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The group inverse and Moore-Penrose inverse of the product of two elements and generalized inverse in a ring with involution

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Abstract. In this paper, using the expression form of the group inverse and Moore-Penrose inverse for the product of two elements, we present some new characterizations of EP elements, partial isometries, SEP elements, normal elements and hermitian elements in a ring with involution.

1. Introduction

EP elements, partial isometries, SEP elements, normal elements and Hermitian elements in rings with involution are characterized by many authors such as [2, 13–23]. For complex matrices, in term of the rank of a matrix, or other finite dimensional methods, these related matrices are discussed [1, 3]. Also, the operator analogues of these notions are explored [4, 5]. In [6], products of EP operators on Hilbert spaces has been studied. In [7], products of EP matrices has been discussed. In [10], products of EP elements in semigroup has been studied. In this paper, we discuss the expression form of group inverse and Moore-Penrose inverse for the product of two elements taken from a given set. Using these group inverses and Moore-Penrose inverses, we give some new and interesting characterizations of EP elements, partial isometries, SEP elements, normal elements and Hermitian elements.

Let R be a ring and $a \in R$. If there exists $a^\# \in R$ such that

$$aa^\# a = a,\ a^\# aa^\# = a^\#,\ aa^\# = a^\# a.$$

The element a is called a group invertible element and $a^\#$ is called a group inverse of a [9, 12, 13], and it is uniquely determined by these equations. We write $R^\#$ to denote the set of all group invertible elements of R.

If a map $* : R \to R$ satisfies

$$(a^*)^* = a,\ (a+b)^* = a^* + b^*,\ (ab)^* = b^* a^*.$$

Then R is said to be an involution ring or a $*-$ring.

Let R be a $*$-ring and $a \in R$. If there exists $a^+ \in R$ such that

$$a = aa^+a,\ a^+ = a^+aa^+,\ (aa^+)^* = aa^+,\ (a^+a)^* = a^+a.$$

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Then a is called a Moore Penrose invertible element, and a^+ is called the Moore Penrese inverse of a [8, 11]. Let R^+ denote the set of all Moore Penrese invertible elements of R.

If $a \in R^\# \cap R^+$ and $a^\# = a^+$, then a is called an EP element.

If $a = aa^*a$, then a is called partial isometry [20]. It is known that $a \in R^+$ is partial isometry if and only if $a^* = a^+$ [15].

The element $a \in R^\# \cap R^+$ is called a SEP element [23] if $a^\# = a^+ = a^*$. Clearly, $a \in R^\# \cap R^+$ is SEP if and only $a \in R^{EP}$ and $a \in R^{PI}$. Where R^{EP} , R^{PI} and R^{SEP} are denoted the set of all EP elements, all PI elements and all SEP elements of R respectively.

If $aa^* = a^*a$, then a is called normal. In [15], it is shown that $a \in R^\# \cap R^+$ is normal if and only if $a^+a^* = a^*a^+$ and $a \in R^{EP}$ if and only if $a^*a^\# = a^\#a^*$. We denote the set of all normal elements of R by R^{Nor}.

If $a = a^*$, then a is called Hermitian. According to [15], $a \in R^\# \cap R^+$ is Hermitian if and only if $(a^\#)^* = a^\#$. We denote the set of all Hermitian elements of R by R^{Her}.

2. The group inverse and Moore-Penrose inverse of product of elements

Let $a \in R^\# \cap R^+$. Taking $\tau_a = \{a, a^\#, (a^+)^*\}$, $\gamma_a = \{a^+, a^*, (a^\#)^*\}$ and $\chi_a = \tau_a \cup \gamma_a$. Clearly, we have $(a^\#)^+ = a^+a^3a^+$ and $(a^+)^\# = (aa^\#)^*a(aa^\#)^*$. The following theorem gives the group inverse and Moore-Penrose inverse of product of two elements in χ_a.

Theorem 2.1. *Let $a \in R^\# \cap R^+$. Then*

(1) If $x \in \tau_a$, then $(xy)^+ = y^+x^\#aa^+$ for each $y \in \chi_a$.

(2) If $x \in \gamma_a$, then $(xy)^+ = y^+x^\#a^+a$ for each $y \in \chi_a$.

(3) $$(xy)^\# = \begin{cases} y^\#x^+aa^\# & , x, y \in \tau_a \\ y^+x^\#aa^+ & , x \in \tau_a,\ y \in \gamma_a \\ y^+x^\#a^+a & , x \in \gamma_a,\ y \in \tau_a \\ y^\#x^+(aa^\#)^*, & x, y \in \gamma_a \end{cases}.$$

Proof. (1) Noting that $yy^+ = \begin{cases} aa^+, & y \in \tau_a \\ a^+a, & y \in \gamma_a \end{cases}$,

$$xaa^+x^\# = xa^+ax^\# = aa^\# = x^\#aa^+x,\ aa^+x = x,$$

and

$$a^+ay^+ = y^+,\ for\ each\ y \in \tau_a,$$

$$aa^+y^+ = y^+,\ for\ each\ y \in \gamma_a.$$

Then

$$(xy)(y^+x^\#aa^+) = x(yy^+)x^\#aa^+ = \begin{cases} xaa^+x^\#aa^+, & y \in \tau_a \\ xa^+ax^\#aa^+, & y \in \gamma_a \end{cases} = aa^\#aa^+ = aa^+,$$

$$(xy)(y^+x^\#aa^+)(xy) = aa^+xy = xy,$$

$$(y^+x^\#aa^+)(xy) = y^+(x^\#aa^+x)y = y^+aa^\#y = \begin{cases} a^+a, & y \in \tau_a \\ aa^+, & y \in \gamma_a \end{cases},$$

$$(y^+x^\#aa^+)(xy)(y^+x^\#aa^+) = \begin{cases} a^+ay^+x^\#aa^+, & y \in \tau_a \\ aa^+y^+x^\#aa^+, & y \in \gamma_a \end{cases} = y^+x^\#aa^+.$$

Hence $(xy)^+ = y^+x^\#aa^+$ for each $y \in \chi_a$.

(2) Noting that $xaa^+x^\# = xa^+ax^\# = (aa^\#)^* = x^\#a^+ax$ for each $x \in \gamma_a$. Then

$$(xy)(y^+x^\#a^+a) = x(yy^+)x^\#a^+a = \begin{cases} xaa^+x^\#a^+a, \; y \in \tau_a \\ xa^+ax^\#a^+a, \; y \in \gamma_a \end{cases} = (aa^\#)^*a^+a = a^+a,$$

$$(xy)(y^+x^\#a^+a)(xy) = a^+a(xy) = xy,$$

$$(y^+x^\#a^+a)(xy) = y^+(x^\#a^+ax)y = y^+(aa^\#)^*y = \begin{cases} a^+a, \; y \in \tau_a \\ aa^+, \; y \in \gamma_a \end{cases},$$

$$(y^+x^\#a^+a)(xy)(y^+x^\#a^+a) = \begin{cases} a^+ay^+x^\#a^+a, \; y \in \tau_a \\ aa^+y^+x^\#a^+a, \; y \in \gamma_a \end{cases} = y^+x^\#a^+a.$$

Hence $(xy)^+ = y^+x^\#a^+a$ for each $y \in \chi_a$.

(3) If $x, y \in \tau_a$, then

$$(xy)(y^\#x^+aa^\#) = x(yy^\#)x^+aa^\# = (xaa^\#x^+)aa^\# = aa^+aa^\# = aa^\#,$$

$$(xy)(y^\#x^+aa^\#)(xy) = aa^\#xy = xy,$$

$$(y^\#x^+aa^\#)(xy) = y^\#(x^+aa^\#x)y = y^\#a^+ay = a^\#a,$$

$$(y^\#x^+aa^\#)(xy)(y^\#x^+aa^\#) = a^\#ay^\#x^+aa^\# = y^\#x^+aa^\#.$$

Hence $(xy)^\# = y^\#x^+aa^\#$.

If $x \in \tau_a$, $y \in \gamma_a$, then the proof of (1) implies $(xy)^\# = y^+x^\#aa^+$.

If $x \in \gamma_a$, $y \in \tau_a$, then the proof of (2) implies $(xy)^\# = y^+x^\#a^+a$.

If $x, y \in \gamma_a$, then $x(aa^\#)^* = x = (aa^\#)^*x$, $x^\# = (aa^\#)^*x^\#$ and $xx^+ = a^+a$, this gives

$$(xy)(y^\#x^+(aa^\#)^*) = x(yy^\#)x^+(aa^\#)^* = x(aa^\#)^*x^+(aa^\#)^* = (aa^\#)^*,$$

$$(y^\#x^+(aa^\#)^*)(xy) = y^\#(x^+(aa^\#)^*x)y = y^\#aa^+y = (aa^\#)^*,$$

$$(xy)(y^\#x^+(aa^\#)^*)(xy) = (aa^\#)^*(xy) = xy,$$

$$(y^\#x^+(aa^\#)^*)(xy)(y^\#x^+(aa^\#)^*) = (aa^\#)^*y^\#x^+(aa^\#)^* = y^\#x^+(aa^\#)^*.$$

Hence $(xy)^\# = y^\#x^+(aa^\#)^*$. □

Using Theorem 2.1, the following theorem gives a new form characterization of generalized inverses.

Theorem 2.2. *Let $a \in R^\# \cap R^+$. Then*

(1) $a \in R^{EP}$ if and only if $(xy)^+ = y^+x^\#a^+a$ for some $x \in \tau_a$ and $y \in \chi_a$.

(2) $a \in R^{PI}$ if and only if $(xy)^+ = y^+x^\#aa^*$ for some $x \in \tau_a$ and $y \in \chi_a$.

(3) $a \in R^{SEP}$ if and only if $(xy)^+ = y^+x^\#a^*a$ for some $x \in \tau_a$ and $y \in \chi_a$.

(4) $a \in R^{Nor}$ if and only if $(xy)^+ = y^+x^\#(a^+)^*a^+a^*a$ for some $x \in \tau_a$ and $y \in \chi_a$.

(5) $a \in R^{Her}$ if and only if $(xy)^+ = y^+x^\#a(a^+)^*$ for some $x \in \tau_a$ and $y \in \chi_a$.

Proof. (1) $\Longrightarrow$ Assume that $a \in R^{EP}$. Then $aa^+ = a^+a$. By Theorem 2.1(1), we have $(xy)^+ = y^+x^\#aa^+ = y^+x^\#a^+a$ for all $x \in \tau_a$ and all $y \in \chi_a$.

$\Longleftarrow$ From the assumption and Theorem 2.1(1), we have

$$y^+x^\#aa^+ = y^+x^\#a^+a, \; for \; some \; x \in \tau_a \; and \; y \in \chi_a.$$

If $y \in \tau_a$, then

$$(aa^+)x^\#aa^+ = (yy^+)x^\#aa^+ = y(y^+x^\#aa^+) = y(y^+x^\#a^+a) = (yy^+)(x^\#a^+a) = aa^+x^\#a^+a,$$

$$aa^+ = aa^\# aa^+ = (xaa^+x^\#)aa^+ = x(aa^+x^\# aa^+) = x(aa^+x^\# a^+a) = (xaa^+x^\#)a^+a = aa^\# a^+a = aa^\#.$$

Hence $a \in R^{EP}$.

If $y \in \gamma_a$, then

$$a^+ax^\# aa^+ = (yy^+)x^\# aa^+ = y(y^+x^\# aa^+) = y(y^+x^\# a^+a) = a^+ax^\# a^+a,$$

$$aa^+ = aa^\# aa^+ = (xa^+ax^\#)aa^+ = x(a^+ax^\# aa^+) = x(a^+ax^\# a^+a) = (xa^+ax^\#)a^+a = aa^\# a^+a = aa^\#.$$

Hence $a \in R^{EP}$.

(2) $\Longrightarrow$ Suppose that $a \in R^{PI}$. Then $a^+ = a^*$. It follows from Theorem 2.1(1), that $(xy)^+ = y^+x^\# aa^+ = y^+x^\# aa^*$ for all $x \in \tau_a$ and all $y \in \chi_a$.

$\Longleftarrow$ By Theorem 2.1(1) and the assumption, one has

$$y^+x^\# aa^+ = y^+x^\# aa^*, \ for\ some\ x \in \tau_a\ and\ y \in \chi_a.$$

Noting that $x \in \tau_a$. Then

$$xyy^+x^\# = \begin{cases} xaa^+x^\#, \ y \in \tau_a \\ xa^+ax^\#, \ y \in \gamma_a \end{cases} = aa^\#.$$

This gives

$$aa^+ = aa^\# aa^+ = xyy^+x^\# aa^+ = xy(y^+x^\# aa^+) = xy(y^+x^\# aa^*) = (xyy^+x^\#)aa^* = aa^\# aa^* = aa^*.$$

Hence $a \in R^{PI}$ by [15, Theorem 1.5.2].

(3) $\Longrightarrow$ Since $a \in R^{SEP}$, $a^\# = a^* = a^+$ and $a^*a = aa^+$ by [15, Theorem 1.5.3]. Hence $(xy)^+ = y^+x^\# a^*a$ by Theorem 2.1(1) for all $x \in \tau_a$ and all $y \in \chi_a$.

$\Longleftarrow$ Using the assumption $(xy)^+ = y^+x^\# a^*a$ and Theorem 2.1(1), we have

$$y^+x^\# aa^+ = y^+x^\# a^*a, \ for\ some\ x \in \tau_a\ and\ y \in \chi_a.$$

This gives

$$\begin{aligned} aa^+ = aa^\# aa^+ &= (xyy^+x^\#)aa^+ = xy(y^+x^\# aa^+) = xy(y^+x^\# a^*a) \\ &= (xyy^+x^\#)a^*a = aa^\# a^*a = (aa^\# a^*a)a^+a = aa^+a^+a. \end{aligned}$$

Hence $a \in R^{EP}$, one gets

$$aa^+ = aa^\# a^*a = a^\# aa^*a = a^+aa^*a = a^*a.$$

Thus $a \in R^{SEP}$ by [15, Theorem 1.5.3].

(4) $\Longrightarrow$ From $a \in R^{Nor}$, we have $aa^* = a^*a$ and $a \in R^{EP}$. This gives

$$(a^+)^*a^+a^*a = (a^+)^*a^+aa^* = (a^+)^*a^* = aa^+.$$

Hence, by Theorem 2.1(1), $(xy)^+ = y^+x^\# aa^+ = y^+x^\#(a^+)^*a^+a^*a$ for $x \in \tau_a$ and $y \in \chi_a$.

$\Longleftarrow$ The assumption and Theorem 2.1(1) give $y^+x^\# aa^+ = y^+x^\#(a^+)^*a^+a^*a$ for some $x \in \tau_a$ and some $y \in \chi_a$. It follows that

$$\begin{aligned} aa^+ = aa^\# aa^+ &= (xyy^+x^\#)aa^+ = xy(y^+x^\# aa^+) = xy(y^+x^\#(a^+)^*a^+a^*a) \\ &= (xyy^+x^\#)(a^+)^*a^+a^*a = aa^\#(a^+)^*a^+a^*a = (a^+)^*a^+a^*a \end{aligned}$$

and

$$a^* = a^*aa^+ = a^*(a^+)^*a^+a^*a = a^+a^*a.$$

Hence $a \in R^{Nor}$ by [15, Theorem 1.3.2].

(5) $\Longrightarrow$ Since $a \in R^{Her}$, $a^+ = a^\# = (a^\#)^* = (a^+)^*$. Hence, by Theorem 2.1(1), one has $(xy)^+ = y^+x^\# aa^+ = y^+x^\# a(a^+)^*$ for all $x \in \tau_a$ and all $y \in \chi_a$.

$\Longleftarrow$ The hypothesis and Theorem 2.1(1) imply

$$y^+x^\# aa^+ = y^+x^\# a(a^+)^*, \ for\ some\ x \in \tau_a\ and\ y \in \chi_a.$$

This induces

$$aa^+ = aa^\# aa^+ = (xyy^+x^\#)aa^+ = xy(y^+x^\# a(a^+)^*) = aa^\# a(a^+)^* = a(a^+)^*.$$

Applying the involution on the equality, one gets

$$aa^+ = a^+a^* = a^+a(a^+a^*) = a^+a(aa^+) = a^+a^2a^+,$$

$$a = aa^+a = a^+a^2a^+a = a^+a^2.$$

Hence $a \in R^{EP}$, this induces $aa^+ = a^+a^* = a^\# a^*$. Thus $a \in R^{Her}$ by [15, Theorem 1.4.2]. □

Corollary 2.3. *Let $a \in R^\# \cap R^+$. Then*
(1) $a \in R^{EP}$ if and only if $aa^+a^*a^+ = a^*a^\# aa^+$.
(2) $a \in R^{SEP}$ if and only if $aa^+a^*a^+ = a^+a^\# aa^+$.
(3) $a \in R^{PI}$ if and only if $a^+a^+a^+ = a^+a^*a^+$.
(4) $a \in R^{Her}$ if and only if $aa^+a^*a^+ = a^+a^2a^+$.
(5) $a \in R^{Nor}$ if and only if $aa^+a^*a^+ = a^+a^*a^+a$.

Proof. (1) $\Longrightarrow$ Since $a \in R^{EP}$, $aa^+ = a^+a$, $a^\# = a^+$, it follows that

$$aa^+a^*a^+ = a^+aa^*a^+ = a^*a^+ = a^*a^+aa^+ = a^*a^\# aa^+.$$

$\Longleftarrow$ By Theorem 2.1(1), we have

$$(a(a^\#)^*)^+ = ((a^\#)^*)^+a^\# aa^+ = ((a^\#)^+)^*a^\# aa^+$$

$$= (a^+a^3a^+)^*a^\# aa^+ = aa^+a^*a^+aa^\# aa^+ = aa^+a^*a^+$$

and

$$(a(a^+)^*)^+ = ((a^+)^*)^+a^\# aa^+ = a^*a^\# aa^+.$$

From the assumption, we have $(a(a^\#)^*)^+ = (a(a^+)^*)^+$, this gives $a(a^\#)^* = a(a^+)^*$. Applying the involution on the equality, one has $a^\# a^* = a^+a^*$. Hence $a \in R^{EP}$ by [15, Theorem 1.2.1].

(2) $\Longrightarrow$ From $a \in R^{SEP}$, we have $a^\# = a^* = a^+$ and $aa^+ = a^+a$. This gives $aa^+a^*a^+ = aa^+a^\# a^+ = a^+aa^\# a^+ = a^+a^\# aa^+$.

$\Longleftarrow$ Noting that

$$(a^2)^+ = a^+a^\# aa^+ \text{ and } (a(a^\#)^*)^+ = aa^+a^*a^+.$$

Hence by the assumption, we have $(a^2)^+ = (a(a^\#)^*)^+$, this gives

$$a^2 = a(a^\#)^*,\ a^+a^2 = a^+a(a^\#)^* = (a^\#)^*,\ a^\# = (a^+a^2)^* = a^*a^+a.$$

Hence $a \in R^{SEP}$ by [15, Theorem 1.5.3].

(3) $\Longrightarrow$ Since $a \in R^{PI}$, $a^+ = a^*$. This implies that $a^+a^+a^+ = a^+a^*a^+$.

$\Longleftarrow$ Assume that $a^+a^+a^+ = a^+a^*a^+$. Then $aa^+a^+a^+ = aa^+a^*a^+$, one gets

$$(aa^+a^+a^+)^+ = (aa^+a^*a^+)^+.$$

Since

$$(a(a^+)^\#)^+ = (a(aa^\#)^*a(aa^\#)^*)^+ = aa^+a^+a^+ \text{ and } (a(a^\#)^*)^+ = aa^+a^*a^+.$$

We yield $a(a^+)^\# = a(a^\#)^*$, that is,

$$a(aa^\#)^*a(aa^\#)^* = a(a^\#)^*.$$

Multiplying the equality on the left by a^+, one has $(aa^\#)^*a(aa^\#)^* = (a^\#)^*$. This induces $(a^+)^\# = (a^\#)^* = (a^*)^\#$, so $a^+ = a^*$. Thus $a \in R^{PI}$.

(4) $\Longrightarrow$ From $a \in R^{Her}$, we have $a = a^*$ and $a \in R^{EP}$. This gives $aa^+a^*a^+ = aa^+aa^+ = aa^+ = a^+a^2a^+$.

$\Longleftarrow$ Suppose that $aa^+a^*a^+ = a^+a^2a^+$. Multiplying the equality on the left by $(aa^\#)^*$, we get

$$a^*a^+ = aa^+a^*a^+.$$

Again multiply the last equality on the right by aa^*, we have

$$a^*a^* = aa^+a^*a^*.$$

This implies that $a^2 = a^3a^+$. Hence $a \in R^{EP}$ by [15, Theorem 1.2.2]. It follows that

$$aa^+a^*a^+ = a^+aa^*a^+ = a^*a^+ \text{ and } a^+a^2a^+ = aa^+ = aa^\#.$$

So $a^*a^+ = aa^\#$. Hence $a \in R^{Her}$ by [15, Theorem 1.4.2].

(5) $\Longrightarrow$ By the hypothesis, we get $aa^* = a^*a$ and $a \in R^{EP}$. This induces $aa^+a^*a^+ = a^+aa^*a^+ = a^+a^*aa^+ = a^+a^*a^+a$.

$\Longleftarrow$ Since $aa^+a^*a^+ = a^+a^*a^+a$, by Theorem 2.1(1) and (2), we have

$$(a(a^\#)^*)^+ = ((a^\#)^*a)^+.$$

This gives $a(a^\#)^* = (a^\#)^*a$. So $a^\#a^* = a^*a^\#$. Thus $a \in R^{Nor}$. □

Theorem 2.4. *Let $a \in R^\# \cap R^+$. Then*

(1) $a \in R^{EP}$ if and only if $xy \in R^{EP}$ for some $x, y \in \tau_a$.

(2) $a \in R^{EP}$ if and only if $xy \in R^{EP}$ for some $x, y \in \gamma_a$.

(3) If $a \in R^{SEP}$, then $xy \in R^{SEP}$ for any $x, y \in \tau_a$.

(4) If $a \in R^{SEP}$, then $xy \in R^{SEP}$ for any $x, y \in \gamma_a$.

Proof. (1) $\Longrightarrow$ Assume that $a \in R^{EP}$ and $x, y \in \tau_a$. Then

$$y^+x^\#aa^+ = y^+x^\#aa^\# = y^+x^\#$$

and

$$y^\#x^+aa^\# = y^\#x^+.$$

We claim that $y^+x^\# = y^\#x^+$.

In fact $a^+ = a^\#$, this gives $((a^+)^*)^\# = ((a^+)^\#)^* = ((a^\#)^\#)^* = a^*$. Hence

$$y^+ = \begin{cases} a^+ & , y = a \\ (a^\#)^+ & , y = a^\# \\ ((a^+)^*)^+ & , y = (a^+)^* \end{cases} = \begin{cases} a^\# , & y = a \\ a & , y = a^\# \\ a^* & , y = (a^+)^* \end{cases} = y^\#.$$

and $x^\# = x^+$. This implies $(xy)^+ = y^+x^\#aa^+ = y^+x^\# = y^\#x^+ = y^\#x^+aa^\# = (xy)^\#$. Hence $xy \in R^{EP}$.

$\Longleftarrow$ Suppose that $xy \in R^{EP}$. Then $(xy)^+ = (xy)^\#$. Since $x, y \in \tau_a$, by Theorem 2.1(1) and (3), we have

$$y^+x^\#aa^+ = y^\#x^+aa^\#.$$

This induces

$$\begin{aligned} aa^+ = aa^\#aa^+ = (xyy^+x^\#)aa^+ &= xy(y^+x^\#aa^+) = xy(y^\#x^+aa^\#) \\ &= (xyy^\#x^+)aa^\# = aa^\#aa^\# = aa^\#. \end{aligned}$$

Hence $a \in R^{EP}$.

(2) $\Longrightarrow$ From $a \in R^{EP}$ and $x, y \in \gamma_a$, we have

$$a^+a = aa^+ = (aa^+)^* = (aa^\#)^*$$

and

$$y^+ = \begin{cases} a & , y = a^+ \\ (a^*)^+ & , y = a^* \\ ((a^\#)^*)^+ & , y = (a^\#)^* \end{cases} = \begin{cases} (a^+)^\#, & y = a^+ \\ (a^*)^\#, & y = a^* \\ a^* & , y = (a^\#)^* \end{cases} = y^\#.$$

This gives

$$(xy)^+ = y^+x^\#a^+a = y^\#x^+(aa^\#)^* = (xy)^\#.$$

Hence $xy \in R^{EP}$.

$\Longleftarrow$ The assumption and Theorem 2.1(2), (3) imply

$$y^+x^\#a^+a = y^\#x^+(aa^\#)^*.$$

It follows that

$$a^+a = (aa^\#)^*a^+a = (xyy^+x^\#)a^+a = (xy)y^+x^\#a^+a$$
$$= (xy)y^\#x^+(aa^\#)^* = (xyy^\#x^+)(aa^\#)^* = (aa^\#)^*(aa^\#)^* = (aa^\#)^*.$$

Hence $a \in R^{EP}$.

(3) Since $a \in R^{SEP}$, $\tau_a \subseteq R^{SEP}$. This implies for any $x, y \in \tau_a$, we have

$$x^* = x^+, \ y^\# = y^*.$$

Noting that $a^+ = a^\# = a^*$. Then

$$(xy)^\# = y^\#x^+aa^\# = y^\#x^+ = y^*x^* = (xy)^*.$$

Hence $xy \in R^{SEP}$.

(4) From $a \in R^{SEP}$, we have $\gamma_a \subseteq R^{SEP}$. This induces that

$$x^* = x^+, \ y^\# = y^* \ for\ any\ x, y \in \gamma_a.$$

Hence

$$(xy)^\# = y^\#x^+(aa^\#)^* = y^\#x^+ = y^*x^* = (xy)^*.$$

Thus $xy \in R^{SEP}$. □

Let $a \in R^{EP}$. Then it is known that $a + 1 - aa^\# \in R^{-1}$ and $(a + 1 - aa^\#)^{-1} = a^\# + 1 - aa^\#$.

Theorem 2.5. *Let $a \in R^\# \cap R^+$. Then*

(1) $xy + 1 - aa^\# \in R^{-1}$ and $(xy + 1 - aa^\#)^{-1} = y^\#x^+aa^\# + 1 - aa^\#$, where $x, y \in \tau_a$.
(2) $a \in R^{PI}$ if and only if $(xy + 1 - aa^\#)^{-1} = y^\#x^+aa^\#a^*a + 1 - aa^\#$ for some $x, y \in \tau_a$.
(3) $a \in R^{EP}$ if and only if $(xy + 1 - aa^\#)^{-1} = y^\#x^+aa^+ + 1 - aa^\#$ for some $x, y \in \tau_a$.
(4) $a \in R^{SEP}$ if and only if $(xy + 1 - aa^\#)^{-1} = y^\#x^+aa^* + 1 - aa^\#$ for some $x, y \in \tau_a$.
(5) $a \in R^{Her}$ if and only if $(xy + 1 - aa^\#)^{-1} = y^\#x^+a(a^\#)^* + 1 - aa^\#$ for some $x, y \in \tau_a$.
(6) $a \in R^{Nor}$ if and only if $(xy + 1 - aa^\#)^{-1} = y^\#x^+aa^*a^\#(a^\#)^* + 1 - aa^\#$ for some $x, y \in \tau_a$.

Proof. (1) Since $x, y \in \tau_a$, $(xy)^\# = y^\#x^+aa^\#$ and $(xy)(xy)^\# = aa^\#$ by Theorem 2.1(3). Hence $(xy + 1 - aa^\#)^{-1} = y^\#x^+aa^\# + 1 - aa^\#$.

(2) $\Longrightarrow$ Suppose that $a \in R^{PI}$. Then $a^\# = a^\#a^*a$ by [15, Theorem 1.5.2]. It follows from (1) that $(xy + 1 - aa^\#)^{-1} = y^\#x^+aa^\# + 1 - aa^\# = y^\#x^+aa^\#a^*a + 1 - aa^\#$ for all $x, y \in \tau_a$.

$\Longleftarrow$ From the assumption and (1), we have

$$y^\#x^+aa^\#a^*a + 1 - aa^\# = (xy + 1 - aa^\#)^{-1} = y^\#x^+aa^\# + 1 - aa^\# \ for\ some\ x, y \in \tau_a.$$

This gives

$$y^\#x^+aa^\#a^*a = y^\#x^+aa^\# \ for\ some\ x, y \in \tau_a.$$

Since $x, y \in \tau_a$, $xyy^{\#}x^+ = xaa^{\#}x^+ = aa^+$, one gets

$$aa^+aa^{\#}a^*a = (xyy^{\#}x^+)aa^{\#}a^*a = xy(y^{\#}x^+aa^{\#}a^*a) = xy(y^{\#}x^+aa^{\#})$$
$$= (xyy^{\#}x^+)aa^{\#} = aa^+aa^{\#} = aa^{\#},$$

e.g.,

$$aa^{\#}a^*a = aa^{\#} \ and \ a = a^2a^{\#} = a^2a^{\#}a^*a = aa^*a.$$

Hence $a \in R^{PI}$.

(3) $\Longrightarrow$ The condition $a \in R^{EP}$ implies $aa^+ = aa^{\#}$. Hence, by (1),

$$(xy + 1 - aa^{\#})^{-1} = y^{\#}x^+aa^+ + 1 - aa^{\#} \ for \ all \ x, y \in \tau_a.$$

$\Longleftarrow$ From the assumption and (1), we have

$$y^{\#}x^+aa^+ + 1 - aa^{\#} = (xy + 1 - aa^{\#})^{-1} = y^{\#}x^+aa^{\#} + 1 - aa^{\#} \ for \ some \ x, y \in \tau_a,$$

e.g.,

$$y^{\#}x^+aa^+ = y^{\#}x^+aa^{\#} \ for \ some \ x, y \in \tau_a.$$

This infers

$$aa^+ = aa^{\#}aa^+ = (xyy^{\#}x^+)aa^+ = xy(y^{\#}x^+aa^+) = xy(y^{\#}x^+aa^{\#})$$
$$= (xyy^{\#}x^+)aa^{\#} = aa^{\#}aa^{\#} = aa^{\#}.$$

Hence $a \in R^{EP}$.

(4) $\Longrightarrow$ From $a \in R^{SEP}$, we have $aa^{\#} = aa^*$ by [15, Theorem 1.5.3]. Hence $(xy+1-aa^{\#})^{-1} = y^{\#}x^+aa^*+1-aa^{\#}$ for all $x, y \in \tau_a$ by (1).

$\Longleftarrow$ The assumption and (1) imply

$$y^{\#}x^+aa^{\#} + 1 - aa^{\#} = (xy + 1 - aa^{\#})^{-1} = y^{\#}x^+aa^* + 1 - aa^{\#} \ for \ some \ x, y \in \tau_a,$$

e.g.,

$$y^{\#}x^+aa^{\#} = y^{\#}x^+aa^* \ for \ some \ x, y \in \tau_a.$$

This induces

$$aa^{\#} = aa^{\#}aa^{\#} = (xyy^{\#}x^+)aa^{\#} = xy(y^{\#}x^+aa^{\#}) = xy(y^{\#}x^+aa^*)$$
$$= (xyy^{\#}x^+)aa^* = aa^{\#}aa^* = aa^*.$$

Hence $a \in R^{SEP}$ by [15, Theorem 1.5.3].

(5) $\Longrightarrow$ Since $a \in R^{Her}$, $a^{\#} = (a^{\#})^*$. By (1), we have

$$(xy + 1 - aa^{\#})^{-1} = y^{\#}x^+aa^{\#} + 1 - aa^{\#} = y^{\#}x^+a(a^{\#})^* + 1 - aa^{\#}.$$

$\Longleftarrow$ By the hypothesis and (1), one yields

$$y^{\#}x^+a(a^{\#})^* = y^{\#}x^+aa^{\#} \ for \ some \ x, y \in \tau_a.$$

Multiplying the equality on the left by xy, one gets

$$a(a^{\#})^* = aa^{\#}.$$

Noting that $(a^{\#})^* = a^+a(a^{\#})^*aa^+$. Then we have

$$a^+ = a^+aa^{\#}aa^+ = a^+a(a^{\#})^*aa^+ = (a^{\#})^*$$

and $aa^+ = a(a^{\#})^* = aa^{\#}$. So $a \in R^{EP}$ and $a^{\#} = a^+ = (a^{\#})^*$. Thus $a \in R^{Her}$.

(6) $\Longrightarrow$ Suppose that $a \in R^{Nor}$. Then $a \in R^{EP}$ and $aa^* = a^*a$, this leads to

$$aa^*a^\#(a^\#)^* = a^*aa^+(a^+)^* = a^+a = aa^+ = aa^\#.$$

Hence, by (1), we are done.

$\Longleftarrow$ From the (1) and the assumption, one obtains

$$y^\#x^+aa^*a^\#(a^\#)^* = y^\#x^+aa^\# \ for \ some \ x, y \in \tau_a.$$

Multiplying the equality on the left by xy, one gets

$$aa^*a^\#(a^\#)^* = aa^\#.$$

Multiplying the last equality on the right by aa^+, one has $aa^\# = aa^+$. Hence $a \in R^{EP}$ and

$$a^+ = a^+aa^+ = a^+aa^\# = a^+(aa^*a^\#(a^\#)^*) = a^*a^\#(a^\#)^*,$$

$$a^\#a^* = a^+a^* = a^*a^\#(a^\#)^*a^* = a^*a^+(aa^\#)^* = a^*a^+ = a^*a^\#.$$

Thus $a \in R^{Nor}$. □

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