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Right e-core inverse and the related generalized inverses in rings

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Abstract. In this paper, some characterizations and properties of right e-core inverses by using right invertible element and $\{1, 3e\}$-inverse are investigated. Meanwhile, some characterizations for a new generalized right e-core inverse which is called right pseudo e-core inverse are also studied. The relationship between right pseudo e-core inverses and right e-core inverses are presented.

1. Introduction

Let $\mathcal{R}$ be an associative ring with the unit 1. An involution $* : \mathcal{R} \to \mathcal{R}$ is an anti-isomorphism which satisfies $(a^*)^* = a$, $(a+b)^* = a^* + b^*$ and $(ab)^* = b^*a^*$ for all $a, b \in \mathcal{R}$. We call $\mathcal{R}$ a $*$-ring if there exists an involution on $\mathcal{R}$. Recall that an element $a \in \mathcal{R}$ is said to be Hermitian if $a^* = a$. And an element $a \in \mathcal{R}$ is an idempotent if $a^2 = a$.

The core inverse of a complex matrix was first introduced by Baksalary and Trenkler [3]. Later, Rakić et al. [11] generalized this concept to the case of an arbitrary $*$-ring. An element $a \in \mathcal{R}$ is core invertible (resp. dual core invertible) if there is an element $x \in \mathcal{R}$ such that

$$axa = a, \quad x\mathcal{R} = a\mathcal{R} \quad (\text{resp. } \mathcal{R}x = \mathcal{R}a), \mathcal{R}x = \mathcal{R}a^* \quad (\text{resp. } x\mathcal{R} = a^*\mathcal{R}).$$

Such an x above is called a core inverse of a. It is unique if it exists and is denoted by $a^{\#}$(resp. $a_{\#}$). Moreover, it was proved in [11] that $a \in \mathcal{R}$ is core invertible if and only if there exists an element $x \in \mathcal{R}$ satisfying the following five equations:

$$axa = a, \quad xax = x, \quad (ax)^* = ax, \quad xa^2 = a, \quad ax^2 = x.$$

Indeed, Xu, Chen and Zhang [14] proved that the above five equations can be deduced to three equations:

$$xa^2 = a, \quad ax^2 = x \quad \text{and} \quad (ax)^* = ax.$$

In [5], Gao and Chen defined the pseudo core inverse by three equations in a $*$-rings, which extend the classical core inverses. An element $a \in \mathcal{R}$ is pseudo core invertible if there exist an $x \in \mathcal{R}$ and a positive integer k satisfying

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$$xa^{k+1} = a^k, \quad ax^2 = x \quad \text{and} \quad (ax)^* = ax.$$

If such an x exists, it is unique and is called a pseudo core inverse of a, and denoted by $a^{\textcircled{\scriptsize D}}$. The smallest positive integer k is called the pseudo core index of a.

Later, Mosić, Deng and Ma [10] introduced the definitions of the e-core inverse and the f-dual core inverse of elements in $*$-rings, which generalized the concepts of the core inverse and the dual core inverse, where e and f are invertible Hermitian elements. Following [10], any element $x \in \mathcal{R}$ is called an e-core inverse (or a weighted core inverse with weight e) of $a \in \mathcal{R}$, if it satisfies

$$axa = a, \quad x\mathcal{R} = a\mathcal{R}, \text{ and } \mathcal{R}x = \mathcal{R}a^*e.$$

Such an e-core inverse x of a is unique if it exists, and is denoted by $a^{e,\textcircled{\#}}$. If $e = 1$ in the above definition, then $a^{e,\textcircled{\#}} = a^{\textcircled{\#}}$ is the ordinary core inverse of a. Moreover, the authors characterized e-core inverse by three equations, that is, a is e-core invertible if and only if there exists $x \in \mathcal{R}$ such that

$$xa^2 = a, \quad ax^2 = x \quad \text{and} \quad (eax)^* = eax.$$

Wang and Mosić [12] introduced the one-sided core inverse, which considered as the special case of right (b, c)-inverse, called it right core inverse in $*$-ring. Then they gave some characterizations for it. Recall that an element $a \in \mathcal{R}$ is said to be right core invertible if there is $x \in \mathcal{R}$ satisfying

$$axa = a, \quad ax^2 = x \quad \text{and} \quad (ax)^* = ax.$$

Later, Wang, Mosić and Gao [13] investigated some properties of right core inverses, and gave new characterizations and expressions for them by using projections and one-sided invertible elements. They also introduced and studied a new generalized right core inverse which is called right pseudo core inverse. An element $a \in \mathcal{R}$ is right pseudo core invertible if there exist $x \in \mathcal{R}$ and positive integer k satisfy

$$axa^k = a^k, \quad ax^2 = x \quad \text{and} \quad (ax)^* = ax.$$

We use the symbols $a_r^{\textcircled{\#}}$ and $a_r^{\textcircled{\scriptsize D}}$ to denote the right core inverse and right pseudo core inverse of a, respectively.

In [15], Zhu and Wang derived the existence criteria and characterizations for the weighted Moore-Penrose, e-core inverse, f-dual core inverse and one-sided inverses along an element in rings. Later they in [16] defined two types of outer generalized inverses, called pseudo e-core inverse and pseudo f-dual core inverse. An element $a \in \mathcal{R}$ is called pseudo e-core invertible (resp. pseudo f-core invertible) if there are $x \in \mathcal{R}$ and positive integer k such that

$$xax = x, \quad x\mathcal{R} = a^k\mathcal{R} \text{ (resp. } \mathcal{R}x = \mathcal{R}a^k), \quad \mathcal{R}x = \mathcal{R}(a^k)^*e \text{ (resp. } fx\mathcal{R} = (a^k)^*\mathcal{R}).$$

Furthermore, they investigated some characterizations and properties for them, and gave the relations between the pseudo e-core inverse and the inverse along an element.

Motivated by the aforementioned above, in this article, we will investigate some characterizations and properties for right e-core inverses by using right invertible element and $\{1, 3e\}$-inverse. Meanwhile, we also study some characterizations for a new generalized right e-core inverse which is called right pseudo e-core inverse. Finally, we present the relationship between right pseudo e-core inverses and right e-core inverses.

Now, we give the main concepts and symbols.

Let $e, f \in \mathcal{R}$ be two invertible Hermitian elements, we say that $a \in \mathcal{R}$ is a weighted Moore-Penrose invertible with weights e, f if there exists an $x \in \mathcal{R}$ satisfying the following four equations (see [1, 2]):

$$(1)\ axa = a, \quad (2)\ xax = x, \quad (3e)\ (eax)^* = eax, \quad (4f)\ (fxa)^* = fxa.$$

If such an x exists, it is unique and called a weighted Moore-Penrose inverse of a, denoted by $a^{\dagger}_{e,f}$. The set of all weighted Moore-Penrose invertible elements of $\mathcal{R}$ with weighted e, f will be denoted by $\mathcal{R}^{\dagger}_{e,f}$. If $e = f = 1$ in the above equations, then $a^{\dagger}_{e,f} = a^{\dagger}$ is the ordinary Moore-Penrose inverse of a. More generally, if a and x satisfy the equations (1) $axa = a$ and (3e) $(eax)^* = eax$, then x is called a $\{1, 3e\}$-inverse of a, and is denoted by $a^{(1,3e)}$. Similarly, if a and x satisfy the equations (1) $axa = a$ and (4f) $(fxa)^* = fxa$, then x is called a $\{1, 4f\}$-inverse of a, and is denoted by $a^{(1,4f)}$. As usual, we denote by $\mathcal{R}^{\{1,3e\}}$ and $\mathcal{R}^{\{1,4f\}}$ the sets of all $\{1, 3e\}$-invertible and $\{1, 4f\}$-invertible elements in $\mathcal{R}$, respectively. If a and x satisfy the equations (1) $axa = a$, (2) $xax = x$, and (5) $ax = xa$, then x is called a group inverse of a, and is denoted by $a^\sharp$. All the group invertible elements of $\mathcal{R}$ is denoted by $\mathcal{R}^\sharp$.

As weaker versions of the (b, c)-invertibility, one-sided (b, c)-invertibility is introduced by Drazin [4]:

Definition 1.1. *Let $a, b, c \in \mathcal{R}$. Then a is called right (resp. left) (b, c)-invertible if $c \in cab\mathcal{R}$ (resp. $b \in \mathcal{R}cab$), or equivalently if there exists $z \in b\mathcal{R}$ such that $caz = c$ (resp. $x \in \mathcal{R}c$ such that $xab = b$), in which case any such z (resp. x) will be called a right (resp. left) (b, c)-inverse of a.*

In [4], Drazin considered some properties of left (or right) (b, c)-inverses under the additional conditions, such as $\mathcal{R}$ is strongly π-regular. In [6], Ke, Višnjić and Chen introduced left and right annihilator (b, c)-inverses and investigated some properties of them and of left (or right) (b, c)-inverses. In [12], the authors studied the properties of left (or right) (b, c)-inverses under the condition cab is regular. As applications, the authors introduced the one-sided core inverse, for the convenience of the reader, the definitions of right core inverses are given again see [13, Definition 1.3].

Definition 1.2. *[13, Definition 1.3] Let $a \in \mathcal{R}$. We say that a is right core invertible if a is right (a, a^*)-invertible.*

Motivated by above definition, the authors introduced the one-sided e-core inverse in [13, Remark 4.12], here we also give the definition.

Definition 1.3. *[13] Let $a \in \mathcal{R}$ and $e \in \mathcal{R}$ be an invertible Hermitian element. An element a is called right e-core invertible if a is right (a, a^*e)-invertible.*

Note that, by Definition 1.3, a is right e-core invertible if and only if $a^*e \in a^*ea^2\mathcal{R}$ if and only if there exists $x \in \mathcal{R}$ such that $x \in a\mathcal{R}$ and $a^*eax = a^*e$. The sets of all right e-core invertible elements of $\mathcal{R}$ will be denoted by $\mathcal{R}_r^{e,\text{#}}$. The symbol $a_r^{e,\text{#}}$ is used to denote the right e-core inverse of a, if $a \in \mathcal{R}_r^{e,\text{#}}$.

Next section we will study the properties of right e-core inverses.

2. Characterizing right e-core inverses by idempotent and one-sided inverse in a $*$-ring

In [9, Theorems 3.3 and 3.4], Li and Chen gave the characterizations and expressions of core inverse of an element by a projection and units. Motivated by this, in this section, we present some equivalent conditions for the existence of right e-core inverses. We will prove that a is right e-core invertible if and only if there exists an idempotent p such that $(ep)^* = ep$, $pa = 0$, and $a^n + p$ is right invertible for $n \geq 1$. Before we start, the following result is needed.

Lemma 2.1. *[7, Theorem 1.4] Let $a \in \mathcal{R}$ and $e \in \mathcal{R}$ be an invertible Hermitian element. Then the following statements are equivalent:*

(i) a is right e-core invertible;

(ii) there exists $x \in \mathcal{R}$ such that $axa = a$, $x = ax^2$ and $(eax)^* = eax$.

Remark 2.2. *In fact, all right e-core inverses $a_r^{e,\text{#}}$ of a satisfy*

$$aa_r^{e,\text{#}}a = a,\ a_r^{e,\text{#}} = a(a_r^{e,\text{#}})^2 \text{ and } (eaa_r^{e,\text{#}})^* = eaa_r^{e,\text{#}}.$$

Moreover, if a is right e-core invertible, then $aa_r^{e,\text{#}}$ is invariant on the choice of $a_r^{e,\text{#}}$. Indeed, assume that x_1 and x_2 are two right e-core inverses of a. Then $eax_1 = (eax_1)^* = x_1^*a^*e = x_1^*(ax_2a)^*e = x_1^*a^*(ax_2)^*e^* = x_1^*a^*(eax_2)^* = x_1^*a^*eax_2 = (eax_1)^*ax_2 = eax_1ax_2 = eax_2$. Since e is invertible, we have $ax_1 = ax_2$. Denote by $a^\pi = 1 - aa_r^{e,\text{#}}$ the idempotent determined by a right e-core inverse of a, if a is right e-core invertible in $\mathcal{R}$.

In [10, Definition 1.1], the authors introduced the concept of weighted-EP elements in a ring with involution, which is a generalization of EP matrices. An element $a \in \mathcal{R}$ is weighted-EP with respect to (e, e) if $a \in \mathcal{R}^{\dagger}_{(e,e)} \cap \mathcal{R}^{\#}$ and $a^{\#} = a^{\dagger}_{(e,e)}$. Moreover, the authors pointed out that $a \in \mathcal{R}$ is e-core invertible if and only if $a \in \mathcal{R}^{\#} \cap \mathcal{R}^{\{1,3e\}}$ in [10, Theorem 2.1]. Using Lemma 2.1 and above remark, we can deduce the following result.

Proposition 2.3. *Let $e \in \mathcal{R}$ be an invertible Hermitian element, and $a \in \mathcal{R}$ be right e-core invertible. If $aa_r^{e,\circledast} = a_r^{e,\circledast}a$, then a is weighted-EP with respect to (e, e) and $a_r^{e,\circledast} = a^{\#} = a^{\dagger}_{(e,e)}$.*

In the following, we will use the symbol $\mathcal{R}_r^{-1}$ to denote the set of all right invertible elements in $\mathcal{R}$. The symbol a_r^{-1} denotes the right inverse of a, if $a \in \mathcal{R}_r^{-1}$. The symbol $r(a)$ (rep. $l(a)$) denotes the right (rep. left) annihilator of $a \in \mathcal{R}$.

In [9], the authors proved that a is core invertible if and only if there exists a projection p such that $pa = 0$, $a^n + p$ is invertible for $n \geq 1$. Here we will give the similar result for right e-core invertible.

Theorem 2.4. *Let $a \in \mathcal{R}$ and $e \in \mathcal{R}$ be an invertible Hermitian element. Then the following statements are equivalent:*
(i) a is right e-core invertible;
(ii) there exists a unique idempotent p such that $(pe)^* = pe$, $pea = 0$ and $u = p + eae^{-1} \in \mathcal{R}_r^{-1}$;
(iii) there exists a unique idempotent p such that $(pe)^* = pe$, $pea = 0$ and $w = eae^{-1}(1 - p) + p \in \mathcal{R}_r^{-1}$.
(iv) there exists a unique idempotent p such that $(ep)^* = ep$, $pa = 0$ and $u = p + a \in \mathcal{R}_r^{-1}$;
(v) there exists a unique idempotent p such that $(ep)^* = ep$, $pa = 0$ and $w = a(1 - p) + p \in \mathcal{R}_r^{-1}$.
In this case, $a_r^{\circledast} = e^{-1}u_r^{-1}(1 - p)e = e^{-1}(1 - p)w_r^{-1}e$.

Proof. (*i*) ⇔ (*ii*) ⇔ (*iii*) For the proofs we refer the reader to [7, Theorem 1.6].

(*i*) ⇒ (*iv*) Suppose that a is right e-core invertible, by Lemma 2.1, there is $x \in \mathcal{R}$ such that $axa = a$, $x = ax^2$ and $(eax)^* = eax$. Let $p = 1 - ax$. Then $p^2 = (1 - ax)^2 = 1 - ax = p$, $ep = e(1 - ax) = e - eax = e^* - (eax)^* = (e - eax)^* = (ep)^*$, $pa = (1 - ax)a = 0$, and $px = (1 - ax)x = 0$. And $(p + a)(x + 1 - xa) = p + ax + a(1 - xa) = p + ax = 1$, this gives $u = p + a \in \mathcal{R}_r^{-1}$.

For the uniqueness of the idempotent, assume that there exist two idempotents p and q satisfy $(ep)^* = ep$, $(eq)^* = eq$, $pa = qa = 0$, $p + a \in \mathcal{R}_r^{-1}$ and $q + a \in \mathcal{R}_r^{-1}$. It is easily seen that $l(1 - p) = l(1 - q) = l(a)$, which implies $p = pq$ and $q = qp$. Hence, $ep = (ep)^* = (epq)^* = q^*(ep)^* = q^*ep = q^*e^*p = (eq)^*p = eqp$, this gives $p = qp = q$ since e is invertible.

(*iv*) ⇒ (*i*) Under hypothesis $pa = 0$ and $p + a \in \mathcal{R}_r^{-1}$, we know $(1 - p)(p + a) = a$ and hence $1 - p = a(p + a)_r^{-1}$. Consider $x = (p + a)_r^{-1}(1 - p)$. Then $ax = a(p + a)_r^{-1}(1 - p) = 1 - p$, which gives that $eax = e(1 - p) = e - ep = e^* - (ep)^* = (e - ep)^* = (eax)^*$, and $axa = (1 - p)a = a$. Note that $p(p + a) = p$, it follows that $p = p(p + a)_r^{-1}$ and hence $(1 - p)(p + a)_r^{-1} = (p + a)_r^{-1} - p$. Therefore, $ax^2 = (1 - p)(p + a)_r^{-1}(1 - p) = [(p + a)_r^{-1} - p](1 - p) = (p + a)_r^{-1}(1 - p) = x$. By Lemma 2.1, we see at once that a is right e-core invertible.

(*i*) ⇒ (*v*) As in the proof of (*i*) ⇒ (*iv*), we also let $p = 1 - ax$. Then $p^2 = p$, $(ep)^* = ep$, $pa = 0$, and $px = 0$. Thus

$$[a(1 - p) + p](x + 1 - ax) = (a^2x + p)(x + 1 - ax) = a^2x^2 + a(ax)(1 - ax) + px + p(1 - ax) = ax + p = 1,$$

that is, $w = a(1 - p) + p \in \mathcal{R}_r^{-1}$.

For the uniqueness of the idempotent, analysis similar to that in the proof of (*i*) ⇒ (*iv*).

(*v*) ⇒ (*i*) Notice that $(1 - p)w = (1 - p)[a(1 - p) + p] = (1 - p)a(1 - p) = a(1 - p)$. Then $1 - p = a(1 - p)w_r^{-1}$. Set $x = (1 - p)w_r^{-1}$. It is clear that $ax = a(1 - p)w_r^{-1} = (1 - p)ww_r^{-1} = 1 - p$, $(eax)^* = (e - ep)^* = e^* - (ep)^* = e - ep = eax$, $axa = (1 - p)a = a$ and $ax^2 = (1 - p)x = (1 - p)^2w_r^{-1} = (1 - p)w_r^{-1} = x$. By Lemma 2.1, we obtain that a is right e-core invertible. □

The following theorem shows that $p + a^n \in \mathcal{R}_r^{-1}$ is true when taking $n \geq 2$ in Theorem 2.4. Before it, we state an auxiliary result.

Lemma 2.5. *[8, Exercise 1.6] Let $a, b \in \mathcal{R}$. Then $1 + ab$ is right invertible if and only if $1 + ba$ is right invertible.*

Theorem 2.6. *Let $a \in \mathcal{R}$ and $e \in \mathcal{R}$ be an invertible Hermitian element and $n \geq 2$. Then the following statements are equivalent:*

(i) *a is right e-core invertible;*

(ii) *there exists a unique idempotent p such that $(pe)^* = pe$, $pea = 0$ and $u = p + ea^n e^{-1} \in \mathcal{R}_r^{-1}$;*

(iii) *there exists a unique idempotent p such that $(pe)^* = pe$, $pea = 0$ and $w = ea^n e^{-1}(1-p) + p \in \mathcal{R}_r^{-1}$;*

(iv) *there exists a unique idempotent p such that $(ep)^* = ep$, $pa = 0$ and $u = p + a^n \in \mathcal{R}_r^{-1}$;*

(v) *there exists a unique idempotent p such that $(ep)^* = ep$, $pa = 0$ and $w = a^n(1-p) + p \in \mathcal{R}_r^{-1}$.*

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) [7, Theorem 1.9] gives these equivalent statements.

(i) $\Rightarrow$ (iv) If a is right e-core invertible, also let $p = 1 - ax$. From the proof of Theorem 2.4 (i) $\Rightarrow$ (iv), we know $p^2 = p$, $(ep)^* = ep$, $pa = 0$, $px = 0$ and $u = p + a = a + 1 - ax \in \mathcal{R}_r^{-1}$. Then $1 + ax(a-1) = a + 1 - ax$ is right invertible. Using Lemma 2.5, it follows that $1 + (a-1)ax = 1 + a^2x - ax$ is right invertible. It is easy to verify that if $n = 2$,

$$a^2 + 1 - ax = (1 + a^2x - ax)(a + 1 - ax)$$

is right invertible. Assume that the result holds for the case $n - 1 (n > 2)$, that is, $a^{n-1} + 1 - ax$ is right invertible, we will prove it for n. Indeed,

$$p + a^n = a^n + 1 - ax = (1 + a^2x - ax)(a^{n-1} + 1 - ax)$$

is right invertible. For the uniqueness of the idempotent, it is similar to (i) $\Rightarrow$ (iv) in Theorem 2.4.

(iv) $\Rightarrow$ (i) From the assumption $pa = 0$ and $u = p + a^n \in \mathcal{R}_r^{-1}$, we get $(1-p)(p + a^n) = a^n$ and $1 - p = a^n(p + a^n)_r^{-1} = a^n u_r^{-1}$. Take $x = a^{n-1}u_r^{-1}$. Then $ax = a^n u_r^{-1} = 1 - p$, $eax = e(1-p) = e - ep = e^* - (ep)^* = (e - ep)^* = (eax)^*$, $axa = (1-p)a = a$ and $ax^2 = (1-p)a^{n-1}u_r^{-1} = a^{n-1}u_r^{-1} = x$. By Lemma 2.1, we obtain a is right e-core invertible.

(i) $\Rightarrow$ (v) We also let $p = 1 - ax$. By the proof of (i) $\Rightarrow$ (iv), we get $p^2 = p$, $(ep)^* = ep$, $pa = 0$, $px = 0$ and $u = p + a^n = a^n + 1 - ax \in \mathcal{R}_r^{-1}$. So $1 + ax(a^n - 1) = a^n + 1 - ax$ is right invertible. Applying Lemma 2.5, $1 + (a^n - 1)ax$ is invertible. Hence $w = a^n(1-p) + p = a^n(1-p) - 1 + p + 1 = 1 + (a^n - 1)(1-p) = 1 + (a^n - 1)ax$ is right invertible.

(v) $\Rightarrow$ (i) Note that $(1-p)w = (1-p)[a^n(1-p) + p] = a^n(1-p)$. Then $1 - p = a^n(1-p)w_r^{-1}$. Take $x = a^{n-1}(1-p)w_r^{-1}$. It is clear that $ax = 1 - p$, $px = 0$, $eax = e(1-p) = e - ep = e - (ep)^* = (e - ep)^* = (eax)^*$, $axa = (1-p)a = a$ and $ax^2 = (1-p)x = x$. Thus, by Lemma 2.1, we know that a is right e-core invertible. $\square$

From Remark 2.2, it is evident that if a is right e-core invertible, then $a^\pi = 1 - aa_r^{e,\circledast}$ is an idempotent determined by a right e-core inverse of a. In the following result, some characterizations of those elements with equal corresponding idempotents are given.

Proposition 2.7. *Let $a, b \in \mathcal{R}_r^{e,\circledast}$. Then the following statements are equivalent:*

(i) *$aa_r^{e,\circledast} = bb_r^{e,\circledast}$;*

(ii) *$a\mathcal{R} = b\mathcal{R}$;*

(iii) *$a^\pi b = 0$ and $a^\pi + b \in \mathcal{R}_r^{-1}$;*

(iv) *$a^\pi b = 0$ and $a^\pi + b(1 - a^\pi) \in \mathcal{R}_r^{-1}$.*

In addition, if one of statements (i)–(iv) holds, then ab is right e-core invertible and $b_r^{e,\circledast}a_r^{e,\circledast}$ is a right e-core inverse of ab.

Proof. (i) $\Rightarrow$ (ii) From $aa_r^{e,\circledast} = bb_r^{e,\circledast}$, we get $a = aa_r^{e,\circledast}a = bb_r^{e,\circledast}a \in b\mathcal{R}$ and $b = bb_r^{e,\circledast}b = aa_r^{e,\circledast}b \in a\mathcal{R}$, which imply $a\mathcal{R} \subseteq b\mathcal{R} \subseteq a\mathcal{R}$, that is $a\mathcal{R} = b\mathcal{R}$.

(ii) $\Rightarrow$ (i) If $a\mathcal{R} = b\mathcal{R}$, there exist $x, y \in \mathcal{R}$ such that $a = bx$ and $b = ay$. Then $bb_r^{e,\circledast}a = bb_r^{e,\circledast}(bx) = bx = a$, and $aa_r^{e,\circledast}b = aa_r^{e,\circledast}(ay) = ay = b$. Thus,

$$eaa_r^{e,\circledast} = e(bb_r^{e,\circledast}a)a_r^{e,\circledast} = (ebb_r^{e,\circledast})^* aa_r^{e,\circledast} = (bb_r^{e,\circledast})^* eaa_r^{e,\circledast} = (bb_r^{e,\circledast})^*(eaa_r^{e,\circledast})^* = (eaa_r^{e,\circledast}bb_r^{e,\circledast})^* = (ebb_r^{e,\circledast})^* = ebb_r^{e,\circledast}.$$

Therefore, $aa_r^{e,\circledast} = bb_r^{e,\circledast}$ since e is invertible.

(i) $\Rightarrow$ (iii) From $aa_r^{e,\oplus} = bb_r^{e,\oplus}$, we have $a^\pi b = (1 - aa_r^{e,\oplus})b = (1 - bb_r^{e,\oplus})b = 0$. Since

$$\begin{aligned}(a^\pi + b)(b_r^{e,\oplus} + 1 - b_r^{e,\oplus}b) &= (1 - aa_r^{e,\oplus} + b)(b_r^{e,\oplus} + 1 - b_r^{e,\oplus}b) = (1 - bb_r^{e,\oplus} + b)(b_r^{e,\oplus} + 1 - b_r^{e,\oplus}b)\\ &= b_r^{e,\oplus} + 1 - b_r^{e,\oplus}b - bb_r^{e,\oplus}b_r^{e,\oplus} - bb_r^{e,\oplus} + bb_r^{e,\oplus}b_r^{e,\oplus}b + bb_r^{e,\oplus} + b - bb_r^{e,\oplus}b\\ &= b_r^{e,\oplus} + 1 - b_r^{e,\oplus}b - b_r^{e,\oplus} - bb_r^{e,\oplus} + b_r^{e,\oplus}b + bb_r^{e,\oplus} + b - b\\ &= 1.\end{aligned}$$

Thus, $a^\pi + b$ is right invertible.

(iii) $\Rightarrow$ (i) Suppose that $a^\pi b = 0$ and $a^\pi + b \in \mathcal{R}_r^{-1}$. Notice that $ea^\pi = e - eaa_r^{e,\oplus} = (e - eaa_r^{e,\oplus})^* = (ea^\pi)^*$, and

$$bb_r^{e,\oplus}a^\pi = e^{-1}ebb_r^{e,\oplus}a^\pi = e^{-1}(ebb_r^{e,\oplus})^*a^\pi = e^{-1}(bb_r^{e,\oplus})^*ea^\pi = e^{-1}(bb_r^{e,\oplus})^*(ea^\pi)^* = e^{-1}(ea^\pi bb_r^{e,\oplus})^* = 0.$$

So we have $(aa_r^{e,\oplus} - bb_r^{e,\oplus})(a^\pi + b) = aa_r^{e,\oplus}a^\pi + aa_r^{e,\oplus}b - bb_r^{e,\oplus}a^\pi - b = aa_r^{e,\oplus}b - b = -a^\pi b = 0$. Therefore $aa_r^{e,\oplus} = bb_r^{e,\oplus}$, since $a^\pi + b$ is right invertible.

(i) $\Rightarrow$ (iv) From $aa_r^{e,\oplus} = bb_r^{e,\oplus}$, we have $a^\pi = 1 - aa_r^{e,\oplus} = 1 - bb_r^{e,\oplus} = b^\pi$, and $a^\pi b = b^\pi b = 0$. Notice that $b^\pi b_r^{e,\oplus} = 0$ and $(b^\pi)^2 = b^\pi$. So we get

$$\begin{aligned}[a^\pi + b(1 - a^\pi)](b_r^{e,\oplus} + b^\pi) &= [b^\pi + b(1 - b^\pi)](b_r^{e,\oplus} + b^\pi) = b^\pi b_r^{e,\oplus} + b^\pi + b(1 - b^\pi)b_r^{e,\oplus} + b(1 - b^\pi)b^\pi\\ &= b^\pi + b(1 - b^\pi)b_r^{e,\oplus} = b^\pi + b(bb_r^{e,\oplus})b_r^{e,\oplus} = b^\pi + bb_r^{e,\oplus} = 1.\end{aligned}$$

Thus, $a^\pi + b(1 - a^\pi)$ is right invertible.

(iv) $\Rightarrow$ (i) If $a^\pi b = 0$, from the proof of (iii) $\Rightarrow$ (i), we know that $bb_r^{e,\oplus}a^\pi = 0$ and $(aa_r^{e,\oplus} - bb_r^{e,\oplus})(a^\pi + b) = 0$. Thus,

$$\begin{aligned}(aa_r^{e,\oplus} - bb_r^{e,\oplus})(a^\pi + b(1 - a^\pi)) &= (aa_r^{e,\oplus} - bb_r^{e,\oplus})(a^\pi + b) - (aa_r^{e,\oplus} - bb_r^{e,\oplus})ba^\pi = -(aa_r^{e,\oplus} - bb_r^{e,\oplus})ba^\pi\\ &= -aa_r^{e,\oplus}ba^\pi + ba^\pi = (1 - aa_r^{e,\oplus})ba^\pi = a^\pi ba^\pi = 0.\end{aligned}$$

Therefore, $aa_r^{e,\oplus} = bb_r^{e,\oplus}$ since $a^\pi + b(1 - a^\pi)$ is right invertible.

The equality $aa_r^{e,\oplus} = bb_r^{e,\oplus}$ gives $abb_r^{e,\oplus}a_r^{e,\oplus} = a(aa_r^{e,\oplus})a_r^{e,\oplus} = aa_r^{e,\oplus}$. Thus $ab(b_r^{e,\oplus}a_r^{e,\oplus})ab = aa_r^{e,\oplus}ab = ab$, $ab(b_r^{e,\oplus}a_r^{e,\oplus})^2 = (ab(b_r^{e,\oplus}a_r^{e,\oplus}))b_r^{e,\oplus}a_r^{e,\oplus} = aa_r^{e,\oplus}b_r^{e,\oplus}a_r^{e,\oplus} = bb_r^{e,\oplus}b_r^{e,\oplus}a_r^{e,\oplus} = b_r^{e,\oplus}a_r^{e,\oplus}$, and $[eab(b_r^{e,\oplus}a_r^{e,\oplus})]^* = (eaa_r^{e,\oplus})^* = eaa_r^{e,\oplus} = eab(b_r^{e,\oplus}a_r^{e,\oplus})$. So, ab is right e-core invertible and $(ab)_r^{e,\oplus} = b_r^{e,\oplus}a_r^{e,\oplus}$. □

More sufficient conditions for the reverse order law of right e-core invertible elements are presented now.

Proposition 2.8. *Let $a, b \in \mathcal{R}_r^{e,\oplus}$. Then the following statements are equivalent:*

(i) $a = abb_r^{e,\oplus}$ and $b = aa_r^{e,\oplus}b$;

(ii) $a^*\mathcal{R} \subseteq eb\mathcal{R} \subseteq ea\mathcal{R}$.

In addition, if one of statements (i)–(ii) holds, then ab is right e-core invertible and $b_r^{e,\oplus}a_r^{e,\oplus}$ is a right e-core inverse of ab.

Proof. (i) $\Rightarrow$ (ii) The assumption $b = aa_r^{e,\oplus}b$ yields $b\mathcal{R} \subseteq a\mathcal{R}$ and $eb\mathcal{R} \subseteq ea\mathcal{R}$. Applying involution to $a = abb_r^{e,\oplus}$, it follows that $a^* = (abb_r^{e,\oplus})^* = (ae^{-1}ebb_r^{e,\oplus})^* = (ebb_r^{e,\oplus})^*(ae^{-1})^* = ebb_r^{e,\oplus}(ae^{-1})^*$, that is, $a^*\mathcal{R} \subseteq eb\mathcal{R}$. Hence, $a^*\mathcal{R} \subseteq eb\mathcal{R} \subseteq ea\mathcal{R}$.

(ii) $\Rightarrow$ (i) Suppose that $a^*\mathcal{R} \subseteq eb\mathcal{R}$ and $eb\mathcal{R} \subseteq ea\mathcal{R}$, then there exist $x, y \in \mathcal{R}$ such that $a^* = ebx$ and $eb = eay$, which give $b = ay$ since e is invertible. So we have $aa_r^{e,\oplus}b = (aa_r^{e,\oplus}a)y = ay = b$. And $(abb_r^{e,\oplus})^* = (ae^{-1}ebb_r^{e,\oplus})^* = (ebb_r^{e,\oplus})^*(ae^{-1})^* = ebb_r^{e,\oplus}(ae^{-1})^* = ebb_r^{e,\oplus}e^{-1}a^* = ebb_r^{e,\oplus}e^{-1}(ebx) = ebx = a^*$, applying involution, we get $a = abb_r^{e,\oplus}$.

From $a = abb_r^{e,\oplus}$, we get $abb_r^{e,\oplus}a_r^{e,\oplus} = aa_r^{e,\oplus}$ and $ab(b_r^{e,\oplus}a_r^{e,\oplus})ab = aa_r^{e,\oplus}ab = ab$. Since $b = aa_r^{e,\oplus}b$, we see $b_r^{e,\oplus} = b(b_r^{e,\oplus})^2 = aa_r^{e,\oplus}b(b_r^{e,\oplus})^2 = aa_r^{e,\oplus}b_r^{e,\oplus}$. Thus, $ab(b_r^{e,\oplus}a_r^{e,\oplus})^2 = (aa_r^{e,\oplus})b_r^{e,\oplus}a_r^{e,\oplus} = b_r^{e,\oplus}a_r^{e,\oplus}$ and $[eab(b_r^{e,\oplus}a_r^{e,\oplus})]^* = (eaa_r^{e,\oplus})^* = eaa_r^{e,\oplus} = eab(b_r^{e,\oplus}a_r^{e,\oplus})$. Therefore, ab is right e-core invertible and $(ab)_r^{e,\oplus} = b_r^{e,\oplus}a_r^{e,\oplus}$. □

Let $p = p^2 \in \mathcal{R}$ be an idempotent. Then we can represent any element $a \in \mathcal{R}$ as

$$a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_p,$$

where $a_{11} = pap$, $a_{12} = pa(1 - p)$, $a_{21} = (1 - p)ap$, $a_{22} = (1 - p)a(1 - p)$.

Now we give matrix representations for a right e-core invertible element and its right e-core inverse.

Theorem 2.9. *Let $a \in \mathcal{R}$ and $e \in \mathcal{R}$ be an invertible Hermitian element. Then the following statements are equivalent:*
(i) *a is right e-core invertible and $x \in \mathcal{R}$ is a right e-core inverse of a;*
(ii) *there exists an idempotent $q \in \mathcal{R}$ such that $(eq)^* = eq$ and*

$$a = \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \end{bmatrix}_q, \qquad x = \begin{bmatrix} x_1 & x_2 \\ 0 & 0 \end{bmatrix}_q, \tag{1}$$

where a_1 is right invertible in $q\mathcal{R}q$, $x_1 = (a_1)_r^{-1}$ and $a_1x_2 = 0$;
(iii) *there exists an idempotent $p \in \mathcal{R}$ such that $(ep)^* = ep$ and*

$$a = \begin{bmatrix} 0 & 0 \\ a_1 & a_2 \end{bmatrix}_p, \qquad x = \begin{bmatrix} 0 & 0 \\ x_1 & x_2 \end{bmatrix}_p,$$

where a_2 is right invertible in $(1-p)\mathcal{R}(1-p)$, $x_2 = (a_2)_r^{-1}$ and $a_2x_1 = 0$.

Proof. (i) $\Rightarrow$ (ii) Suppose that a is right e-core invertible and $x \in \mathcal{R}$ is a right e-core inverse of a, by Lemma 2.1, we have $axa = a, ax^2 = x, (eax)^* = eax$. Let $q = ax$. Then $q^2 = (ax)(ax) = ax = q$, $eq = eax = (eax)^* = (eq)^*$, $qa = (ax)a = a$ and $qx = (ax)x = x$, which imply $(1-q)a = 0$ and $(1-q)x = 0$. Thus, $a = \begin{bmatrix} qaq & qa(1-q) \\ (1-q)aq & (1-q)a(1-q) \end{bmatrix}_q = \begin{bmatrix} qaq & qa(1-q) \\ 0 & 0 \end{bmatrix}_q = \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \end{bmatrix}_q$, and $x = \begin{bmatrix} x_1 & x_2 \\ 0 & 0 \end{bmatrix}_q$, that is, a and x are represented as in (1). Since $a_1 = qaq = aq = a^2x$ and $x_1 = qxq = xax$, we get $a_1x_1 = (a^2x)xax = axax = ax = q$, that is, x_1 is a right inverse of a_1 in $q\mathcal{R}q$. By $x_2 = qx(1-q) = (ax)x(1-ax) = x(1-ax)$, we have $a_1x_2 = (a^2x)x(1-ax) = ax(1-ax) = 0$.

(ii) $\Rightarrow$ (i) Because $ax = \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \end{bmatrix}_q \begin{bmatrix} x_1 & x_2 \\ 0 & 0 \end{bmatrix}_q = \begin{bmatrix} a_1x_1 & a_1x_2 \\ 0 & 0 \end{bmatrix}_q = \begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix}_q = q$, we can verify that x satisfies $(eax)^* = (eq)^* = eq = eax$, $axa = a$ and $x = ax^2$. Using Lemma 2.1, we deduce that a is right e-core invertible and x is a right e-core inverse of a.

(i) $\Leftrightarrow$ (iii) This equivalence follows similarly as (i) $\Leftrightarrow$ (ii) for $p = 1 - ax$.

Indeed, let $p = 1-ax$. Then $p^2 = (1-ax)^2 = 1-ax-ax-axax = 1-ax$, $pa = 0$, $px = 0$, $(ep)^* = (e(1-ax))^* = (e-eax)^* = e-(eax)^* = e-eax = ep$, $a = \begin{bmatrix} pap & pa(1-p) \\ (1-p)ap & (1-p)a(1-p) \end{bmatrix}_p = \begin{bmatrix} 0 & 0 \\ (1-p)ap & (1-p)a(1-p) \end{bmatrix}_p = \begin{bmatrix} 0 & 0 \\ a_1 & a_2 \end{bmatrix}_p$, and $x = \begin{bmatrix} 0 & 0 \\ x_1 & x_2 \end{bmatrix}_p$. Since $a_1 = (1-p)ap = ap = a(1-ax) = a - a^2x$ and $a_2 = a(1-p) = a^2x$, $x_1 = xp = x - xax$, $x_2 = x(1-p) = xax$, we get $a_2x_2 = (a^2x)xax = axax = ax = 1-p$, that is, x_2 is a right inverse of a_2 in $(1-p)\mathcal{R}(1-p)$. And $a_2x_1 = (a^2x)x(1-ax) = ax(1-ax) = 0$.

Conversely, as $ax = \begin{bmatrix} 0 & 0 \\ a_2x_1 & a_2x_2 \end{bmatrix}_p = \begin{bmatrix} 0 & 0 \\ 0 & 1-p \end{bmatrix}_p = 1-p$, it is easy to prove that a is right e-core invertible and x is a right e-core inverse of a. □

Notice that p and q, which appear in Theorem 2.9, are invariant on the choice of x. We present one decomposition of a right e-core invertible element which is also invariant on the choice of right e-core inverse.

Proposition 2.10. *Let $a \in \mathcal{R}$ be right e-core invertible. Then $a = a_1 + a_2$, where*
(i) *a_1 is right e-core invertible,*
(ii) *$a_2^2 = 0$,*
(iii) *$a_2a_1 = 0$.*
In addition, $a^2a_r^{e,\circledast}$ is right e-core invertible and $a_r^{e,\circledast}aa_r^{e,\circledast}$ is a right e-core inverse of $a^2a_r^{e,\circledast}$.

Proof. Suppose that a is right e-core invertible, and x is a right e-core inverse of a. Let $a_1 = a^2x$ and $a_2 = a-a^2x$. We have $a = a_1+a_2$, where $a_2a_1 = (a-a^2x)a^2x = a(a^2x-a^2x) = 0$ and $a_2^2 = a(1-ax)a(1-ax) = a(a-a)(1-ax) = 0$.

Set $y = xax$. Since $a_1y = a^2x^2ax = axax = ax$, $(ea_1y)^* = (eax)^* = eax = ea_1y$, $a_1ya_1 = (ax)a^2x = a^2x = a_1$ and $a_1y^2 = (ax)xax = xax = y$. Hence, a_1 is right e-core invertible and y is a right e-core inverse of a_1. □

3. More characterizations of right e-core inverses

From Lemma 2.1 and Remark 2.2, we know that if a is right e-core invertible and x is a right e-core inverse of a, then $a \in \mathcal{R}^{\{1,3e\}}$. Since $a = axa = a(ax^2)a = a^2x^2a$, it gives that $a\mathcal{R} \subseteq a^2\mathcal{R}$. Since $a^2\mathcal{R} \subseteq a\mathcal{R}$, we have $a\mathcal{R} = a^2\mathcal{R}$. So we have the following result.

Theorem 3.1. *Let $a \in \mathcal{R}$ and $e \in \mathcal{R}$ be an invertible Hermitian element. Then the following statements are equivalent:*
(i) a is right e-core invertible;
(ii) $a \in \mathcal{R}^{\{1,3e\}}$ and $a\mathcal{R} = a^2\mathcal{R}$;
$(ii)'$ $a \in \mathcal{R}^{\{1,3e\}}$ and $a\mathcal{R} \subseteq a^2\mathcal{R}$;
(iii) $a \in \mathcal{R}^{\{1,3e\}}$ and $a\mathcal{R} = a^n\mathcal{R}$ for any $n \geq 2$;
$(iii)'$ $a \in \mathcal{R}^{\{1,3e\}}$ and $a\mathcal{R} \subseteq a^n\mathcal{R}$ for any $n \geq 2$;
(iv) $a \in \mathcal{R}^{\{1,3e\}}$ and $a\mathcal{R} = a^n\mathcal{R}$ for some $n \geq 2$;
$(iv)'$ $a \in \mathcal{R}^{\{1,3e\}}$ and $a\mathcal{R} \subseteq a^n\mathcal{R}$ for some $n \geq 2$;
(v) a^n is right e-core invertible and $a\mathcal{R} = a^n\mathcal{R}$ for any $n \geq 2$;
$(v)'$ a^n is right e-core invertible and $a\mathcal{R} \subseteq a^n\mathcal{R}$ for any $n \geq 2$;
(vi) a^n is right e-core invertible and $a\mathcal{R} = a^n\mathcal{R}$ for some $n \geq 2$;
$(vi)'$ a^n is right e-core invertible and $a\mathcal{R} \subseteq a^n\mathcal{R}$ for some $n \geq 2$;
(vii) $a^n \in \mathcal{R}^{\{1,3e\}}$ and $aR = a^k\mathcal{R}$ for any $n \geq 2$ and $k > n$;
$(vii)'$ $a^n \in \mathcal{R}^{\{1,3e\}}$ and $aR \subseteq a^k\mathcal{R}$ for any $n \geq 2$ and $k > n$;
$(viii)$ $a^n \in \mathcal{R}^{\{1,3e\}}$ and $aR = a^k\mathcal{R}$ for some $n \geq 2$ and $k > n$;
$(viii)'$ $a^n \in \mathcal{R}^{\{1,3e\}}$ and $aR \subseteq a^k\mathcal{R}$ for some $n \geq 2$ and $k > n$.
In this case, for any $n \geq 2$,

$$(a^n)_r^{e,\circledast} = (a_r^{e,\circledast})^n \quad \text{and} \quad a_r^{e,\circledast} = a^{n-1}(a^n)_r^{e,\circledast} = a^{n-1}(a^n)^{(1,3e)}.$$

Proof. $(i) \Rightarrow (ii) \Rightarrow (ii)'$ and $(iii) \Rightarrow (iii)' \Rightarrow (iv)' \Rightarrow (iv)$ It is clear.

$(ii)' \Rightarrow (iii)$ Since $a\mathcal{R} \subseteq a^2\mathcal{R}$, we have $a = a^2t$ for some $t \in \mathcal{R}$. Then we get

$$a = aat = a(a^2t)t = a^3t^2 = a^2at^2 = a^2(a^2t)t^2 = a^4t^3 = \cdots = a^nt^{n-1}.$$

This means $a\mathcal{R} \subseteq a^n\mathcal{R}$. And obviously, $a^n\mathcal{R} \subseteq a\mathcal{R}$. So $a\mathcal{R} = a^n\mathcal{R}$ for any $n \geq 2$.

$(iv) \Rightarrow (i)$ The condition $a \in \mathcal{R}^{\{1,3e\}}$ implies that $p = 1 - eaa^{(1,3e)}e^{-1}$ is an idempotent, $pea = 0$ and $pe = e - eaa^{(1,3e)} = (pe)^*$. Since $a\mathcal{R} = a^n\mathcal{R}$, we have $a = a^nt$ for some $t \in \mathcal{R}$. Next we will prove that $p + eae^{-1} \in \mathcal{R}_r^{-1}$ and then, by Theorem 2.4, we get that a is right e-core invertible. Indeed, $(p + eae^{-1})(1 + ea^{n-1}ta^{(1,3e)}e^{-1} - ea^{n-1}te^{-1}) = p + eae^{-1} + ea^nta^{(1,3e)}e^{-1} - ea^nte^{-1} = p + eaa^{(1,3)}e^{-1} = 1$.

$(i) \Rightarrow (v)$ Suppose that x is a right e-core inverse of a and $n \geq 2$. Then, from

$$ax = a(ax^2) = a^2x^2 = \cdots = a^nx^n,$$

we get $(ea^nx^n)^* = (eax)^* = eax = ea^nx^n$. Moreover, it is easy to get $a^nx^na^n = axa^n = a^n$ and $a^n(x^n)^2 = axx^n = (ax^2)x^{n-1} = x^n$. Hence, by Lemma 2.1, we obtain that a^n is right e-core invertible and $(a^n)_r^{e,\circledast} = x^n = (a_r^{e,\circledast})^n$. Since (i) is equivalent to (iii), it follows that $a\mathcal{R} = a^n\mathcal{R}$ for any $n \geq 2$.

$(v) \Rightarrow (v)' \Rightarrow (vi)' \Rightarrow (vi)$ This is obvious.

$(vi) \Rightarrow (i)$ Let $x = a^{n-1}(a^n)_r^{e,\circledast}$ and $a = a^nt$ for some $t \in \mathcal{R}$. Firstly, we observe that $ax = a^n(a^n)_r^{e,\circledast}$, which gives $(eax)^* = [ea^n(a^n)_r^{e,\circledast}]^* = ea^n(a^n)_r^{e,\circledast} = eax$. Further, $axa = a^n(a^n)_r^{e,\circledast}a = a^n(a^n)_r^{e,\circledast}a^nt = a^nt = a$ and $ax^2 = a^n(a^n)_r^{e,\circledast}a^{n-1}(a^n)_r^{e,\circledast} = a^n(a^n)_r^{e,\circledast}a^nta^{n-2}(a^n)_r^{e,\circledast} = a^nta^{n-2}(a^n)_r^{e,\circledast} = a^{n-1}(a^n)_r^{e,\circledast} = x$. So, by Lemma 2.1, a is right e-core invertible and $a_r^{e,\circledast} = x = a^{n-1}(a^n)_r^{e,\circledast}$.

$(i) \Rightarrow (vii) \Rightarrow (vii)'$ Consequently, by previous proofs.

$(vii)' \Rightarrow (viii) \Rightarrow (viii)'$ It is obvious.

$(viii)' \Rightarrow (i)$ Let $x = a^{n-1}(a^n)^{(1,3e)}$ and $a = a^kt$ for some $t \in \mathcal{R}$. It is clear that $(eax)^* = [ea^n(a^n)^{(1,3e)}]^* = ea^n(a^n)^{(1,3e)} = eax$. Further, by $k > n$, $axa = a^n(a^n)^{(1,3e)}a = a^n(a^n)^{(1,3e)}a^kt = a^n(a^n)^{(1,3e)}a^na^{k-n}t = a^na^{k-n}t = a^kt = a$ and $ax^2 = a^n(a^n)^{(1,3e)}a^{n-1}(a^n)^{(1,3e)} = a^n(a^n)^{(1,3e)}a^kta^{n-2}(a^n)^{(1,3e)} = a^kta^{n-2}(a^n)^{(1,3e)} = a^{n-1}(a^n)^{(1,3e)} = x$. So, by Lemma 2.1, a is right e-core invertible and $a_r^{e,\circledast} = x = a^{n-1}(a^n)^{(1,3e)}$. □

By [16, Lemma 3.15], it is well known that $a \in \mathcal{R}^{\{1,3e\}}$ is equivalent to $\mathcal{R}a = \mathcal{R}a^*ea$. And obviously $\mathcal{R}a^*ea \subseteq \mathcal{R}a$, so $\mathcal{R}a = \mathcal{R}a^*ea$ is equivalent to $\mathcal{R}a \subseteq \mathcal{R}a^*ea$. Thus, by Theorem 3.1, we have the following result.

Proposition 3.2. *Let $a \in \mathcal{R}$ and $e \in \mathcal{R}$ be an invertible Hermitian element. Then the following statements are equivalent:*
(*i*) *a is right e-core invertible;*
(*ii*) $\mathcal{R}a = \mathcal{R}a^*ea$ *and* $a\mathcal{R} = a^2\mathcal{R}$;
(*iii*) $\mathcal{R}a \subseteq \mathcal{R}a^*ea$ *and* $a\mathcal{R} \subseteq a^2\mathcal{R}$;
(*iv*) $\mathcal{R}a = \mathcal{R}a^*ea$ *and* $a\mathcal{R} = a^n\mathcal{R}$ *for any* $n \geq 2$;
(*v*) $\mathcal{R}a \subseteq \mathcal{R}a^*ea$ *and* $a\mathcal{R} \subseteq a^n\mathcal{R}$ *for any* $n \geq 2$;
(*vi*) $\mathcal{R}a = \mathcal{R}a^*ea$ *and* $a\mathcal{R} = a^n\mathcal{R}$ *for some* $n \geq 2$;
(*vii*) $\mathcal{R}a \subseteq \mathcal{R}a^*ea$ *and* $a\mathcal{R} \subseteq a^n\mathcal{R}$ *for some* $n \geq 2$;
(*viii*) $\mathcal{R}a^n = \mathcal{R}(a^n)^*ea^n$ *and* $aR = a^k\mathcal{R}$ *for any* $n \geq 2$ *and* $k > n$;
(*ix*) $\mathcal{R}a^n \subseteq \mathcal{R}(a^n)^*ea^n$ *and* $aR \subseteq a^k\mathcal{R}$ *for any* $n \geq 2$ *and* $k > n$;
(*x*) $\mathcal{R}a^n = \mathcal{R}(a^n)^*ea^n$ *and* $aR = a^k\mathcal{R}$ *for some* $n \geq 2$ *and* $k > n$;
(*xi*) $\mathcal{R}a^n \subseteq \mathcal{R}(a^n)^*ea^n$ *and* $aR \subseteq a^k\mathcal{R}$ *for some* $n \geq 2$ *and* $k > n$.

In the following, we will give some characterizations for a right e-core invertible element by using $\mathcal{R}a \subseteq \mathcal{R}(a^*)^nea$, where $n \geq 2$. For $n = 2$, that is if $\mathcal{R}a \subseteq \mathcal{R}(a^*)^2ea$, we know $a = t(a^*)^2ea$ for some $t \in \mathcal{R}$, then $a^* = a^*ea^2t^*$. It gives that $t(a^*)^2 = ta^*a^* = ta^*(a^*ea^2t^*) = (ta^*a^*ea)at^* = a^2t^*$. This implies that $a = t(a^*)^2ea = a^2t^*ea$. Hence, $\mathcal{R}a \subseteq \mathcal{R}(a^*)^2ea$ gives that $\mathcal{R}a \subseteq \mathcal{R}a^*ea$ and $a\mathcal{R} \subseteq a^2\mathcal{R}$. This means that $\mathcal{R}a = \mathcal{R}(a^*)^2ea$ implies that a is right e-core invertible by Proposition 3.2.

Proposition 3.3. *Let $a \in \mathcal{R}$ and $e \in \mathcal{R}$ be an invertible Hermitian element and $n \geq 2$. Then the following statements are equivalent:*
(*i*) *a is right e-core invertible;*
(*ii*) $\mathcal{R}a = \mathcal{R}(a^*)^nea$;
(*iii*) $\mathcal{R}a \subseteq \mathcal{R}(a^*)^nea$;

Proof. (*ii*) ⇔ (*iii*) Obviously.

(*i*) ⇒ (*iii*) If a is right e-core invertible, by Proposition 3.2(iv), we know $\mathcal{R}a = \mathcal{R}a^*ea$ and $a\mathcal{R} = a^n\mathcal{R}$, so $a = a^nt$ for some $t \in \mathcal{R}$. Thus $a^* = t^*(a^*)^n$. The condition $\mathcal{R}a = \mathcal{R}a^*ea$ gives that $a = sa^*ea$ for some $s \in \mathcal{R}$, then $a = st^*(a^*)^nea$. This means that $\mathcal{R}a \subseteq \mathcal{R}(a^*)^nea$.

(*iii*) ⇒ (*i*) By assumption $\mathcal{R}a \subseteq \mathcal{R}(a^*)^nea$, we have $a = t(a^*)^nea$ for some $t \in \mathcal{R}$, and then $a^* = a^*ea^nt^*$. It gives that $t(a^*)^n = t(a^*)^{n-1}a^* = t(a^*)^{n-1}(a^*ea^nt^*) = [t(a^*)^nea]a^{n-1}t^* = a^nt^*$. This implies that $a = t(a^*)^nea = a^nt^*ea$. Hence, $\mathcal{R}a \subseteq \mathcal{R}(a^*)^nea$ implies that $\mathcal{R}a \subseteq \mathcal{R}a^*ea$ and $a\mathcal{R} \subseteq a^n\mathcal{R}$. Therefore, a is right e-core invertible by Proposition 3.2. □

Note that a^* is left (ea, a^*) invertible if and only if $\mathcal{R}ea = \mathcal{R}(a^*)^2ea$. Then we can obtain the following result.

Proposition 3.4. *Let $a \in \mathcal{R}$ and $e \in \mathcal{R}$ be an invertible Hermitian element. Then a is right (a, a^*e) invertible if and only if a^* is left (ea, a^*) invertible.*

Proof. If a is right (a, a^*e) invertible, by Definition 1.3, a is right e-core invertible, applying Proposition 3.3, we have $\mathcal{R}(a^*)^2ea = \mathcal{R}a$. Then $\mathcal{R}ea \subseteq \mathcal{R}a = \mathcal{R}(a^*)^2ea \subseteq \mathcal{R}ea$, i.e. $\mathcal{R}ea = \mathcal{R}(a^*)^2ea$, so a^* is left (ea, a^*) invertible.

Conversely, if a^* is left (ea, a^*) invertible, we know $\mathcal{R}ea = \mathcal{R}(a^*)^2ea$, which gives that $ea = t(a^*)^2ea$ for some $t \in \mathcal{R}$. Then $a = e^{-1}t(a^*)^2ea \in \mathcal{R}ea$ as e is invertible, which yields $\mathcal{R}a \subseteq \mathcal{R}ea \subseteq \mathcal{R}a$, i.e. $\mathcal{R}a = \mathcal{R}ea = \mathcal{R}(a^*)^2ea$, so a is right (a, a^*e) invertible. □

Remark 3.5. *Notice that if $aR = a^2R$, we have a^* is left (ea, a^*) invertible if and only if $(a^*)^n$ is left (ea, a^*) invertible. Indeed, a^* is left (ea, a^*) invertible* $\overset{Proposition 3.4}{\Longleftrightarrow}$ *a is right (a, a^*e) invertible* $\overset{Theorem 3.1}{\Longleftrightarrow}$ *a^n is right (a, a^*e) invertible* $\overset{Proposition 3.4}{\Longleftrightarrow}$ *$(a^n)^*$ is left (ea, a^*) invertible.*

4. The related generalized core inverses

In this section, some new characterizations of (generalized) e-core inverses are given. Through these characterizations, we can clearly find the relationship between these generalized inverses. In what follows, we assume that $e \in \mathcal{R}$ is an invertible Hermitian element.

Theorem 4.1. *Let $a, e \in \mathcal{R}$ and $k \geq 1$. Then the following statements are equivalent:*
(i) *a is e-core invertible;*
(ii) *there exists $x \in \mathcal{R}$ such that $xa^2 = a$, $x^k = ax^{k+1}$ and $(ea^kx^k)^* = ea^kx^k$;*
(iii) *there exists $x \in \mathcal{R}$ such that $xa^2 = a$, $x^k = ax^{k+1}$ and $(eax)^* = eax$.*

Proof. If $k = 1$, by [10, Theorem 2.1], it is clear that (i) ⇔ (ii) ⇔ (iii). Let us assume that $k \geq 2$ in the rest of the proof.

(i) ⇒ (ii) and (iii) If a is e-core invertible, let $x = a^{e,\circledast}$. By [10, Theorem 2.1], we obtain $xa^2 = a$, $x = ax^2$ and $(eax)^* = eax$. For $k \geq 2$, it is easy to get $ax^{k+1} = ax^2x^{k-1} = x^k$. Notice that $ax = a(ax^2) = a^2x^2 = a^2(ax^2)x = \cdots = a^kx^k$, then $(ea^kx^k)^* = (eax)^* = eax = ea^kx^k$.

(ii) ⇒ (i) Suppose that there exists $x \in \mathcal{R}$ such that $xa^2 = a$, $x^k = ax^{k+1}$ and $(ea^kx^k)^* = ea^kx^k$. Let $z = a^{k-1}x^k$. Then $a = xa^2 = xaa = x(xa^2)a = x^2a^3 = \cdots = x^{k-1}a^k = x^ka^{k+1} = (ax^{k+1})a^{k+1} = a(x^k)xa^{k+1} = a(ax^{k+1})xa^{k+1} = a^2(x^k)x^2a^{k+1} = a^2(ax^{k+1})x^2a^{k+1} = \cdots = a^{k-1}x^{k+(k-1)}a^{k+1} = (a^{k-1}x^k)(x^{k-1}a^k)a = zaa = za^2$, and $z = a^{k-1}x^k = a^{k-1}(ax^{k+1}) = a^kx^{k+1} = a^kx(ax^{k+1}) = a^kx(xa^2)x^{k+1} = a^kx^2a^2x^{k+1} = a^kx^2(xa^2)ax^{k+1} = a^kx^3a^3x^{k+1} = \cdots = a^kx^ka^kx^{k+1} = a^kx^ka^{k-1}(ax^{k+1}) = a^kx^ka^{k-1}x^k = az^2$. Note that $az = a^kx^k$, thus $(eaz)^* = eaz$. Hence $a^{e,\circledast} = z = a^{k-1}x^k$.

(iii) ⇒ (i) Note that $a = xa^2 = x(xa^2)a = x^2a^3 = x^2(xa^2)a^2 = x^3a^4 = \cdots = x^ka^{k+1}$. Write $z = xax$. Then $az = axax = ax(x^ka^{k+1})x = (ax^{k+1})a^{k+1}x = x^ka^{k+1}x = ax$, which gives that $(eaz)^* = eaz$. It is easy to get $za^2 = xa(xa^2) = xa^2 = a$. Moreover, $az^2 = axz = ax(xax) = ax^2(x^ka^{k+1})x = (ax^{k+1})xa^{k+1}x = x^{k+1}a^{k+1}x = xax = z$. This implies that $a^{e,\circledast} = xax$. □

Proposition 4.2. *Let $a, e, x \in \mathcal{R}$ and $k \geq 1$. Then the following statements are equivalent:*
(i) *x is the e-core inverse of a;*
(ii) *$xa^2 = a$, $xax = x$, $x^k = ax^{k+1}$ and $(eax)^* = eax$;*
(iii) *$xa^2 = a$, $x^{k+1}a^{k+1}x = x$, $x^k = x^ka^{k+1}x^{k+1}$ and $(ex^ka^{k+1}x)^* = ex^ka^{k+1}x$.*

Proof. (i) ⇒ (ii) If x is the e-core inverse of a, we know $xa^2 = a$, $x = ax^2$ and $(eax)^* = eax$, then $x^k = xx^{k-1} = ax^{k+1}$, and $ax = a(ax^2) = a^2x^2$. Thus $xax = x(a^2x^2) = (xa^2)x^2 = ax^2 = x$.

(ii) ⇒ (i) It suffices to prove $ax^2 = x$. Indeed, by $xa^2 = a$, we get $xa = x(xa^2) = x^2a^2 = \cdots = x^ka^k$. Thus, $x = xax = x^ka^kx = ax^{k+1}a^kx = ax(x^ka^k)x = axxax = ax^2$.

(ii) ⇔ (iii) Since $a = xa^2$ implies $a = x^ka^{k+1}$, this equivalence is obvious. □

Recall in [10, Theorem 2.4] that a is weighted-EP with respect to (e, e) if and only if there exists $x \in \mathcal{R}$ such that $(exa)^* = exa, xa^2 = a, ax^2 = x$, which are also equivalent to that $(eax)^* = eax$, $a^2x = a$, $x^2a = x$. In the following result, we will change the condition $(eax)^* = eax$ in Theorem 4.1 into $(exa)^* = exa$. It is interesting that a is weighted-EP with respect to (e, e).

Theorem 4.3. *Let $a, e \in \mathcal{R}$ and $k \geq 1$. Then the following statements are equivalent:*
(i) *a is weighted-EP with respect to (e, e);*
(ii) *there exists $x \in \mathcal{R}$ such that $xa^2 = a$, $x^k = ax^{k+1}$ and $(exa)^* = exa$;*
(iii) *there exists $x \in \mathcal{R}$ such that $xa^2 = a$, $x^k = x^ka^{k+1}x^{k+1}$ and $(ex^{k+1}a^{k+1})^* = ex^{k+1}a^{k+1}$;*
(iv) *there exists $x \in \mathcal{R}$ such that $a^2x = a$, $x^k = x^{k+1}a$ and $(eax)^* = eax$;*
(v) *there exists $x \in \mathcal{R}$ such that $a^2x = a$, $x^k = x^{k+1}a^{k+1}x^k$ and $(ea^{k+1}x^{k+1})^* = ea^{k+1}x^{k+1}$.*

Proof. (i) ⇒ (ii) If a is weighted-EP with respect to (e, e), we know that $(exa)^* = exa, xa^2 = a, ax^2 = x$ for some $x \in \mathcal{R}$. Thus $x^k = xx^{k-1} = ax^{k+1}$.

(ii) ⇒ (i) Note that $a = xa^2 = x(xa^2)a = x^2a^3 = \cdots = x^ka^{k+1}$. Write $z = xax$. Then $az = axax = ax(x^ka^{k+1})x = (ax^{k+1})a^{k+1}x = x^ka^{k+1}x = ax$, and $za = xaxa = xax(x^ka^{k+1}) = x(ax^{k+1})a^{k+1} = x(x^ka^{k+1}) = xa$, which imply that $(eza)^* = (exa)^* = exa = eaz$. It is easy to get $za^2 = xa(xa^2) = xa^2 = a$. Moreover,

$az^2 = axz = ax(xax) = ax^2(x^ka^{k+1})x = (ax^{k+1})xa^{k+1}x = x^{k+1}a^{k+1}x = xax = z$. Hence, a is weighted-EP with respect to (e, e).

$(ii) \Leftrightarrow (iii)$ Since $a = xa^2$ implies $a = x^ka^{k+1}$, it is clear.

$(i) \Rightarrow (iv)$ Suppose that a is weighted-EP with respect to (e, e), then we have $(eax)^* = eax, a^2x = a, x^2a = x$ for some $x \in \mathcal{R}$. Thus $x^{k+1}a = x^{k-1}(x^2a) = x^k$.

$(iv) \Rightarrow (i)$ Note that $a = a^2x = a(a^2x)x = a^3x^2 = \cdots = a^{k+1}x^k$. Let $z = xax$. Then $az = axax = (a^{k+1}x^k)xax = a^{k+1}(x^{k+1}a)x = a^{k+1}x^{k+1} = ax$, and $za = xaxa = x(a^{k+1}x^k)xa = xa^{k+1}(x^{k+1}a) = xa^{k+1}x^k = xa$. So $(eaz)^* = (eax)^* = eax = eaz$, and $a^2z = a^2x = a$. Furthermore, $z^2a = zxa = (xax)xa = x(a^{k+1}x^k)x^2a = xa^{k+1}x(x^{k+1}a) = xa^{k+1}x^{k+1} = xax = z$. Hence, a is weighted-EP with respect to (e, e).

$(iv) \Leftrightarrow (v)$ The condition $a = a^2x$ implies $a = a^{k+1}x^k$, thus the equivalence is obvious. □

In [16], Zhu and Wang introduced the concept of pseudo e-core inverse in $*$-rings.

Definition 4.4. [16] *Let $a, e \in \mathcal{R}$. The pseudo e-core inverse of a, denoted by $a^{e,\textcircled{D}}$, is the unique solution to system*

$$xa^{k+1} = a^k \text{ for some } k \geq 1,\ ax^2 = x \text{ and } (eax)^* = eax.$$

The authors introduced the one-sided pseudo e-core inverse in [13, Remark 4.12], here we also present the definition.

Definition 4.5. *[13] Let $a, e \in \mathcal{R}$. Then a is called right pseudo e-core invertible if there exist $x \in \mathcal{R}$ and some positive integer k such that $axa^k = a^k$, $x = ax^2$ and $(eax)^* = eax$.*

We use the symbol $a_r^{e,\textcircled{D}}$ to denote the right pseudo e-core inverse of a, if a is right pseudo e-core invertible.

Next we will characterize pseudo e-core invertible elements.

Theorem 4.6. *Let $a, e \in \mathcal{R}$ and $k \geq 1$. Then the following are equivalent:*

(i) a is pseudo e-core invertible;

(ii) there exists $x \in \mathcal{R}$ such that $xa^{k+1} = a^k$, $ax^{k+1} = x^k$ and $(ea^kx^k)^* = ea^kx^k$;

(iii) there exists $x \in \mathcal{R}$ such that $a^kx^{k+1}a^{k+1} = a^k$, $ax^2 = x$ and $(ea^{k+1}x^{k+1})^* = ea^{k+1}x^{k+1}$.

Proof. $(i) \Rightarrow (ii)$ By Definition 4.4, there exists $x \in \mathcal{R}$ such that $xa^{k+1} = a^k$, $ax^2 = x$ and $(eax)^* = eax$, which give that $ax^{k+1} = ax^2x^{k-1} = x^k$, and $ax = a(ax^2) = a^2x^2 = a^2(ax^2)x = a^3x^3 = \cdots = a^kx^k$. So $(ea^kx^k)^* = (eax)^* = eax = ea^kx^k$.

$(ii) \Rightarrow (i)$ By the assumption, let $z = a^{k-1}x^k$. Then $az = a^kx^k$, and $(eaz)^* = (ea^kx^k)^* = ea^kx^k = eaz$. Notice that $a^k = xa^{k+1} = x(xa^{k+1})a = x^2a^{k+2} = \cdots = x^{k-1}a^{2k-1} = x^ka^{2k}$, and $x^k = ax^{k+1} = a(ax^{k+1})x = a^2x^{k+2} = \cdots = a^{k-1}x^{2k-1} = a^kx^{2k}$, which imply that $z = a^{k-1}x^k = a^{k-1}(ax^{k+1}) = a^kx^{k+1} = \cdots = a^{2k}x^{2k+1} = a^k(x^ka^{2k})x^{2k+1} = a^kx^ka^k(a^kx^{2k})x = a^kx^ka^kx^{k+1} = a^kx^ka^{k-1}x^k = az^2$, and $a^k = x^ka^{2k} = (a^{k-1}x^{2k-1})a^{2k} = (a^{k-1}x^k)(x^{k-1}a^{2k-1})a = za^{k+1}$. These yield that $z = a^{k-1}x^k$ is a pseudo e-core inverse of a, and a is pseudo e-core invertible.

$(ii) \Leftrightarrow (iii)$ The equality $x = ax^2$ gives $x = a^kx^{k+1}$ and so the rest is clear. □

In the following result, we will reveal the relationship between right pseudo e-core inverses and right e-core inverses.

Theorem 4.7. *Let $a, e \in \mathcal{R}$. Then the following statements are equivalent:*

(i) a is right pseudo e-core invertible;

(ii) a^k is right e-core invertible for some positive integer k.

Proof. $(i) \Rightarrow (ii)$ If a is right pseudo e-core invertible, then we can check that $z = (a_r^{e,\textcircled{D}})^k$ is a right e-core inverse of a^k. Indeed, the condition $a_r^{e,\textcircled{D}} = a(a_r^{e,\textcircled{D}})^2$ yields $a_r^{e,\textcircled{D}} = a^{k-1}(a_r^{e,\textcircled{D}})^k$. Thus $a^kz = a^k(a_r^{e,\textcircled{D}})^k = a(a^{k-1}(a_r^{e,\textcircled{D}})^k) = aa_r^{e,\textcircled{D}}$. Therefore, $(ea^kz)^* = (eaa_r^{e,\textcircled{D}})^* = eaa_r^{e,\textcircled{D}} = ea^kz$, $a^kza^k = aa_r^{e,\textcircled{D}}a^k = a^k$ and $a^kz^2 = aa_r^{e,\textcircled{D}}z = aa_r^{e,\textcircled{D}}(a_r^{e,\textcircled{D}})^k = (a(a_r^{e,\textcircled{D}})^2)(a_r^{e,\textcircled{D}})^{k-1} = (a_r^{e,\textcircled{D}})^k = z$.

$(ii) \Rightarrow (i)$ If a^k is right e-core invertible for some positive integer k, then we can check that $y = a^{k-1}(a^k)_r^{e,\textcircled{\#}}$ is a right pseudo e-core inverse of a. Indeed, $ay = a^k(a^k)_r^{e,\textcircled{\#}}$, $aya^k = a^k(a^k)_r^{e,\textcircled{\#}}a^k = a^k$, $ay^2 = a^k(a^k)_r^{e,\textcircled{\#}}y = a^k(a^k)_r^{e,\textcircled{\#}}a^{k-1}(a^k)_r^{e,\textcircled{\#}} = a^k(a^k)_r^{e,\textcircled{\#}}a^{k-1}\{a^k[(a^k)_r^{e,\textcircled{\#}}]^2\} = a^k(a^k)_r^{e,\textcircled{\#}}a^ka^{k-1}[(a^k)_r^{e,\textcircled{\#}}]^2 = a^ka^{k-1}[(a^k)_r^{e,\textcircled{\#}}]^2 = a^{k-1}a^k[(a^k)_r^{e,\textcircled{\#}}]^2 = a^{k-1}(a^k)_r^{e,\textcircled{\#}} = y$, and $(eay)^* = (ea^kx)^* = ea^kx = eay$. □

Next we characterize right pseudo e-core invertible elements by using Theorem 4.7.

Theorem 4.8. *Let $a, e \in \mathcal{R}$. Then the following statements are equivalent:*
(i) *a is right pseudo e-core invertible;*
(ii) *$a^k \in \mathcal{R}^{\{1,3e\}}$ and $a^k\mathcal{R} = a^{k+1}\mathcal{R}$ for some positive integer k;*
(iii) *$a^k \in \mathcal{R}^{\{1,3e\}}$ and $a^k\mathcal{R} \subseteq a^{k+1}\mathcal{R}$ for some positive integer k;*
(iv) *$\mathcal{R}a^k = \mathcal{R}(a^k)^*ea^k$ and $a^k\mathcal{R} = a^{k+1}\mathcal{R}$ for some positive integer k;*
(v) *$\mathcal{R}a^k \subseteq \mathcal{R}(a^k)^*ea^k$ and $a^k\mathcal{R} \subseteq a^{k+1}\mathcal{R}$ for some positive integer k;*
(vi) *$\mathcal{R}a^k = \mathcal{R}(a^*)^{k+1}ea^k$ for some positive integer k;*
(vii) *$\mathcal{R}a^k \subseteq \mathcal{R}(a^*)^{k+1}ea^k$ for some positive integer k.*

Proof. a is right pseudo e-core invertible $\overset{Theorem 4.7}{\Longleftrightarrow}$ a^k is right e-core invertible for some positive integer k $\overset{Theorem 3.1}{\Longleftrightarrow}$ $a^k \in \mathcal{R}^{\{1,3e\}}$ and $a^k\mathcal{R} = a^{k+1}\mathcal{R}$ for some positive integer k $\overset{Theorem 3.1}{\Longleftrightarrow}$ (iii) $\overset{Proposition 3.2}{\Longleftrightarrow}$ (iv) $\overset{Proposition 3.2}{\Longleftrightarrow}$ (v) $\overset{Proposition 3.3}{\Longleftrightarrow}$ (vi) $\overset{Proposition 3.3}{\Longleftrightarrow}$ (vii). □

Finally, the matrix representations of right pseudo e-core invertible element and its right pseudo e-core inverse are presented in the following theorem.

Theorem 4.9. *Let $a, e \in \mathcal{R}$. Then the following statements are equivalent:*
(i) *a is right pseudo e-core invertible and $x \in \mathcal{R}$ is a right pseudo e-core inverse of a;*
(ii) *there exists an idempotent $q \in \mathcal{R}$ such that $(eq)^* = eq$ and*

$$a = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}_q, \qquad x = \begin{bmatrix} x_1 & x_2 \\ 0 & 0 \end{bmatrix}_q, \tag{2}$$

where a_1 is right invertible in $q\mathcal{R}q$, $x_1 = (a_1)_r^{-1}$, $a_1x_2 = 0$, $a_3x_1 = 0$, $a_3x_2 = 0$ and $qa^k = a^k$ for some $k \geq 1$;
(iii) *there exists an idempotent $p \in \mathcal{R}$ such that $(ep)^* = ep$ and*

$$a = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}_p, \qquad x = \begin{bmatrix} 0 & 0 \\ x_1 & x_2 \end{bmatrix}_p, \tag{3}$$

where a_4 is right invertible in $(1-p)\mathcal{R}(1-p)$, $x_2 = (a_4)_r^{-1}$, $a_2x_1 = 0$, $a_2x_2 = 0$, $a_4x_1 = 0$ and $pa^k = 0$ for some $k \geq 1$.

Proof. (i) $\Rightarrow$ (ii) If a is right pseudo e-core invertible and $x \in \mathcal{R}$ is a right pseudo e-core inverse of a, by Definition 4.5, we have $axa^k = a^k$, $x = ax^2$ and $(eax)^* = eax$ for some $k \geq 1$. Note that $ax = a(ax^2) = a^2x^2 = \cdots = a^kx^k$, which gives $axax = ax(a^kx^k) = (axa^k)x^k = a^kx^k = ax$. For $q = ax$, we get $q^2 = axax = ax = q$, $(eq)^* = (eax)^* = eax = eq$, $qa^k = a^k$ and $qx = x$ implying (3). Since

$$\begin{bmatrix} a_1x_1 & a_1x_2 \\ a_3x_1 & a_3x_2 \end{bmatrix}_q = ax = q = \begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix}_q,$$

the rest is clear.

(ii) $\Rightarrow$ (i) Because $ax = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}_q \begin{bmatrix} x_1 & x_2 \\ 0 & 0 \end{bmatrix}_q = \begin{bmatrix} a_1x_1 & a_1x_2 \\ a_3x_1 & a_3x_2 \end{bmatrix}_q = \begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix}_q = q$, we can prove this implication by elementary computations.

(i) $\Leftrightarrow$ (iii) This equivalence follows similarly as (i) $\Leftrightarrow$ (ii) for $p = 1 - ax$. □

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