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Non homogeneous dual wavelet frames and oblique extension principles in $H^s(\mathbb{K})$

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Abstract. In this paper, we introduce the notion of nonhomogeneous dual wavelet frames in Sobolev spaces over local fields. We provide the complete characterization of nonhomogeneous dual wavelet frames on local fields. Furthermore, we obtain a mixed oblique extension principle for such frames.

1. Introduction

The wavelet transform is a simple mathematical tool that cuts up data or functions into different frequency components, and then studies each components with a resolution matched to its scale. The main feature of the wavelet transform is to hierarchically decompose general functions, as a signal or a process, into a set of approximation functions with different scales. One of the important factor behind the stable decomposition of a signal for analysis or transmission is related to the type of representation used for its spanning set (representation system). A careful choice of the spanning set enables us to solve a variety of analysis tasks. During the last two decades, many researchers have contributed in the designing and time-frequency analysis of these representation systems for the various spaces, namely, finite and infinite abelian groups, Euclidean spaces, locally compact abelian groups. Nonhomogeneous dual wavelet frames admit fast wavelet transform as compared to homogeneous ones and possesses more designing freedom than homogeneous ones. Han [18–20] studied nonhomogeneous dual wavelet frames in $L^2(\mathbb{R}^d)$. Nonhomogeneous dual wavelet frames in $\left(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d)\right)$ were studied by various authors [13, 15–17].

During the last decade, there is a substantial body of work that has been concerned with the construction of wavelets and frames on local fields. For example, R. L. Benedetto and J. J. Benedetto [12] developed a wavelet theory for local fields and related groups. They did not develop the multiresolution analysis (MRA) approach, their method is based on the theory of wavelet sets and only allows the construction of wavelet functions whose Fourier transforms are characteristic functions of some sets. Jiang et al. [22] pointed out a method for constructing orthogonal wavelets on local field $\mathbb{K}$ with a constant generating sequence and derived necessary and sufficient conditions for a solution of the refinement equation to generate a multiresolution analysis of $L^2(\mathbb{K})$. Later on, Li and Jiang [23] have obtained a necessary condition and a set of sufficient conditions for the wavelet system $\left\{\psi_{j,k} =: q^{j/2}\psi\left(\mathfrak{p}^{-j}x - u(k)\right) : j,k \in \mathbb{N}_0\right\}$ to be a tight wavelet frame on local fields in the frequency domain. Ahmad and his collaborators in the series of papers

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investigated frame theory on local fields and obtained various interesting results [2–10, 24–27]. Continuing our investigation of frames on local fields, our main goal in this paper is to develop the theory of nonhomogeneous dual wavelet frames in the settings of local fields and provide their complete characterization.We also derive a mixed oblique extension principle (MOEP) for such frames.

The paper is structured as follows. In section 2, we discuss the preliminaries on local fields, definition of Sobolov spaces and the notion of nonhomogeneous dual wavelet frames on these fields. In Section 3, we provide the complete characterization of nonhomogeneous dual wavelet frames in Sobolev spaces over local fields. Section 4 is devoted to the derivation of mixed oblique extension principle for nonhomogeneous dual wavelet frames.

2. Preliminaries on Local Fields

2.1. Local Fields

In this paper, we use the symbols $\mathbb{N}, \mathbb{N}_0$ and $\mathbb{Z}$ to denote the sets of natural, non-negative integers and integers, respectively. A local field $\mathbb{K}$ is a locally compact, non-discrete and totally disconnected field. If it is of characteristic zero, then it is a field of p-adic numbers $\mathbb{Q}_p$ or its finite extension. If $\mathbb{K}$ is of positive characteristic, then $\mathbb{K}$ is a field of formal Laurent series over a finite field $GF(p^c)$. If $c = 1$, it is a p-series field, while for $c \neq 1$, it is an algebraic extension of degree c of a p-series field. Let $\mathbb{K}$ be a fixed local field with the ring of integers $\mathfrak{D} = \{x \in \mathbb{K} : |x| \leq 1\}$. Since $\mathbb{K}^+$ is a locally compact Abelian group, we choose a Haar measure dx for $\mathbb{K}^+$. A local field $\mathbb{K}$ is endowed with non–Archimedean norm $|\cdot| : \mathbb{K} \to \mathbb{R}^+$ satisfying

(a) $|x| = 0$ if and only if $x = 0$;

(b) $|x\, y| = |x||y|$ for all $x, y \in \mathbb{K}$;

(c) $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in \mathbb{K}$.

Property (c) is called the ultrametric inequality. Let $\mathfrak{B} = \{x \in \mathbb{K} : |x| < 1\}$ be the prime ideal of the ring of integers $\mathfrak{D}$ in $\mathbb{K}$. Then, the residue space $\mathfrak{D}/\mathfrak{B}$ is isomorphic to a finite field $GF(q)$, where $q = p^c$ for some prime p and $c \in \mathbb{N}$. Since $\mathbb{K}$ is totally disconnected and $\mathfrak{B}$ is both prime and principal ideal, so there exist a prime element $\mathfrak{p}$ of $\mathbb{K}$ such that $\mathfrak{B} = \langle \mathfrak{p} \rangle = \mathfrak{p}\mathfrak{D}$. Let $\mathfrak{D}^* = \mathfrak{D} \setminus \mathfrak{B} = \{x \in \mathbb{K} : |x| = 1\}$. Clearly, $\mathfrak{D}^*$ is a group of units in $\mathbb{K}^*$ and if $x \neq 0$, then can write $x = \mathfrak{p}^n y, y \in \mathfrak{D}^*$. Moreover, if $\mathcal{U} = \{a_m : m = 0, 1, \dots, q-1\}$ denotes the fixed full set of coset representatives of $\mathfrak{B}$ in $\mathfrak{D}$, then every element $x \in \mathbb{K}$ can be expressed uniquely as $x = \sum_{\ell=k}^{\infty} c_\ell\, \mathfrak{p}^\ell$ with $c_\ell \in \mathcal{U}$. Recall that $\mathfrak{B}$ is compact and open, so each fractional ideal $\mathfrak{B}^k = \mathfrak{p}^k\mathfrak{D} = \left\{x \in \mathbb{K} : |x| < q^{-k}\right\}$ is also compact and open and is a subgroup of K^+. We use the notation in Taibleson's book [28].

Let χ be a fixed character on $\mathbb{K}^+$ that is trivial on $\mathfrak{D}$ but non-trivial on $\mathfrak{B}^{-1}$. Therefore, χ is constant on cosets of $\mathfrak{D}$ so if $y \in \mathfrak{B}^k$, then $\chi_y(x) = \chi(x, y), x \in \mathbb{K}$. Suppose that χ_u is any character on $\mathbb{K}^+$, then the restriction $\chi_u|\mathfrak{D}$ is a character on $\mathfrak{D}$. Moreover, as characters on $\mathfrak{D}, \chi_u = \chi_v$ if and only if $u - v \in \mathfrak{D}$. Hence, if $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representative of $\mathfrak{D}$ in $\mathbb{K}^+$, then, as it was proved in [28], the set $\left\{\chi_{u(n)} : n \in \mathbb{N}_0\right\}$ of distinct characters on $\mathfrak{D}$ is a complete orthonormal system on $\mathfrak{D}$.

We now impose a natural order on the sequence $\{u(n)\}_{n=0}^{\infty}$. We have $\mathfrak{D}/\mathfrak{B} \cong GF(q)$ where $GF(q)$ is a c-dimensional vector space over the field $GF(p)$. We choose a set $\{1 = \zeta_0, \zeta_1, \zeta_2, \dots, \zeta_{c-1}\} \subset \mathfrak{D}^*$ such that $\text{span}\left\{\zeta_j\right\}_{j=0}^{c-1} \cong GF(q)$. For $n \in \mathbb{N}_0$ satisfying

$$0 \leq n < q, \quad n = a_0 + a_1 p + \cdots + a_{c-1}p^{c-1}, \quad 0 \leq a_k < p, \quad \text{and } k = 0, 1, \dots, c-1,$$

we define

$$u(n) = (a_0 + a_1\zeta_1 + \cdots + a_{c-1}\zeta_{c-1})\, \mathfrak{p}^{-1}. \tag{2.1}$$

Also, for $n = b_0 + b_1 q + b_2 q^2 + \cdots + b_s q^s,\ n \in \mathbb{N}_0,\ 0 \le b_k < q, k = 0, 1, 2, \ldots, s$, we set

$$u(n) = u(b_0) + u(b_1)\mathfrak{p}^{-1} + \cdots + u(b_s)\mathfrak{p}^{-s}. \tag{2.2}$$

This defines $u(n)$ for all $n \in \mathbb{N}_0$. In general, it is not true that $u(m+n) = u(m) + u(n)$. But, if $r, k \in \mathbb{N}_0$ and $0 \le s < q^k$, then $u(rq^k + s) = u(r)\mathfrak{p}^{-k} + u(s)$. Further, it is also easy to verify that $u(n) = 0$ if and only if $n = 0$ and $\{u(\ell) + u(k) : k \in \mathbb{N}_0\} = \{u(k) : k \in \mathbb{N}_0\}$ for a fixed $\ell \in \mathbb{N}_0$.

Let a and b be any two fixed elements in $\mathbb{K}$. Then, for any prime $\mathfrak{p}$ and $m, n \in \mathbb{N}_0$, we define the unitary operators on $L^2(\mathbb{K})$ by:

$$T_{u(n)a} f(x) = f\big(x - u(n)a\big), \qquad \text{(Translation)}$$
$$D_{\mathfrak{p}} f(x) = \sqrt{q} f\left(\mathfrak{p}^{-1}x\right), \qquad \text{(Dilation).}$$

2.2. *Fourier Transforms on Local Fields*

The Fourier transform of $f \in L^1(\mathbb{K})$ is denoted by $\hat{f}(\xi)$ and defined by

$$\mathcal{F}\Big\{f(x)\Big\} = \hat{f}(\xi) = \int_{\mathbb{K}} f(x)\overline{\chi_\xi(x)}\, dx. \tag{2.3}$$

It is noted that

$$\hat{f}(\xi) = \int_{\mathbb{K}} f(x)\, \overline{\chi_\xi(x)} dx = \int_{\mathbb{K}} f(x)\chi(-\xi x)\, dx. \tag{2.4}$$

The properties of Fourier transforms on local field $\mathbb{K}$ are much similar to those of on the classical field $\mathbb{R}$. In fact, the Fourier transform on local fields of positive characteristic have the following properties:

- The map $f \to \hat{f}$ is a bounded linear transformation of $L^1(\mathbb{K})$ into $L^\infty(\mathbb{K})$, and $\big\|\hat{f}\big\|_\infty \le \big\|f\big\|_1$.
- If $f \in L^1(\mathbb{K})$, then $\hat{f}$ is uniformly continuous.
- If $f \in L^1(\mathbb{K}) \cap L^2(\mathbb{K})$, then $\big\|\hat{f}\big\|_2 = \big\|f\big\|_2$.

The Fourier transform of a function $f \in L^2(\mathbb{K})$ is defined by

$$\hat{f}(\xi) = \lim_{k\to\infty} \hat{f}_k(\xi) = \lim_{k\to\infty} \int_{|x|\le q^k} f(x)\overline{\chi_\xi(x)}\, dx, \tag{2.5}$$

where $f_k = f\,\mathbf{\Phi}_{-k}$ and $\mathbf{\Phi}_k$ is the characteristic function of $\mathfrak{B}^k$. Furthermore, if $f \in L^2(\mathfrak{D})$, then we define the Fourier coefficients of f as

$$\hat{f}\big(u(n)\big) = \int_{\mathfrak{D}} f(x)\overline{\chi_{u(n)}(x)}\, dx. \tag{2.6}$$

The series $\sum_{n\in\mathbb{N}_0} \hat{f}\big(u(n)\big)\chi_{u(n)}(x)$ is called the Fourier series of f. From the standard L^2-theory for compact Abelian groups, we conclude that the Fourier series of f converges to f in $L^2(\mathfrak{D})$ and Parseval's identity holds:

$$\big\|f\big\|_2^2 = \int_{\mathfrak{D}} \big|f(x)\big|^2 dx = \sum_{n\in\mathbb{N}_0} \Big|\hat{f}\big(u(n)\big)\Big|^2. \tag{2.7}$$

For $j \in \mathbb{N}_0$, let $\mathcal{N}_j$ denote a full collection of coset representatives of $\mathbb{N}_0/q^j\mathbb{N}_0$, i.e.,

$$\mathcal{N}_j = \left\{0, 1, 2, \ldots, q^j - 1\right\}, \quad j \geq 0.$$

Then, $\mathbb{N}_0 = \bigcup_{n \in \mathcal{N}_j} \left(n + q^j\mathbb{N}_0\right)$, and for any distinct $n_1, n_2 \in \mathcal{N}_j$, we have $\left(n_1 + q^j\mathbb{N}_0\right) \cap \left(n_2 + q^j\mathbb{N}_0\right) = \emptyset$. Thus, every non-negative integer k can uniquely be written as $k = rq^j + s$, where $r \in \mathbb{N}_0, s \in \mathcal{N}_j$.

We denote the test function space on $\mathbb{K}$ by $\Omega(\mathbb{K})$, i.e., each function f in $\Omega(\mathbb{K})$ is a finite linear combination of functions of the form $\mathbf{\Phi}_k(x - h), h \in \mathbb{K}, k \in \mathbb{Z}$, where $\mathbf{\Phi}_k$ is the characteristic function of $\mathfrak{B}^k$. Then, it is clear that $\Omega(\mathbb{K})$ is dense in $L^p(\mathbb{K}), 1 \leq p < \infty$, and each function in $\Omega(\mathbb{K})$ is of compact support and so is its Fourier transform. The space $\Omega'(\mathbb{K})$ of continuous linear functional on $\Omega(\mathbb{K})$ is called the space of distributions.

Definition 2.1. For $s \in \mathbb{R}$, we denote by $H^s(\mathbb{K})$ as the space of all $f \in \Omega'(\mathbb{K})$ such that

$$\widehat{\gamma}^{s/2}\widehat{f}(\xi) \in L^2(\mathbb{K}), \;\; \textit{where}\; \widehat{\gamma}(\xi) = \max(1, |\xi|).$$

The space $H^s(\mathbb{K})$ is a linear space equipped with the inner product

$$\langle f, g\rangle_{H^s(\mathbb{K})} = \int_{\mathbb{K}} \widehat{f(\xi)}\overline{\widehat{g(\xi)}}\widehat{\gamma}^s(\xi)\, d\xi, \;\; f, g \in H^s(\mathbb{K}),$$

which induces the norm

$$\left\|f\right\|^2_{H^s(\mathbb{K})} = \int_{\mathbb{K}} \left|\widehat{f(\xi)}\right|^2 \widehat{\gamma}^s(\xi)\, d\xi.$$

The space $\Omega(\mathbb{K})$ is dense in $H^s(\mathbb{K})$. For each $g \in H^{-s}(\mathbb{K})$,

$$\langle f, g\rangle = \int_{\mathbb{K}} \hat{f}(\xi)\overline{\widehat{g(\xi)}}\, d\xi,$$

is a linear continuous functional in $H^s(\mathbb{K})$. The spaces $H^s(\mathbb{K})$ and $H^{-s}(\mathbb{K})$ form pairs of dual spaces. For functions $f, g : \mathbb{K} \to \mathbb{C}$, define

$$[f, g]_s(\xi) = \sum_{k \in \mathbb{N}_0} f\left(\xi + u(k)\right) \overline{g\left(\xi + u(k)\right)}\widehat{\gamma}^s(\xi + u(k)), \;\; s \in \mathbb{R}.$$

The spectrum $\sigma_s(f)$ is given by

$$\sigma_s(f) = \left\{\xi \in \mathfrak{B} : [\hat{f}, \hat{f}]_s(\xi) > 0\right\}.$$

For a distribution f, $j \in \mathbb{Z}, k \in \mathbb{N}_0$, $s \in \mathbb{R}$, we write

$$f_{j,k} = q^{j/2} f\left(\mathfrak{p}^{-j}\xi - u(k)\right) \;\; \textit{and} \;\; f^s_{j,k} = q^{js} f\left(\mathfrak{p}^{-j}\xi - u(k)\right).$$

2.4. Nonhomogeneous Dual Wavelet Frames on Local Fields

For $s \in \mathbb{R}$, let $\{\varphi, \psi_\ell\}_{\ell=1}^L \subseteq H^s(\mathbb{K})$ and $\left\{\widetilde{\varphi}, \widetilde{\psi}_\ell\right\}_{\ell=1}^L \subseteq H^{-s}(\mathbb{K})$,we define the following two nonhomogeneous wavelet systems in $H^s(\mathbb{K})$ and $H^{-s}(\mathbb{K})$, respectively:

$$\begin{aligned} \mathcal{W}^s(\varphi; \Psi) =& \mathcal{W}^s\left(\varphi; \psi_1, \psi_2, \ldots, \psi_L\right) \\ =& \left\{\varphi_{0,k} : k \in \mathbb{N}_0\right\} \cup \left\{\psi^s_{\ell,j,k} : j \in \mathbb{N}_0, k \in \mathbb{N}_0, 1 \leq \ell \leq L\right\} \end{aligned} \tag{2.8}$$

and

$$\begin{aligned}\mathcal{W}^{-s}(\widetilde{\varphi};\widetilde{\Psi}) =& \mathcal{W}^{-s}\left(\widetilde{\varphi};\widetilde{\psi}_1,\widetilde{\psi}_2,\cdots.\widetilde{\psi}_L\right)\\ &= \{\widetilde{\varphi}_{0,k}: k\in\mathbb{N}_0\}\cup\left\{\widetilde{\psi}^{-s}_{\ell,j,k}: j\in\mathbb{N}_0, k\in\mathbb{N}_0, 1\le\ell\le L\right\}.\end{aligned} \tag{2.9}$$

We say that $\mathcal{W}^s(\varphi,\Psi)$ is a *nonhomogeneous wavelet frame* in $H^s(\mathbb{K})$ if there exists two positive constants A, B such that

$$A\|f\|^2_{H^s(\mathbb{K})}\le\sum_{k\in\mathbb{N}_0}\left|\langle f,\varphi_{0,k}\rangle_{H^s(\mathbb{K})}\right|^2+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{N}_0}\left|\langle f,\psi^s_{\ell,j,k}\rangle_{H^s(\mathbb{K})}\right|^2\le B\|f\|^2_{H^s(\mathbb{K})} \tag{2.10}$$

where A, B are called *frame bounds*; it is called a *nonhomogeneous wavelet Bessel sequence* in $H^s(\mathbb{K})$ if the right hand inequality in (2.10) holds, where B is called a *Bessel bound*. Furthermore, we say that $\left(\mathcal{W}^s(\varphi;\Psi),\mathcal{W}^{-s}(\widetilde{\varphi};\widetilde{\Psi})\right)$ is a pair of *nonhomogeneous wavelet dual frames* in $(H^s(\mathbb{K}),H^{-s}(\mathbb{K}))$ if $\mathcal{W}^s(\varphi;\Psi)$ and $\mathcal{W}^{-s}(\widetilde{\varphi};\widetilde{\Psi})$ are Bessel sequences in $H^s(\mathbb{K})$ and $H^{-s}(\mathbb{K})$ respectively, and

$$\langle f,g\rangle=\sum_{k\in\mathbb{N}_0}\langle f,\widetilde{\varphi}_{0,k}\rangle\langle\varphi_{0,k},g\rangle+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{N}_0}\langle f,\widetilde{\psi}^{-s}_{\ell,j,k}\rangle\langle\psi^s_{\ell,j,k},g\rangle \tag{2.11}$$

holds for all $f\in H^s(\mathbb{K})$ and $g\in H^{-s}(\mathbb{K})$.

If $\left(\mathcal{W}^s(\varphi;\Psi),\mathcal{W}^{-s}(\widetilde{\varphi};\widetilde{\Psi})\right)$ is a pair of *dual frames* in $(H^s(\mathbb{K}),H^{-s}(\mathbb{K}))$, then it follows from (2.11) that the series

$$f=\sum_{k\in k}\langle f,\widetilde{\varphi}_{0,k}\rangle\varphi_{0,k}+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in k}\langle f,\widetilde{\psi}^{-s}_{\ell,j,k}\rangle\ \psi^s_{\ell,j,k}$$

and

$$g=\sum_{k\in k}\langle g,\varphi_{0,k}\rangle\widetilde{\varphi}_{0,k}+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in k}\langle f,\psi^s_{\ell,j,k}\rangle\ \widetilde{\psi}^{-s}_{\ell,j,k},$$

converging unconditionally in $H^s(\mathbb{K})$ and $H^{-s}(\mathbb{K})$ respectively.

Definition 2.2. Define a function $\kappa:\mathbb{N}_0\to\mathbb{N}_0$ by

$$\kappa(k)=\sup\left\{j\in\mathbb{Z}:\mathfrak{p}^{-j}u(k)\in\mathbb{N}_0\right\}.$$

It immediately follows that $\kappa(0)=+\infty$. By the definition, we have the following propositions:

Proposition 2.1. $\left\{\mathfrak{p}^{-\kappa(k)-1}u(k):k\in\mathbb{N}\right\}=\bigcup_{k\in\mathcal{N}_j\setminus\{0\}}\left(\mathbb{N}_0+u(k)\right).$

Proposition 2.2. For $\varphi\in H^s(\mathbb{K})$ and $\widetilde{\varphi}\in H^{-s}(\mathbb{K})$, we have

$$\begin{aligned}&\sigma(\varphi)\cap\tau\left(\sigma(\widetilde{\varphi})-u(\nu)\right)\\ &\quad=\left\{\xi\in\mathfrak{B}:\hat{\varphi}(\xi+u(k))\overline{\hat{\widetilde{\varphi}}(\xi+u(\nu)+u(n))}\ne 0\,\textit{for some}\ k,n\in\mathbb{N}_0,\nu\in\mathcal{N}_j\right\},\end{aligned}$$

where τ is a mapping from $\mathbb{K}$ to $\mathfrak{D}$ defined as $\tau(x)=x-u(k)$ for $x\in\mathfrak{D}+u(k)$ with $k\in\mathbb{N}$.

Proposition 2.3. Let $\{\alpha_k\}_{k\in\mathbb{N}_0}$ and $\{\beta_k\}_{k\in\mathbb{N}_0}$ be two sequences and $\displaystyle\sum_{j=0}^{\infty}\sum_{k\in\mathbb{N}}|\alpha_{q^jk}\beta_k|<\infty$, then

$$\sum_{j=0}^{\infty}\sum_{k\in\mathbb{N}}\alpha_{q^jk}\beta_k=\sum_{k\in\mathbb{N}}\sum_{j=0}^{\kappa(k)}\alpha_k\beta_{q^{-j}k}.$$

3. Characterization of Nonhomogeneous Dual Wavelet Frames

In this section, we establish the characterization of nonhomogeneous dual wavelet frames on Sobolev spaces over local fields of positive characteristic. In order to establish these results we need various results which we will state as lemmas. Define

$$\mathcal{F}(\Psi)=\left\{\psi_{\ell,j,k}:j\in\mathbb{Z},k\in\mathbb{N}_0,1\leq\ell\leq L\right\} \tag{3.1}$$

and

$$\mathcal{F}(\widetilde{\Psi})=\left\{\widetilde{\psi}_{\ell,j,k}:j\in\mathbb{Z},k\in\mathbb{N}_0,1\leq\ell\leq L\right\}. \tag{3.2}$$

Bownik [11] obtained the following important characterization for dual wavelet frames.

Proposition 3.1. Let $\mathcal{F}(\Psi)$ and $\mathcal{F}(\widetilde{\Psi})$ be Bessel sequences in $L^2(K)$. Then $\left(\mathcal{F}(\Psi),\mathcal{F}(\widetilde{\Psi})\right)$ is a pair of dual frames in $L^2(K)$ if and only if, for every $k\in\mathbb{N}_0$,

$$\sum_{\ell=1}^{L}\sum_{j=-\infty}^{\kappa(k)}\psi_\ell(\mathfrak{p}^{-j}\xi)\overline{\widehat{\widetilde{\psi}}_\ell\left(\mathfrak{p}^{-j}(\xi+u(k))\right)}=\delta_{0,k}\ a.e.\ \xi\in\mathbb{K}. \tag{3.3}$$

By a standard argument, we have the following result.

Lemma 3.1. For $s\in\mathbb{R}$. Define $\mathcal{P}$ by

$$\widehat{\mathcal{P}f}(\xi)=\widehat{\gamma}^{\frac{s}{2}}(\xi)\widehat{f}(\xi)$$

for $f\in H^s(\mathbb{K})$. then we have

(i) $\mathcal{P}$ is a unitary operator both from $H^s(\mathbb{K})$ onto $L^2(\mathbb{K})$ and from $L^2(\mathbb{K})$ onto $H^{-s}(\mathbb{K})$;

(ii)

$$\begin{aligned}(\widehat{\mathcal{P}f_{j,k}})(\xi)&=q^{-j/2}\widehat{\gamma}^{\frac{s}{2}}(\xi)\ \overline{\chi_{u(k)}(\mathfrak{p}^j\xi)}\widehat{f}(\mathfrak{p}^j\xi)\\&=\left\{\frac{\widehat{\gamma}(\xi)}{\widehat{\gamma}(\mathfrak{p}^j\xi)}\right\}^{s/2}[(\widehat{\mathcal{P}f})_{j,k}](\xi)\end{aligned}$$

for $f\in H^s(\mathbb{K})$.

Lemma 3.2. [See [11]] For $\psi\in L^2(\mathbb{K}),\left\{T_{u(k)}\psi:k\in\mathbb{N}_0\right\}$ is a Bessel sequence in $L^2(\mathbb{K})$ with Bessel bound B if and only if

$$[\widehat{\psi},\widehat{\psi}]_0(\xi)\leq B\ a.e.\ \xi\in\mathfrak{B}.$$

By Lemma 3.1 (i) and 3.2, we have

Lemma 3.3. Let $s\in\mathbb{R}$ and $\psi\in H^s(\mathbb{K}),\left\{T_{u(k)}\psi:k\in\mathbb{N}_0\right\}$ is a Bessel sequence in $H^s(\mathbb{K})$ with Bessel bound B if and only if

$$[\widehat{\psi},\widehat{\psi}]_s(\xi)\leq B\ a.e.\xi\in\mathfrak{B}.$$

The idea of the following lemma is borrowed from [29] and can be proved analogously.

Lemma 3.4. Let S be a bounded set in $\mathbb{K}$. Then there exist finite sets $F_1\subset\mathbb{N}_0$ and $F_2\subset\mathbb{N}$ such that

$$S\cap(S+\mathfrak{p}^{-j}u(k))=\emptyset\ for\ (j,k)\in(\mathbb{N}_0\times\mathbb{N})\setminus F_1\times F_2. \tag{3.4}$$

Lemma 3.5. For a given $s\in\mathbb{R}$, $j\in\mathbb{Z}$ and $\psi\in H^s(\mathbb{K})$, we have

(i) $\left\{\psi^s_{j,k} : k \in \mathbb{N}_0\right\}$ is a Bessel sequence in $H^s(\mathbb{K})$ with Bessel bound B_j if $\left\{T_{u(k)}\psi : k \in \mathbb{N}_0\right\}$ is a Bessel sequence in $H^s(\mathbb{K})$ with Bessel bound B, where

$$B_j = q^{-2js} B \sup_{\xi\in\mathbb{K}} \left\{\frac{\widehat{\gamma}(\xi)}{\hat{\gamma}(\mathfrak{p}^{-j}\xi)}\right\}^s;$$

(ii) $\left\{T_{u(k)}\psi : k \in \mathbb{N}_0\right\}$ is a Bessel sequence in $H^s(\mathbb{K})$ with Bessel bound

$$B_j = q^{2js} B \sup_{\xi\in\mathbb{K}} \left\{\frac{\widehat{\gamma}(\mathfrak{p}^{-j}\xi)}{\widehat{\gamma}(\xi)}\right\}^s$$

if $\left\{\psi^s_{j,k} : k \in \mathbb{N}_0\right\}$ is a Bessel sequence in $H^s(\mathbb{K})$ with Bessel bound B_j.

Proof. (i) Since $\left\{T_{u(k)}\psi : k \in \mathbb{N}_0\right\}$ is a Bessel sequence in $H^s(\mathbb{K})$ with Bessel bound B, therefore by Lemma 3.1 (i), $\{\mathcal{P}T_{u(k)}\psi : k \in \mathbb{N}_0\}$ is a Bessel sequence in $L^2(\mathbb{K})$ with Bessel bound B. It is easy to check that $\mathcal{P}T_{u(k)} = T_{u(k)}\mathcal{P}$ for $k \in \mathbb{N}_0$. So $\{T_{u(k)}\mathcal{P}\psi : k \in \mathbb{N}_0\}$ is a Bessel sequence in $L^2(\mathbb{K})$ with Bessel bound B. It follows that $\{(\mathcal{P}\psi)^s_{j,k} : k \in \mathbb{N}_0\}$ is a Bessel sequence in $L^2(\mathbb{K})$ with Bessel bound $q^{-2js}B$. By Lemma 3.1(ii), we have

$$(\widehat{\mathcal{P}\psi^s_{j,k}})(\xi) = \left\{\frac{\widehat{\gamma}(\xi)}{\widehat{\gamma}(\mathfrak{p}^{-j}\xi)}\right\}^{s/2} [(\widehat{\mathcal{P}\psi)^s_{j,k}}](\xi)$$

for each $k \in \mathbb{N}_0$. Thus

$$\begin{aligned}
\sum_{k\in\mathbb{N}_0} \left|\left\langle f, \mathcal{P}\psi^s_{j,k}\right\rangle_{L^2(\mathbb{K})}\right|^2 &= \sum_{k\in\mathbb{N}_0} \left|\left\langle \hat{f}, (\widehat{\mathcal{P}\psi^s_{j,k}})\right\rangle_{L^2(\mathbb{K})}\right|^2 \\
&= \sum_{k\in\mathbb{N}_0} \left|\left\langle \widehat{f}(\xi) \left\{\frac{\widehat{\gamma}(\xi)}{\widehat{\gamma}(\mathfrak{p}^{-j}\xi)}\right\}^{s/2},\ [(\widehat{\mathcal{P}\psi)^s_{j,k}}](\xi)\right\rangle_{L^2(\mathbb{K}}\right|^2 \\
&\leq B_j \|\widehat{f}\|^2_{L^2(\mathbb{K})} \\
&= B_j \|f\|^2_{L^2(\mathbb{K})}
\end{aligned}$$

for $f \in L^2(\mathbb{K})$ by the Plancherel theorem. So $\{\mathcal{P}\psi^s_{j,k} : k \in \mathbb{N}_0\}$ is a Bessel sequence in $L^2(\mathbb{K})$ with Bessel bound B_j, and the lemma therefore follows by Lemma 3.1 (i).

(ii) we have

$$(\widehat{D_{\mathfrak{p}^j}\mathcal{P}T_{u(k)}\psi})(\xi) = q^{js} (\widehat{\mathcal{P}\psi^s_{j,k}})(\xi) \left\{\frac{\widehat{\gamma}(\mathfrak{p}^{-j}\xi)}{\widehat{\gamma}(\xi)}\right\}^{s/2},$$

and thus

$$\sum_{k\in\mathbb{N}_0} \left|\left\langle \widehat{f}, (\widehat{\mathcal{P}\psi^s_{j,k}})\right\rangle_{L^2(\mathbb{K})}\right|^2 = q^{js} \sum_{k\in\mathbb{N}_0} \left|\left\langle \widehat{f}(\xi) \left\{\frac{\widehat{\gamma}(\mathfrak{p}^{-j}\xi)}{\widehat{\gamma}(\xi)}\right\}^{s/2}, (\widehat{\mathcal{P}\psi^s_{j,k}})(\xi)\right\rangle_{L^2(\mathbb{K})}\right|^2$$

for $f \in L^2(\mathbb{K})$. Then, by the same procedure as in (i), we can prove that $\{D_{\mathfrak{p}^j}\mathcal{P}T_{u(k)}\psi : k \in \mathbb{N}_0\}$ is a Bessel sequence in $L^2(\mathbb{K})$ with Bessel bound $B = q^{2js} B_j \sup_{\xi\in\mathbb{K}} \left\{\frac{\widehat{\gamma}(\mathfrak{p}^{-j}\xi)}{\widehat{\gamma}(\xi)}\right\}^s$. This implies that $\{\mathcal{P}T_{u(k)}\psi : k \in \mathbb{N}_0\}$ is a Bessel sequence in $L^2(\mathbb{K})$ with Bessel bound B due to $D_{\mathfrak{p}^j}$ being unitary, and thus $\{T_{u(k)}\psi : k \in \mathbb{N}_0\}$ is a Bessel sequence in $H^s(\mathbb{K})$ with Bessel bound B by Lemma 3.1 (i).

Lemma 3.6. Let $s \in \mathbb{R}$, and $\{e_i\}_{i\in I}$ a sequence in $H^s(\mathbb{K})$. Then $\{e_i\}_{i\in I}$ is a Bessel sequence in $H^s(\mathbb{K})$ with Bessel bound B if and only if

$$\sum_{i\in I} |\langle f, e_i\rangle|^2 \leq B\|f\|^2_{H^{-s}(\mathbb{K})} \, for \ f \in H^{-s}(\mathbb{K}). \tag{3.5}$$

Proof: By Lemma 3.1(i), $\{e_i\}_{i\in I}$ is a Bessel sequence in $H^s(\mathbb{K})$ with Bessel bound B if and only if

$$\sum_{i\in I} |\langle f, \mathcal{P}e_i\rangle|^2 \leq B\|f\|^2_{L^2(\mathbb{K})} \, for \ f \in L^2(\mathbb{K}). \tag{3.6}$$

Also observe that

$$\sum_{i\in I} |\langle f, \mathcal{P}e_i\rangle|^2 = \sum_{i\in I} |\langle \mathcal{P}f, e_i\rangle|^2,$$

and

$$\|f\|^2_{L^2(\mathbb{K})} = \|\mathcal{P}f\|^2_{H^{-s}(\mathbb{K})}$$

by Lemma 3.1 (i). It follows that(3.6) is equivalent to

$$\sum_{i\in I} |\langle f, \mathcal{P}e_i\rangle|^2 \leq B\|\mathcal{P}f\|^2_{H^{-s}(\mathbb{K})} \, for \ f \in L^2(\mathbb{K}).$$

This leads to the lemma since $\mathcal{P}$ is a unitary operator from $L^2(\mathbb{K})$ onto $H^{-s}(\mathbb{K})$ by Lemma 3.1(i).

As an immediate consequence of Lemma 3.7, we have the following lemma:

Lemma 3.7. Let $s_1 \in \mathbb{R}$, *and* $\{e_i\}_{i\in I}$ a Bessel sequence in $H^{s_1}(\mathbb{K})$. Then $\{e_i\}_{i\in I}$ is a Bessel sequence in $H^{s_2}(\mathbb{K})$ *for* $s_2 < s_1$.

Recently, Ahmad and Sheikh [10] studied dual wavelet frames on local fields of positive characteristic and obtained various results similar to some of the results in this paper but the norm of the Sobolev space which is used in [10] is not a non-Archimedean norm which is not consistent in the domain of local fields where as in this paper we have used a non-Archimedean norm of the Sobolev spaces.

Lemma 3.8. Let $s \in \mathbb{R}$, $\psi \in H^{-s}(\mathbb{K})$. Then for $f \in H^s(\mathbb{K}), k \in k$, the k-th Fourier coefficient of $\left[q^{j/2}\widehat{f}\left(\mathfrak{p}^j\xi\right), \widehat{\psi}(\xi)\right]_0$ is $\left\langle f, \psi_{j,k}\right\rangle$. Furthermore, if $\left\{\psi_{j,k} : k \in \mathbb{N}_0\right\}$ is a Bessel Sequence in $H^{-s}(\mathbb{K})$, then we have

$$\left[q^{j/2}\widehat{f}\left(\mathfrak{p}^j\xi\right), \widehat{\psi}(\xi)\right]_0 = \sum_{k\in\mathbb{N}_0} \left\langle f, \psi_{j,k}\right\rangle \overline{\chi_{u(k)}(\xi)} \tag{3.7}$$

Proof. Since $f \in H^s(\mathbb{K})$ and $\psi \in H^{-s}(\mathbb{K})$, we have $\widehat{f}\left(\mathfrak{p}^j\xi\right)\overline{\widehat{\psi}(\xi)} \in L^1(\mathbb{K})$, and by Plancherel theorem we have

$$\begin{aligned}
&\int_{\mathfrak{D}} \left[q^{j/2}\widehat{f}\left(\mathfrak{p}^j\xi\right), \widehat{\psi}(\xi)\right]_0 \chi_{u(k)}(\xi)d\xi \\
&=q^{j/2} \int_{\mathfrak{D}} \sum_{k\in\mathbb{N}_0} \widehat{f}\left(\mathfrak{p}^j(\xi + u(k))\right) \overline{\widehat{\psi}\left(\xi + u(k)\right)} \chi_{u(k)}(\xi)d\xi \\
&=q^{j/2} \int_{\mathbb{K}} \widehat{f}\left(\mathfrak{p}^j\xi\right) \overline{\widehat{\psi}(\xi)} \chi_{u(k)}(\xi)d\xi \\
&=q^{-j/2} \int_{\mathbb{K}} \widehat{f}(\xi)\overline{\widehat{\psi}\left(\mathfrak{p}^j\xi\right)} \chi_{u(k)}\left(\mathfrak{p}^j\xi\right)d\xi \\
&= \int_{\mathbb{K}} \widehat{f}(\xi)\overline{[\psi_{j,k}(.)]^\wedge(\xi)}d\xi \\
&=\langle f, \psi_{j,k}\rangle.
\end{aligned} \tag{3.8}$$

If $\left\{\psi_{j,k} : k \in \mathbb{N}_0\right\}$ is a Bessel sequence in $H^{-s}(\mathbb{K})$, then $\{\langle f, \psi_{j,k}\rangle\}_{k\in\mathbb{N}_0} \in \ell^2(\mathbb{N}_0)$, and hence (3.7) follows by (3.8). This completes the proof of the lemma.

It is well known that the condition $\sum_{\ell=1}^{\infty}\sum_{j\in\mathbb{Z}}|\widehat{\psi_\ell}(\mathfrak{p}^j\xi)|^2 \in L^\infty(\mathbb{K})$ is necessary for $\mathcal{F}(\Psi)$ to be a Bessel sequence in $L^2(\mathbb{K})$. To establish a similar necessary condition for $\mathcal{W}^s(\varphi;\Psi)$ to be a Bessel sequence in $H^s(\mathbb{K})$, we need the following lemma:

Lemma 3.9. Given $s \in \mathbb{R}$, let $\{T_{u(k)}\varphi : k \in \mathbb{N}_0\} \cup \{T_{u(k)}\psi_\ell : k \in \mathbb{N}_0, 1 \le \ell \le L\}$ be a Bessel sequence in $H^s(\mathbb{K})$, then

$$\begin{aligned}
&\sum_{k\in\mathbb{N}_0}|\langle g,\varphi_{0,k}\rangle|^2 + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{N}_0}|\langle g,\psi^s_{\ell,j,k}\rangle|^2\\
&\quad= \int_{\mathbb{K}}|\widehat{g}(\xi)|^2\left\{|\widehat{\varphi}(\xi)|^2 + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}q^{-2js}|\widehat{\psi_\ell}(\mathfrak{p}^{-j}\xi)|^2\right\}d\xi\\
&\quad+ \int_{\mathbb{K}}\overline{\widehat{g}(\xi)}\sum_{k\in\mathbb{N}}\widehat{g}(\xi+u(k))\left\{\widehat{\varphi}(\xi)\overline{\widehat{\varphi}(\xi+u(k))} + \sum_{\ell=1}^{L}\sum_{j=0}^{\kappa(k)}q^{-2js}\widehat{\psi_\ell}(\mathfrak{p}^{-j}\xi)\overline{\widehat{\psi_\ell}(\mathfrak{p}^{-j}(\xi+u(k)))}\right\}d\xi
\end{aligned}\tag{3.8}$$

for $g \in \Omega(\mathbb{K})$.

Proof. By Lemma 3.8, we have

$$\begin{aligned}
&\sum_{k\in\mathbb{N}_0}|\langle g,\varphi_{0,k}\rangle|^2 + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{N}_0}|\langle g,\psi^s_{\ell,j,k}\rangle|^2\\
&\quad= \int_{\mathfrak{B}}\left|\sum_{k\in\mathbb{N}_0}\widehat{g}(\xi+u(k))\overline{\widehat{\varphi}(\xi+u(k))}\right|^2 d\xi + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}q^{j(1-2s)}\int_{\mathfrak{B}}\left|\sum_{k\in\mathbb{N}_0}\widehat{g}(\mathfrak{p}^j(\xi+u(k)))\overline{\widehat{\psi_\ell}(\xi+u(k))}\right|^2 d\xi\\
&\quad= \int_{\mathfrak{B}}\left\{\sum_{k\in\mathbb{N}_0}(\widehat{\varphi}(\xi+u(k))\overline{\widehat{g}(\xi+u(k))}\right\}\left\{\sum_{k\in\mathbb{N}_0}\widehat{g}(\xi+u(k))\overline{\widehat{\varphi}(\xi+u(k))}\right\}d\xi\\
&\quad+ \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}q^{j(1-2s)}\int_{\mathfrak{B}}\left\{\sum_{k\in\mathbb{N}_0}\widehat{\psi_\ell}(\xi+u(k))\overline{\widehat{g}(\mathfrak{p}^j(\xi+u(k)))}\right\}\left\{\sum_{k\in\mathbb{N}_0}\widehat{g}(\mathfrak{p}^j(\xi+u(k)))\overline{\hat{\psi}_\ell(\xi+u(k))}\right\}d\xi
\end{aligned}\tag{3.9}$$

Write

$$F_0(\xi) = \sum_{k\in\mathbb{N}_0}\widehat{g}(\xi+u(k))\overline{\widehat{\varphi}(\xi+u(k))}$$

and

$$F_{\ell,j}(\xi) = \sum_{k\in\mathbb{N}_0}\widehat{g}(\mathfrak{p}^j(\xi+u(k)))\overline{\widehat{\psi_\ell}(\xi+u(k))}.$$

Then

$$|F_0(\xi)| \le [\widehat{g},\hat{g}]^{1/2}_{-s}(\xi)\,[\widehat{\varphi},\widehat{\varphi}]^{1/2}_{s}(\xi),$$

$$|F_{\ell,j}(\xi)| \le \left[\widehat{g}(\mathfrak{p}^j\xi),\widehat{g}(\mathfrak{p}^j\xi)\right]^{1/2}_{-s}(\xi)[\widehat{\psi_\ell},\widehat{\psi_\ell}]^{1/2}_{s}(\xi),$$

and thus $F_0, F_{\ell,j} \in L^\infty(\mathfrak{D})$ by Lemma 3.3 since $\{T_{u(k)}g : k \in \mathbb{N}_0\}$ and $\{T_{u(k)}g_1 : k \in \mathbb{N}_0\}$ with $\hat{g}_l(\xi) = \hat{g}(\mathfrak{p}^j\xi)$ are

Bassel sequences in $H^{-s}(\mathbb{K})$ if $g \in \Omega$. It follows that

$$\int_{\mathfrak{B}} \sum_{k\in\mathbb{N}_0} |\overline{\widehat{g}(\xi+u(k))}\widehat{\varphi}(\xi+u(k))F_0(\xi)|d\xi$$
$$\leq \|F_0\|_{L^\infty(\mathfrak{B})} \int_{\mathfrak{B}} [\widehat{g},\widehat{g}]_{-s}^{1/2}(\xi)[\widehat{\varphi},\widehat{\varphi}]_s^{1/2}(\xi)d\xi < \infty,$$

and thus

$$\int_{\mathfrak{B}} \left\{\sum_{k\in\mathbb{N}_0} (\widehat{\varphi}(\xi+u(k))\overline{\widehat{g}(\xi+u(k))}\right\}\left\{\sum_{k\in\mathbb{N}_0} \widehat{g}(\xi+u(k))\overline{\widehat{\varphi}(\xi+u(k))}\right\}d\xi$$
$$= \int_{\mathbb{K}} \widehat{\varphi}(\xi)\overline{\widehat{g}(\xi)} \sum_{k\in\mathbb{N}_0} \widehat{g}(\xi+u(k))\overline{\widehat{\varphi}(\xi+u(k))}d\xi \tag{3.10}$$

by the Fubini-Tonelli Theorem.

Similarly, we also have

$$\int_{\mathfrak{B}} \left\{\sum_{k\in\mathbb{N}_0} \widehat{\psi}_\ell(\xi+u(k))\overline{\widehat{g}(\mathfrak{p}^j(\xi+u(k)))}\right\}\left\{\sum_{k\in\mathbb{N}_0} \widehat{g}(\mathfrak{p}^j(\xi+u(k)))\overline{\widehat{\psi}_\ell(\xi+u(k))}\right\}$$
$$= \int_{\mathbb{K}} \widehat{\psi}_\ell(\xi)\overline{\widehat{g}(\mathfrak{p}^j\xi)} \sum_{k\in\mathbb{N}_0} \widehat{g}(\mathfrak{p}^j(\xi+u(k)))\overline{\widehat{\psi}_\ell(\xi+u(k))}d\xi. \tag{3.11}$$

By (3.9)-(3.11), we have

$$\sum_{k\in\mathbb{N}_0} |\langle g,\varphi_{0,k}\rangle|^2 + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{N}_0} |\langle g,\psi^s_{\ell,j,k}\rangle|^2$$
$$= \int_{\mathbb{K}} \widehat{\varphi}(\xi)\overline{\widehat{g}(\xi)} \sum_{k\in\mathbb{N}_0} \widehat{g}(\xi+u(k))\overline{\widehat{\varphi}(\xi+u(k))}d\xi$$
$$+ \sum_{\ell=1}^{L}\sum_{j=0}^{\infty} q^{j(1-2s)} \int_{\mathbb{K}} \widehat{\psi}_l(\xi)\overline{\widehat{g}(\mathfrak{p}^j\xi)} \sum_{k\in\mathbb{N}_0} \widehat{g}(\mathfrak{p}^j(\xi+u(k)))\overline{\widehat{\psi}_\ell(\xi+u(k))}d\xi. \tag{3.12}$$

Let us check every part in (3.12). Observe that

$$\int_{\mathbb{K}} |\widehat{\varphi}(\xi)\overline{\widehat{g}(\xi)}| \sum_{k\in\mathbb{N}_0} |\widehat{g}(\xi+u(k))\overline{\widehat{\varphi}(\xi+u(k))}|d\xi$$
$$\leq \int_{supp(\widehat{g})} \left\{\sum_{k\in\mathbb{N}_0} |\widehat{g}(\xi+u(k))\overline{\widehat{\varphi}(\xi+u(k))}|\right\}^2 d\xi$$
$$\leq \int_{supp(\widehat{g})} [\widehat{g},\widehat{g}]_{-s}(\xi)\,[\widehat{\varphi},\widehat{\varphi}]_s(\xi)\,d\xi$$

and that $[\widehat{g},\widehat{g}]_{-s}(\xi)[\widehat{\varphi},\widehat{\varphi}]_s(\xi) \in L^\infty(\mathbb{K})$ by Lemma (3.3). It follows that

$$\int_{\mathbb{K}} |\widehat{\varphi}(\xi)\overline{\widehat{g}(\xi)}| \sum_{k\in\mathbb{N}_0} |\widehat{g}(\xi+u(k))\overline{\widehat{\varphi}(\xi+u(k))}|\,d\xi < \infty,$$

and thus

$$\int_{\mathbb{K}} \widehat{\varphi}(\xi)\overline{\widehat{g}(\xi)} \sum_{k\in\mathbb{N}_0} \widehat{g}(\xi+u(k))\overline{\widehat{\varphi}(\xi+u(k))}d\xi$$

$$= \int_{\mathbb{K}} |\widehat{\varphi}(\xi)|^2|\widehat{g}(\xi)|^2 d\xi + \int_{\mathbb{K}} \widehat{\varphi}(\xi)\overline{\widehat{g}(\xi)} \sum_{0\neq k\in\mathbb{N}_0} \widehat{g}(\xi+u(k))\overline{\widehat{\varphi}(\xi+u(k))}d\xi. \tag{3.13}$$

Now, we turn to the second part. Take S in Lemma 3.4 as a compact set in $\mathbb{K}$ such that $supp(\widehat{g}) \subset S$. Then there exist finite sets $F_1 \subset \mathbb{N}_0$ and $F_2 \subset \mathbb{N}$ such that

$$S \cap (S + \mathfrak{p}^j u(k)) = \varphi \; for \; (j,k) \in (\mathbb{N}_0 \times \mathbb{N}) \setminus F_1 \times F_2. \tag{3.14}$$

It follows that

$$\sum_{\ell=1}^{L}\sum_{j=0}^{\infty} q^{j(1-2s)} \int_{\mathbb{K}} \widehat{\psi_\ell}(\xi)\overline{\widehat{g}(\mathfrak{p}^j\xi)} \sum_{k\in\mathbb{N}_0} \widehat{g}(\mathfrak{p}^j\xi + \mathfrak{p}^j u(k)))\overline{\widehat{\psi_\ell}(\xi+u(k))}d\xi$$

$$= \sum_{\ell=1}^{L}\sum_{j\in F_1} m^{j(1-2s)} \int_{\mathbb{K}} \widehat{\psi_\ell}(\xi)\overline{\widehat{g}(\mathfrak{p}^j\xi)} \sum_{k\in F_2} \widehat{g}(\mathfrak{p}^j(\xi+u(k)))\overline{\widehat{\psi_\ell}(\xi+u(k))}d\xi. \tag{3.15}$$

Write $G = \bigcup_{k\in F_2\cup\{0\}} \left(\bigcup_{j\in F_1} \mathfrak{p}^{-j}S + u(k)\right)$. Then we have

$$\begin{aligned}
&\int_{\mathbb{K}} |\widehat{\psi_\ell}(\xi)\overline{\widehat{g}(\mathfrak{p}^j\xi)}\widehat{g}(\mathfrak{p}^j(\xi+u(k)))\overline{\widehat{\psi_\ell}(\xi+u(k))}|d\xi \\
&\quad\le \|\widehat{g}\|^2_{L^\infty(\mathbb{K})} \int_{\mathfrak{p}^{-j}S} |\widehat{\psi_\ell}(\xi)\widehat{\psi_\ell}(\xi+u(k))|d\xi \\
&\quad\le \|\widehat{g}\|^2_{L^\infty(\mathbb{K})} \left\{\int_{\mathfrak{p}^{-j}S} |\widehat{\psi_\ell}(\xi)|^2 d\xi\right\}^{\frac{1}{2}} \left\{\int_{\mathfrak{p}^{-j}S} |\widehat{\psi_l}(\xi+u(k))|^2 d\xi\right\}^{\frac{1}{2}} \\
&\quad\le \|\widehat{g}\|^2_{L^\infty(\mathbb{K})} \int_{K} |\widehat{\psi_\ell}(\xi)|^2 d\xi
\end{aligned}$$

For each $(j,k) \in F_1 \times F_2$. Also observe that $1 \le \{\max_{\xi\in G} \widehat{\gamma}^{-s}(\xi)\}\, \widehat{\gamma}^s(\xi)$ for $\xi \in G$. It follows that

$$\begin{aligned}
&\int_{\mathbb{K}} |\widehat{\psi_\ell}(\xi)\overline{\widehat{g}(\mathfrak{p}^j\xi)}\widehat{g}(\mathfrak{p}^j(\xi+u(k)))\overline{\widehat{\psi_\ell}(\xi+u(k))}|d\xi \\
&\qquad\le \{max_{\xi\in G}\widehat{\gamma}^{-s}(\xi)\}\, \|\widehat{g}\|^2_{L^\infty(\mathbb{K})} \int_{G} |\widehat{\psi_\ell}(\xi)|^2 \widehat{\gamma}^s(\xi)d\xi \\
&\qquad\le \{max_{\xi\in G}\widehat{\gamma}^{-s}(\xi)\}\, \|\widehat{g}\|^2_{L^\infty(\mathbb{K})} \|\widehat{\psi_\ell}(\xi)\|^2_{H^s(\mathbb{K})} \\
&\qquad< \infty.
\end{aligned} \tag{3.16}$$

So, collecting (3.14) - (3.16), we have

$$\sum_{\ell=1}^{L}\sum_{j=0}^{\infty} q^{j(1-2s)} \int_{\mathbb{K}} \widehat{\psi_\ell}(\xi)\overline{\widehat{g}(\mathfrak{p}^j\xi)} \sum_{k\in\mathbb{N}_0} \widehat{g}(\mathfrak{p}^j\xi + \mathfrak{p}^j k))\overline{\widehat{\psi_\ell}(\xi+u(k))}d\xi$$

$$= \int_{\mathbb{K}} \sum_{\ell=1}^{L}\sum_{j=0}^{\infty} q^{j(1-2s)} |\widehat{\psi_\ell}(\xi)|^2 |\widehat{g}(\mathfrak{p}^j\xi)|^2 d\xi + \int_{\mathbb{K}} \sum_{\ell=1}^{L}\sum_{j=0}^{\infty} q^{j(1-2s)} \overline{\widehat{g}(\mathfrak{p}^j\xi)\widehat{\psi_\ell}(\xi)} \sum_{k\in\mathbb{N}} \widehat{g}(\mathfrak{p}^j\xi + \mathfrak{p}^j u(k)))\overline{\widehat{\psi_\ell}(\xi+u(k))}d\xi$$

$$= \int_{\mathbb{K}} \sum_{\ell=1}^{L} \sum_{j=0}^{\infty} q^{-2js} |\widehat{\psi_\ell}(\mathfrak{p}^{-j}\xi)|^2 |\widehat{g}(\xi)|^2 d\xi + \int_{\mathbb{K}} \sum_{l=1}^{L} \sum_{j=0}^{\infty} q^{-2js} \overline{\widehat{g}(\xi)\widehat{\psi_\ell}(\mathfrak{p}^{-j}\xi)} \sum_{k\in\mathbb{N}} \widehat{g}(\xi + \mathfrak{p}^j u(k))) \overline{\widehat{\psi_\ell}(\mathfrak{p}^{-j}\xi + u(k))} d\xi$$

$$= \int_{\mathbb{K}} \sum_{\ell=1}^{L} \sum_{j=0}^{\infty} q^{-2js} |\widehat{\psi_\ell}(\mathfrak{p}^{j}\xi)|^2 |\widehat{g}(\xi)|^2 d\xi + \int_{\mathbb{K}} \overline{\widehat{g}(\xi)} \sum_{k\in\mathbb{N}} \widehat{g}(\xi + u(k)) \sum_{\ell=1}^{L} \sum_{j=0}^{\kappa(k)} q^{-2js} \overline{\widehat{\psi_\ell}(\mathfrak{p}^{j}(\xi + u(k)))} d\xi, \tag{3.17}$$

where Proposition 1.3 is used in the last equality. Equation (3.8) therefore follows by (3.12), (3.13) and(3.17).

Lemma 3.10 Let $s \in \mathbb{R}$, $\{\varphi, \psi_\ell\}_{\ell=1}^{L} \subseteq H^s(\mathbb{K})$. Then the system $\mathcal{W}^s(\varphi;\Psi)$ given by (3.1) is a Bessel sequence in $H^s(\mathbb{K})$ with Bessel bound B, then

$$|\widehat{\varphi}(\xi)|^2 + \sum_{\ell=1}^{L} \sum_{j=0}^{\infty} q^{-2js} \left|\widehat{\psi_\ell}\left(\mathfrak{p}^{-j}\xi\right)\right|^2 \leq B\, \widehat{\gamma}^{-s}(\xi), \tag{3.18}$$

holds a.e on $\mathbb{K}$.

Proof. Since $\mathcal{W}^s(\varphi;\Psi)$ be a Bessel sequence in $H^s(\mathbb{K})$ with Bessel bound B, we have

$$\sum_{k\in\mathbb{N}_0} |\langle g, \varphi_{0,k}\rangle|^2 + \sum_{\ell=1}^{L} \sum_{j=0}^{\infty} \sum_{k\in\mathbb{N}_0} |\langle g, \psi^s_{\ell,j,k}\rangle|^2 \leq B\|g\|^2_{H^{-s}(\mathbb{K})} \, for \; g \in H^{-s}(\mathbb{K}) \tag{3.19}$$

by Lemma 3.6. Next, we prove the lemma by contradiction. Suppose (3.18) does not hold, i.e. there exists $E \subset \mathbb{K}$ with $|E| > 0$ such that

$$|\widehat{\varphi}(\xi)|^2 + \sum_{\ell=1}^{L} \sum_{j=0}^{\infty} q^{-2js} |\widehat{\psi_\ell}(\mathfrak{p}^{j}\xi)|^2 > B\widehat{\gamma}^{-s}(\xi) \; on \; E.$$

It follows that

$$|\widehat{\varphi}(\xi)|^2 + \sum_{\ell=1}^{L} \sum_{j=0}^{\infty} q^{-2js} |\widehat{\psi_\ell}(\mathfrak{p}^{j}\xi)|^2 > B\widehat{\gamma}^{-s}(\xi)$$

on some $E' = E \cap (\mathfrak{B} + u(k_0))$ with $|E'| > 0$ and $k_0 \in \mathbb{N}_0$. Define g by $\widehat{g}(\xi) = \widehat{\gamma}^{s/2}(\xi)\varphi_{E'}(\xi)$. Then $g \in \Omega(\mathbb{K})$ and thus $g \in H^{-s}(\mathbb{K})$, and

$$\|g\|^2_{H^{-s}(\mathbb{K})} = |E'|. \tag{3.20}$$

Applying Lemma 3.9 to such g, we have

$$\begin{aligned} \sum_{k\in\mathbb{N}_0} |\langle g, \varphi_{0,k}\rangle|^2 &+ \sum_{\ell=1}^{L} \sum_{j=0}^{\infty} \sum_{k\in\mathbb{N}_0} |\langle g, \psi^s_{\ell,j,k}\rangle|^2 \\ &= \int_{\mathbb{K}} |\widehat{g}(\xi)|^2 \left\{ |\widehat{\varphi}(\xi)|^2 \sum_{\ell=1}^{L} \sum_{j=0}^{\infty} q^{-2js} |\widehat{\psi_\ell}(\mathfrak{p}^{-j}\xi|^2) \right\} d\xi \\ &> B|E'|, \end{aligned}$$

and thus

$$\sum_{k\in\mathbb{N}_0} |\langle g, \varphi_{0,k}\rangle|^2 + \sum_{\ell=1}^{L} \sum_{j=0}^{\infty} \sum_{k\in\mathbb{N}_0} |\langle g, \psi^s_{\ell,j,k}\rangle|^2 > B\|g\|^2_{H^{-s}(\mathbb{K})}$$

by (3.20). It contradicts (3.19). The proof is completed.

Lemma 3.11. Given $s \in \mathbb{R}$, let $\mathcal{W}^s(\varphi;\Psi)$ and $\mathcal{W}^{-s}(\widetilde{\varphi};\widetilde{\Psi})$ be Bessel sequences in $H^s(\mathbb{K})$ and $H^{-s}(\mathbb{K})$, respectively. Then

$$\begin{aligned}
&\sum_{k\in\mathbb{N}_0}\langle f,\widetilde{\varphi}(\xi-u(k))\rangle\langle\varphi(\xi-u(k)),g\rangle+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{N}_0}\langle f,\widetilde{\psi}^{-s}_{\ell,j,k}\rangle\langle\psi^{s}_{\ell,j,k},g\rangle\\
&\quad=\int_{\mathbb{K}}\widehat{f}(\xi)\overline{\widehat{g}(\xi)}\left\{\widehat{\varphi}(\xi)\overline{\widehat{\widetilde{\varphi}}(\xi)}+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\widehat{\psi}_\ell(\mathfrak{p}^{-j}\xi)\overline{\widehat{\widetilde{\psi}}_\ell(\mathfrak{p}^{-j}\xi)}\right\}d\xi\\
&\qquad+\int_{\mathbb{K}}\overline{\widehat{g}(\xi)}\sum_{k\in\mathbb{N}}\widehat{f}(\xi+u(k))\left\{\widehat{\varphi}(\xi)\overline{\widehat{\widetilde{\varphi}}(\xi+u(k))}\right.\\
&\qquad\qquad\left.+\sum_{\ell=1}^{L}\sum_{j=0}^{\kappa(k)}\widehat{\psi}_\ell(\mathfrak{p}^{-j}\xi)\overline{\widehat{\widetilde{\psi}}_\ell(\mathfrak{p}^{-j}(\xi+u(k)))}\right\}d\xi
\end{aligned}\tag{3.21}$$

for $f,g\in\Omega(\mathbb{K})$.

Proof: Since $\mathcal{W}^s(\varphi;\Psi)$ and $\mathcal{W}^{-s}(\widetilde{\varphi};\widetilde{\Psi})$ are Bessel sequences in $H^s(\mathbb{K})$ and $H^{-s}(\mathbb{K})$, respectively, the left hand side of (3.21) is well-defined. By the same procedure as in Lemma 3.9, we can prove that

$$\begin{aligned}
&\sum_{k\in\mathbb{N}_0}\langle f,\widetilde{\varphi}(\xi-u(k))\rangle\langle\varphi(\xi-u(k)),g\rangle+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{N}_0}\langle f,\widetilde{\psi}^{-s}_{\ell,j,k}\rangle\langle\psi^{s}_{\ell,j,k},g\rangle\\
&\quad=\int_{\mathbb{K}}\sum_{k\in\mathbb{N}_0}\widehat{f}(\xi+u(k))\overline{\widehat{\widetilde{\varphi}}(\xi+u(k))}\widehat{\varphi}(\xi)\overline{\widehat{g}(\xi)}d\xi\\
&\qquad\qquad+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}q^j\int_{\mathbb{K}}\sum_{k\in\mathbb{N}_0}\widehat{f}(\mathfrak{p}^j(\xi+u(k)))\overline{\widehat{\widetilde{\psi}}_\ell(\xi+u(k))}\widehat{\psi}(\xi)\overline{\widehat{g}(\mathfrak{p}^j\xi)}d\xi.
\end{aligned}\tag{3.22}$$

Observe that

$$\sum_{k\in\mathbb{N}_0}|\widehat{f}(\xi+k)\overline{\widehat{\widetilde{\varphi}}(\xi+u(k))}||\widehat{\varphi}(\xi)\overline{\widehat{g}(\xi)}|\leq[\widehat{f},\widehat{f}]_s^{\frac{1}{2}}(\xi)[\widehat{\widetilde{\varphi}},\widehat{\widetilde{\varphi}}]_{-s}^{\frac{1}{2}}(\xi)[\widehat{g},\widehat{g}]_{-s}^{\frac{1}{2}}(\xi)[\widehat{\varphi},\widehat{\varphi}]_s^{\frac{1}{2}}(\xi),$$

which belongs to $L^\infty(\mathbb{K})$ by Lemma 3.3. It follows that

$$\begin{aligned}
&\int_{\mathbb{K}}\sum_{k\in\mathbb{N}_0}|\widehat{f}(\xi+u(k))\overline{\widehat{\widetilde{\varphi}}(\xi+u(k))}\widehat{\varphi}(\xi)\overline{\widehat{g}(\xi)}|d\xi\\
&\qquad\qquad=\int_{supp(\widehat{g})}\sum_{k\in\mathbb{N}_0}|\widehat{f}(\xi+u(k))\overline{\widehat{\widetilde{\varphi}}(\xi+u(k))}\widehat{\varphi}(\xi)\overline{\widehat{g}(\xi)}|d\xi<\infty,
\end{aligned}$$

and thus

$$\begin{aligned}
&\int_{\mathbb{K}}\sum_{k\in\mathbb{N}_0}\widehat{f}(\xi+u(k))\overline{\widehat{\widetilde{\varphi}}(\xi+u(k))}\widehat{\varphi}(\xi)\overline{\widehat{g}(\xi)}d\xi\\
&\quad=\int_{\mathbb{K}}\widehat{f}(\xi)\overline{\widehat{g}(\xi)}\widehat{\varphi}(\xi)\overline{\widehat{\widetilde{\varphi}}(\xi)}d\xi+\int_{\mathbb{K}}\overline{\widehat{g}(\xi)}\sum_{k\in\mathbb{N}}\widehat{f}(\xi+u(k))\widehat{\varphi}(\xi)\overline{\widehat{\widetilde{\varphi}}(\xi+u(k))}.
\end{aligned}\tag{3.23}$$

By the Cauchy-Schwarz inequality and Lemma 3.10, we have

$$\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\left|\widehat{\psi_\ell}(\mathfrak{p}^{-j}\xi)\overline{\widehat{\widetilde{\psi_\ell}}(\mathfrak{p}^{-j}\xi)}\right| \le \left\{\sum_{\ell=1}^{L}\sum_{j=0}^{\infty} q^{-2js}|\widehat{\psi_\ell}(\mathfrak{p}^{-j}\xi)|^2\right\}^{\frac{1}{2}}\left\{\sum_{\ell=1}^{L}\sum_{j=0}^{\infty} q^{2js}|\widehat{\widetilde{\psi_\ell}}(\mathfrak{p}^{-j}\xi)|^2\right\}^{\frac{1}{2}} \le B_1B_2,$$

where B_1 and B_2 are Bessel bounds of $\mathcal{W}^s(\varphi;\Psi)$ and $\mathcal{W}^{-s}(\widetilde{\varphi};\widetilde{\Psi})$ respectively. It follows that

$$\int_{\mathbb{K}}|\widehat{f}(\xi)\overline{\widehat{g}(\xi)}|\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\left|\widehat{\psi_\ell}(\mathfrak{p}^{-j}\xi)\overline{\widehat{\widetilde{\psi_\ell}}(\mathfrak{p}^{-j}\xi)}\right|d\xi \le B_1B_2\left|supp(\widehat{f})\cap supp(\widehat{g})\right|\|\widehat{f}\|_{L^\infty(\mathbb{K})}\|\widehat{g}\|_{L^\infty(\mathbb{K})}<\infty. \tag{3.24}$$

Take S as a compact set in $\mathbb{K}$ such that $supp(\widehat{f})\cup supp(\widehat{g})\subset S$. By Lemma 3.4, there exist finite sets $F_1\subset\mathbb{N}_0$ and $F_2\subset\mathbb{N}$ such that

$$S\cup(S+\mathfrak{p}^ju(k))=\varphi \text{ for } (j,k)\in(\mathbb{N}_0\times\mathbb{N})\setminus F_1\times F_2. \tag{3.25}$$

Write $G=\cup_{k\in F_2\cup\{0\}}\left(\cup_{j\in F_1}\mathfrak{p}^{-j}S+u(k)\right)$. By the same procedure as in Lemma 3.9, we have

$$\begin{aligned}
&\int_{\mathbb{K}}\left|\overline{\widehat{g}(\mathfrak{p}^j\xi)}\widehat{f}(\mathfrak{p}^j\xi+\mathfrak{p}^ju(k))\widehat{\psi_\ell}(\xi)\overline{\widehat{\widetilde{\psi_\ell}}(\xi+u(k))}\right|d\xi\\
&\le\|\widehat{g}\|_{L^\infty(\mathbb{K})}\|\widehat{f}\|_{L^\infty(\mathbb{K})}\left\{\int_{\mathfrak{p}^{-j}S}|\widehat{\psi_\ell}(\xi)|^2d\xi\right\}^{\frac{1}{2}}\left\{\int_{\mathfrak{p}^{-j}S}|\widehat{\widetilde{\psi}}_\ell(\xi+u(k))|^2d\xi\right\}^{\frac{1}{2}}\\
&\le\|\widehat{g}\|_{L^\infty(\mathbb{K})}\|\widehat{f}\|_{L^\infty(\mathbb{K})}\left\{\int_{G}|\widehat{\psi_\ell}(\xi)|^2d\xi\right\}^{\frac{1}{2}}\left\{\int_{G}|\widehat{\widetilde{\psi}}_\ell(\xi)|^2d\xi\right\}^{\frac{1}{2}}\\
&\le\|\widehat{g}\|_{L^\infty(\mathbb{K})}\|\widehat{f}\|_{L^\infty(\mathbb{K})}\left\{\max_{\xi\in G}\widehat{\gamma}^{-s/2}(\xi)\right\}\left\{\max_{\xi\in G}\widehat{\gamma}^{s/2}(\xi)\right\}\|\psi_\ell\|_{H^s(\mathbb{K})}\|\widetilde{\psi_\ell}\|_{H^{-s}(\mathbb{K})}\\
&<\infty.
\end{aligned}\tag{3.26}$$

for $(j,k)\in F_1\times F_2$. By (3.25) and (3.26) , we have

$$\begin{aligned}
&\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}q^j\int_{\mathbb{K}}\sum_{k\in\mathbb{N}}\widehat{f}(\mathfrak{p}^j(\xi+u(k)))\overline{\widehat{\widetilde{\psi_\ell}}(\xi+u(k))}\widehat{\psi_\ell}(\xi)\overline{\widehat{g}(\mathfrak{p}^j\xi)}d\xi\\
&=\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\int_{\mathbb{K}}\overline{\widehat{g}(\xi)}\widehat{\psi_\ell}(\mathfrak{p}^{-j}\xi)\sum_{k\in\mathbb{N}}\widehat{f}(\xi+\mathfrak{p}^ju(k))\overline{\widehat{\widetilde{\psi_\ell}}(\mathfrak{p}^{-j}\xi+u(k))}d\xi\\
&=\int_{\mathbb{K}}\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\overline{\widehat{g}(\xi)}\widehat{\psi_\ell}(\mathfrak{p}^{-j}\xi)\sum_{k\in\mathbb{N}}\widehat{f}(\xi+\mathfrak{p}^ju(k))\overline{\widehat{\widetilde{\psi_\ell}}(\mathfrak{p}^{-j}\xi+u(k))}d\xi\\
&=\int_{\mathbb{K}}\overline{\widehat{g}(\xi)}\sum_{k\in\mathbb{N}}\widehat{f}(\xi+u(k))\sum_{\ell=1}^{L}\sum_{j=0}^{\kappa(k)}\widehat{\psi_\ell}(\mathfrak{p}^{-j}\xi)\overline{\widehat{\widetilde{\psi_\ell}}(\mathfrak{p}^{-j}(\xi+u(k)))}d\xi.
\end{aligned}\tag{3.27}$$

Collecting (3.22)-(3.24) and (3.27), we obtain (3.21). The proof is completed.

Theorem 3.1. Suppose that the system $\mathcal{W}^s(\varphi;\Psi)$ given by (3.1) is a Bessel sequence in $H^s(\mathbb{K})$ and the system $\mathcal{W}^{-s}(\widetilde{\varphi};\widetilde{\Psi})$ given by (3.2) is a Bessel sequence in $H^{-s}(\mathbb{K})$. Then the necessary and sufficient condition for

$\left(\mathcal{W}^s(\varphi;\Psi),\mathcal{W}^{-s}(\widetilde{\varphi};\widetilde{\Psi})\right)$ to be a pair of dual frames in $(H^s(\mathbb{K}),H^{-s}(\mathbb{K}))$ is

$$\widehat{\varphi}(\xi)\overline{\widehat{\widetilde{\varphi}}(\xi+u(k))}\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\widehat{\psi}_\ell\left(\mathfrak{p}^{-j}\xi\right)\overline{\widehat{\widetilde{\psi}}_\ell\left(\mathfrak{p}^{-j}(\xi+u(k))\right)}=\delta_{0,k}\ a.e\ \xi\in\mathbb{K}. \tag{3.28}$$

Proof. Since by the definition, $\left(\mathcal{W}^s(\varphi;\Psi),\mathcal{W}^{-s}(\widetilde{\varphi};\widetilde{\Psi})\right)$ is a pair of dual frames in $(H^s(\mathbb{K}),H^{-s}(\mathbb{K}))$ if and only if

$$\sum_{k\in\mathbb{N}_0}\langle f,\widetilde{\varphi}_{0,k}\rangle\langle\varphi_{0,k},g\rangle+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{N}_0}\left\langle f,\widetilde{\psi}^{-s}_{\ell,j,k}\right\rangle\left\langle\psi^{s}_{\ell,j,k},g\right\rangle$$
$$=\langle f,g\rangle,\ for\ f,g\in\Omega. \tag{3.29}$$

By Lemma 3.11, (3.29) can be rewritten as

$$\int_{\mathbb{K}}\widehat{f}(\xi)\overline{\widehat{g}(\xi)}\left\{\widehat{\varphi}(\xi)\overline{\widehat{\widetilde{\varphi}}(\xi)}+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\widehat{\psi}_\ell\left(\mathfrak{p}^{-j}\xi\right)\overline{\widehat{\widetilde{\psi}}\left(\mathfrak{p}-j\xi\right)}\right\}d\xi$$
$$+\int_{\mathbb{K}}\overline{\widehat{g}(\xi)}\sum_{k\in\mathbb{N}}\widehat{f}(\xi+u(k))\left\{\widehat{\varphi}(\xi)\overline{\widehat{\widetilde{\varphi}}(\xi+u(k))}\right.$$
$$\left.+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\widehat{\psi}_\ell\left(\mathfrak{p}^{-j}\xi\right)\overline{\widehat{\widetilde{\psi}}\left(\mathfrak{p}^{-j}(\xi+u(k))\right)}\right\}d\xi$$
$$=\int_{\mathbb{K}}\widehat{f}(\xi)\overline{\widehat{g}(\xi)}d\xi. \tag{3.30}$$

Clearly, (3.28) implies (3.30). Next, we prove the converse implication. Suppose (3.30) holds. By the Cauchy-Schwarz inequality, we have

$$\left|\widehat{\varphi}(\xi)\overline{\widehat{\widetilde{\varphi}}(\xi+u(k))}\right|+\sum_{\ell=1}^{L}\sum_{j=0}^{\kappa(k)}\left|\widehat{\psi}_\ell(\mathfrak{p}^{-j}\xi)\overline{\widehat{\widetilde{\psi}}_\ell(\mathfrak{p}^{-j}(\xi+u(k)))}\right|$$
$$\leq\left\{|\widehat{\varphi}(\xi)|^2+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}q^{-2js}|\widehat{\psi}_\ell(\mathfrak{p}^{-j}\xi)|^2\right\}^{1/2}$$
$$\times\left\{\left|\widehat{\widetilde{\varphi}}(\xi+u(k))\right|^2+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}q^{2js}\left|\widehat{\widetilde{\psi}}_\ell(\mathfrak{p}^{-j}(\xi+u(k)))\right|^2\right\}^{1/2}$$
$$\leq B_1B_2\widehat{\gamma}^{-s}(\xi)\ \widehat{\gamma}^{s}(\xi+u(k))$$
$$\leq B_1B_2\sup_{\xi\in\mathbb{K}}\widehat{\gamma}^{-s}(\xi)\ \widehat{\gamma}^{s}(\xi+u(k))$$
$$=C_k<\infty.$$

for each $k\in\mathbb{N}_0$ by Lemma 3.10, where B_1 and B_2 are Bessel bounds of $\mathcal{W}^s(\varphi;\Psi)$ and $\mathcal{W}^{-s}(\widetilde{\varphi};\widetilde{\Psi})$ respectively. Thus the series

$$\widehat{\varphi}(\xi)\overline{\widehat{\widetilde{\varphi}}(\xi+u(k))}+\sum_{\ell=1}^{L}\sum_{j=0}^{\kappa(k)}\widehat{\psi}_\ell(\mathfrak{p}^{-j}\xi)\overline{\widehat{\widetilde{\psi}}_\ell(\mathfrak{p}^{-j}(\xi+u(k)))}$$

converges absolutely to a function in $L^\infty(\mathbb{K})$ a.e. on $\mathbb{K}$ for each $k \in \mathbb{N}_0$. It follows that almost every point in $\mathbb{K}$ is a Lebesgue point of all

$$\widehat{\varphi}(\xi)\overline{\widehat{\widetilde{\varphi}}(\xi + u(k))} + \sum_{\ell=1}^{L}\sum_{j=0}^{\kappa(k)} \widehat{\psi_\ell}(\mathfrak{p}^{-j}\xi)\overline{\widehat{\widetilde{\psi}_\ell}(\mathfrak{p}^{-j}(\xi + u(k)))}$$

with $k \in \mathbb{N}_0$. Let $\xi_0 \in \mathbb{K}$ be such a point. For $0 < \epsilon < \frac{1}{2}$, take f and g such that

$$\widehat{f}(\xi) = \widehat{g}(\xi) = \frac{\chi_{B(\xi_0,\epsilon)}(.)}{\sqrt{|B(\xi_0,\epsilon)|}}$$

in (3.30), where $B(\xi_0,\epsilon) = \{\xi \in \mathbb{K} : \|\xi - \xi_0\|_2 < \xi\}$. Then

$$\frac{1}{|B(\xi_0,\epsilon)|}\int_{B(\xi_0,\epsilon)} \left\{\widehat{\varphi}(\xi)\overline{\widehat{\widetilde{\varphi}}(\xi)} + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty} \widehat{\psi_\ell}(\mathfrak{p}^{-j}\xi)\overline{\widehat{\widetilde{\psi}_\ell}(\mathfrak{p}^{-j}\xi)}\right\} d\xi = 1,$$

and letting $\epsilon \to 0$ we obtain

$$\widehat{\varphi}(\xi_0)\overline{\widehat{\widetilde{\varphi}}(\xi_0)} + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty} \widehat{\psi_l}(\mathfrak{p}^{-j}\xi_0)\overline{\widehat{\widetilde{\psi}_\ell}(\mathfrak{p}^{-j}\xi_0)} = 1.$$

For $k_0 \in \mathbb{N}$, take f and g such that

$$\hat{f}(\xi + u(k_0)) = \hat{g}(\xi) = \frac{\chi_{B(\xi_0,\epsilon)}(\xi)}{\sqrt{|B(\xi_0,\epsilon)|}}$$

in (3.30), where $0 < \epsilon < \frac{1}{2}$. Then

$$\frac{1}{|B(\xi_0,\epsilon)|}\int_{B(\xi_0,\epsilon)} \left\{\widehat{\varphi}(\xi)\overline{\widehat{\widetilde{\varphi}}(\xi + u(k_0))} + \sum_{\ell=1}^{L}\sum_{j=0}^{\kappa(k_0)} \widehat{\psi_\ell}(\mathfrak{p}^{-j}\xi)\overline{\widehat{\widetilde{\psi}_\ell}(\mathfrak{p}^{-j}(\xi + u(k_0)))}\right\} d\xi = 0,$$

and letting $\epsilon \to 0$ we obtain

$$\widehat{\varphi}(\xi)\overline{\widehat{\widetilde{\varphi}}(\xi + u(k_0))} + \sum_{\ell=1}^{L}\sum_{j=0}^{\kappa(k_0)} \widehat{\psi_\ell}(\mathfrak{p}^{-j}\xi)\overline{\widehat{\widetilde{\psi}_\ell}(\mathfrak{p}^{-j}(\xi + u(k_0)))} = 0.$$

By the arbitrariness of ξ_0 and $k_0 \in \mathbb{N}$, we obtain (3.28). The proof is completed.

4. 4. Mixed Oblique Extension Principle for Nonhomogeneous Dual wavelet Frames

In this section, using Theorem 3.1 we derive an MOEP for nonhomogeneous dual wavelet frames in $(H^s(\mathbb{K}), H^{-s}(\mathbb{K}))$ under the following assumptions:

Assumption 4.1: $\varphi \in H^s(\mathbb{K})$ and $\widetilde{\varphi} \in H^{-s}(\mathbb{K})$ are $\mathfrak{p}$-refinable with symbols in $L^\infty(\mathfrak{B})$, i.e there exist $\widehat{a}, \widehat{\widetilde{a}} \in L^\infty(\mathfrak{B})$ such that

$$\widehat{\varphi}(\mathfrak{p}\xi) = \widehat{a}(\xi)\widehat{\varphi}(\xi) \ \ and \ \ \widehat{\widetilde{\varphi}}(\mathfrak{p}\xi) = \widehat{\widetilde{a}}(\xi)\widehat{\widetilde{\varphi}}(\xi) \ \ a.e. \ on \ \mathbb{K}. \tag{4.1}$$

Assumption 4.2: $\lim_{j\to\infty} \widehat{\varphi}(\mathfrak{p}^{-j}\xi)\overline{\widehat{\widetilde{\varphi}}(\mathfrak{p}^{-j}\xi)} = 1$ *a.e. on* $\mathbb{K}$.

Given a positive integer L, let φ, $\widetilde{\varphi}$ satisfy Assumption 4.1 and $b_\ell, \widetilde{b_\ell} \in L^\infty(\mathfrak{B})$ *with* $1 \le l \le L$, and define ψ_ℓ and $\widetilde{\psi_\ell}$ by

$$\widehat{\psi_\ell}(\mathfrak{p}\xi) = \widehat{b_\ell}(\xi)\widehat{\varphi}(\xi) \text{ and } \widehat{\widetilde{\psi_\ell}}(\mathfrak{p}\xi) = \widehat{\widetilde{b_\ell}}(\xi)\widehat{\widetilde{\varphi}}(\xi) \tag{4.2}$$

We begin with some lemmas.

Lemma 4.1. For a given $s \in \mathbb{R}$ and $\varphi \in H^s(\mathbb{K})$, let φ satisfy Assumption 4.1, and ψ_l *with* $1 \le \ell \le L$ defined as in (4.2). Assume that

(i) $[\widehat{\varphi}, \widehat{\varphi}]_t \in L^\infty(\mathbb{K})$ for some $t > s$;

(ii) there exist a nonnegative number α with $\alpha > -s$ and a positive constant C such that

$$\sum_{\ell=1}^{L} |\widehat{b_\ell}(\xi)|^2 \le C \ \min\left(1, |\xi|_2^{2\alpha}\right) \ a.e. \ on \ \mathbb{K}. \tag{4.3}$$

Then $\mathcal{W}^s(\varphi; \Psi)$ is a Bessel sequence in $H^s(\mathbb{K})$.

Proof: For the case s = 0, take $0 < s_0 < min\{t, \alpha\}$. Then the conditions (i) and (ii) hold for $s = s_0$. By Lemma 3.7, the lemma holds for s=0 if it holds for $s = s_0$. So, to finish the proof, we only need to prove the lemma for $s = 0$. By Lemma 3.6, it is enough to prove that there exists a positive constant C such that

$$\sum_{k\in\mathbb{N}_0} |\langle g, \varphi_{0,k}\rangle|^2 + \sum_{l=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{N}_0} |\langle g, \psi^s_{l,j,k}\rangle|^2 \le C\|g\|^2_{H^{-s}(\mathbb{K}).} \tag{4.4}$$

Using Lemma 3.8, we have

$$\begin{aligned}\sum_{k\in\mathbb{N}_0} \left|\langle g, \varphi(\xi - u(k))\rangle\right|^2 &+ \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{N}_0} |\langle g, \psi^s_{\ell,j,k}\rangle|^2 \\ &= \int_{\mathfrak{B}} \left|[\widehat{g}, \widehat{\varphi}]_0\,(\xi)\right|^2 d\xi + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty} q^{j(1-2s)} \int_{\mathfrak{B}} \left|[\widehat{g}(\mathfrak{p}^j\xi), \widehat{\psi_\ell}]_0\,(\xi)\right|^2 d\xi.\end{aligned}$$

Also observe that

$$|[\widehat{g}, \widehat{\varphi}]_0(\xi)|^2 \le [\widehat{g}, \widehat{g}]_{-s}(\xi)[\widehat{\varphi}, \widehat{\varphi}]_s(\xi) \le [\widehat{\varphi}, \widehat{\varphi}]_t(\xi)[\widehat{g}, \widehat{g}]_{-s}(\xi).$$

It follows that

$$\begin{aligned}\sum_{k\in\mathbb{N}_0} |\langle g, \varphi(\xi - u(k))\rangle|^2 &+ \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{N}_0} |\langle g, \psi^s_{\ell,j,k}\rangle|^2 \\ &\le \|[\widehat{\varphi}, \widehat{\varphi}]_t\|_{L^\infty(\mathbb{K})}\|g\|^2_{H^{-s}(\mathbb{K})} + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty} q^{j(1-2s)} \int_{\mathfrak{B}} |[\widehat{g}(\mathfrak{p}^j\xi), \widehat{\psi_\ell}]_0(\xi)|^2 d\xi.\end{aligned} \tag{4.5}$$

To finish the proof, next we prove that there exists a positive constant C such that

$$\sum_{\ell=1}^{L}\sum_{j=0}^{\infty} q^{j(1-2s)} \int_{\mathfrak{B}} |[\widehat{g}(\mathfrak{p}^j\xi), \widehat{\psi_\ell}]_0(\xi)|^2 d\xi \le C\|g\|^2_{H^{-s}(\mathbb{K})} \ \ for \ \ g \in H^{-s}(\mathbb{K}). \tag{4.6}$$

Now, we have

$$\begin{aligned}\int_{\mathfrak{B}}\left|[\widehat{g}(\mathfrak{p}^{j}\cdot),\widehat{\psi_\ell}]_0(\xi)\right|^2 d\xi &= \int_{\mathfrak{B}}\left|\sum_{k\in\mathbb{N}_0}\widehat{g}(\mathfrak{p}^{j}(\xi+u(k)))\overline{\widehat{b_\ell}(\mathfrak{p}^{-1}(\xi+u(k)))\widehat{\varphi}(\mathfrak{p}^{-1}(\xi+u(k)))}\right|^2 d\xi\\
&= \int_{\mathfrak{B}}\left|\sum_{\nu\in\mathcal{N}_j}\overline{\widehat{b_\ell}(\mathfrak{p}^{-1}\xi+u(\nu))}[\widehat{g}(\mathfrak{p}^{j+1}\cdot),\widehat{\varphi}]_0(\mathfrak{p}^{-1}\xi+u(\nu))\right|^2 d\xi\\
&\le q\sum_{\nu\in\mathcal{N}_j}\int_{\mathfrak{B}}|\widehat{b_\ell}(\mathfrak{p}^{-1}\xi+u(\nu))\left[\widehat{g}(\mathfrak{p}^{j+1}\cdot),\widehat{\varphi}(\cdot)\right]_0(\mathfrak{p}^{-1}\xi+u(\nu))|^2 d\xi\\
&= q^2\int_{\mathfrak{B}}\left|\widehat{b_\ell}(\xi)\left[\widehat{g}(\mathfrak{p}^{j+1}\xi),\widehat{\varphi}(\xi)\right]_0\right|^2 d\xi\\
&\le q^2\int_{\mathfrak{B}}|\widehat{b_\ell}(\xi)|^2\left[\widehat{g}(\mathfrak{p}^{j+1}\cdot),\widehat{g}(\mathfrak{p}^{j+1}\cdot)\right]_{-t}(\xi)\left[\widehat{\varphi},\widehat{\varphi}\right]_t(\xi)\,d\xi\\
&\le q^2\left\|[\widehat{\varphi},\widehat{\varphi}]_t\right\|_{L^\infty(\mathbb{K})}\int_{\mathfrak{B}}|\widehat{b_\ell}(\xi)|^2\left[\widehat{g}(\mathfrak{p}^{j+1}\cdot),\widehat{g}(\mathfrak{p}^{j+1}\cdot)\right]_{-t}(\xi)\,d\xi\\
&= q^2\left|[\widehat{\varphi},\widehat{\varphi}]_t\right|_{L^\infty(\mathbb{K})}\int_{\mathbb{K}}|\widehat{b_\ell}(\xi)|^2|\widehat{g}(\mathfrak{p}^{j+1}\xi)|^2\widehat{\gamma}^{-t}(\xi)\,d\xi\\
&= q^{1-j}\left\|[\widehat{\varphi},\widehat{\varphi}]_t\right\|_{L^\infty(\mathbb{K})}\int_{\mathbb{K}}|\widehat{b_\ell}(\mathfrak{p}^{-j-1}\xi)|^2|\widehat{g}(\xi)|^2\widehat{\gamma}^{-t}(\mathfrak{p}^{-j-1}\xi)\,d\xi.\end{aligned}\tag{4.7}$$

This leads to

$$\begin{aligned}&\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}q^{j(1-2s)}\int_{\mathfrak{B}}\left|[\widehat{g}(\mathfrak{p}^{j}\cdot),\widehat{\psi_\ell}]_0(\xi)\right|^2 d\xi\\
&\qquad\le q\left\|[\widehat{\varphi},\widehat{\varphi}]_t\right\|_{L^\infty(\mathbb{K})}\int_{\mathbb{K}}|\widehat{g}(\xi)|^2\widehat{\gamma}^{-s}(\xi)\Delta_{s,t}(\xi)d\xi,\end{aligned}$$

where

$$\Delta_{s,t}(\xi)=\sum_{j=0}^{\infty}q^{-2js}\widehat{\gamma}^{s}(\xi)\sum_{\ell=1}^{L}|\widehat{b_\ell}(\mathfrak{p}^{-j-1}\xi)|^2\widehat{\gamma}^{-t}(\mathfrak{p}^{-j-1}\xi)\tag{4.8}$$

So (4.6) holds if $\Delta_{s,t}\in L^\infty(\mathbb{K})$. Clearly, $\Delta_{s,t}\in L^\infty(\mathbb{K})$ when $s<0$. When $s>0$, we have

$$\Delta_{s,t}(\xi)\le\sum_{j=0}^{\infty}q^{-2js}(1+\varrho_1^2|\xi|^2)^s\sum_{\ell=1}^{L}|\widehat{b_l}(\mathfrak{p}^{-j-1}\xi)|^2\left(1+q^{-2j-2}\varrho_2^2|\xi|^2\right)^{-t}\le CB_{s,t}(\xi)$$

and thus $\Delta_{s,t}\in L^\infty(\mathbb{K})$. The proof is completed.

Based on Theorem 3.1, the following theorem gives an MOEP for such dual frames.

Theorem 4.1. For a given $s\in\mathbb{R}$, let $\varphi\in H^s(\mathbb{K})$ and $\widetilde{\varphi}\in H^{-s}(\mathbb{K})$ satisfy Assumption 4.1 and 4.2, and ψ_ℓ and $\widetilde{\psi_\ell}$ with $1\le\ell\le L$ defined as in (4.2). Assume that

(i) $[\widehat{\varphi},\widehat{\varphi}]_t\in L^\infty(\mathbb{K})$ for some $t>s$, $[\widehat{\widetilde{\varphi}},\widehat{\widetilde{\varphi}}]_{t'}\in L^\infty(\mathbb{K})$ for some $t'>-s$;

(ii) there exist two non-negative numbers α and $\widetilde{\alpha}$ with $\alpha>-s$ and $\widetilde{\alpha}>s$, and positive constant C such that

$$\sum_{\ell=1}^{L}|\widehat{b_\ell}(\xi)|^2\le C\min\left(1,|\xi|_2^{2\alpha}\right),\sum_{\ell=1}^{L}|\widehat{\widetilde{b}_\ell}(\xi)|^2\le C\min\left(1,|\xi|_2^{2\widetilde{\alpha}}\right)\ a.e.\ on\ \mathbb{K}.\tag{4.9}$$

And assume that $\theta \in L^\infty(\mathfrak{B})$ and η is defined by $\widehat{\eta}(\xi) = \theta(\xi)\widehat{\varphi}(\xi)$ *a.e. on* $\mathbb{K}$. Then $\left(\mathcal{W}^s(\eta;\Psi), \mathcal{W}^{-s}(\widetilde{\varphi};\widetilde{\Psi})\right)$ is a pair of dual frames in $(H^s(\mathbb{K}), H^{-s}(\mathbb{K}))$ if and only if

$$\lim_{j\to\infty} \theta(\mathfrak{p}^{-j}\xi) = 1 \; a.e \; \xi \in \mathbb{K}; \tag{4.10}$$

$$\theta(\mathfrak{p}\xi)\widehat{a}(\xi)\overline{\widehat{\widetilde{a}}(\xi + u(\nu))} + \sum_{\ell=1}^{L} \widehat{b_\ell}(\xi)\overline{\widehat{\widetilde{b}_\ell}(\xi + u(\nu))}$$
$$= \theta(\xi)\delta_{0,\nu} \;\; a.e. \;\; on \;\; \sigma(\varphi) \cap \tau(\sigma(\widetilde{\varphi}) - u(\nu)) \; with \; \nu \in \mathcal{N}_j. \tag{4.11}$$

Proof. By Lemma 4.1, $\mathcal{W}^s(\varphi;\Psi)$ and $\mathcal{W}^{-s}(\widetilde{\varphi};\widetilde{\Psi})$ are Bessel sequences in $H^s(\mathbb{K})$ and $H^{-s}(\mathbb{K})$, respectively. In particular, $\left\{\psi_{\ell,j,k} : j \in Z_+, k \in \mathbb{N}_0, 1 \le \ell \le L\right\}$ and $\left\{T_{u(k)}\varphi : k \in \mathbb{N}_0\right\}$ are Bessel sequences in $H^s(\mathbb{K})$. By Lemma 3.3 and the definition of η, $\{T_{u(k)}\eta : k \in \mathbb{N}_0\}$ is a Bessel sequence in $H^s(\mathbb{K})$. It follows that $\mathcal{W}^s(\eta;\Psi)$ is a Bessel sequence in $H^s(\mathbb{K})$. So $\left(\mathcal{W}^s(\eta;\Psi), \mathcal{W}^{-s}(\widetilde{\varphi};\widetilde{\Psi})\right)$ is a pair of dual frames in $(H^s(\mathbb{K}), H^{-s}(\mathbb{K}))$ if and only if, for every $n \in \mathbb{N}_0$,

$$\theta(\xi)\widehat{\varphi}(\xi)\overline{\widehat{\widetilde{\varphi}}(\xi + u(\eta))} + \sum_{\ell=1}^{L}\sum_{j=0}^{\kappa(n)} \widehat{\psi_\ell}(\mathfrak{p}^{-j}\xi)\overline{\widehat{\widetilde{\psi}_\ell}(\mathfrak{p}^{-j}(\xi + u(n)))} = \delta_{0,n} \;\; for \; a.e. \; \xi \in \mathbb{K}. \tag{4.12}$$

by Theorem 3.1. Next, we show that (4.12) is equivalent to (4.10) and (4.11).

First, we suppose that (4.10) and (4.11) hold. For $n \in \mathbb{N}_0$, we calculate

$$\theta(\xi)\widehat{\varphi}(\xi)\overline{\widehat{\widetilde{\varphi}}(\xi + u(\eta))} + \sum_{\ell=1}^{L}\sum_{j=0}^{\kappa(n)} \widehat{\psi_\ell}(\mathfrak{p}^{-j}\xi)\overline{\widehat{\widetilde{\psi}_\ell}(\mathfrak{p}^{-j}(\xi + u(n)))}$$

in this way: using Assumption 4.1 to $\widehat{\varphi}(\xi)\overline{\widehat{\widetilde{\varphi}}(\xi + u(n))}$, and (4.2) to the $j = 0$ term of

$$\sum_{\ell=1}^{L}\sum_{j=0}^{\kappa(n)} \widehat{\psi_\ell}(\mathfrak{p}^{-j}\xi)\overline{\widehat{\widetilde{\psi}_\ell}(\mathfrak{p}^{-j}(\xi + u(n)))},$$

we have

$$\theta(\xi)\widehat{\varphi}(\xi)\overline{\widehat{\widetilde{\varphi}}(\xi + u(n))} + \sum_{\ell=1}^{L}\sum_{j=0}^{\kappa(n)} \widehat{\psi_\ell}(\mathfrak{p}^{-j}\xi)\overline{\widehat{\widetilde{\psi}_\ell}(\mathfrak{p}^{-j}(\xi + u(n)))}$$
$$= \widehat{\varphi}(\mathfrak{p}^{-1}\xi)\overline{\widehat{\widetilde{\varphi}}(\mathfrak{p}^{-1}(\xi + n))}\left\{\theta(\xi)\widehat{a}(\mathfrak{p}^{-1}\xi)\overline{\widehat{\widetilde{a}}(\mathfrak{p}^{-1}\xi)} + \sum_{\ell=1}^{L}\widehat{b_\ell}(\mathfrak{p}^{-1}\xi)\overline{\widehat{\widetilde{b}_\ell}(\mathfrak{p}^{-1}\xi)}\right\}$$
$$+ \sum_{\ell=1}^{L}\sum_{j=1}^{\kappa(n)} \widehat{\psi_\ell}(\mathfrak{p}^{-j}\xi)\overline{\widehat{\widetilde{\psi}_\ell}(\mathfrak{p}^{-j}(\xi + u(n)))},$$

and thus

$$\theta(\xi)\widehat{\varphi}(\xi)\overline{\widehat{\widetilde{\varphi}}(\xi + u(n))} + \sum_{\ell=1}^{L}\sum_{j=0}^{\kappa(n)} \widehat{\psi_\ell}(\mathfrak{p}^{-j}\xi)\overline{\widehat{\widetilde{\psi}_\ell}(\mathfrak{p}^{-j}(\xi + u(n)))}$$
$$= \theta(\mathfrak{p}^{-1}\xi)\widehat{\varphi}(\mathfrak{p}^{-1}\xi)\overline{\widehat{\widetilde{\varphi}}(\mathfrak{p}^{-1}(\xi + u(n)))} + \sum_{\ell=1}^{L}\sum_{j=1}^{\kappa(n)} \widehat{\psi_l}(\mathfrak{p}^{-j}\xi)\overline{\widehat{\widetilde{\psi}_\ell}(\mathfrak{p}^{-j}(\xi + u(n)))}$$

by (4.11). Then applying the same procedure to $\widehat{\varphi}(\mathfrak{p}^{-1}\xi)\overline{\widehat{\widetilde{\varphi}}(\mathfrak{p}^{-1}(\xi+u(n)))}$ and $j=1$ term of $\sum_{\ell=1}^{L}\sum_{j=0}^{\kappa(n)}\widehat{\psi}_\ell(\mathfrak{p}^{-j}\xi)\overline{\widehat{\widetilde{\psi}}_\ell(\mathfrak{p}^{-j}(\xi+u(n)))}$, we obtain that

$$\begin{aligned}\theta(\xi)\widehat{\varphi}(\xi)\overline{\widehat{\widetilde{\varphi}}(\xi+u(n))}&+\sum_{\ell=1}^{L}\sum_{j=0}^{\kappa(n)}\widehat{\psi}_\ell(\mathfrak{p}^{-j}\xi)\overline{\widehat{\widetilde{\psi}}_\ell(\mathfrak{p}^{-j}(\xi+u(n)))}\\&=\theta(\mathfrak{p}^{-2}\xi)\widehat{\varphi}(\mathfrak{p}^{-2}\xi)\overline{\widehat{\widetilde{\varphi}}(\mathfrak{p}^{-2}(\xi+u(n)))}+\sum_{\ell=1}^{L}\sum_{j=2}^{\kappa(n)}\widehat{\psi}_\ell(\mathfrak{p}^{-j}\xi)\overline{\widehat{\widetilde{\psi}}_\ell(\mathfrak{p}^{-j}(\xi+u(n)))}.\end{aligned}$$

Applying the same procedure $\kappa(n)+1$ times, we finally obtain that

$$\begin{aligned}\theta(\xi)\widehat{\varphi}(\xi)\overline{\widehat{\widetilde{\varphi}}(\xi+u(n))}&+\sum_{\ell=1}^{L}\sum_{j=0}^{\kappa(n)}\widehat{\psi}_\ell(\mathfrak{p}^{-j}\xi)\overline{\widehat{\widetilde{\psi}}_\ell(\mathfrak{p}^{-j}(\xi+u(n)))}\\&=\widehat{\varphi}(\mathfrak{p}^{-\kappa(n)-1}\xi)\overline{\widehat{\widetilde{\varphi}}(\mathfrak{p}^{-\kappa(n)-1}(\xi+u(n)))}\Bigg\{\theta(\mathfrak{p}^{-\kappa(n)-1}\xi)\widehat{a}(\mathfrak{p}^{-\kappa(n)-1}\xi)\overline{\widehat{\widetilde{a}}(\mathfrak{p}^{-\kappa(n)-1}(\xi+u(n)))}\\&\qquad+\sum_{\ell=1}^{L}\sum_{j=2}^{\kappa(n)-1}\widehat{b}_\ell(\mathfrak{p}^{-\kappa(n)-1}\xi)\overline{\widehat{\widetilde{b}}_\ell(\mathfrak{p}^{-\kappa(n)-1}(\xi+u(n)))}\Bigg\}.\end{aligned}$$

Also observe that $\mathfrak{p}^{-\kappa(n)-1}u(n)\in\mathbb{N}_0+u(\nu)$ for some $\nu\in\mathcal{N}_j$ by Proposition 1.1. It follows that

$$\begin{aligned}\theta(\xi)\widehat{\varphi}(\xi)\overline{\widehat{\widetilde{\varphi}}(\xi+u(n))}&+\sum_{\ell=1}^{L}\sum_{j=0}^{\kappa(n)}\widehat{\psi}_\ell(\mathfrak{p}^{-j}\xi)\overline{\widehat{\widetilde{\psi}}_\ell(\mathfrak{p}^{-j}(\xi+u(n)))}\\&=\widehat{\varphi}(\mathfrak{p}^{-\kappa(n)-1}\xi)\overline{\widehat{\widetilde{\varphi}}(\mathfrak{p}^{-\kappa(n)-1}(\xi+n))}\Bigg\{\theta(\mathfrak{p}^{-\kappa(n)-1}\xi)\widehat{a}(\mathfrak{p}^{-\kappa(n)-1}\xi)\overline{\widehat{\widetilde{a}}(\mathfrak{p}^{-\kappa(n)-1}\xi+u(\nu))}\\&\qquad+\sum_{\ell=1}^{L}\sum_{j=2}^{\kappa(n)-1}\widehat{b}_\ell(\mathfrak{p}^{-\kappa(n)-1}\xi)\overline{\widehat{\widetilde{b}}_\ell(\mathfrak{p}^{-\kappa(n)-1}\xi+u(\nu))}\Bigg\}\\&=0\end{aligned}$$

Similarly, for $n=0$, we have

$$\begin{aligned}\theta(\xi)\widehat{\varphi}(\xi)\overline{\widehat{\widetilde{\varphi}}(\xi)}&+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\widehat{\psi}_\ell(\mathfrak{p}^{-j}\xi)\overline{\widehat{\widetilde{\psi}}_\ell(\mathfrak{p}^{-j}\xi)}\\&=\theta(\mathfrak{p}^{-N}\xi)\widehat{\varphi}(\mathfrak{p}^{-N}\xi)\overline{\widehat{\widetilde{\varphi}}(\mathfrak{p}^{-N}\xi)}+\sum_{\ell=1}^{L}\sum_{j=N}^{\infty}\widehat{\psi}_\ell(\mathfrak{p}^{-j}\xi)\overline{\widehat{\widetilde{\psi}}_\ell(\mathfrak{p}^{-j}(\xi))}\end{aligned}\tag{4.13}$$

for all $N\in\mathbb{N}$. By Lemma 3.10 and the Cauchy-Schwarz inequality, the series $\sum_{\ell=1}^{L}\sum_{j=N}^{\infty}\widehat{\psi}_\ell(\mathfrak{p}^{-j}\xi)\overline{\widehat{\widetilde{\psi}}_\ell(\mathfrak{p}^{-j}(\xi))}$ converges absolutely for a.e. $\xi\in\mathbb{K}$. So we have

$$\lim_{N\to\infty}\sum_{\ell=1}^{L}\sum_{j=N}^{\infty}\widehat{\psi}_\ell(\mathfrak{p}^{-j}\xi)\overline{\widehat{\widetilde{\psi}}_\ell(\mathfrak{p}^{-j}(\xi))}=0\text{ for a.e. }\xi\in\mathbb{K}$$

Letting $N \to \infty$ in (4.13), we have

$$\theta(\xi)\widehat{\varphi}(\xi)\overline{\widehat{\widehat{\varphi}}(\xi)} + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\widehat{\psi_\ell}(\mathfrak{p}^{-j}\xi)\overline{\widehat{\widehat{\psi_\ell}}(\mathfrak{p}^{-j}\xi)} = 1 \ \textit{for a.e. } \xi \in \mathbb{K}$$

by (4.10) and Assumption 4.2.

Next, we turn to the converse implication, i.e. (4.12) implies (4.10) and (4.11). Suppose (4.12) holds. First we fix $\nu \in \mathcal{N}_j \setminus \{0\}$ and $\xi \in \sigma(\varphi) \cap \tau(\sigma(\widehat{\widehat{\varphi}}) - u(\nu))$. Then there exist $r, s \in \mathbb{N}_0$ such that

$$\widehat{\varphi}(\xi + u(r))\overline{\widehat{\widehat{\varphi}}(\xi + u(\nu) + u(s))} \neq 0 \tag{4.15}$$

by Proposition 1.2. Taking $n = \mathfrak{p}(s - r) + \mathfrak{p}\nu$ (this yields $\kappa(n) = 0$), replacing ξ by $\mathfrak{p}(\xi + u(r))$ in (4.12), and using the $\mathbb{N}_0$ periodicity of $\theta, \widehat{a}, \widehat{\widehat{a}}, \widehat{b_\ell}$ and $\widehat{\widehat{b_\ell}}$, we have

$$\begin{aligned} 0 &= \theta(\mathfrak{p}\xi)\widehat{\varphi}(\mathfrak{p}(\xi + u(r)))\overline{\widehat{\widehat{\varphi}}(\mathfrak{p}(\xi + u(s)) + \mathfrak{p}\nu)} + \sum_{\ell=1}^{L}\widehat{\psi_\ell}(\mathfrak{p}(\xi + u(r)))\overline{\widehat{\widehat{\psi_\ell}}(\mathfrak{p}(\xi + u(s)) + \mathfrak{p}u(\nu))} \\ &= \widehat{\varphi}(\xi + u(r))\overline{\widehat{\widehat{\varphi}}(\xi + u(\nu) + u(s))}\left\{\theta(\mathfrak{p}\xi)\widehat{a}(\xi)\overline{\widehat{\widehat{a}}(\xi + u(\nu))}\sum_{\ell=1}^{L}\widehat{b_\ell}(\xi)\overline{\widehat{\widehat{b_\ell}}(\xi + u(\nu))}\right\}. \end{aligned} \tag{4.16}$$

It follows (4.15) that

$$\theta(\mathfrak{p}\xi)\widehat{a}(\xi)\overline{\widehat{\widehat{a}}(\xi + u(\nu))} + \sum_{\ell=1}^{L}\widehat{b_\ell}(\xi)\overline{\widehat{\widehat{b_\ell}}(\xi + u(\nu))} = 0.$$

Let us check the case $\nu = 0$. Fix $\xi \in \sigma(\varphi) \cap \sigma(\widehat{\widehat{\varphi}})$. Then there exist $r, s \in \mathbb{N}_0$ such that

$$\widehat{\varphi}(\xi + u(r))\overline{\widehat{\widehat{\varphi}}(\xi + u(s))} \neq 0. \tag{4.17}$$

If $r \neq s$, then replacing ξ by $\xi + u(r)$ and taking $n = s - r$ in (4.12), we have

$$\theta(\xi)\widehat{\varphi}(\xi + u(r))\overline{\widehat{\widehat{\varphi}}(\xi + u(s))} + \sum_{\ell=1}^{L}\sum_{j=0}^{\kappa(s-r)}\widehat{\psi_\ell}(\mathfrak{p}^{-j}(\xi + u(r)))\overline{\widehat{\widehat{\psi_\ell}}(\mathfrak{p}^{-j}(\xi + u(s)))} = 0. \tag{4.18}$$

Replacing ξ by $\mathfrak{p}(\xi + u(r))$ and n by $\mathfrak{p}(s - r)$ in (4.12), we have

$$\begin{aligned} 0 &= \theta(\mathfrak{p}\xi)\widehat{\varphi}(\mathfrak{p}(\xi + u(r)))\overline{\widehat{\widehat{\varphi}}(\mathfrak{p}(\xi + u(s)))} + \sum_{\ell=1}^{L}\sum_{j=0}^{\kappa(\mathfrak{p}(s-r))}\widehat{\psi_\ell}(\mathfrak{p}^{-j+1}(\xi + u(r)))\overline{\widehat{\widehat{\psi_\ell}}(\mathfrak{p}^{-j+1}(\xi + u(s)))} \\ &= \widehat{\varphi}(\xi + u(r))\overline{\widehat{\widehat{\varphi}}(\xi + u(s))}\left\{\theta(\mathfrak{p}\xi)\widehat{a}(\xi)\overline{\widehat{\widehat{a}}(\xi)} + \sum_{\ell=1}^{L}\widehat{b_\ell}(\xi)\overline{\widehat{\widehat{b_\ell}}(\xi)}\right\} \\ &\quad + \sum_{\ell=1}^{L}\sum_{j=0}^{\kappa(s-r)}\widehat{\psi_\ell}(\mathfrak{p}^{-j}(\xi + u(r)))\overline{\widehat{\widehat{\psi_\ell}}(\mathfrak{p}^{-j}(\xi + u(s)))}. \end{aligned} \tag{4.19}$$

Collecting (4.17) - (4.19), we obtain that

$$\theta(\xi) = \theta(\mathfrak{p}\xi)\widehat{a}(\xi)\overline{\widehat{\widehat{a}}(\xi)} + \sum_{\ell=1}^{L}\widehat{b_\ell}(\xi)\overline{\widehat{\widehat{b_\ell}}(\xi)}.$$

If $s = r = k$, taking $n = 0$ and replacing ξ by $\xi + u(k)$ in (4.12), we have

$$\theta(\xi)\widehat{\varphi}(\xi + u(k))\overline{\widehat{\widetilde{\varphi}}(\xi + u(k))} + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\widehat{\psi_\ell}(\mathfrak{p}^{-j}(\xi + u(k)))\overline{\widehat{\widetilde{\psi}_\ell}(\mathfrak{p}^{-j}(\xi + u(k)))} = 1. \tag{4.20}$$

Replacing ξ by $\mathfrak{p}(\xi + u(k))$ and n by 0 in (4.12), we have

$$\begin{aligned} 1 &= \theta(\mathfrak{p}\xi)\widehat{\varphi}(\mathfrak{p}(\xi + u(k)))\overline{\widehat{\widetilde{\varphi}}(\mathfrak{p}(\xi + u(k)))} + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\widehat{\psi_\ell}(\mathfrak{p}^{-j+1}(\xi + u(k)))\overline{\widehat{\widetilde{\psi}_\ell}(\mathfrak{p}^{-j+1}(\xi + u(k)))} \\ &= \widehat{\varphi}(\xi + u(k))\overline{\widehat{\widetilde{\varphi}}(\xi + u(k))}\left\{\theta(\mathfrak{p}\xi)\widehat{a}(\xi)\overline{\widehat{\widetilde{a}}(\xi)} + \sum_{\ell=1}^{L}\widehat{b_\ell}(\xi)\overline{\widehat{\widetilde{b}_\ell}(\xi)}\right\} \\ &\qquad + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\widehat{\psi_\ell}(\mathfrak{p}^{-j}(\xi + u(k)))\overline{\widehat{\widetilde{\psi}_\ell}(\mathfrak{p}^{-j}(\xi + u(k)))}. \end{aligned} \tag{4.21}$$

Collecting (4.17), (4.20) and (4.21), we obtain

$$\theta(\xi) = \theta(\mathfrak{p}\xi)\widehat{a}(\xi)\overline{\widehat{\widetilde{a}}(\xi)} + \sum_{\ell=1}^{L}\widehat{b_\ell}(\xi)\overline{\widehat{\widetilde{b}_\ell}(\xi)}.$$

Now, we prove (4.10). Taking $n = 0$ in (4.12), we have

$$\theta(\xi)\widehat{\varphi}(\xi)\overline{\widehat{\widetilde{\varphi}}(\xi)} + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\widehat{\psi_\ell}(\mathfrak{p}^{-j}\xi)\overline{\widehat{\widetilde{\psi}_\ell}(\mathfrak{p}^{-j}\xi)} = 1. \tag{4.22}$$

Observe that although (4.13) and (4.14) are in the part of the proof where (4.10) and (4.11) are assumed to hold, that they don't follow from (4.10). Combining (4.22) with (4.13) and (4.14), we conclude that

$$\lim_{N\to\infty}\theta(\mathfrak{p}^{-N}\xi)\widehat{\varphi}(\mathfrak{p}^{-N}\xi)\overline{\widehat{\widetilde{\varphi}}(\mathfrak{p}^{-N}\xi)} = 1 \ \ \textit{for a.e.}\ \ \xi \in \mathbb{K},$$

and, by the Assumption 4.2, we obtain (4.10). The proof is completed.

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