{\frac{1}{u}+\frac{1}{v}-\frac{1}{x}}\right) \le f\left(\frac{vx}{v+x}\right) + f\left(\frac{uv}{(u+2v)x-uv}\right) \tag{58}$$

holds with the choices $x_1 = x$, $x_2 = \frac{1}{\frac{1}{u}+\frac{1}{v}-\frac{1}{x}}$, $y_1 = \frac{vx}{v+x}$ and $y_2 = \frac{uv}{(u+2v)x-uv}$.
The inequality

$$f\left(\frac{vx}{v+x}\right) \le \frac{1}{2}\left[f\left(\frac{vx}{tv+(1-t)x}\right) + f\left(\frac{vx}{tx+(1-t)v}\right)\right] \tag{59}$$

holds with the choices $x_1 = x_2 = \frac{vx}{v+x}$, $y_1 = \frac{vx}{tv+(1-t)x}$ and $y_2 = \frac{vx}{tx+(1-t)v}$.
The inequality

$$f\left(\frac{uv}{(u+2v)x-uv}\right) \le \frac{1}{2}\left[f\left(\frac{uvx}{(1-t)vx+t((u+v)x-uv)}\right) + f\left(\frac{uvx}{tvx+(1-t)((u+v)x-uv)}\right)\right] \tag{60}$$

holds with the choices $x_1 = x_2 = \frac{uv}{(u+2v)x-uv}$, $y_1 = \frac{uvx}{tvx+(1-t)((u+v)x-uv)}$ and $y_2 = \frac{uvx}{(1-t)vx+t((u+v)x-uv)}$.
The inequality

$$\frac{1}{2}\left[f\left(\frac{vx}{tv+(1-t)x}\right)+f\left(\frac{vx}{tx+(1-t)v}\right)\right] \le \frac{f(u)+f(x)}{2} \tag{61}$$

holds with the choices $x_1 = \frac{vx}{tx+(1-t)v}$, $x_2 = \frac{vx}{tv+(1-t)x}$, $y_1 = u$ and $y_2 = x$.
The inequality

$$\frac{1}{2}\left[f\left(\frac{uvx}{(1-t)vx+t((u+v)x-uv)}\right)+f\left(\frac{uvx}{tvx+(1-t)((u+v)x-uv)}\right)\right] \le \frac{f(v)+f\left(\frac{1}{\frac{1}{u}+\frac{1}{v}-\frac{1}{x}}\right)}{2} \tag{62}$$

holds with the choices $x_1 = \frac{uvx}{tvx+(1-t)((u+v)x-uv)}$, $x_2 = \frac{uvx}{(1-t)vx+t((u+v)x-uv)}$, $y_1 = \frac{1}{\frac{1}{u}+\frac{1}{v}-\frac{1}{x}}$ and $y_2 = v$.
Multiplying the inequalities (58)-(62) by $\frac{p(x)}{x^2}$ and integrating them over t on $\left[0, \frac{1}{2}\right]$ over x on $\left[u, \frac{2uv}{u+v}\right]$ and using identities (54)-(57), we derive (44).
(ii) Using substitution rules for integration and the assumptions on p, we have the following identities:

$$\begin{aligned} V_p(t) &= \int_u^{\frac{2uv}{u+v}} f\left(\frac{vx}{tx+(1-t)v}\right)\frac{p(x)}{x^2}dx + \int_{\frac{2uv}{u+v}}^{v} f\left(\frac{ux}{tx+(1-t)u}\right)\frac{p(x)}{x^2}dx \\ &= \int_u^{\frac{2uv}{u+v}} \left[f\left(\frac{vx}{tx+(1-t)v}\right)+f\left(\frac{uvx}{tvx+(1-t)((u+v)x-uv)}\right)\right]\frac{p(x)}{x^2}dx \end{aligned} \tag{63}$$

and

$$\begin{aligned} \mathcal{L}_p(t) &= \frac{1}{2}\left[\int_u^{\frac{2uv}{u+v}} f\left(\frac{vx}{tx+(1-t)v}\right)\frac{p(x)}{x^2}dx \right.\\ &\left.+\int_{\frac{2uv}{u+v}}^{v} f\left(\frac{ux}{tx+(1-t)u}\right)\frac{p(x)}{x^2}dx\right] + \frac{1}{2}\left[\int_u^{\frac{2uv}{u+v}} f\left(\frac{ux}{tx+(1-t)u}\right)\frac{p(x)}{x^2}dx\right.\\ &\left.+\int_{\frac{2uv}{u+v}}^{v} f\left(\frac{vx}{tx+(1-t)v}\right)\frac{p(x)}{x^2}dx\right] = \frac{1}{2}V_p(t) + \frac{1}{2}\int_u^{\frac{2uv}{u+v}}\left[f\left(\frac{ux}{tx+(1-t)u}\right)\right.\\ &\left.+f\left(\frac{uvx}{tux+(1-t)((u+v)x-uv)}\right)\right]\frac{p(x)}{x^2}dx \end{aligned} \tag{64}$$

for all $t \in [0,1]$.
By choosing $x_1 = \frac{uvx}{tux+(1-t)((u+v)x-uv)}$, $x_2 = \frac{ux}{tx+(1-t)u}$, $y_1 = \frac{vx}{tx+(1-t)v}$, $y_2 = \frac{uvx}{tvx+(1-t)((u+v)x-uv)}$ in Lemma 2.5, we observe that the inequality

$$\begin{aligned} &f\left(\frac{ux}{tx+(1-t)u}\right)+f\left(\frac{uvx}{tux+(1-t)((u+v)x-uv)}\right) \\ &\le f\left(\frac{vx}{tx+(1-t)v}\right)+f\left(\frac{uvx}{tvx+(1-t)((u+v)x-uv)}\right) \end{aligned} \tag{65}$$

holds for all $t \in [0,1]$ and $x \in \left[u, \frac{2uv}{u+v}\right]$.
Multiplying the inequality (65) by $\frac{p(x)}{x^2}$, integrating both sides over x on $\left[u, \frac{2uv}{u+v}\right]$ and using identities (63) and (64), we derive the first inequality of (45). The second and third inequalities of (45) can be obtained by

the convexity of f and (21). Using substitution rules for integration and the hypothesis of p, we have the following identity:

$$\begin{aligned}\mathcal{N}(t) &= \frac{1}{2}\int_u^v \left[f\left(\frac{2vx}{2xt+(1-t)(v+x)}\right)+f\left(\frac{2uvx}{2uvxt+(1-t)((u+2v)x-uv)}\right)\right]\frac{p(x)}{x^2}dx\\ &= \int_u^{\frac{2uv}{u+v}}\left[f\left(\frac{vx}{tx+(1-t)v}\right)+f\left(\frac{uvx}{tvx+(1-t)((u+v)x-uv)}\right)\right]\frac{p\left(\frac{vx}{2v-x}\right)}{x^2}dx\\ &= \int_u^{\frac{4uv}{u+3v}}\left[f\left(\frac{vx}{tx+(1-t)v}\right)+f\left(\frac{2uvx}{2tux+(1-t)((3u+v)x-2uv)}\right)\right.\\ &\qquad\left.+f\left(\frac{2uvx}{2tvx+(1-t)((v-u)x-2uv)}\right)+f\left(\frac{uvx}{tvx+(1-t)((u+v)x-uv)}\right)\right]\frac{p\left(\frac{vx}{2v-x}\right)}{x^2}dx \end{aligned} \tag{66}$$

for all $t\in[0,1]$.
By Lemma 2.4, the following inequalities hold for all $t\in[0,1]$ and $x\in\left[u,\frac{4uv}{u+3v}\right]$:
The inequality

$$f\left(\frac{vx}{tx+(1-t)v}\right)+f\left(\frac{2uvx}{2tux+(1-t)((3u+v)x-2uv)}\right)\le f(v)+f\left(\frac{2uv}{2tu+(1-t)(u+v)}\right) \tag{67}$$

holds for $x_1=\frac{vx}{tx+(1-t)v}$, $x_2=\frac{2uvx}{2tux+(1-t)((3u+v)x-2uv)}$, $y_1=v$ and $y_2=\frac{2uv}{2tu+(1-t)(u+v)}$.
The inequality

$$\begin{aligned}&f\left(\frac{2uvx}{2tvx+(1-t)((v-u)x-2uv)}\right)+f\left(\frac{uvx}{tvx+(1-t)((u+v)x-uv)}\right)\\ &\qquad\le f(u)+f\left(\frac{2uv}{2tv+(1-t)(u+v)}\right)\end{aligned} \tag{68}$$

holds for $x_1=\frac{2uvx}{2tvx+(1-t)((v-u)x-2uv)}$, $x_2=\frac{uvx}{tvx+(1-t)((u+v)x-uv)}$, $y_1=u$ and $y_2=\frac{2uv}{2tv+(1-t)(u+v)}$.
Multiplying the inequalities (67)-(68) by $\frac{p\left(\frac{vx}{2v-x}\right)}{x^2}$ and integrating them over x on $\left[u,\frac{4uv}{u+3v}\right]$ and using (66), we have

$$\mathcal{N}(t)\le\frac{1}{2}\left[\frac{f(u)+f(v)}{2}+G_1(t)\right]\int_u^v\frac{p(x)}{x^2}dx \tag{69}$$

for all $t\in[0,1]$. The second inequality in (46) is a consequence of (69).
Applying Lemma 2.4, we observe that the inequality

$$\begin{aligned}&f\left(\frac{2uv}{2tu+(1-t)(u+v)}\right)+f\left(\frac{2uv}{2tv+(1-t)(u+v)}\right)\\ &\qquad\le f\left(\frac{vx}{tx+(1-t)v}\right)+f\left(\frac{uvx}{tvx+(1-t)((u+v)x-uv)}\right)\end{aligned} \tag{70}$$

holds for all $t\in[0,1]$ and $x\in\left[u,\frac{2uv}{u+v}\right]$ when $x_1=\frac{2uv}{2tu+(1-t)(u+v)}$, $x_2=\frac{2uv}{2tv+(1-t)(u+v)}$, $y_1=\frac{vx}{tx+(1-t)v}$ and $y_2=\frac{uvx}{tvx+(1-t)((u+v)x-uv)}$.
Multiplying the inequalities (70) by $\frac{p\left(\frac{vx}{2v-x}\right)}{x^2}$ and integrating them over x on $\left[u,\frac{2uv}{u+v}\right]$ and using the first part of the identity (66), we get (46).

(iii) Integrating by parts, we have

$$\frac{uv}{v-u}\int_{u}^{\frac{2uv}{u+v}}\frac{1}{x^{2}}\left(\frac{1}{v}-\frac{1}{x}\right)\left[x^{2}f'(x)-\frac{f'\left(\frac{1}{\frac{1}{u}+\frac{1}{v}-\frac{1}{x}}\right)}{\left(\frac{1}{u}+\frac{1}{v}-\frac{1}{x}\right)^{2}}\right]dx=\left(\frac{v-u}{uv}\right)\int_{u}^{v}\frac{f(x)}{x^{2}}dx-f\left(\frac{2uv}{u+v}\right). \tag{71}$$

Using substitution rules for integration, we have the following identity:

$$\frac{uv}{v-u}\int_{u}^{v}\frac{f(x)}{x^{2}}dx=\frac{uv}{v-u}\int_{u}^{\frac{2uv}{u+v}}\frac{1}{x^{2}}\left[f(x)+f\left(\frac{1}{\frac{1}{u}+\frac{1}{v}-\frac{1}{x}}\right)\right]dx. \tag{72}$$

Since $f:[u,v]\to\mathbb{R}$ is harmonic convex on $[u,v]$, hence $g:\left[\frac{1}{v},\frac{1}{u}\right]$ defined by $g(x)=f\left(\frac{1}{x}\right)$ is convex on $\left[\frac{1}{v},\frac{1}{u}\right]$. Using the convexity of p and $p(x)\geq 0$ on $[u,v]$, the inequality

$$\begin{aligned}&\left[g\left(t\frac{1}{v}+(1-t)x\right)-g(x)\right]p\left(\frac{1}{x}\right)+\left[g\left(t\frac{1}{v}+(1-t)\left(\frac{1}{u}+\frac{1}{v}-x\right)\right)-g\left(\frac{1}{u}+\frac{1}{v}-x\right)\right]p\left(\frac{1}{x}\right)\\&\geq t\left(\frac{1}{v}-x\right)g'(x)p\left(\frac{1}{x}\right)+t\left(x-\frac{1}{v}\right)g'\left(\frac{1}{u}+\frac{1}{v}-x\right)p\left(\frac{1}{x}\right)=t\left(x-\frac{1}{v}\right)\left[g'\left(\frac{1}{u}+\frac{1}{v}-x\right)-g'(x)\right]p\left(\frac{1}{x}\right)\end{aligned} \tag{73}$$

holds for all $t\in[0,1]$ and $x\in\left[\frac{1}{v},\frac{u+v}{2uv}\right]$.
The inequality (73) can be re-written as

$$\begin{aligned}&\left[f\left(\frac{vx}{tx+(1-t)v}\right)-f(x)\right]\frac{p(x)}{x^{2}}+\left[f\left(\frac{uvx}{uxt+(1-t)((u+v)x-uv)}\right)-f\left(\frac{1}{\frac{1}{u}+\frac{1}{v}-\frac{1}{x}}\right)\right]\frac{p(x)}{x^{2}}\\&\geq x^{2}t\left(\frac{1}{x}-\frac{1}{v}\right)f'(x)\frac{p(x)}{x^{2}}-t\left(\frac{1}{x}-\frac{1}{v}\right)\frac{f'\left(\frac{1}{\frac{1}{u}+\frac{1}{v}-\frac{1}{x}}\right)}{\left(\frac{1}{u}+\frac{1}{v}-\frac{1}{x}\right)^{2}}\frac{p(x)}{x^{2}}\\&=t\left(\frac{1}{x}-\frac{1}{v}\right)\left[x^{2}f'(x)-\frac{f'\left(\frac{1}{\frac{1}{u}+\frac{1}{v}-\frac{1}{x}}\right)}{\left(\frac{1}{u}+\frac{1}{v}-\frac{1}{x}\right)^{2}}\right]\frac{p(x)}{x^{2}}\geq t\left(\frac{1}{x}-\frac{1}{v}\right)\left[x^{2}f'(x)-\frac{f'\left(\frac{1}{\frac{1}{u}+\frac{1}{v}-\frac{1}{x}}\right)}{\left(\frac{1}{u}+\frac{1}{v}-\frac{1}{x}\right)^{2}}\right]\frac{1}{x^{2}}\inf_{x\in[u,v]}p(x)\end{aligned} \tag{74}$$

for all $t\in[0,1]$ and $x\in\left[u,\frac{2uv}{u+v}\right]$.
Integrating the above inequality over x on $\left[u,\frac{2uv}{u+v}\right]$, multiplying both sides by $\frac{uv}{v-u}$ and using (20), (63), (72) and (74), we derive (47).
We also observe that

$$\frac{g\left(\frac{1}{v}\right)-g\left(\frac{u+v}{2uv}\right)}{2}\int_{\frac{1}{v}}^{\frac{1}{u}}p\left(\frac{1}{x}\right)dx\leq\frac{1}{2}\left(\frac{1}{v}-\frac{u+v}{2uv}\right)g'\left(\frac{1}{v}\right)\int_{\frac{1}{v}}^{\frac{1}{u}}p\left(\frac{1}{x}\right)dx=\left(\frac{u-v}{4uv}\right)g'\left(\frac{1}{v}\right)\int_{\frac{1}{v}}^{\frac{1}{u}}p\left(\frac{1}{x}\right)dx \tag{75}$$

and

$$\frac{g\left(\frac{1}{u}\right)-g\left(\frac{u+v}{2uv}\right)}{2}\int_{\frac{1}{v}}^{\frac{1}{u}}p\left(\frac{1}{x}\right)dx\leq\frac{1}{2}\left(\frac{1}{u}-\frac{u+v}{2uv}\right)g'\left(\frac{1}{v}\right)\int_{\frac{1}{v}}^{\frac{1}{u}}p\left(\frac{1}{x}\right)dx=\left(\frac{v-u}{4uv}\right)g'\left(\frac{1}{u}\right)\int_{\frac{1}{v}}^{\frac{1}{u}}p\left(\frac{1}{x}\right)dx. \tag{76}$$

Adding (75) and (76), we get

$$\frac{g\left(\frac{1}{u}\right)+g\left(\frac{1}{v}\right)}{2}\int_{\frac{1}{v}}^{\frac{1}{u}}p\left(\frac{1}{x}\right)dx-g\left(\frac{u+v}{2uv}\right)\int_{\frac{1}{v}}^{\frac{1}{u}}p\left(\frac{1}{x}\right)dx\leq\frac{(v-u)\left(g'\left(\frac{1}{u}\right)-g'\left(\frac{1}{v}\right)\right)}{4}\int_{\frac{1}{v}}^{\frac{1}{u}}p\left(\frac{1}{x}\right)dx. \tag{77}$$

The inequality (77) is equivalent to

$$\frac{f(u)+f(v)}{2}\int_u^v \frac{p(x)}{x^2}dx - f\left(\frac{2uv}{u+v}\right)\int_u^v \frac{p(x)}{x^2}dx \le \frac{(v-u)\left(v^2 f'(v)-u^2 f'(u)\right)}{4}\int_u^v \frac{p(x)}{x^2}dx. \quad (78)$$

Finally, (48)-(53) follow from (33), (34), (36), (40), (45) and (78). □

Corollary 2.13. *If $p(x)=\frac{uv}{v-u}$, $x\in[u,v]$, then Hermite-Hadamard-type inequalities that are obvious consequences of Theorem 2.12 are given as follows:*

(i) The inequalities

$$\frac{uv}{v-u}\int_u^v \frac{f(x)}{x^2}dx \le \frac{2uv}{v-u}\left[\int_u^{\frac{4uv}{u+3v}} \frac{f(x)}{x^2}dx + \int_{\frac{4uv}{3u+v}}^v \frac{f(x)}{x^2}dx\right]$$
$$\le \int_0^1 V(t)\,dt \le \frac{1}{2}\left[\frac{uv}{v-u}\int_u^v \frac{f(x)}{x^2}dx + \frac{f(u)+f(v)}{2}\right] \quad (79)$$

hold.

(ii) The inequalities

$$\mathcal{L}(t) \le V(t) \le (1-t)\frac{uv}{v-u}\int_u^v \frac{f(x)}{x^2}dx + t\cdot\frac{f(u)+f(v)}{2} \le \frac{f(u)+f(v)}{2} \quad (80)$$

and

$$0 \le V(t) - G_1(t) \le \frac{f(u)+f(v)}{2} - V(t) \quad (81)$$

hold for all $t\in[0,1]$.

(iii) If f is differentiable on $[u,v]$, then we have the inequalities

$$0 \le t\frac{uv}{v-u}\left[\frac{1}{v-u}\int_u^v \frac{f(x)}{x^2}dx - f\left(\frac{2uv}{u+v}\right)\right] \le V(t) - \frac{uv}{v-u}\int_u^v \frac{f(x)}{x^2}dx, \quad (82)$$

$$0 \le V(t) - f\left(\frac{2uv}{u+v}\right) \le \frac{(v-u)\left(v^2 f'(v)-u^2 f'(u)\right)}{4uv}, \quad (83)$$

$$0 \le \mathcal{L}(t) - U(t) \le \frac{(v-u)\left(v^2 f'(v)-u^2 f'(u)\right)}{4uv}, \quad (84)$$

$$0 \le V(t) - \mathcal{L}(t) \le \frac{(v-u)\left(v^2 f'(v)-u^2 f'(u)\right)}{4uv}, \quad (85)$$

and

$$0 \le V(t) - U(t) \le \frac{(v-u)\left(v^2 f'(v)-u^2 f'(u)\right)}{4uv}, \quad (86)$$

hold for all $t\in[0,1]$.

Remark 2.14. *The inequalities (44) give a new refinement of the Fejér type inequalities (21).*

Remark 2.15. *The inequalities (45) refine the Fejér-type inequalities (36).*

In the next theorem, we point out some inequalities for the functions $G_1, Q, U_p, V_p, \mathcal{S}_p$ considered above.

Theorem 2.16. *Let $f, p, G_1, Q, U_p, V_p, \mathcal{S}_p$ be defined as above. Then the following Fejér type inequalities hold:*

(i) The inequalities

$$U_p(t) \le Q(t)\int_u^v \frac{p(x)}{x^2}dx \le \frac{f(u)+f(v)}{2}\int_u^v \frac{p(x)}{x^2}dx, \tag{87}$$

hold for $t \in \left[0, \frac{1}{3}\right]$ and

$$f\left(\frac{2uv}{u+v}\right)\int_u^v \frac{p(x)}{x^2}dx \le Q(t)\int_u^v \frac{p(x)}{x^2}dx \le V_p(t), \tag{88}$$

inequalities hold for $t \in \left[\frac{1}{3}, 1\right]$.

(ii) The inequalities

$$0 \le \mathcal{S}_p(t) \le G_1(t)\int_u^v \frac{p(x)}{x^2}dx \le \frac{1}{2}\left[\frac{f(u)+f(v)}{2} + Q(t)\right]\int_u^v \frac{p(x)}{x^2}dx + \mathcal{S}_p(t), \tag{89}$$

hold for all $t \in [0,1]$.

Proof. (i) Here we consider the two cases.

Case 1. $t \in \left[0, \frac{1}{3}\right]$

Using substitution rules for integration and the hypothesis of p, we have the following identity:

$$U_p(t) = \int_u^{\frac{2uv}{u+v}} \left[f\left(\frac{2uvx}{2uvt + (1-t)(u+v)x}\right) + f\left(\frac{2uvx}{2((u+v)x - uv)t + (1-t)(u+v)x}\right)\right]\frac{p(x)}{x^2}dx. \tag{90}$$

We observe that the following inequality is a result of usage of Lemma 2.4:

The inequality

$$f\left(\frac{2uvx}{2uvt + (1-t)(u+v)x}\right) + f\left(\frac{2uvx}{2((u+v)x - uv)t + (1-t)(u+v)x}\right) \le f\left(\frac{uv}{(1-t)v + tu}\right) + f\left(\frac{uv}{(1-t)u + tv}\right) \tag{91}$$

holds for $x_1 = \frac{2uvx}{2uvt+(1-t)(u+v)x}$, $x_2 = \frac{2uvx}{2((u+v)x-uv)t+(1-t)(u+v)x}$, $y_1 = \frac{uv}{(1-t)v+tu}$, $y_2 = \frac{uv}{(1-t)u+tv}$ in Lemma 2.4, where $t \in \left[0, \frac{1}{3}\right]$ and $x \in \left[u, \frac{2uv}{u+v}\right]$.

Multiplying the inequality (91) by $\frac{p(x)}{x^2}$, integrating both sides over x on $\left[u, \frac{2uv}{u+v}\right]$ and using identity (90), we derive the first inequality of (87). From Lemma 2.6, we get that

$$\sup_{t\in\left[0,\frac{1}{2}\right]} Q(t) = \frac{f(u)+f(v)}{2}.$$

Thus the second inequality in (87) is established.

Case 2. $t \in \left[\frac{1}{3}, 1\right]$

By choosing $x_1 = \frac{uv}{(1-t)v+tu}$, $x_2 = \frac{uv}{(1-t)u+tv}$, $y_1 = \frac{vx}{tx+(1-t)v}$, $y_2 = \frac{2uvx}{((u+v)x-uv)(1-t)+tv}$ in Lemma 2.6, where $t \in \left[\frac{1}{3}, 1\right]$ and $x \in \left[u, \frac{2uv}{u+v}\right]$, we get

$$f\left(\frac{uv}{(1-t)v + tu}\right) + f\left(\frac{uv}{(1-t)u + tv}\right) \le f\left(\frac{2uvx}{u(x+v)(1-t) + x(u+v)t}\right) + f\left(\frac{2uvx}{u(x+v)(1-t) + x(u+v)t}\right). \tag{92}$$

Multiplying the inequality (92) by $\frac{p(x)}{x^2}$, integrating both sides over x on $\left[u, \frac{2uv}{u+v}\right]$ and using identity (63), we derive the second inequality of (88). From Lemma 2.6, we get that

$$\inf_{t\in\left[\frac{1}{2},1\right]} Q(t) = f\left(\frac{2uv}{u+v}\right).$$

Thus the first inequality in (88) is also achieved.

(ii) Using substitution rules for integration and the hypothesis of p, we have the following identity

$$\begin{aligned}
2\mathcal{S}_p &= \int_u^{\frac{2uv}{u+v}} \left[f\left(\frac{vx}{tx+(1-t)v}\right) + f\left(\frac{ux}{tx+(1-t)u}\right)\right] \frac{p\left(\frac{vx}{2v-x}\right)}{x^2} dx \\
&+ \int_{\frac{2uv}{u+v}}^{v} \left[f\left(\frac{vx}{tx+(1-t)v}\right) + f\left(\frac{ux}{tx+(1-t)u}\right)\right] \frac{p\left(\frac{ux}{2u-x}\right)}{x^2} dx \\
&= \int_u^{\frac{2uv}{u+v}} \left[f\left(\frac{vx}{tx+(1-t)v}\right) + f\left(\frac{uvx}{tux+(1-t)((u+v)x-uv)}\right)\right. \\
&\left. + f\left(\frac{ux}{tx+(1-t)u}\right) + f\left(\frac{uvx}{tvx+(1-t)((u+v)x-uv)}\right)\right] \frac{p\left(\frac{ux}{2u-x}\right)}{x^2} dx \\
&= \int_u^{\frac{4uv}{u+3v}} \left[f\left(\frac{vx}{tx+(1-t)v}\right) + f\left(\frac{2uvx}{2tux+(1-t)((3u+v)x-2uv)}\right)\right. \\
&+ f\left(\frac{2uvx}{2tux+(1-t)((v-u)x+2uv)}\right) + f\left(\frac{uvx}{tux+(1-t)((u+v)x-uv)}\right) \\
&+ f\left(\frac{2uvx}{2tvx+(1-t)((v-u)x+2uv)}\right) + f\left(\frac{2uvx}{2tvx+(1-t)((3u+v)x-2uv)}\right) \\
&\left. + f\left(\frac{uvx}{tvx+(1-t)((u+v)x-uv)}\right) + f\left(\frac{ux}{tx+(1-t)u}\right)\right] \frac{p\left(\frac{ux}{2u-x}\right)}{x^2} dx. \quad (93)
\end{aligned}$$

By using Lemma 2.4, we observe that the following inequality holds for all $t \in [0,1]$ and $x \in \left[u, \frac{4uv}{u+3v}\right]$:
The inequality

$$f\left(\frac{vx}{tx+(1-t)v}\right) + f\left(\frac{2uvx}{2tux+(1-t)((3u+v)x-2uv)}\right) \le f(v) + f\left(\frac{2uv}{2ut+(1-t)(u+v)}\right) \quad (94)$$

holds for $x_1 = \frac{vx}{tx+(1-t)v}$, $x_2 = \frac{2uvx}{2tux+(1-t)((3u+v)x-2uv)}$, $y_1 = v$ and $y_2 = \frac{2uv}{2ut+(1-t)(u+v)}$.
The inequality

$$\begin{aligned}
&f\left(\frac{2uvx}{2tux+(1-t)((v-u)x+2uv)}\right) + f\left(\frac{uvx}{tux+(1-t)((u+v)x-uv)}\right) \\
&\qquad \le f\left(\frac{2uv}{tu+(1-t)(u+v)}\right) + f\left(\frac{uv}{tu+(1-t)v}\right) \quad (95)
\end{aligned}$$

holds for $x_1 =, \frac{2uvx}{2tux+(1-t)((v-u)x+2uv)}$, $x_2 = \frac{uvx}{tux+(1-t)((u+v)x-uv)}$, $y_1 = \frac{2uv}{tu+(1-t)(u+v)}$ and $y_2 = \frac{uv}{tu+(1-t)v}$.
The inequality

$$\begin{aligned}
&f\left(\frac{ux}{tx+(1-t)u}\right) + f\left(\frac{2uvx}{2tux+(1-t)((3u+v)x-2uv)}\right) \\
&\qquad \le f\left(\frac{uv}{tv+(1-t)u}\right) + f\left(\frac{2uv}{2vt+(1-t)(u+v)}\right) \quad (96)
\end{aligned}$$

holds for $x_1=\frac{ux}{tx+(1-t)u}$, $x_2=\frac{2uvx}{2tux+(1-t)((3u+v)x-2uv)}$, $y_1=\frac{uv}{tv+(1-t)u}$ and $y_2=\frac{2uv}{2vt+(1-t)(u+v)}$.
The inequality

$$f\left(\frac{2uvx}{2tvx+(1-t)((v-u)x+2uv)}\right)+f\left(\frac{uvx}{tvx+(1-t)((u+v)x-uv)}\right) \leq f\left(\frac{2uv}{2vt+(1-t)(u+v)}\right)+f(u) \quad (97)$$

holds for $x_1=\frac{2uvx}{2tvx+(1-t)((v-u)x+2uv)}$, $x_2=\frac{uvx}{tvx+(1-t)((u+v)x-uv)}$, $y_1=\frac{2uv}{2vt+(1-t)(u+v)}$ and $y_2=u$.
Multiplying the inequalities (94)-(97) by $\frac{p\left(\frac{ux}{2u-x}\right)}{x^2}$ and integrating them over x on $\left[u,\frac{2uv}{u+v}\right]$ and using identity (93), we get

$$2\mathcal{S}_p(t)\leq G_1(t)\int_u^v\frac{p(x)}{x^2}dx\leq\frac{1}{2}\left[\frac{f(u)+f(v)}{2}+Q(t)\right]\int_u^v\frac{p(x)}{x^2}dx, \quad (98)$$

for all $t\in[0,1]$. Using (40) and (98), we derive (89).
□

Corollary 2.17. *Let $p(x)=\frac{uv}{v-u}$, $x\in[u,v]$ in Theorem 2.16. Then $I_1(t)=V(t)$, $t\in[0,1]$ and therefore we observe that*

(i) The inequalities

$$U(t)\leq Q(t)\leq\frac{f(u)+f(v)}{2}, \quad (99)$$

hold for $t\in\left[0,\frac{1}{3}\right]$ and

$$f\left(\frac{2uv}{u+v}\right)\leq Q(t)\leq V(t), \quad (100)$$

inequalities hold for $t\in\left[\frac{1}{3},1\right]$.

(ii) The inequalities

$$0\leq\mathcal{L}(t)\leq G_1(t)\int_u^v\frac{p(x)}{x^2}dx\leq\frac{1}{2}\left[\frac{f(u)+f(v)}{2}+Q(t)\right]+\mathcal{L}(t), \quad (101)$$

hold for all $t\in[0,1]$.

The following Fejér-type inequalities can be deduced from Theorems 21, 34, 2.10, 2.11, 2.12, 2.16, Corollary 2.9 and Lemma 2.6 and we omit their proofs.

Theorem 2.18. *Let f, p, U_p, V_p, G_1, K, $\mathcal{L}_p$, $\mathcal{S}_p$ be defined as above. Then, the following inequalities hold for all $t\in[0,1]$:*

$$\begin{aligned}f\left(\frac{2uv}{u+v}\right)\int_u^v\frac{p(x)}{x^2}dx&\leq U_p(t)\leq G_1(t)\int_u^v\frac{p(x)}{x^2}dx\\&\leq\mathcal{S}_p(t)\leq(1-t)\int_u^v\left[f\left(\frac{2ux}{u+x}\right)+f\left(\frac{2xv}{x+v}\right)\right]\frac{p(x)}{x^2}dx\\&\quad+t\cdot\frac{f(u)+f(v)}{2}\int_u^v\frac{p(x)}{x^2}dx\leq\frac{f(u)+f(v)}{2}\int_u^v\frac{p(x)}{x^2}dx \quad (102)\end{aligned}$$

and

$$f\left(\frac{2uv}{u+v}\right)\int_u^v \frac{p(x)}{x^2}dx \le K(t) \le G_1(t)\int_u^v \frac{p(x)}{x^2}dx \le \mathcal{L}_p(t) \le V_p(t) \le (1-t)\int_u^v \frac{f(x)p(x)}{x^2}dx + t\cdot\frac{f(u)+f(v)}{2}\int_u^v \frac{p(x)}{x^2}dx \le \frac{f(u)+f(v)}{2}\int_u^v \frac{p(x)}{x^2}dx. \quad (103)$$

Theorem 2.19. *Let f, p, U_p, G_1, K, $\mathcal{Q}$ be defined as above. Then, the following inequality holds for all $t\in\left[0,\frac{1}{4}\right]$:*

$$f\left(\frac{2uv}{u+v}\right)\int_u^v \frac{p(x)}{x^2}dx \le U_p(t) \le U_p(2t) \le G_1(2t)\int_u^v \frac{p(x)}{x^2}dx \le \mathcal{Q}(t)\int_u^v \frac{p(x)}{x^2}dx \le \frac{f(u)+f(v)}{2}\int_u^v \frac{p(x)}{x^2}dx \quad (104)$$

and

$$f\left(\frac{2uv}{u+v}\right)\int_u^v \frac{p(x)}{x^2}dx \le K(t) \le K(2t) \le G_1(2t)\int_u^v \frac{p(x)}{x^2}dx \le \mathcal{Q}(t)\int_u^v \frac{p(x)}{x^2}dx \le \frac{f(u)+f(v)}{2}\int_u^v \frac{p(x)}{x^2}dx. \quad (105)$$

Theorem 2.20. *Let f, p, U_p, V_p, G_1, $\mathcal{Q}$, $\mathcal{L}_p$, $\mathcal{S}_p$ be defined as above. Then, the following inequalities hold for all $t\in\left[\frac{1}{4},\frac{1}{3}\right]$:*

$$f\left(\frac{2uv}{u+v}\right)\int_u^v \frac{p(x)}{x^2}dx \le U_p(t) \le \mathcal{Q}(t)\int_u^v \frac{p(x)}{x^2}dx \le G_1(2t)\int_u^v \frac{p(x)}{x^2}dx \le \mathcal{L}_p(2t) \le V_p(2t) \le (1-2t)\int_u^v \frac{f(x)p(x)}{x^2}dx + 2t\cdot\frac{f(u)+f(v)}{2}\int_u^v \frac{p(x)}{x^2}dx \le \frac{f(u)+f(v)}{2}\int_u^v \frac{p(x)}{x^2}dx \quad (106)$$

and

$$f\left(\frac{2uv}{u+v}\right)\int_u^v \frac{p(x)}{x^2}dx \le U_p(t) \le \mathcal{Q}(t)\int_u^v \frac{p(x)}{x^2}dx \le G_1(2t)\int_u^v \frac{p(x)}{x^2}dx \le \mathcal{S}_p(2t) \le (1-2t)\int_u^v \frac{1}{2}\left[f\left(\frac{2ux}{u+x}\right)+f\left(\frac{2xv}{x+v}\right)\right]\frac{p(x)}{x^2}dx + 2t\cdot\frac{f(u)+f(v)}{2}\int_u^v \frac{p(x)}{x^2}dx \le \frac{f(u)+f(v)}{2}\int_u^v \frac{p(x)}{x^2}dx. \quad (107)$$

Theorem 2.21. *Let f, p, U_p, V_p, G_1, $\mathcal{Q}$, $\mathcal{L}_p$, $\mathcal{S}_p$ be defined as above. Then, the following inequalities hold for all $t\in\left[\frac{1}{3},\frac{1}{2}\right]$:*

$$f\left(\frac{2uv}{u+v}\right)\int_u^v \frac{p(x)}{x^2}dx \le \mathcal{Q}(t)\int_u^v \frac{p(x)}{x^2}dx \le G_1(2t)\int_u^v \frac{p(x)}{x^2}dx \le \mathcal{L}_p(2t) \le V_p(2t) \le (1-2t)\int_u^v \frac{f(x)p(x)}{x^2}dx + 2t\cdot\frac{f(u)+f(v)}{2}\int_u^v \frac{p(x)}{x^2}dx \le \frac{f(u)+f(v)}{2}\int_u^v \frac{p(x)}{x^2}dx, \quad (108)$$

$$f\left(\frac{2uv}{u+v}\right)\int_u^v \frac{p(x)}{x^2}dx \le \mathcal{Q}(t)\int_u^v \frac{p(x)}{x^2}dx \le G_1(2t)\int_u^v \frac{p(x)}{x^2}dx \le \mathcal{S}_p(2t)$$
$$\le (1-2t)\int_u^v \frac{1}{2}\left[f\left(\frac{2ux}{u+x}\right)+f\left(\frac{2xv}{x+v}\right)\right]\frac{p(x)}{x^2}dx$$
$$+2t\cdot\frac{f(u)+f(v)}{2}\int_u^v \frac{p(x)}{x^2}dx \le \frac{f(u)+f(v)}{2}\int_u^v \frac{p(x)}{x^2}dx \quad (109)$$

and

$$f\left(\frac{2uv}{u+v}\right)\int_u^v \frac{p(x)}{x^2}dx \le \mathcal{Q}(t)\int_u^v \frac{p(x)}{x^2}dx$$
$$\le G_1(2t)\int_u^v \frac{p(x)}{x^2}dx \le V_p(t) \le V_p(2t) \le (1-2t)\int_u^v \frac{f(x)p(x)}{x^2}dx$$
$$+2t\cdot\frac{f(u)+f(v)}{2}\int_u^v \frac{p(x)}{x^2}dx \le \frac{f(u)+f(v)}{2}\int_u^v \frac{p(x)}{x^2}dx. \quad (110)$$

Theorem 2.22. *Let f, p, U_p, V_p, G_1, $\mathcal{Q}$, $\mathcal{L}_p$, $\mathcal{S}_p$ be defined as above. Then, the following inequalities hold for all $t\in\left[\frac{1}{2},\frac{2}{3}\right]$:*

$$f\left(\frac{2uv}{u+v}\right)\int_u^v \frac{p(x)}{x^2}dx \le \mathcal{Q}(t)\int_u^v \frac{p(x)}{x^2}dx \le G_1(2(1-t))\int_u^v \frac{p(x)}{x^2}dx \le \mathcal{L}_p(2(1-t))$$
$$\le V_p(2(1-t)) \le (2t-1)\int_u^v \frac{f(x)p(x)}{x^2}dx$$
$$+2(1-t)\cdot\frac{f(u)+f(v)}{2}\int_u^v \frac{p(x)}{x^2}dx \le \frac{f(u)+f(v)}{2}\int_u^v \frac{p(x)}{x^2}dx \quad (111)$$

and

$$f\left(\frac{2uv}{u+v}\right)\int_u^v \frac{p(x)}{x^2}dx \le \mathcal{Q}(t)\int_u^v \frac{p(x)}{x^2}dx \le G_1(2(1-t))\int_u^v \frac{p(x)}{x^2}dx \le \mathcal{S}_p(2(1-t))$$
$$\le (2t-1)\int_u^v \frac{1}{2}\left[f\left(\frac{2ux}{u+x}\right)+f\left(\frac{2xv}{x+v}\right)\right]\frac{p(x)}{x^2}dx$$
$$+2(1-t)\cdot\frac{f(u)+f(v)}{2}\int_u^v \frac{p(x)}{x^2}dx \le \frac{f(u)+f(v)}{2}\int_u^v \frac{p(x)}{x^2}dx. \quad (112)$$

Theorem 2.23. *Let f, p, U_p, V_p, G_1, $\mathcal{Q}$, $\mathcal{L}_p$, $\mathcal{S}_p$ be defined as above. Then, the following inequalities hold for all $t\in\left[\frac{2}{3},\frac{3}{4}\right]$:*

$$f\left(\frac{2uv}{u+v}\right)\int_u^v \frac{p(x)}{x^2}dx \le \mathcal{Q}(t)\int_u^v \frac{p(x)}{x^2}dx \le G_1(2(1-t))\int_u^v \frac{p(x)}{x^2}dx \le G_1(t)\int_u^v \frac{p(x)}{x^2}dx$$
$$\le \mathcal{L}_p(t) \le V_p(t) \le (1-t)\int_u^v \frac{f(x)p(x)}{x^2}dx$$
$$+t\cdot\frac{f(u)+f(v)}{2}\int_u^v \frac{p(x)}{x^2}dx \le \frac{f(u)+f(v)}{2}\int_u^v \frac{p(x)}{x^2}dx \quad (113)$$

and

$$
\begin{aligned}
&f\left(\frac{2uv}{u+v}\right)\int_u^v \frac{p(x)}{x^2}dx \le \mathcal{Q}(t)\int_u^v \frac{p(x)}{x^2}dx \le G_1(2(1-t))\int_u^v \frac{p(x)}{x^2}dx \le G_1(t)\int_u^v \frac{p(x)}{x^2}dx \\
&\le \mathcal{S}_p(t) \le (1-t)\int_u^v \frac{1}{2}\left[f\left(\frac{2ux}{u+x}\right)+f\left(\frac{2xv}{x+v}\right)\right]\frac{p(x)}{x^2}dx \\
&\qquad\qquad \le t\cdot\frac{f(u)+f(v)}{2}\int_u^v \frac{p(x)}{x^2}dx \le \frac{f(u)+f(v)}{2}\int_u^v \frac{p(x)}{x^2}dx. \qquad (114)
\end{aligned}
$$

Theorem 2.24. *Let f, p, U_p, V_p, G_1, $\mathcal{Q}$, $\mathcal{L}_p$, $\mathcal{S}_p$ be defined as above. Then, the following inequalities hold for all $t\in\left[\frac{3}{4},1\right]$:*

$$
\begin{aligned}
&f\left(\frac{2uv}{u+v}\right)\int_u^v \frac{p(x)}{x^2}dx \le U_p(2(1-t)) \le G_1(2(1-t))\int_u^v \frac{p(x)}{x^2}dx \\
&\le \mathcal{Q}(t)\int_u^v \frac{p(x)}{x^2}dx \le V_p(t) \le (1-t)\int_u^v \frac{f(x)p(x)}{x^2}dx \\
&\qquad\qquad + t\cdot\frac{f(u)+f(v)}{2}\int_u^v \frac{p(x)}{x^2}dx \le \frac{f(u)+f(v)}{2}\int_u^v \frac{p(x)}{x^2}dx \qquad (115)
\end{aligned}
$$

and

$$
\begin{aligned}
&f\left(\frac{2uv}{u+v}\right)\int_u^v \frac{p(x)}{x^2}dx \le K(2(1-t)) \le G_1(2(1-t))\int_u^v \frac{p(x)}{x^2}dx \\
&\le \mathcal{Q}(t)\int_u^v \frac{p(x)}{x^2}dx \le V_p(t) \le (1-t)\int_u^v \frac{f(x)p(x)}{x^2}dx \\
&\qquad\qquad + t\cdot\frac{f(u)+f(v)}{2}\int_u^v \frac{p(x)}{x^2}dx \le \frac{f(u)+f(v)}{2}\int_u^v \frac{p(x)}{x^2}dx. \qquad (116)
\end{aligned}
$$

Corollary 2.25. *Let f, $\mathcal{Q}$, G_1, U, V, $\mathcal{L}$ be defined as above and $p(x)=\frac{uv}{v-u}$, then we have:*

(i) The inequalities

$$
f\left(\frac{2uv}{u+v}\right) \le U(t) \le U(2t) \le G_1(2t) \le \mathcal{Q}(t) \le \frac{f(u)+f(v)}{2} \qquad (117)
$$

hold for all $t\in\left[0,\frac{1}{4}\right]$.

(ii) The inequalities

$$
\begin{aligned}
&f\left(\frac{2uv}{u+v}\right) \le U(t) \le \mathcal{Q}(t) \le G_1(2t) \le \mathcal{L}(2t) \le V(2t) \\
&\qquad\qquad \le \left(\frac{1-2t}{v-u}\right)\int_u^v f(x)\,dx + 2t\cdot\frac{f(u)+f(v)}{2} \le \frac{f(u)+f(v)}{2} \qquad (118)
\end{aligned}
$$

hold for all $t\in\left[\frac{1}{4},\frac{1}{3}\right]$.

(iii) The inequalities

$$
\begin{aligned}
&f\left(\frac{2uv}{u+v}\right) \le \mathcal{Q}(t) \le G_1(2t) \le \mathcal{L}(2t) \le V(2t) \\
&\qquad\qquad \le \left(\frac{1-2t}{v-u}\right)\int_u^v \frac{f(x)}{x^2}dx + 2t\cdot\frac{f(u)+f(v)}{2} \le \frac{f(u)+f(v)}{2} \qquad (119)
\end{aligned}
$$

and

$$f\left(\frac{2uv}{u+v}\right) \leq Q(t) \leq V(t) \leq V(2t) \leq \left(\frac{1-2t}{v-u}\right)\int_u^v \frac{f(x)}{x^2}dx + 2t\cdot\frac{f(u)+f(v)}{2} \leq \frac{f(u)+f(v)}{2} \quad (120)$$

hold for all $t \in \left[\frac{1}{3}, \frac{1}{2}\right]$.

(iv) The inequalities

$$f\left(\frac{2uv}{u+v}\right) \leq Q(t) \leq G_1(2(1-t)) \leq \mathcal{L}(2(1-t)) \leq V(2(1-t)) \leq \left(\frac{2t-1}{v-u}\right)\int_u^v \frac{f(x)}{x^2}dx + 2(1-t)\cdot\frac{f(u)+f(v)}{2} \leq \frac{f(u)+f(v)}{2} \quad (121)$$

hold for all $t \in \left[\frac{1}{2}, \frac{2}{3}\right]$.

(v) The inequalities

$$f\left(\frac{2uv}{u+v}\right) \leq Q(t) \leq G_1(2(1-t)) \leq G_1(t) \leq \mathcal{L}(t) \leq V(t) \leq \left(\frac{1-t}{v-u}\right)\int_u^v \frac{f(x)}{x^2}dx + t\cdot\frac{f(u)+f(v)}{2} \leq \frac{f(u)+f(v)}{2} \quad (122)$$

hold for all $t \in \left[\frac{2}{3}, \frac{3}{4}\right]$.

(vi) The inequalities

$$f\left(\frac{2uv}{u+v}\right) \leq U(2(1-t)) \leq G_1(2(1-t)) \leq Q(t) \leq V(t) \leq \left(\frac{1-t}{v-u}\right)\int_u^v \frac{f(x)}{x^2}dx + t\cdot\frac{f(u)+f(v)}{2} \leq \frac{f(u)+f(v)}{2} \quad (123)$$

hold for all $t \in \left[\frac{3}{4}, 1\right]$.

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