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On the Eneström-Kakeya theorem for quaternionic polynomial

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Abstract. The main purpose of this paper is to extend various results of Eneström-Kakeya type from the complex to quaternionic setting by virtue of a maximum modulus theorem and the structure of the zero sets in the newly developed theory of regular functions and polynomials of a quaternionic variable. Our findings generalise several newly proven conclusions concerning the distribution of zeros of a quaternionic polynomial.

1. Introduction

In mathematics, polynomial zeros have a long and illustrious history. The study of zeros of complex polynomials is an old theme in analytic theory of polynomials, has spawned a vast amount of research over the past millennium includes its applications both within and outside of mathematics. In addition to having numerous applications, this study has been the inspiration for much theoretical research (including being the initial motivation for modern algebra). Algebraic and analytic methods for finding zeros of a polynomial, in general, can be quite complicated, so it is desirable to put some restrictions on polynomials. This motivated the study of identifying suitable regions in the complex plane containing the zeros of a polynomial when their coefficients are restricted with special conditions. The subject dates back to around the time when the geometric representation of complex numbers was introduced into mathematics, and the first contributors to the subject were Gauss and Cauchy. The following elegant result concerning the distribution of zeros of a polynomial when its coefficients are restricted is known in the literature as Eneström-Kakeya theorem (see [4], [11], [14]).

Theorem 1.1. *If $p(z) = \sum\limits_{v=0}^{n} a_v z^v$, is a polynomial of degree n (where z is a complex variable) with real coefficients satisfying*

$$a_n \geq a_{n-1} \geq ... \geq a_1 \geq a_0 \geq 0,$$

then all the zeros of $p(z)$ lie in

$$|z| \leq 1.$$

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When G. Eneström was researching an issue in the theory of pension funds, it appears that he was the first to come to a conclusion of this kind. In essence, the aforementioned discovery was published for the first time in a little paper by Eneström [3]. Later, Eneström made the significant parts of his earlier paper accessible to the international mathematical community and mentioned it in his publications of 1893-95. Independently, in 1912, the result was obtained by S. Kakeya [10] with a purely geometrical approach and in a more general form. Kakeya precisely established the more general conclusion that is listed below.

Theorem 1.2. *If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n with real and positive coefficients, then all the zeros of $p(z)$ lie in the annulus*

$$R_1 \leq |z| \leq R_2,$$

where

$$R_1 = \min_{0 \leq v \leq n-1} a_v / a_{v+1}$$

and

$$R_2 = \max_{0 \leq v \leq n-1} a_v / a_{v+1}.$$

The Eneström-Kakeya theorem has been expanded in a number of ways, including to complex coefficients with constrained arguments, and is particularly significant in the research of the stability of numerical methods for differential equations. In the literature, for example see ([1], [8], [9]), there exist various extensions and generalizations of the Eneström-Kakeya theorem. We refer the reader to the comprehensive books of Marden [11] and Milovanović et al. [14] for an exhaustive survey of extensions and refinements of this well-known result. In 1967, Joyal et al. [9] published a result which might be considered the foundation of the studies which we are currently studying. According to Theorem 1.1, the Eneström-Kakeya theorem applies to polynomials with non-negative coeficients that form a monotone sequence. By eliminating the non-negativity condition, Joyal et al. generalised Theorem 1.1 retaining monotonicity. They did this by showing the following outcome.

Theorem 1.3. *If $p(z) = \sum\limits_{v=0}^{n} a_v z^v$, is a polynomial of degree n (where z is a complex variable) with real coefficients satisfying*

$$a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0,$$

then all the zeros of $p(z)$ lie in

$$|z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|}.$$

Of course, when $a_0 \geq 0$, Theorem 1.3 reduces to Theorem 1.1. Since the second half of the 19th century, numerous estimates for the placement of zeros in terms of coefficients have been thoroughly researched, with a focus on the distribution of zeros of the algebraic polynomials with restricted coefficients and significant advancements have been made. The Eneström-Kakeya theorem and its various generalizations as mentioned above are the classic and significant examples of this kind. Provided such a richness of the complex setting, a natural question is to ask what kind of results in the quaternionic setting can be obtained. In this paper we consider this problem and present extensions to the quaternionic setting of some classical results of Eneström-Kakeya type as discussed above.

2. Background

Let's introduce some introductory information on quaternions that will be helpful in the follow-up in order to introduce the framework in which we will operate. Quaternions are essentially a four-dimensional extension of complex numbers (one real and three imaginary parts) which Sir Rowan William Hamilton first examined and perfected in 1843. This number system of quaternions is denoted by $\mathbb{H}$ in honor of Hamilton. This theory of quaternions is by now very well developed in many different directions, and we refer the reader to [7], [12], [13] and [15] for the basic features of quaternions and quaternionic functions. Before we proceed further, we need to introduce some preliminaries on quaternions. The set of quaternions denoted by $\mathbb{H}$ is a noncommutative division ring. It consists of elements of the form $q = \alpha+\beta i+\gamma j+\delta k,\ \alpha, \beta, \gamma, \delta \in \mathbb{R}$, where the imaginary units i, j, k satisfy $i^2 = j^2 = k^2 = ijk = -1,\ ij = -ji = k, jk = -kj = i, ki = -ik = j$. Every element $q = \alpha + \beta i + \gamma j + \delta k \in \mathbb{H}$ is composed by the real part $\mathrm{Re}(q) = \alpha$ and the imaginary part $\mathrm{Im}(q) = \beta i + \gamma j + \delta k$. The conjugate of q is denoted by $\overline{q}$ and is defined as $\overline{q} = \alpha - \beta i - \gamma j - \delta k$ and the norm of q is $|q| = \sqrt{q\overline{q}} = \sqrt{\alpha^2 + \beta^2 + \gamma^2 + \delta^2}$. The inverse of each non zero element q of $\mathbb{H}$ is given by $q^{-1} = |q|^{-2}\overline{q}$.

Very recently, Carney et al. [2] demonstrated the following generalisation of Theorem 1.1 for the quaternionic polynomial $p(q)$. They proved the following outcome more succinctly.

Theorem 2.1. *If $p(q) = \sum_{v=0}^{n} q^v a_v$, is a polynomial of degree n (where q is a quaternionic variable) with real coefficients satisfying*

$$a_n \geq a_{n-1} \geq ... \geq a_1 \geq a_0 \geq 0,$$

then all the zeros of $p(q)$ lie in

$$|q| \leq 1.$$

They also demonstrated the following result, which is identical to Theorem 1.3 but instead of polynomials with monotone increasing real coefficients, it considers quaternionic polynomials with monotone increasing real and imaginary parts and thus giving the quaternionic analogue of Theorem 1.3.

Theorem 2.2. *If $p(q) = \sum_{v=0}^{n} q^v a_v$, is a polynomial of degree n (where q is a quaternionic variable) with quaternionic coefficients, where $a_v = \alpha_v + \beta_v i + \gamma_v j + \delta_v k$ for $v = 0, 1, 2, ..., n$, satisfying*

$$\alpha_n \geq \alpha_{n-1} \geq ... \geq \alpha_1 \geq \alpha_0,$$
$$\beta_n \geq \beta_{n-1} \geq ... \geq \beta_1 \geq \beta_0,$$
$$\gamma_n \geq \gamma_{n-1} \geq ... \geq \gamma_1 \geq \gamma_0,$$
$$\delta_n \geq \delta_{n-1} \geq ... \geq \delta_1 \geq \delta_0,$$

then all the zeros of $p(q)$ lie in

$$|q| \leq \frac{(|\alpha_0| - \alpha_0 + \alpha_n) + (|\beta_0| - \beta_0 + \beta_n) + (|\gamma_0| - \gamma_0 + \gamma_n) + (|\delta_0| - \delta_0 + \delta_n)}{|a_n|}.$$

Additionally, they demonstrated the following outcome using Lemma 4.2:

Theorem 2.3. *If $p(q) = \sum_{v=0}^{n} q^v a_v$ is a polynomial of degree n with quaternionic coefficients and quaternionic variable. Let b be a non-zero quaternion and suppose $\angle(a_v, b) \leq \theta \leq \pi/2$ for some θ and $v = 0, 1, 2, ..., n$. Assume*

$$|a_n| \geq |a_{n-1}| \geq ... \geq |a_0|,$$

then all the zeros of $p(q)$ lie in

$$|q| \leq \cos\theta + \sin\theta + \frac{2\sin\theta}{|a_n|}\sum_{v=0}^{n-1}|a_v|.$$

Meanwhile, Tripathi [16] established the following generalisation of Theorem 2.2 in addition to demonstrating a few additional results.

Theorem 2.4. *Let* $p(q) = \sum_{v=0}^{n} q^v a_v$*, be a polynomial of degree* n *(where* q *is a quaternionic variable) with quaternionic coefficients, where* $a_v = \alpha_v + \beta_v i + \gamma_v j + \delta_v k$ *for* $v = 0, 1, 2, ..., n$*, satisfying*

$$\begin{aligned}
&\alpha_n \geq \alpha_{n-1} \geq ... \geq \alpha_l,\\
&\beta_n \geq \beta_{n-1} \geq ... \geq \beta_l,\\
&\gamma_n \geq \gamma_{n-1} \geq ... \geq \gamma_l,\\
&\delta_n \geq \delta_{n-1} \geq ... \geq \delta_l,
\end{aligned}$$

for $0 \leq l \leq n$*. Then all the zeros of* $p(q)$ *lie in*

$$\left|q\right| \leq \frac{1}{|a_n|}\Big[|\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + (\alpha_n - \alpha_l) + (\beta_n - \beta_l) + (\gamma_n - \gamma_l) + (\delta_n - \delta_l) + M_l\Big],$$

$$\text{where } M_l = \sum_{v=1}^{l}\Big[|\alpha_v - \alpha_{v-1}| + |\beta_v - \beta_{v-1}| + |\gamma_v - \gamma_{v-1}| + |\delta_v - \delta_{v-1}|\Big].$$

3. Main Results

In this section, we state our main results. We begin with the following result:

Theorem 3.1. *If* $p(q) = \sum_{v=0}^{n} q^v a_v$*, is a quaternionic polynomial of degree* n *with real coefficients* $a_v,\ v = 0, 1, 2, ..., n$ *and for some* $k_v \geq 0,\ v = 0, 1, 2, ..., r,\ 0 \leq r \leq n-1$*, we have*

$$k_0 + a_n \geq k_1 + a_{n-1} \geq k_2 + a_{n-2} \geq ... \geq k_r + a_{n-r} \geq a_{n-r-1} \geq ... \geq a_1 \geq a_0,$$

then all the zeros of p *lie in*

$$|q| \leq \frac{1}{|a_n|}\left\{\sum_{v=0}^{r} |k_v - k_{v+1}| + (k_0 + a_n) - a_0 + |a_0|\right\}.$$

If we take $k_v = 0,\ v = 0, 1, 2, ..., r$ and $r = n - 1$ in Theorem 3.1, we obtain the following result which is an extension of Theorem 1.3 from the complex to quaternionic setting.

Corollary 3.2. *If* $p(q) = \sum_{v=0}^{n} q^v a_v$*, is a quaternionic polynomial of degree* n *with real coefficients* $a_v,\ v = 0, 1, 2, ..., n$*, and satisfying*

$$a_n \geq a_{n-1} \geq a_{n-2} \geq ... \geq a_1 \geq a_0,$$

then all the zeros of p *lie in*

$$|q| \leq \frac{1}{|a_n|}(a_n - a_0 + |a_0|).$$

Setting $a_0 > 0$ in Corollary 3.2, we get Theorem 2.1. It can been easily seen that Corollary 3.2 is also obtained in the form of a result (see [13], Corollary 1 (for μ=1)).

Theorem 3.3. *If* $p(q) = \sum_{v=0}^{n} q^v a_v$, *is a quaternionic polynomial of degree n with quaternionic coefficients* $a_v = \alpha_v + \beta_v i + \gamma_v j + \delta_v k$ *for* $v = 0, 1, 2, ..., n$, *and for some* $k_v \geq 0,\ v = 0, 1, 2, ..., r, 0 \leq r \leq n-1$, *we have*

$$k_0 + \alpha_n \geq k_1 + \alpha_{n-1} \geq k_2 + \alpha_{n-2} \geq ... \geq k_r + \alpha_{n-r} \geq \alpha_{n-r-1} \geq ... \geq \alpha_1 \geq \alpha_0,$$

then all the zeros of p lie in

$$|q| \leq \frac{1}{|a_n|}\Big\{(k_0 + \alpha_n) - \alpha_0 + |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + \sum_{v=0}^{r} |k_v - k_{v+1}| + L\Big\},$$

where

$$L = \sum_{v=0}^{n-1}\Big\{|\beta_{v+1} - \beta_v| + |\gamma_{v+1} - \gamma_v| + |\delta_{v+1} - \delta_v|\Big\}.$$

Applying Theorem 3.3 to the polynomial $p(q)$ having real coefficients, i.e., $\beta = \gamma = \delta = 0$, we get Theorem 3.1.

Theorem 3.4. *If* $p(q) = \sum_{v=0}^{n} q^v a_v$ *is a polynomial of degree n with quaternionic coefficients and quaternionic variable. Let b be a non-zero quaternion and suppose* $\angle(a_v, b) \leq \theta \leq \pi/2$ *for some* θ, $v = 0, 1, 2, ..., n$ *and for some number* $k_v \geq 0,\ v = 0, 1, 2, ..., r,\ 0 \leq r \leq n-1$. *Assume*

$$|k_0 + a_n| \geq |k_1 + a_{n-1}| \geq |k_2 + a_{n-2}| \geq ... \geq |k_r + a_{n-r}| \geq |a_{n-r-1}| \geq ... \geq |a_1| \geq |a_0|,$$

then all the zeros of p lie in

$$|q| \leq \frac{1}{|a_n|}\Big\{|a_0| + \sum_{v=0}^{r} |k_v - k_{v+1}| + (|k_0 + a_n| - |a_0|)\cos\theta + (|k_0 + a_n| + |a_0|)\sin\theta + 2\sin\theta \sum_{v=1}^{n} |k_v + a_{n-v}|\Big\}.$$

If we take $k_v = 0,\ v = 0, 1, 2, ..., r$ and $r = n-1$ in Theorem 3.4, we obtain the following result similar to Theorem 2.3.

Corollary 3.5. *If* $p(q) = \sum_{v=0}^{n} q^v a_v$ *is a polynomial of degree n with quaternionic coefficients and quaternionic variable. Let b be a non-zero quaternion and suppose* $\angle(a_v, b) \leq \theta \leq \pi/2$ *for some* θ, $v = 0, 1, 2, ..., n$. *Assume*

$$|a_n| \geq |a_{n-1}| \geq |a_{n-2}| \geq ... \geq |a_1| \geq |a_0|,$$

then all the zeros of p lie in

$$|q| \leq \frac{1}{|a_n|}\Big\{|a_0| + (|a_n| - |a_0|)\cos\theta + (|a_n| + |a_0|)\sin\theta + 2\sin\theta \sum_{v=1}^{n} |a_{n-v}|\Big\}.$$

It can been easily seen that Corollary 3.5 is also obtained in the form of a result (see [13], Theorem 2 for (μ=1)).

4. Auxiliary Results

We need the following lemmas for the proofs of the main results. The first lemma is due to Gentili and Stoppato [5].

Lemma 4.1. *If* $f(q) = \sum_{v=0}^{\infty} q^v a_v$ *and* $g(q) = \sum_{v=0}^{\infty} q^v b_v$ *be two given quaternionic power series with radii of convergence greater than R. The regular product of* $f(q)$ *and* $g(q)$ *is defined as* $(f \star g)(q) = \sum_{v=0}^{\infty} q^v c_v$, *where* $c_v = \sum_{k=0}^{v} a_k b_{v-k}$. *Let* $|q_0| < R$, *then* $(f \star g)(q_0) = 0$ *if and only if either* $f(q_0) = 0$ *or* $f(q_0) \neq 0$ *implies* $g(f(q_0)^{-1} q_0 f(q_0)) = 0$.

The following lemma is due to Carney et al. [2].

Lemma 4.2. *Let* $q_1,\ q_2 \in \mathbb{H}$, *where* $q_1 = \alpha_1 + i\beta_1 + j\gamma_1 + k\delta_1$ *and* $q_2 = \alpha_2 + i\beta_2 + j\gamma_2 + k\delta_2$, $\angle(q_1, q_2) = 2\theta' \leq 2\theta$, *and* $|q_1| \leq |q_2|$, *then*

$$|q_2 - q_1| \leq (|q_2| - |q_1|)\cos\theta + (|q_2| + |q_1|)\sin\theta.$$

5. Proof of Main Results

Proof of Theorem 3.1. Consider the polynomial

$$f(q) = \sum_{v=1}^{n} q^v(a_v - a_{v-1}) + a_0.$$

Let $p(q) \star (1-q) = f(q) - q^{n+1}a_n$, therefore by Lemma 4.1, $p(q) \star (1-q) = 0$ if and only if either $p(q) = 0$ or $p(q) \neq 0$ implies $p(q)^{-1}qp(q) - 1 = 0$, that is, $p(q)^{-1}qp(q) = 1$. If $p(q) \neq 0$, then $q = 1$. Therefore, the only zeros of $p(q) \star (1-q)$ are $q = 1$ and the zeros of $p(q)$.
For $|q| = 1$, we have

$$\begin{aligned}
|f(q)| &= |q^n(a_n - a_{n-1}) + \ldots + q^{n-r}(a_{n-r} - a_{n-r-1}) + \ldots + q(a_1 - a_0) + a_0| \\
&= \Big| q^n\Big[(k_0 + a_n) - (k_1 + a_{n-1}) - (k_0 - k_1)\Big] + q^{n-1}\Big[(k_1 + a_{n-1}) - (k_2 + a_{n-2}) - (k_1 - k_2)\Big] \\
&\quad + \ldots + q^{n-r+2}\Big[(k_{r-2} + a_{n-r+2}) - (k_{r-1} + a_{n-r+1}) - (k_{r-2} - k_{r-1})\Big] \\
&\quad + q^{n-r+1}\Big[(k_{r-1} + a_{n-r+1}) - (k_r + a_{n-r}) - (k_{r-1} - k_r)\Big] \\
&\quad + q^{n-r}\Big[(k_r + a_{n-r}) - (k_{r+1} + a_{n-r-1}) - (k_r - k_{r+1})\Big] + \ldots + q(a_1 - a_0) + a_0 \Big| \\
&= \Big| - q^n(k_0 - k_1) + q^n\Big[(k_0 + a_n) - (k_1 + a_{n-1})\Big] + q^{n-1}\Big[(k_1 + a_{n-1}) - (k_2 + a_{n-2})\Big] \\
&\quad - q^{n-1}(k_1 - k_2) + \ldots + q^{n-r+2}\Big[(k_{r-2} + a_{n-r+2}) - (k_{r-1} + a_{n-r+1})\Big] \\
&\quad - q^{n-r+2}(k_{r-2} - k_{r-1}) + q^{n-r+1}\Big[(k_{r-1} + a_{n-r+1}) - (k_r + a_{n-r})\Big] - q^{n-r+1}(k_{r-1} - k_r) \\
&\quad + q^{n-r}\Big[(k_r + a_{n-r}) - (k_{r+1} + a_{n-r-1})\Big] - q^{n-r}(k_r - k_{r+1}) + \ldots + q(a_1 - a_0) + a_0 \Big| \\
&\leq |k_0 - k_1| + (k_0 + a_n) - (k_1 + a_{n-1}) + (k_1 + a_{n-1}) - (k_2 + a_{n-2}) + |k_1 - k_2| \\
&\quad + \ldots + (k_{r-2} + a_{n-r+2}) - (k_{r-1} + a_{n-r+1}) + |k_{r-2} - k_{r-1}| + (k_{r-1} + a_{n-r+1}) - (k_r + a_{n-r}) \\
&\quad + |k_{r-1} - k_r| + (k_r + a_{n-r}) - (k_{r+1} + a_{n-r-1}) + |k_r - k_{r+1}| + \ldots + a_2 - a_1 + a_1 - a_0 + |a_0| \\
&= \sum_{v=0}^{r} |k_v - k_{v+1}| + (k_0 + a_n) - a_0 + |a_0|.
\end{aligned}$$

Since

$$\max_{|q|=1}\left|q^n \star f\left(\frac{1}{q}\right)\right| = \max_{|q|=1}\left|f\left(\frac{1}{q}\right)\right| = \max_{|q|=1}|f(q)|,$$

therefore, $q^n \star f\left(\frac{1}{q}\right)$ has the same bound on $|q| = 1$ as $f(q)$, that is

$$\left|q^n \star f\left(\frac{1}{q}\right)\right| \leq \sum_{v=0}^{r} |k_v - k_{v+1}| + (k_0 + a_n) - a_0 + |a_0| \quad \text{for} \quad |q| = 1.$$

Applying maximum modulus theorem ([6], Theorem 3.4), it follows that

$$\Big|q^n \star f\Big(\frac{1}{q}\Big)\Big| \leq \sum_{v=0}^{r} |k_v - k_{v+1}| + (k_0 + a_n) - a_0 + |a_0| \quad \text{for} \quad |q| \leq 1.$$

Replacing q by $\frac{1}{q}$, we get for $|q| \geq 1$

$$|f(q)| \leq \Big\{\sum_{v=0}^{r} |k_v - k_{v+1}| + (k_0 + a_n) - a_0 + |a_0|\Big\}|q|^n. \tag{1}$$

But $|p(q) \star (1 - q)| = |f(q) - q^{n+1}a_n| \geq |a_n||q|^{n+1} - |f(q)|$.
Using (1), we have for $|q| \geq 1$

$$|p(q) \star (1 - q)| \geq |a_n||q|^{n+1} - \Big\{\sum_{v=0}^{r} |k_v - k_{v+1}| + (k_0 + a_n) - a_0 + |a_0|\Big\}|q|^n.$$

This implies that $|p(q) \star (1 - q)| > 0$, i.e., $p(q) \star (1 - q) \neq 0$ if

$$|q| > \frac{1}{|a_n|}\Big\{\sum_{v=0}^{r} |k_v - k_{v+1}| + (k_0 + a_n) - a_0 + |a_0|\Big\}.$$

Since the only zeros of $p(q) \star (1 - q)$ are $q = 1$ and the zeros of $p(q)$. Therefore, $p(q) \neq 0$ for

$$|q| > \frac{1}{|a_n|}\Big\{\sum_{v=0}^{r} |k_v - k_{v+1}| + (k_0 + a_n) - a_0 + |a_0|\Big\}.$$

Hence all the zeros of $p(q)$ lie in

$$|q| \leq \frac{1}{|a_n|}\Big\{\sum_{v=0}^{r} |k_v - k_{v+1}| + (k_0 + a_n) - a_0 + |a_0|\Big\}.$$

This completes the proof of Theorem 3.1.

Proof of Theorem 3.3. Consider the polynomial

$$f(q) = \sum_{v=1}^{n} q^v (a_v - a_{v-1}) + a_0.$$

Let $p(q) \star (1 - q) = f(q) - q^{n+1}a_n$, therefore by Lemma 4.1, $p(q) \star (1 - q) = 0$ if and only if either $p(q) = 0$ or $p(q) \neq 0$ implies $p(q)^{-1}qp(q) - 1 = 0$, that is, $p(q)^{-1}qp(q) = 1$. If $p(q) \neq 0$, then $q = 1$. Therefore, the only zeros of $p(q) \star (1 - q)$ are $q = 1$ and the zeros of $p(q)$.
For $|q| = 1$, we have

$$\begin{aligned}|f(q)| = \Big| & q^n(\alpha_n - \alpha_{n-1}) + ... + q^{n-r}(\alpha_{n-r} - \alpha_{n-r-1}) + ... + q(\alpha_1 - \alpha_0) + \alpha_0 \\ & + i\Big\{q^n(\beta_n - \beta_{n-1}) + q^{n-1}(\beta_{n-1} - \beta_{n-2}) + ... + q(\beta_1 - \beta_0) + \beta_0\Big\} \\ & + j\Big\{q^n(\gamma_n - \gamma_{n-1}) + q^{n-1}(\gamma_{n-1} - \gamma_{n-2}) + ... + q(\gamma_1 - \gamma_0) + \gamma_0\Big\} \\ & + k\Big\{q^n(\delta_n - \delta_{n-1}) + q^{n-1}(\delta_{n-1} - \delta_{n-2}) + ... + q(\delta_1 - \delta_0) + \delta_0\Big\}\Big|\end{aligned}$$

$$
\begin{aligned}
&= \Big| - q^n(k_0 - k_1) + q^n\Big[(k_0 + \alpha_n) - (k_1 + \alpha_{n-1})\Big] + q^{n-1}\Big[(k_1 + \alpha_{n-1}) - (k_2 + \alpha_{n-2})\Big] \\
&- q^{n-1}(k_1 - k_2) + \dots + q^{n-r+2}\Big[(k_{r-2} + \alpha_{n-r+2}) - (k_{r-1} + \alpha_{n-r+1})\Big] - q^{n-r+2}(k_{r-2} - k_{r-1}) \\
&+ q^{n-r+1}\Big[(k_{r-1} + \alpha_{n-r+1}) - (k_r + \alpha_{n-r})\Big] - q^{n-r+1}(k_{r-1} - k_r) \\
&+ q^{n-r}\Big[(k_r + \alpha_{n-r}) - (k_{r+1} + \alpha_{n-r-1})\Big] - q^{n-r}(k_r - k_{r+1}) + \dots + q(\alpha_1 - \alpha_0) + \alpha_0 \\
&+ i\Big\{q^n(\beta_n - \beta_{n-1}) + q^{n-1}(\beta_{n-1} - \beta_{n-2}) + \dots + q(\beta_1 - \beta_0) + \beta_0\Big\} \\
&+ j\Big\{q^n(\gamma_n - \gamma_{n-1}) + q^{n-1}(\gamma_{n-1} - \gamma_{n-2}) + \dots + q(\gamma_1 - \gamma_0) + \gamma_0\Big\} \\
&+ k\Big\{q^n(\delta_n - \delta_{n-1}) + q^{n-1}(\delta_{n-1} - \delta_{n-2}) + \dots + q(\delta_1 - \delta_0) + \delta_0\Big\}\Big| \\
&\le |k_0 - k_1| + (k_0 + \alpha_n) - (k_1 + \alpha_{n-1}) + (k_1 + \alpha_{n-1}) - (k_2 + \alpha_{n-2}) + |k_1 - k_2| \\
&+ \dots + (k_{r-2} + \alpha_{n-r+2}) - (k_{r-1} + \alpha_{n-r+1}) + |k_{r-2} - k_{r-1}| + (k_{r-1} + \alpha_{n-r+1}) - (k_r + \alpha_{n-r}) \\
&+ |k_{r-1} - k_r| + (k_r + \alpha_{n-r}) - (k_{r+1} - \alpha_{n-r-1}) + |k_r - k_{r+1}| + \alpha_2 - \alpha_1 \\
&+ \alpha_1 - \alpha_0 + |\alpha_0| + \Big\{|\beta_n - \beta_{n-1}| + |\beta_{n-1} - \beta_{n-2}| + \dots + |\beta_1 - \beta_0| + |\beta_0|\Big\} \\
&+ \Big\{|\gamma_n - \gamma_{n-1}| + |\gamma_{n-1} - \gamma_{n-2}| + \dots + |\gamma_1 - \gamma_0| + |\gamma_0|\Big\} \\
&+ \Big\{|\delta_n - \delta_{n-1}| + |\delta_{n-1} - \delta_{n-2}| + \dots + |\delta_1 - \delta_0| + |\delta_0|\Big\} \\
&= (k_0 + \alpha_n) - \alpha_0 + |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + \sum_{v=0}^{n-1}\Big\{|\beta_{v+1} - \beta_v| + |\gamma_{v+1} - \gamma_v| + |\delta_{v+1} - \delta_v|\Big\} \\
&+ \sum_{v=0}^{r} |k_v - k_{v+1}| \\
&= (k_0 + \alpha_n) - \alpha_0 + |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + \sum_{v=0}^{r} |k_v - k_{v+1}| + L,
\end{aligned}
$$

where

$$L = \sum_{v=0}^{n-1}\Big\{|\beta_{v+1} - \beta_v| + |\gamma_{v+1} - \gamma_v| + |\delta_{v+1} - \delta_v|\Big\}.$$

Since

$$\max_{|q|=1}\Big|q^n \star f\Big(\frac{1}{q}\Big)\Big| = \max_{|q|=1}\Big|f\Big(\frac{1}{q}\Big)\Big| = \max_{|q|=1}|f(q)|,$$

therefore, $q^n \star f\big(\frac{1}{q}\big)$ has the same bound on $|q| = 1$ as $f(q)$, that is

$$\Big|q^n \star f\Big(\frac{1}{q}\Big)\Big| \le (k_0 + \alpha_n) - \alpha_0 + |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + \sum_{v=0}^{r} |k_v - k_{v+1}| + L \quad \text{for} \quad |q| = 1.$$

After few steps as in the proof of Theorem 3.1, we conclude that all the zeros of $p(q)$ lie in

$$|q| \le \frac{1}{|a_n|}\Big\{(k_0 + \alpha_n) - \alpha_0 + |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + \sum_{v=0}^{r} |k_v - k_{v+1}| + L\Big\}.$$

This completes the proof of Theorem 3.3.

Proof of Theorem 3.4. Consider the polynomial

$$f(q)=\sum_{v=1}^{n} q^{v}(a_v-a_{v-1})+a_0.$$

Let $p(q)\star(1-q)=f(q)-q^{n+1}a_n$, therefore by Lemma 4.1, $p(q)\star(1-q)=0$ if and only if either $p(q)=0$ or $p(q)\neq 0$ implies $p(q)^{-1}qp(q)-1=0$, that is, $p(q)^{-1}qp(q)=1$. If $p(q)\neq 0$, then $q=1$. Therefore, the only zeros of $p(q)\star(1-q)$ are $q=1$ and the zeros of $p(q)$.

For $|q|=1$, we have

$$\begin{aligned}
|f(q)| &= |q^n(a_n-a_{n-1})+\ldots+q^{n-r}(a_{n-r}-a_{n-r-1})+\ldots+q(a_1-a_0)+a_0|\\
&= \Big|q^n\Big[(k_0+a_n)-(k_1+a_{n-1})-(k_0-k_1)\Big]+q^{n-1}\Big[(k_1+a_{n-1})-(k_2+a_{n-2})-(k_1-k_2)\Big]\\
&\quad +\ldots+q^{n-r+2}\Big[(k_{r-2}+a_{n-r+2})-(k_{r-1}+a_{n-r+1})-(k_{r-2}-k_{r-1})\Big]\\
&\quad +q^{n-r+1}\Big[(k_{r-1}+a_{n-r+1})-(k_r+a_{n-r})-(k_{r-1}-k_r)\Big]\\
&\quad +q^{n-r}\Big[(k_r+a_{n-r})-(k_{r+1}+a_{n-r-1})-(k_r-k_{r+1})\Big]+\ldots+q(a_1-a_0)+a_0\Big|\\
&= \Big|-q^n(k_0-k_1)+q^n\Big[(k_0+a_n)-(k_1+a_{n-1})\Big]+q^{n-1}\Big[(k_1+a_{n-1})-(k_2+a_{n-2})\Big]\\
&-q^{n-1}(k_1-k_2)+\ldots+q^{n-r+2}\Big[(k_{r-2}+a_{n-r+2})-(k_{r-1}+a_{n-r+1})\Big]-q^{n-r+2}(k_{r-2}-k_{r-1})\\
&+q^{n-r+1}\Big[(k_{r-1}+a_{n-r+1})-(k_r+a_{n-r})\Big]-q^{n-r+1}(k_{r-1}-k_r)\\
&+q^{n-r}\Big[(k_r+a_{n-r})-(k_{r+1}+a_{n-r-1})\Big]-q^{n-r}(k_r-k_{r+1})+\ldots+q(a_1-a_0)+a_0\Big|\\
&\leq |k_0-k_1|+|(k_0+a_n)-(k_1+a_{n-1})|+|(k_1+a_{n-1})-(k_2+a_{n-2})|\\
&+|k_1-k_2|+\ldots+|(k_{r-2}+a_{n-r+2})-(k_{r-1}+a_{n-r+1})|+|k_{r-2}-k_{r-1}|\\
&+|(k_{r-1}+a_{n-r+1})-(k_r+a_{n-r})|+|k_{r-1}-k_r|\\
&+|(k_r+a_{n-r})-(k_{r+1}+a_{n-r-1})|+|k_r-k_{r+1}|+|a_2-a_1|+|a_1-a_0|+|a_0|.
\end{aligned}$$

Now using Lemma 4.2, it follows that

$$\begin{aligned}
|f(q)| &\leq \sum_{v=0}^{r}|k_v-k_{v+1}|+(|k_0+a_n|-|k_1+a_{n-1}|)\cos\theta+(|k_0+a_n|+|k_1+a_{n-1}|)\sin\theta\\
&+(|k_1+a_{n-1}|-|k_2+a_{n-2}|)\cos\theta+(|k_1+a_{n-1}|+|k_2+a_{n-2}|)\sin\theta+\ldots\\
&+(|k_{r-2}+a_{n-r+2}|-|k_{r-1}+a_{n-r+1}|)\cos\theta+(|k_{r-2}+a_{n-r+2}|+|k_{r-1}+a_{n-r+1}|)\sin\theta\\
&+(|k_{r-1}+a_{n-r+1}|-|k_r+a_{n-r}|)\cos\theta+(|k_{r-1}+a_{n-r+1}|+|k_r+a_{n-r}|)\sin\theta\\
&+(|k_r+a_{n-r}|-|k_{r+1}+a_{n-r-1}|)\cos\theta+(|k_r+a_{n-r}|+|k_{r+1}+a_{n-r}|)\sin\theta\\
&+(|a_2|-|a_1|)\cos\theta+(|a_2|+|a_1|)\sin\theta+(|a_1|-|a_0|)\cos\theta+(|a_1|+|a_0|)\sin\theta+|a_0|\\
&=|a_0|+\sum_{v=0}^{r}|k_v-k_{v+1}|+(|k_0+a_n|-|a_0|)\cos\theta+(|k_0+a_n|+|a_0|)\sin\theta+2\sin\theta\sum_{v=1}^{n}|k_v+a_{n-v}|.
\end{aligned}$$

Since

$$\max_{|q|=1}\Big|q^n\star f\Big(\frac{1}{q}\Big)\Big|=\max_{|q|=1}\Big|f\Big(\frac{1}{q}\Big)\Big|=\max_{|q|=1}|f(q)|,$$

therefore, $q^n \star f\left(\frac{1}{q}\right)$ has the same bound on $|q| = 1$ as $f(q)$, that is

$$\left|q^n \star f\left(\frac{1}{q}\right)\right|$$
$$\leq |a_0| + \sum_{v=0}^{r} |k_v - k_{v+1}| + (|k_0 + a_n| - |a_0|)\cos\theta + (|k_0 + a_n| + |a_0|)\sin\theta + 2\sin\theta \sum_{v=1}^{n} |k_v + a_{n-v}|.$$

Now, proceeding similarly as in the proof of Theorem 3.1, it follows that all the zeros of $p(q)$ lie in

$$|q| \leq \frac{1}{|a_n|}\left\{|a_0| + \sum_{v=0}^{r} |k_v - k_{v+1}| + (|k_0 + a_n| - |a_0|)\cos\theta + (|k_0 + a_n| + |a_0|)\sin\theta + 2\sin\theta \sum_{v=1}^{n} |k_v + a_{n-v}|\right\}.$$

This completes the proof of Theorem 3.4.

6. Conclusions

Some fresh finidings on the Eneström-Kakeya theorem for quaternionic polynomials has been established that are benefical in determining the regions containing all the zeros of a polynomial.

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