A study of triple sequence spaces in fuzzy anti-normed linear spaces

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Abstract. The proposed article aims to study some topological properties of triple real sequence spaces with respect to fuzzy-anti-normed linear space (FANLS). In this work the algebra of fuzzy $I_{\lambda}$-limit and fuzzy $I_{\lambda}$-anti limit, where $0 < \lambda < 1$ and $I$ is an ideal on $\mathbb{N}^3$, of triple real sequences along with an interesting example have been studied. Furthermore, the completeness of a special kind of sequence space with respect to fuzzy anti-norm has been examined.

1. Introduction


In 2000, P. Kostyrko [13] introduced the notion of ideal convergence ($I$-convergence) of real sequences, which generalizes the concept of statistical convergence of sequences. The idea of statistical convergence was first examined by Fast [6] and Schoenberg [17] individually in their own way, which elaborates the concepts of ordinary convergence to some higher context. The concept of statistical convergence is based on the technique of natural density of the set with respect to the set of natural numbers $\mathbb{N}$. Let $F$ be the subset of $\mathbb{N}$. Then the natural density of $F$ is defined as

$$\delta(F) = \lim_{p \to \infty} \frac{1}{p} \cdot \text{card} \{ q \leq p : q \in F \ \text{and} \ p \in \mathbb{N} \}.$$
Later, various scholars examined the concept of ideal convergence in their own approach and generalized it (see [5]). Some authors have recently presented the concepts of statistical convergence and ideal convergence in the framework of fuzzy theory [9, 12, 14, 15].


We now recall some preliminaries which will be used later on. An ideal $I$ on $N$ is a collection of subsets of $N$ that meet the requirements, (i) $\emptyset \in I$, (ii) $G_1, G_2 \in I \Rightarrow G_1 \cup G_2 \in I$, (iii) $G_1 \in I$ and $G_2 \subseteq G_1 \Rightarrow G_2 \in I$. An ideal $I$ is called non-trivial if $N$ does not lie in $I$, $I \neq \emptyset$, and ideal $I$ is called admissible if it contains all finite subsets of $N$. There is another family of subsets of the set $N$, we denote it by $I^*$ and termed a filter on $N$ that meets the requirements, (i) $\emptyset \notin I^*$, (ii) $G_1, G_2 \in I^* \Rightarrow G_1 \cap G_2 \in I^*$, (iii)$G_1 \in I^*$ and $G_2 \supseteq G_1 \Rightarrow G_2 \in I^*$. Corresponding to each ideal $I$ there exist a filter $F(I)$ defined as

$$F(I) = \{ M \subseteq N : M = N - F \text{ and } F \in I \}$$

Throughout the work $I$ denotes a triple ideal on $N^3$ and $N^3$ deals the triple product $N \times N \times N$ and $L$ stands for unit interval $[0, 1]$.

We now outline the work in the article as follows. Section 1, is the introduction part containing brief history and some concepts, Section 2, contains the basic definitions and results which are useful in the work, Section 3, contains some new definitions with examples, and Section 4, is devoted to main theorems and results that established the relationships between studied notions in Section 3.

2. Background

**Definition 2.1.** [18] A binary operation $\odot : L^2 \to L$ meeting the requirements

(i) $\odot$ is commutative.

(ii) $\odot$ is associative.

(iii) $s_1 \odot s_2 \leq s_3 \odot s_4$ if $s_1 \leq s_3$ and $s_2 \leq s_4$ for every $s_1, s_2, s_3, s_4 \in [0, 1]$.

(iv) $s \odot 0 = s$ for all $s \in [0, 1]$.

is defined as $t$-conorm.

**Remark 2.2.** [16]

(i) For any $0 < \epsilon_1 < 1$, we can find $0 < \epsilon_2 < 1$ such that $\epsilon_2 \odot \epsilon_2 \leq \epsilon_1$.

(ii) For any $0 < \epsilon, \epsilon_3 < 1$, and $\epsilon > \epsilon_3$, we can find $0 < \epsilon_4 < 1$ such that $\epsilon \geq \epsilon_3 \odot \epsilon_4$.

**Definition 2.3.** [11] A 3-tuple object $(\mathcal{V}, \beta, \odot)$, where $\mathcal{V}$ is a linear space, $\beta : \mathcal{V} \times \mathcal{R} \to \mathcal{R}$, gives fuzzy anti-norm and $\odot$ is $t$-conorm, is defined as a fuzzy anti-norm linear space (FANLS) if for all $u, v \in \mathcal{V}$, the following requirements meets.

(i) $\beta(u, \tau) = 1$ for all $-\infty < \tau < 0$.

(ii) $\beta(u, \tau) = 0$ if $u = \theta$, where $\theta$ is zero element of $\mathcal{V}$.

(iii) $\beta(au, \tau) = \beta\left(u, \frac{\tau}{|a|}\right)$ if $a \neq 0$.

(iv) $\beta(u + v, \tau + \theta) \leq \beta(u, \tau) \odot \beta(v, \theta)$.

(v) $\lim_{\tau \to \infty} \beta(u, \tau) = 0$.

**Remark 2.4.** [10] $\beta(x, \tau) : \mathcal{V} \times \mathcal{R} \to \mathcal{L}$ is a non increasing function of $\tau$ for each $u$.

**Remark 2.5.** [11] A fuzzy norm can be induced by a norm. As an example, let $(\mathcal{V}, |||.)|||$ be a normed linear space and $\odot$ be a $t$ co-norm defined as $u \odot v = \max \{u, v\}$. Now define $\beta : \mathcal{V} \times \mathcal{R} \to \mathcal{L}$ such that

$$\beta(u, \tau) = \frac{|||u|||}{\tau + |||u|||}, \text{ if } \tau > 0$$

$$\beta(u, \tau) = 1, \text{ if } \tau \leq 0$$
Definition 2.6. [10] A sequence \( u = (u_n)_{n \geq 1} \) of the elements of FANLS \( (V, \beta, \diamond) \) is said to be \( \beta \)-convergent to a point \( \xi \), if for all \( \epsilon > 0 \) and \( \tau > 0 \) there exists \( n_\epsilon \in \mathbb{N} \) such that \( \beta(u_n - \xi, \tau) < \epsilon \) \( \forall \ n \geq n_\epsilon \).

Equivalently,

\[
\lim_{n \to \infty} \beta(u_n - \xi, \tau) = 0. \tag{1}
\]

We denote the case as \( u \xrightarrow[\beta]{\infty} \xi \).

3. Fuzzy \( I_\lambda \)-anti-convergence of triple sequences

Definition 3.1. Let \( I \) be an ideal on \( \mathbb{N} \). Then a triple sequence \( u = (u_{kji}) \) of the elements of FANLS \( (V, \beta, \diamond) \) is called fuzzy ideally \( \beta \)-convergent \( (I_\beta - \text{cgt}) \) to \( \xi \), if for all \( \epsilon > 0 \) and \( \tau > 0 \)

\[
\{(k, j, i) \in \mathbb{N}^3 : \beta(u_{kji} - \xi, \tau) < \epsilon \} \in I.
\]

We denote the case as, \( I_\beta - \lim u = \xi \) and \( \xi \) is defined as fuzzy \( I_\beta \)-limit of \( u \).

Definition 3.2. Let \( I \) be an ideal on \( \mathbb{N} \). Then for \( 0 < \lambda < 1 \), a triple sequence \( u = (u_{kji}) \) of the elements of FANLS \( (V, \beta, \diamond) \) is said to be fuzzy \( I_\lambda \)-anti-convergent \( (I_\beta - \text{anti-cgt}) \) to \( \xi \), if for all \( \epsilon > 0 \) and \( \tau > 0 \)

\[
\{(k, j, i) \in \mathbb{N}^3 : \beta(u_{kji} - \xi, \tau) < 1 - \lambda \} \in I.
\]

We denote the case as, \( I_\lambda - \lim u = \xi \) and \( \xi \) is defined as fuzzy \( I_\lambda \)-limit of \( u \).

Definition 3.3. Let \( I \) defines an ideal on \( \mathbb{N} \). Then a triple sequence \( u = (u_{kji}) \) of the elements of FANLS \( (V, \beta, \diamond) \) is said to be fuzzy ideally anti-\( \beta \)-convergent \( (I_\beta - \text{anti-cgt}) \) to \( \xi \), if for all \( \epsilon > 0 \) and \( \tau > 0 \)

\[
\{(k, j, i) \in \mathbb{N}^3 : \beta(u_{kji} - \xi, \tau) < \epsilon \} \in \mathcal{F}(I).
\]

We denote the case as, \( u \xrightarrow[a-I_\beta]{\text{anti}} \xi \), and \( \xi \) is known as fuzzy anti \( I_\beta \)-limit of \( u \).

Definition 3.4. Let \( I \) be an ideal on \( \mathbb{N} \). Then for \( 0 < \lambda < 1 \), a triple sequence \( u = (u_{kji}) \) of the elements of FANLS \( (V, \beta, \diamond) \) is said to be fuzzy \( I_\lambda \)-anti convergent \( (I_\beta - \text{anti-conv}) \) to \( \xi \), if for all \( \epsilon > 0 \) and \( \tau > 0 \)

\[
\{(k, j, i) \in \mathbb{N}^3 : \beta(u_{kji} - \xi, \tau) < 1 - \lambda \} \in \mathcal{F}(I).
\]

We denote the case, \( u \xrightarrow[a-I_\beta]{\text{anti}} \xi \), and \( \xi \) is known as fuzzy \( I_\lambda \)-anti limit of \( u \).

Example 3.5. Let \( I = \{ E \subseteq \mathbb{N}^3 : \delta(E) = 0 \} \) be an ideal on \( \mathbb{N}^3 \), and let \( (\mathcal{R}^3, \beta, \diamond) \) be a FANLS with idempotent \( t \)-conorm \( \diamond \) and \( \beta(u, \tau) = \frac{\|u\|}{\|\tau\|} \). Let \( u = (u_{kji}) \) be a triple real sequence of the elements of \( (\mathcal{R}^3, \beta, \diamond) \) defined as

\[
u = u_{kji} = \begin{cases} \frac{1}{3}, & \text{if } k, j, i \text{ are perfect cube} \\ \frac{1}{7}, & 0 \\ \frac{1}{1}, & -1 \end{cases} \quad \text{Otherwise}
\]

Then,

\[
fuzzy - I_\lambda - \lim(u) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}; \quad \text{and} \quad fuzzy - I_\lambda \text{ anti - lim}(u) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.
\]
Proof. Let
\[ F = \{(r,q,p) \in N^3 : r = k^3, q = \beta^3, p = \beta^3 \quad \text{and} \quad k, j, i \in N\}, \]
then natural density of \( F = \delta(F) = 0 \) since
\[ 0 \leq \delta(F) = \lim_{r,q,p \to \infty} \frac{1}{rqp} |F| \leq \lim_{r,q,p \to \infty} \frac{1}{rqp} \sqrt{rqp} = 0. \]
(7)

Now for an arbitrary \( \tau > 0 \) and \( 0 < \lambda < 1 \), there exists some \((r_0,q_0,p_0) \in F\) such that for all \((r,q,p) \in F\) where \( r \geq r_0, \quad q \geq q_0, \quad p \geq p_0\), we get
\[ \beta(u_{rqp} - (0,0,0)^t, \tau) = \frac{\sqrt{\frac{1}{r^2} + \frac{1}{q^2} + \frac{1}{p^2}}}{\tau + \sqrt{\frac{1}{r^2} + \frac{1}{q^2} + \frac{1}{p^2}}} = 0 < 1 - \lambda. \]
(8)

Furthermore the set,
\[ F_1 = \{(r,q,p) \in N^3 : \beta(u_{rqp} - (0,0,0)^t, \tau) < 1 - \lambda\} \subseteq F. \]
(9)
Thus, \( F_1 \in I \) (since \( F \in I \)).
Hence,
\[ \text{fuzzy} - I_\lambda - \text{lim}(u) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \]
(10)
on the other hand, if \((k, j, i) \notin F\) then for an arbitrary \( \tau > 0 \) and \( 0 < \lambda < 1 \),
\[ \beta(u_{kji} - (0,1,-1)^t, \tau) = \frac{0}{\tau + 0} = 0 < 1 - \lambda. \]
(11)
Here, we observe that, the set
\[ \{(k, j, i) \in N^3 : \beta(u_{kji} - (0,1,-1)^t, \tau) < 1 - \lambda\} = N^3 - F \in \mathcal{F}(I) \]
(12)
hence,
\[ \text{fuzzy} - I_\lambda - \text{anti} - \text{lim}(u) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}. \]
(13)
\[
\square
\]
\textbf{Definition 3.6.} Let \( I \) be an ideal on \( N^3 \). Then for \( 0 < \lambda < 1 \), a triple sequence \( u = (u_{kji}) \) of the elements of \textbf{FANLS} \((\mathcal{V}, \beta, \phi)\) is called fuzzy \( I_\lambda - \text{Cauchy sequence} \) if for all \( \tau > 0 \), there exist \( r, q, p \in N \) such that for all \( k \geq r, \quad j \geq q \) and \( i \geq p \) the set
\[ \{(k, j, i) \in N^3 : \beta(u_{kji} - u_{rqp}, \tau) < 1 - \lambda\} \subseteq I. \]
(14)
\textbf{Definition 3.7.} Let \( I \) be an ideal on \( N^3 \). Then for \( 0 < \lambda < 1 \), a triple sequence \( u = (u_{kji}) \) of the elements \textbf{FANLS} \((\mathcal{V}, \beta, \phi)\) is called fuzzy \( I_\lambda - \text{anti Cauchy sequence} \), if for all \( \tau > 0 \) there exist \( r, q, p \in N \) such that for all \( k \geq r, \quad j \geq q \) and \( i \geq p \) the set
\[ \{(k, j, i) \in N^3 : \beta(u_{kji} - u_{rqp}, \tau) < 1 - \lambda\} \in \mathcal{F}(I). \]
(15)
Definition 3.8. Let \( I \) be an ideal on \( N^3 \). Then for \( 0 < \lambda < 1 \), a triple sequence \( u = (u_{kji}) \) of the elements of \( \text{FANLS} \) \((V, \beta, \circ)\) is called fuzzy \( I_\lambda \)-complete, if every fuzzy \( I_\lambda \)-Cauchy sequence in \( V \) is fuzzy \( I_\lambda \)-\( \circ \)-convergent in \( V \).

Definition 3.9. Let \( I \) be an ideal on \( N^3 \). Then for \( 0 < \lambda < 1 \), a triple sequence \( u = (u_{kji}) \) of the elements of \( \text{FANLS} \) \((V, \beta, \circ)\) is called fuzzy \( I_\lambda \)-anti complete, if every fuzzy \( I_\lambda \)-anti Cauchy sequence in \( V \) is fuzzy anti \( I_\lambda \)-\( \circ \)-convergent in \( V \).

Remark 3.10. Let \((V, \beta, \circ)\) be a fuzzy anti norm linear space then an open ball \( B^{\beta, r}(u) \), centred at \( u \) with radius \( r \) with respect to t-conorm \( \circ \) is the collection of triple sequences \( (w_{kji}) \in I_\infty^3 \) such that \( \beta(u_{kji} - w_{kji}, r) < r \)
or, \[
B^{\beta, r}(u) = \{(w_{kji}) \in I_\infty^3 : \beta(u_{kji} - w_{kji}, r) < r\},
\]
where \( I_\infty^3 \) stands for the space of triple bounded sequences.

Definition 3.11. Let \( I \) be an ideal on \( N^3 \). Then a triple sequence \( u = (u_{kji}) \) of the elements of \( \text{FANLS} \) \((V, \beta, \circ)\) is called \( I \)-convergent to a point \( \xi \) with fuzzy anti-norm \( \beta \), if for each \( \epsilon > 0 \) and \( \tau > 0 \), we have
\[
(k, j, i) \in N^3 : \beta(u_{kji} - \xi, \tau) \geq \epsilon \}
\]
We denote the case as \((\beta) - I - \lim u \rightarrow \xi\).

4. Main Theorems

Theorem 4.1. Let \( I \) be an ideal on \( N^3 \). If a triple sequence \( u = (u_{kji}) \) of the elements of \( \text{FANLS} \) \((V, \beta, \circ)\) is \( I \)-convergent to \( \xi \), then \( u = (u_{kji}) \) fuzzy \( I_\beta \)-anti convergent to \( \xi \).

Proof. Let \((\beta) - I - \lim u = \xi \), then for each \( \epsilon > 0 \) and \( \tau > 0 \) we obtain
\[
M = \{(k, j, i) \in N^3 : \beta(u_{kji} - \xi, \tau) \geq \epsilon \} \subseteq I,
\]
which implies
\[
M^c = \{(k, j, i) \in N^3 : \beta(u_{kji} - \xi, \tau) < \epsilon \} = N^3 - M \in F(I)
\]
Hence, \( u^{\alpha} \rightarrow I_\xi \). \( \square \)

Theorem 4.2. Let \( I \) be an ideal on \( N \), triple sequence \( u = (u_{kji}) \) of the elements of \( \text{FANLS} \) \((V, \beta, \circ)\) is fuzzy anti-\( I_\beta \) convergent to \( \xi \) if and only if it is fuzzy \( I_\lambda \)-anti \( \circ \)-convergent to \( \xi \).

Proof. Let \( u = (u_{kji}) \) be fuzzy-\( I_\beta \)-anti \( \circ \)-convergent to \( \xi \). Then for every \( \epsilon > 0 \) and \( \tau > 0 \), we have
\[
A = \{(k, j, i) \in N^3 : \beta(u_{kji} - \xi, \tau) < \epsilon \} \subseteq F(I).
\]
For every \( \epsilon > 0 \), we can choose \( 0 < \lambda < 1 \) such that \( \epsilon < 1 - \lambda \) and
\[
\{(k, j, i) \in N^3 : \beta(u_{kji} - \xi, \tau) < \epsilon \} \subseteq \{(k, j, i) \in N^3 : \beta(u_{kji} - \xi, \tau) < 1 - \lambda \}.
\]
\[
\Rightarrow \{(k, j, i) \in N^3 : \beta(u_{kji} - \xi, \tau) < 1 - \lambda \} \subseteq F(I)
\]
Conversely let \( u = (u_{kji}) \) be fuzzy-\( I_\lambda \)-anti \( \circ \)-convergent to \( \xi \), then for every \( 0 < \lambda < 1 \) and \( \tau > 0 \), we have
Proof. For every $0 < \lambda < 1$ we can find $\epsilon > 0$ such that $1 - \lambda < \epsilon$ and
\[
\left\{ (k, j, i) \in \mathbb{N}^3 : \beta(u_{kji} - \xi, \tau) < 1 - \lambda \right\} \subseteq \left\{ (k, j, i) \in \mathbb{N}^3 : \beta(u_{kji} - \xi, \tau) < \epsilon \right\}.
\]
Hence,
\[
\Rightarrow \left\{ (k, j, i) \in \mathbb{N}^3 : \beta(u_{kji} - \xi, \tau) < \epsilon \right\} \in \mathcal{F}(I).
\]
\[
\square
\]

**Theorem 4.3.** Let $I$ be an ideal on $\mathbb{N}$. If a triple sequence $u = (u_{kji})$ of the elements of FANLS $(\mathcal{V}, \beta, \circ)$ with idempotent $t$-conorm $\circ$, is fuzzy $\mathcal{I}_\lambda$-anti cgt to $\xi$, then fuzzy $\mathcal{I}_\lambda$-anti limit $\xi$ is unique.

**Proof.** Let the fuzzy-$\mathcal{I}_\lambda$-anti lim$(u) = \xi$. If possible, we suppose there is another limit $\eta$ such that fuzzy $\mathcal{I}_\lambda$-anti lim$(u) = \eta$. Now for every $0 < \lambda < 1$ and $\tau > 0$ we have,
\[
A = \left\{ (k, j, i) \in \mathbb{N}^3 : \beta(u_{kji} - \xi, \tau) < 1 - \lambda \right\} \in \mathcal{F}(I)
\]
and
\[
A_1 = \left\{ (k, j, i) \in \mathbb{N}^3 : \beta(u_{kji} - \eta, \tau) < 1 - \lambda \right\} \in \mathcal{F}(I).
\]
Let $(l, m, n) \in A \cap A_1$. Now since $\circ$ is idempotent $t$-conorm then for every $0 < \lambda < 1$, we have $(1 - \lambda) \circ (1 - \lambda) < 1 - \lambda$. Hence,
\[
\beta(\xi - \eta, \tau) \leq \beta(\xi - u_{lmn} + u_{lmn}, \tau) = \beta(\xi - u_{lmn}, \frac{\tau}{2}) \circ \beta(u_{lmn}, \frac{\tau}{2}) < (1 - \lambda) \circ (1 - \lambda) < 1 - \lambda, \quad \forall \, \lambda.
\]
Now we can find $\epsilon > 0$ with respect to every $0 < \lambda < 1$ such that $1 - \lambda < \epsilon$. Thus we would have
\[
\beta(\xi - \eta, \tau) < \epsilon \Rightarrow \xi - \eta = \theta \Rightarrow \xi = \eta.
\]
\[
\square
\]

**Theorem 4.4.** Let $I$ be an ideal on $\mathbb{N}$. If the two triple sequences $u = (u_{kji})$ and $v = (v_{kji})$ of the elements FANLS $(\mathcal{V}, \beta, \circ)$ with idempotent $t$-conorm $\circ$, are such that fuzzy $\mathcal{I}_\lambda$-anti lim$(u) = \xi$ and fuzzy anti-$\mathcal{I}_\lambda$-anti lim($v$) = $\eta$, then fuzzy $\mathcal{I}_\lambda$-anti-$\text{lim}(u + v) = \xi + \eta$.

**Proof.** For $0 < \lambda < 1$ and $\tau > 0$, we have
\[
A_1 = \left\{ (k, j, i) \in \mathbb{N}^3 : \beta(u_{kji} - \xi, \frac{\tau}{2}) < 1 - \lambda \right\} \in \mathcal{F}(I)
\]
and
\[
A_2 = \left\{ (k, j, i) \in \mathbb{N}^3 : \beta(v_{kji} - \eta, \frac{\tau}{2}) < 1 - \lambda \right\} \in \mathcal{F}(I).
\]
Let $(l, m, n) \in A_1 \cap A_2$. Now for every $0 < \lambda < 1$, we have $(1 - \lambda) \circ (1 - \lambda) < 1 - \lambda$ and then
\[
\beta(u_{lmn} + v_{lmn} - (\xi + \eta), \tau) \leq \beta(u_{lmn} - \xi, \frac{\tau}{2}) \circ \beta(v_{lmn} - \eta, \frac{\tau}{2}) < (1 - \lambda) \circ (1 - \lambda) < 1 - \lambda, \quad \forall \, \lambda.
\]
Hence we can find
\[
A_1 \cap A_2 = \left\{ (k, j, i) \in \mathbb{N}^3 : \beta(u_{kji} + v_{kji} - (\xi + \eta), \frac{\tau}{2}) < 1 - \lambda \right\} \in \mathcal{F}(I)
\]
which means, fuzzy $\mathcal{I}$-anti lim$(u + v) = \xi + \eta$. \[
\square
\]
Theorem 4.5. Let $I$ be an ideal on $\mathcal{N}^3$. If the triple sequences $u = (u_{kji})$ of the elements of $\text{FANLS} (\mathcal{V}, \beta, \circ)$ with idempotent $t$-conorm $\circ$, is fuzzy-$I_{\lambda} - \text{anti}$ converges to $\xi$, then fuzzy-$I_{\lambda} - \text{anti lim}(au) = \alpha \xi$, where $\alpha$ is a scalar.

Proof. Case I. If $\alpha = 0$, then theorem is obvious.
Case II. If $\alpha \neq 0$ and for every $0 < \lambda < 1$ and $\tau > 0$, we have
\[
A = \{(k, j, i) \in \mathcal{N}^3 : \beta \left( u_{kji} - \xi, \tau \right) < 1 - \lambda \} \in \mathcal{F}(I).
\]
Since $\tau > 0$ is arbitrary, therefore we can replace it by $\frac{1}{\lambda \lambda}$, in (29) where $t > 0$ and $\alpha \neq 0$, which follows
\[
A = \{(k, j, i) \in \mathcal{N}^3 : \beta \left( u_{kji} - \xi, \frac{t}{\lambda \lambda} \right) < 1 - \lambda \} \in \mathcal{F}(I).
\]
Equivalently,
\[
A = \{(k, j, i) \in \mathcal{N}^3 : \beta \left( au_{kji} - \alpha \xi, t \right) < 1 - \lambda \} \in \mathcal{F}(I).
\]
Since $t > 0$ was arbitrary, hence the theorem is proved. \hfill \Box

Theorem 4.6. Let $I$ be an ideal on $\mathcal{N}^3$. If the triple sequences $u = (u_{kji}) \in I_{\infty}$ of the elements of $\text{FANLS} (\mathcal{V}, \beta, \circ)$ with idempotent $t$-conorm $\circ$, is fuzzy anti-$I_{\beta}$ convergent to $\xi$, then fuzzy anti-$I_{\beta} - \text{anti lim}(u^2) = \xi^2$.

Proof. Let $M > 0$ such that $\sup(u_{kji}) = M$. For all $k, j, i \in N$ and for every $\epsilon > 0$ and $\tau > 0$, we have
\[
A = \{(k, j, i) \in \mathcal{N}^3 : \beta \left( u_{kji} - \xi, \tau \right) < \epsilon \} \in \mathcal{F}(I).
\]
Now for $\tau > 0$, let
\[
A_1 = \{(k, j, i) \in \mathcal{N}^3 : \beta \left( u_{kji}^2 - \xi^2, \tau \right) < \epsilon \}.
\]
We now wish to prove that $A_1 \in \mathcal{F}(I)$. For this it suffices to show $A \subset A_1$.
Let $(l, m, n) \in A$. Then $\beta \left( u_{lmn} - \xi, \tau \right) < \epsilon$ and
\[
\beta \left( u_{lmn}^2 - \xi^2, \tau \right) = \beta \left( (u_{lmn} - \xi)(u_{lmn} + \xi), \tau \right) = \beta \left( u_{lmn} - \xi, \frac{\tau}{u_{lmn} + \xi} \right)
\]
\[
= \beta \left( u_{lmn} - \xi, \frac{\tau}{M + \xi} \right) = \beta \left( u_{lmn} - \xi, \frac{\tau}{M + \xi} \right) < \epsilon \text{ by } (32)
\]
where, $t = \frac{\tau}{M + \xi} > 0$. Therefore $(l, m, n) \in A_1$, which implies that
\[
A_1 = \{(k, j, i) \in \mathcal{N}^3 : \beta \left( u_{kji}^2 - \xi^2, \tau \right) < \epsilon \} \in \mathcal{F}(I).
\]
\hfill \Box

Theorem 4.7. Let $I$ be an ideal on $\mathcal{N}^3$. If the triple sequences $u = (u_{kji}) \in I_{\infty}$, $u_{kji} > 0$ of the elements of $\text{FANLS} (\mathcal{V}, \beta, \circ)$ with idempotent $t$-conorm $\circ$, is fuzzy-$I_{\beta} - \text{anti}$ convergent to $\xi$, then
\[
\text{fuzzy-}I_{\beta} - \text{anti lim} \left( \frac{1}{u} \right) = \frac{1}{\xi}.
\]
Proof. Let fuzzy-$I_{\beta} - \text{anti lim}(u) = \xi$, and $u = (u_{kji}) \in I_{\infty}$. Then there exists some $M > 0$ such that $\sup(u_{kji}) = M$ for all $k, j, i \in N$. Then for every $\epsilon > 0$ and $\tau > 0$, we have
\[
A = \{(k, j, i) \in \mathcal{N}^3 : \beta \left( u_{kji} - \xi, \tau \right) < \epsilon \} \in \mathcal{F}(I).
\]
Let
\[
A_1 = \left\{ (k, j, i) \in \mathcal{N}^3 : \beta \left( \frac{1}{u_{kji}} - \frac{1}{\xi}, \tau \right) < \epsilon \right\}.
\]  

We now wish to prove that \( A_1 \subset \mathcal{F}(I) \). For the requirement it is sufficient to prove \( A \subset A_1 \).

Let \((l, m, n) \in A\). Then \( \beta (u_{lmn} - \xi, \tau) < \epsilon \). Thus
\[
\beta \left( \frac{1}{u_{lmn}} - \frac{1}{\xi}, l \right) = \beta \left( \frac{u_{lmn} - \xi}{u_{lmn}}, l \right) = \beta (u_{lmn} - \xi, l | u_{lmn} \xi |)
\]
\[
= \beta (u_{lmn} - \xi, tM \xi ) = \beta (u_{lmn} - \xi, \tau) < \epsilon, \text{ by (36)}
\]
where \( \tau = tM \xi > 0 \). Hence, we obtain \((l, m, n) \in A_1\), which implies that
\[
A_1 = \left\{ (k, j, i) \in \mathcal{N}^3 : \beta \left( \frac{1}{u_{kji}} - \frac{1}{\xi}, \tau \right) < \epsilon \right\} \in \mathcal{F}(I).
\]  

\[ \Box \]

**Remark 4.8.** Let \((V, \beta, \circ)\) be a FANLS with idempotent \( t\)-conorm \( \circ \). Then the triple sequence \( u = (u_{kji})\) of terms of \((X, \beta, \circ)\) is said to be a Cauchy sequence if for all \( \tau > 0 \) and \( \epsilon > 0 \) there exist \( k_0, j_0, i_0 \in \mathcal{N} \), such that for all \( r, k \geq k_0 \), \( q, j \geq j_0 \) and \( p, i \geq i_0 \), we have
\[
\beta \left( u_{rpi} - u_{kji}, \tau \right) < \epsilon.
\]  

**Remark 4.9.** By \( \Omega^{F_3}_{\lambda} \), we denote the space of all triple real bounded fuzzy \( I_\lambda \) – convergent sequences, where \( 0 < \lambda < 1 \) and \( I \) is an ideal on \( \mathcal{N} \) of the elements of FANLS \((V, \beta, \circ)\) with idempotent \( t\)-conorm \( \circ \).

Moreover,
\[
\Omega^{F_3}_{\lambda} = \left\{ (u_{kji}) \in \mathbb{P}^3_\infty : \exists u \in \mathcal{R} : \{(k, j, i) \in \mathcal{N}^3 : \beta \left( u_{kji} - u, \tau \right) < 1 - \lambda \} \in I \right\}.
\]  

**Theorem 4.10.** Let \((X, \beta, \circ)\) be a FANLS with idempotent \( t\)-conorm \( \circ \). Then the space \( \Omega^{F_3}_{\lambda} \) defined above is complete with respect to fuzzy anti norm \( \beta \), where \( \beta \) is given by (Remark 2.5).

**Proof.** Let \((u_{kji}^n)\) be a Cauchy sequence in \( \Omega^{F_3}_{\lambda} \). We now wish to prove that there exists \( (u_{kji}) \in \Omega^{F_3}_{\lambda} \) such that for all \( \epsilon > 0 \) and \( \tau > 0 \), there exists some \( n_\epsilon \in \mathcal{N} \) and for \( n \geq n_\epsilon \), we have
\[
\beta \left( (u_{kji}^n) - u_{kji}, \tau \right) < \epsilon.
\]  

Since \((u_{kji}^n)\) is a Cauchy sequence, then there exists some \( n_\epsilon \in \mathcal{N} \) such that for all \( n, m \geq n_\epsilon \), we have
\[
\| u_{kji}^n - u_{kji}^m \| < \epsilon,
\]
which implies that
\[
\beta \left( (u_{kji}^n) - u_{kji}^m, \tau \right) < \epsilon \quad \text{for all} \quad \epsilon > 0.
\]

Since \((u_{kji}^n)\) bounded real triple sequence and \( \mathcal{R} \) is complete with respect to \( \max |\cdot| \), therefore there exists a \((u_{kji})\) such that
\[
\lim_{n \to \infty} u_{kji}^n = u_{kji}.
\]
and hence
\[ \beta\left(\left(u_{kji}^n - u_{kji}, \tau\right) \right) < \epsilon \]  (46)

We now prove that \((u_{kji}) \in \Omega^I_{\lambda}\). For this, it is sufficient to show that there exist \(u \in R\) such that fuzzy-
\[
I_{\lambda} \lim \left(u_{kji}\right) = u.
\]
Let
\[
\{(k, j, i) \in N^3 : \beta\left(u_{kji}^n - u_{kji}, \tau\right) < 1 - \lambda\} \in I.
\]  (47)

Take \(p > n_{c}\). And then
\[ \beta\left(u_{kji}^p - u_{kji}, \tau\right) < \epsilon < 1 - \lambda. \]  (48)

So, for every \(\epsilon > 0\), we can choose \(0 < \lambda < 1\) such that \(\epsilon < 1 - \lambda\). Let

\[
E = \left\{(k, j, i) \in N^3 : \beta\left(u_{kji}^p - u_{kji}, \tau\right) < 1 - \lambda\right\}
\]  (49)

and
\[
F = \left\{(k, j, i) \in N^3 : \beta\left(u_{kji}^p - u_{p}, \tau\right) < 1 - \lambda\right\} \in I.
\]  (50)

The sequence \((u_n)\) is convergent to \(u\) with respect to \(\beta\). Then there exists \(n_c \in N\) such that for all \(p \geq n_c\), we have
\[ \beta\left(u_{p} - u, \tau\right) < \epsilon \text{ for all } \epsilon > 0. \]  (51)

Now we prove that \(E \subseteq F\). Let \((l, m, n) \in E\). Then
\[ \beta\left(u_{lmn} - u, \tau\right) < 1 - \lambda, \]  (52)

and
\[
\beta\left(u_{lmn}^p - u_{p}, 3\tau\right) = \beta\left(u_{lmn}^p - u_{lmn} + u_{lmn} - u + u_{p}, 3\tau\right) \\
\leq \beta\left(u_{lmn}^p - u_{lmn}, \tau\right) \circ \beta\left(u_{lmn} - u, \tau\right) \circ \beta\left(u_{p} - u, \tau\right) \\
< (1 - \lambda) \circ (1 - \lambda) \circ (1 - \lambda) \\
= (1 - \lambda). \]  (53)

Since \(\tau > 0\) was arbitrary, for all \(t > 0\) and \(t = 3\tau\), therefore
\[ \beta\left(u_{lmn}^p - u_{p}, t\right) < 1 - \lambda \]  (54)

thus \((l, m, n) \in F\), and hence \(E \subseteq F\). \(\Box\)

**Conclusion**

Our proposed work establishes more general results on the algebra of limits of ideally convergent sequences spaces with respect to fuzzy anti-norm linear space and discusses some examples to elaborate the notion of the spaces. The study also includes the completeness property of a special sequence space that is original.
Data Availability

No data is used in this study.

Conflict of Interest

The authors declare that there is no conflict of interest in the publication of this article.

References