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On variants of θ-Menger spaces

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Abstract. In this paper we further study θ-Menger, θ-almost Menger and θ-weakly Menger properties [13] and investigate their relationships with other selective covering properties. We prove that in extremally disconnected semi-regular spaces, the properties viz. Menger, semi-Menger, α-Menger, θ-Menger, midly Menger are equivalent; and every finite power of a space X has the selection property $S_{fin}(\theta\text{-}\mathcal{O}, \theta\text{-}\mathcal{O})$ if and only if X has the property $S_{fin}(\theta\text{-}\Omega, \theta\text{-}\Omega)$.

1. Introduction and preliminaries

The Menger property [19] is a classical covering property: a space X is said to have the Menger property if for each sequence $(\mathcal{A}_k : k \in \mathbb{N})$ of open covers of X there is a sequence $(\mathcal{B}_k : k \in \mathbb{N})$ such that for each k, $\mathcal{B}_k$ is a finite subset of $\mathcal{A}_k$ and $X = \bigcup_{k\in\mathbb{N}} \cup\mathcal{B}_k$. After that several weak variants of the Menger covering property occurred in the mathematical literature such as almost Menger and weakly Menger ([2, 11]). A new variant of the Menger property is also formed when the open cover is replaced by a cover of generalized open sets.

A subset A of a space X is said to be:

- θ-open if for each element $x \in A$, there is an open subset B of space X such that $x \in B \subset Cl(B) \subset A$ [32];
- α-open if $A \subset Int(Cl(Int(A)))$, or equivalently, if there is an open subset B of space X such that $B \subset A \subset Int(Cl(B))$ [22];
- semi-open if there exists an open subset B of space X such that $B \subset A \subset Cl(B)$, or equivalently, if $A \subset Cl(Int(A))$ [18] and $SO(X)$ denotes the set of all semi-open sets.

Clearly, we have the following implications:

$$\text{clopen} \Rightarrow \theta\text{-open} \Rightarrow \text{open} \Rightarrow \alpha\text{-open} \Rightarrow \text{semi-open}.$$

Using the semi-open sets Sabah et.al [26] defined the semi-Menger property which is stronger than the Menger property and Kočinac [12] introduced mildy Menger spaces using the clopen sets. Recently, Kočinac [13] introduced and investigated the covering properties θ-Menger and α-Menger using θ-open sets and α-open sets, respectively.

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A space X is said to have semi-Menger [26] (resp., α-Menger [13], θ-Menger [13], mildly-Menger [12]) if for each sequence $(\mathcal{A}_k : k \in \mathbb{N})$ of semi-open (resp., α-open, θ-open, clopen) covers of X there exists a sequence $(\mathcal{B}_k : k \in \mathbb{N})$, where $\mathcal{B}_k$ is a finite subset of $\mathcal{A}_k$ for each k, such that $X = \bigcup_{k\in\mathbb{N}} \cup\mathcal{B}_k$.

The above properties are also written in a form of selection principle. Let $\mathcal{A}$ and $\mathcal{B}$ be the collections of subsets of a space X. Then a space X satisfies the selection principle : $S_{fin}(\mathcal{A}, \mathcal{B})$ (resp., $U_{fin}(\mathcal{A}, \mathcal{B})$) if for each sequence $(\mathcal{A}_k : k \in \mathbb{N})$ in $\mathcal{A}$ there exists a sequence $(\mathcal{B}_k : k \in \mathbb{N})$, where for each k, $\mathcal{B}_k$ is a finite subset of $\mathcal{A}_k$ such that $\bigcup_{k\in\mathbb{N}} \mathcal{B}_k \in \mathcal{B}$ (resp., $\{\cup\mathcal{B}_k : k = 1, 2, 3, ...\}$ is in $\mathcal{B}$) (see, [14, 15]). Let $\mathcal{O}$, $\mathcal{CO}$, θ-$\mathcal{O}$, α-$\mathcal{O}$, s-$\mathcal{O}$, denote the collection of all open, clopen, θ-open, α-open, semi-open covers of a space X, respectively. Then the Menger, mildly Menger, θ-Menger, α-Menger, semi-Menger, property of X is the property $S_{fin}(\mathcal{O}, \mathcal{O})$, $S_{fin}(\mathcal{CO}, \mathcal{CO})$, $S_{fin}(\theta\text{-}\mathcal{O}, \theta\text{-}\mathcal{O})$, $S_{fin}(\alpha\text{-}\mathcal{O}, \alpha\text{-}\mathcal{O})$, $S_{fin}(\text{s-}\mathcal{O}, \text{s-}\mathcal{O})$, respectively.

In 1968, Velichko [32] introduced θ-closure operator to study H-closed spaces. For a subset A of a space X, the θ-closure of A denoted by $Cl_\theta(A)$ and defined as $\mathrm{Cl}_\theta(A) = \{x \in X :$ for each neighbourhood U of x, $Cl(U) \cap A \neq \phi\}$. A subset A of space X is called θ-closed if $Cl_\theta(A) = A$ and A is θ-open if its complement is θ-closed. Many papers have been published on θ-closure operator (see [3, 6, 7, 16, 21]). Recently, using the θ-closure operator Kočinac [13, Remark 3.6] generalized almost Menger and weakly Menger properties namely introduced θ-almost Menger and θ-weakly Menger properties, respectively. In this paper, we further continue the study of θ-Menger, θ-almost Menger and θ-weakly Menger properties.

2. θ-Menger spaces

For a topological space X, we denote:

1. Ω: the collection of ω covers of X: An open cover $\mathcal{A}$ of X is a ω-cover if no element of $\mathcal{A}$ contains X, and each finite subset of X is contained in some element of $\mathcal{A}$.
2. θ-Ω: the collection of θ-ω-covers of X: A cover $\mathcal{A}$ of X is a θ-ω-cover if it is a θ-open cover of X such that no element of $\mathcal{A}$ contains X, and each finite subset of X is a subset of some element of $\mathcal{A}$.
3. Γ: the collection of γ-covers of X: An infinite open cover $\mathcal{A}$ of X is γ-cover if, for each x in X, the set $\{U \in \mathcal{A} : x \notin U\}$ is finite.
4. θ-Γ: the collection of θ-γ-covers of X: An infinite θ-open cover $\mathcal{A}$ of X is θ-γ-cover if, for each x in X, the set $\{U \in \mathcal{A} : x \notin U\}$ is finite.

Theorem 2.1. *For a space X the following statements are equivalent:*

1. *X has the θ-Menger property;*
2. *X satisfies $S_{fin}(\theta\text{-}\Omega, \theta\text{-}\mathcal{O})$.*

Proof. 1 ⇒ 2 It follows from the fact that each θ-ω-cover of X is a θ-open cover of X.

2 ⇒ 1 Let $(\mathcal{A}_k : k \in \mathbb{N})$ be a sequence of θ-open covers of X. Let $\mathbb{N} = Y_1 \cup Y_2 \cup ... \cup Y_m \cup ...$ be a partition of $\mathbb{N}$ into countably many pairwise disjoint infinite subsets. For each k, let $\mathcal{B}_k$ contained all sets of the form $A_{k_1} \cup A_{k_2} \cup ... \cup A_{k_n}$, $k_1 \leq ... \leq k_n$, $k_i \in Y_k$, $A_{k_i} \in \mathcal{A}_k$, $i \leq n$, $n \in \mathbb{N}$. Then for each k, $\mathcal{B}_k$ is a θ-ω-cover of X. Applying $S_{fin}(\theta\text{-}\Omega, \theta\text{-}\mathcal{O})$ on the sequence $(\mathcal{B}_k : k \in \mathbb{N})$, there is a sequence $(\mathcal{C}_k : k \in \mathbb{N})$, where for each k, $\mathcal{C}_k$ is a finite subset of $\mathcal{B}_k$ such that $X = \bigcup_{k\in\mathbb{N}} \cup\{C : C \in \mathcal{C}_k\}$. Suppose $\mathcal{C}_k = \{C_k^1, ...C_k^{m_k}\}$, then from the construction, each $C_k^i = A_k^{k_{i_1}} \cup ..... \cup A_k^{k_{i_n}}$. Therefore for each k, we can construct a finite subset $\mathcal{A}'_k$ of $\mathcal{A}_k$ such that $\cup\mathcal{C}_k \subseteq \cup\mathcal{A}'_k$. Hence X has the θ-Menger property. □

Sheepers et al. ([31, Theorem 3.9] proved the following equivalence.

Theorem 2.2. ([31]) *For a space X the following statements are equivalent:*

1. *Every finite power of X has property $S_{fin}(\mathcal{O}, \mathcal{O})$;*
2. *X has property $S_{fin}(\Omega, \Omega)$.*

We prove the similar result for θ-Menger spaces in Theorem 2.7. In the following example we also observe that square of θ-Menger space need not be θ-Menger.

Example 2.3. There exists a θ-Menger space X such that X^2 is not θ-Menger.

Proof. Let $i : S \to \mathbb{R}$ be the identity map from the Sorgenfrey line S to the real line $\mathbb{R}$. If $A \subset \mathbb{R}$, then denote $A_S = i^{-1}(A)$. In [17], Lelek proved that L_S has the Menger property for each Lusin set L in $\mathbb{R}$. He also stated that if $(L \times L) \cap \{(a, b) : a + b = 0\}$ is an uncountable set, then $L_S \times L_S$ does not have the Menger property. Then L_S is a θ-Menger space but ${L_S}^2$ is not θ-Menger, $L_S \times L_S$ being a regular space and in regular spaces the θ-Menger property coincides with the Menger property. □

Recall that a space X is called extremally disconnected if the closure of each open set in X is open [25].

Lemma 2.4. *Let X be an extremally disconnected space and A be a θ-open set in the product space X^k, $k \in \mathbb{N}$, then for each $(a_1, a_2, ...a_k) \in A$, there exists θ-open set U_i containing a_i such that $(a_1, a_2, ..., a_k) \in \prod_{i=1}^{k} U_i \subseteq A$.*

Proof. Let A be a θ-open set of X^k, $n \in \mathbb{N}$. Then for each $(a_1, a_2, ...a_k) \in A$ there exists open set U_i containing a_i such that $(a_1, a_2, ..., a_k) \in \prod_{i=1}^{k} U_i \subseteq \prod_{i=1}^{k} Cl(U_i) \subseteq A$, moreover $Cl(U_i)$ is θ-open, X being an extremally disconnected. □

Proposition 2.5. *Let X be an extremally disconnected space. For each θ-ω-cover $\mathcal{A}$ of X^k, $k \in \mathbb{N}$, there exists a θ-ω-cover $\mathcal{B}$ of X such that the θ-open cover $\{B^k : B \in \mathcal{B}\}$ of X^k refines $\mathcal{A}$.*

Proof. Let $\mathcal{A}$ be a θ-ω-cover of X^k. Let F be a finite subset of X, thus F^k is a finite subset of X^k. Then there is a θ-open set $A \in \mathcal{A}$ such that $F^k \subset A$. Since X is extremally disconnected, from Lemma 2.4, for each $(x_1, ..., x_k,) \in F^k$, there is a θ-open set A_{x_i} containing x_i such that $(x_1, ..., x_k) \in \prod_{i=1}^{k} A_{x_i} \subset A$. For each $x \in F$, consider A_x is the intersection of all A_{x_i} containing x. Let $B_F = \cup_{x \in F} A_x$. Then B_F is a θ-open set of X containing F, thus $F^k \subset B_F^k \subset A$. Put $\mathcal{B} = \{B_F : F$ is a finite subset of $X\}$. Then $\mathcal{B}$ is a required θ-ω-cover of X such that the θ-open cover $\{B^k : B \in \mathcal{B}\}$ refines $\mathcal{A}$. □

Theorem 2.6. *Let X be an extremally disconnected space. If X has the property $S_{fin}(\theta\text{-}\Omega, \theta\text{-}\Omega)$, then for each $n \in \mathbb{N}$, X^n also has this property.*

Proof. Let $(\mathcal{A}_k, : k \in \mathbb{N})$ be a sequence of θ-ω-covers of X^n. Then by Proposition 2.5, for each k, there exists a θ-ω-cover $\mathcal{B}_k$ of X such that $\{B^n : B \in \mathcal{B}_k\}$ refines $\mathcal{A}_k$. Now apply the condition $S_{fin}(\theta\text{-}\Omega, \theta\text{-}\Omega)$ of X on the sequence $(\mathcal{B}_k : k \in \mathbb{N})$, for each k, there exists a finite subset C_k of $\mathcal{B}_k$ such that $\bigcup_{k=1}^{\infty} C_k$ forms θ-ω-cover of X. Since for each k, $\{B^n : B \in \mathcal{B}_k\}$ refines $\mathcal{A}_k$, for each $C \in C_k$ there is a $A \in \mathcal{A}_k$ such that $C^n \subset A$. Hence for each k, we can find a finite subset $\mathcal{A}'_k$, of $\mathcal{A}_k$ such that $\bigcup_{k=1}^{\infty} \mathcal{A}'_k$ forms a θ-ω-cover of X^n. □

Theorem 2.7. *For an extremally disconnected space X the following statements are equivalent:*

1. *Every finite power of X has the property $S_{fin}(\theta\text{-}\mathcal{O}, \theta\text{-}\mathcal{O})$;*
2. *X has the property $S_{fin}(\theta\text{-}\Omega, \theta\text{-}\Omega)$.*

Proof. (1) ⇒ (2) Already done by Kočinac [13, Theorem 3.12].

(2) ⇒ (1) The result follows directly from Theorem 2.6 and Theorem 2.1. □

In [29] Scheepers proved that for a Lindelöf space, $S_{fin}(\mathcal{O}, \mathcal{O}) = U_{fin}(\Gamma, \mathcal{O})$. In the following theorem, we show the similar result for θ-Lindelöf spaces. First, recall that a space X is called θ-compact (resp., θ-Lindelöf) if each θ-open cover of X has finite (resp., countable) subcover.

Theorem 2.8. *For a θ-Lindelöf space X, $S_{fin}(\theta\text{-}\mathcal{O}, \theta\text{-}\mathcal{O})$ if and only if $U_{fin}(\theta\text{-}\Gamma, \theta\text{-}\mathcal{O})$.*

Proof. The forward part is obvious. For the converse part, assume that X satisfies $U_{fin}(\theta\text{-}\Gamma, \theta\text{-}\mathcal{O})$. Let $(\mathcal{A}_k : k \in \mathbb{N})$ be a sequence of θ-open covers of X. We may assume that X is not θ-compact and for each $k \in \mathbb{N}$, $\mathcal{A}_k$ is countably infinite with no finite subset which covers X. For each k, let $\mathcal{A}_k = (A_k^n : n = 1, 2, 3, ...)$. Thus $\mathcal{B}_k = \{B_m : B_m = \bigcup_{n=1}^{m} A_k^n, m \in \mathbb{N}\}$ forms θ-γ-cover of X. Since X satisfying $U_{fin}(\theta\text{-}\Gamma, \theta\text{-}\mathcal{O})$, choose a finite set C_k of $\mathcal{B}_k$, such that $\{\cup C_k : k \in \mathbb{N}\}$ is a θ-open cover of X. Then for each k, disassembling the members of C_k. Thus for each k, we can find a finite subset $\mathcal{A}'_k$ of $\mathcal{A}_k$ such that $\bigcup_{k=1}^{\infty} \mathcal{A}'_k$ forms θ-open cover of X. □

Definition 2.9. ([13, Definition 5.2]) A space X is called nearly Menger if for each sequence $(\mathcal{A}_k : k \in \mathbb{N})$ of open covers of X there is a sequence $(\mathcal{B}_k : k \in \mathbb{N})$ such that for each $\mathcal{B}_k$ is a finite subset of $\mathcal{A}_k$ and $X = \bigcup_{k\in\mathbb{N}} \bigcup\{Int(Cl(B)) : B \in \mathcal{B}_k\}$.

Each nearly-Menger space is almost Menger and Kočinac [13, Theorem 3.5] showed that an almost Menger space is θ-Menger. Thus we have the following implications:

$$\text{semi-Menger} \Rightarrow \alpha\text{-Menger} \Rightarrow \text{Menger} \Rightarrow \text{nearly Menger} \Rightarrow \theta\text{-Menger} \Rightarrow \text{mildy Menger}.$$

But the reverse implications do not hold in general. In the following example we show that a θ-Menger space need not be nearly-Menger. For the details about other reverse implications see [13, 24]

Example 2.10. Let $X = \{a, b, c_i, a_{ij}, b_{ij} : i \in A, j \in \mathbb{N}\}$, where $A = [0, \Omega)$ and Ω is the smallest uncountable ordinal number. We topololize X as follows: $B^n_{c_i} = \{c_i, a_{ij}, b_{ij}\}_{j\geq n}$, $B^\alpha_a = \{a, a_{ij}\}_{i\geq\alpha, j\in\mathbb{N}}$ and $B^\alpha_b = \{b, b_{ij}\}_{i\geq\alpha, j\in\mathbb{N}}$ are the fundamental system of neighborhoods of the points c_i, a, b respectively, and $\{a_{ij}\}$, $\{b_{ij}\}$ are isolated points. Then the space X is not nearly Menger ([24, Example 2.6]). Now, we show that the space X is θ-Menger. Let $(\mathcal{U}_k : k \in \mathbb{N})$ be a sequence of θ-open covers of X. For each k and each $x \in X$ there is an open set U'_x such that $x \in U'_x \subset Cl(U'_x) \subset U$ for some $U \in \mathcal{U}_k$. Put $\mathcal{U}'_k = \{U'_x : x \in X\}$. Then $(\mathcal{U}'_k : k \in \mathbb{N})$ is a sequence of open covers of X. For fixed $k_1 \in \mathbb{N}$, there are open sets U'_a, U'_b in $\mathcal{U}'_{k_1}$ such that $a \in U'_a$ and $b \in U'_b$. Thus $\exists$ α_1, $\alpha_2 \in A$ such that $B^{\alpha_1}_a \subset U'_a$, $B^{\alpha_2}_b \subset U'_b$. It is clear that the set $X \setminus (\overline{B^{\alpha_1}_a \cup B^{\alpha_2}_b})$ is countable, hence the set $X \setminus (\overline{U'_a \cup U'_b})$ is also countable. Thus we can find a sequence $(\mathcal{V}'_k : k \in \mathbb{N} \setminus k_1)$, such that, $k \in \mathbb{N} \setminus k_1$, $\mathcal{V}'_k$ is a finite subset of $\mathcal{U}'_k$ and $X \setminus (\overline{U'_a \cup U'_b}) \subset \bigcup_{k\in\mathbb{N}\setminus k_1} \cup\{Cl(V') : V' \in \mathcal{V}'_k\}$. Fixed $\mathcal{V}'_{k_1} = \{U'_a, U'_b\}$. Then we have a sequence $(\mathcal{V}'_k : k \in \mathbb{N})$, where for each k, $\mathcal{V}'_k$ is a finite subset of $\mathcal{U}'_k$ such that $X = \bigcup_{k\in\mathbb{N}} \cup\{Cl(V') : V' \in \mathcal{V}'_k\}$. For each $V' \in \mathcal{V}'_k$ we can find a $U_{V'} \in \mathcal{U}_k$ such that $V' \subset Cl(V') \subset U_{V'}$. Let $\mathcal{W}_k = \{U_{V'} : V' \in \mathcal{V}'_k\}$, then for each k, $\mathcal{W}_k$ is a finite subset of $\mathcal{U}_k$ and $X = \bigcup_{k\in\mathbb{N}} \cup\mathcal{W}_k$. Hence X is a θ-Menger space.

Recall that a space X is called semi-regular [4] if for each element $x \in X$ and for each semi-closed set U such that $x \notin U$, there exist disjoint semi-open subsets A and B of X such that $x \in A$ and $U \subset B$.

Lemma 2.11. ([4]) *For a space X the following statements are equivalent:*

(i) *X is semi-regular;*

(ii) *For each element $x \in X$ and $A \in SO(X)$ with $x \in A$, there is a $B \in SO(X)$ such that $x \in B \subset sCl(B) \subset A$, $sCl(A)$ denotes the semi-closure of A.*

Now we prove that for an extremally disconnected semi regular space, all the above mentioned variants of Menger property are equivalent.

Theorem 2.12. *For extremally disconnected semi-regular spaces X, the following statements are equivalent:*

1. *X is semi-Menger;*
2. *X is α-Menger;*
3. *X is Menger;*
4. *X is nearly Menger;*
5. *X is θ-Menger;*
6. *X is mildly Menger.*

Proof. Obviously (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6).

For (6) $\Rightarrow$ (1), let $(\mathcal{A}_k : k \in \mathbb{N})$ be a sequence of semi-open covers of X. Then for each $x \in X$ there is a $B_{k,x} \in SO(X)$ such that $x \in B_{k,x} \subset sCl(B_{k,x}) \subset A$ for some $A \in \mathcal{A}_k$, X being a semi-regular space. Let for $k \in \mathbb{N}$, $\mathcal{B}_k = \{B_{k,x} : x \in X\}$. Then $(\mathcal{B}_k : k \in \mathbb{N})$, is a sequence of semi-open covers of X. Since X is extremally disconnected, from [8, Proposition 4.1], we have $B \subset Int(Cl(B))$ for each $B \in SO(X)$. Further, $Cl(Int(Cl(B)))$ is a clopen subset of X for each $B \in SO(X)$. Put $\mathcal{C}_k = \{Cl(Int(Cl(B))) : B \in \mathcal{B}_k\}$. Thus $(\mathcal{C}_k : k \in \mathbb{N})$ is a sequence of clopen covers of X. As X is mildly Menger, there is a sequence $(\mathcal{C}'_k : k \in \mathbb{N})$, where $\mathcal{C}'_k$ is a finite subset of $\mathcal{C}_k$ for each $k \in \mathbb{N}$ such that $\bigcup_{k\in\mathbb{N}} \cup\mathcal{C}'_k = X$. Note that, $Int(Cl(A)) \subset sCl(A)$ for each subset A of space X and by extremal disconnectedness of X, $sCl(A) = Cl(A)$ for each $A \in SO(X)$. Then from the above construction, for each $C' \in \mathcal{C}'_k$ there is a $A_{C'} \in \mathcal{A}_k$ such that $C' \subset A_{C'}$. Thus for $k \in \mathbb{N}$, let $\mathcal{A}'_k = \{A_{C'} : C' \in \mathcal{C}'_k\}$. Hence for each k, $\mathcal{A}'_k$ is a finite subset of $\mathcal{A}_k$ such that $\bigcup_{k\in\mathbb{N}} \cup\mathcal{A}'_k = X$. This means that X is semi-Menger. □

In the following examples, we show that the extremally disconnectedness and semi-regularity are necessary conditions in Theorem 2.12.

Example 2.13. Consider the real line $\mathbb{R}$ with usual topology. Then clearly, $\mathbb{R}$ is not an extremally disconnected space but it is semi-regular mildly Menger space being a regular Menger space. On the other hand, $\mathbb{R}$ is not semi-Menger [26].

Example 2.14. Let X be an uncountable cofinite space, that is uncountable set X with cofinite topology. Then X is an extremally disconnected mildly Menger space. On the other hand X it not semi-Menger, since a semi-open cover $\{X \setminus \{x\} : x \in X\}$ has no countable subcover.

Since in extremally disconnected spaces, zero-dimensionality is equivalent to semi-regularity [25, Theorem 6.4], we have the following corollary:

Corollary 2.15. *For extremally disconnected, zero-dimensional spaces X, the following statements are equivalent:*

1. *X is semi-Menger;*
2. *X is α-Menger;*
3. *X is Menger;*
4. *X is nearly Menger;*
5. *X is θ-Menger;*
6. *X is mildly Menger.*

A space X is called S-paracompact [1] if for each open cover of X has a locally finite semi-open refinement. A Hausdorff S-paracompact space X is semi-regular [1, Corollary 2.3]. Hence all the properties mentioned in Theorem 2.12 are also equivalent for an extremally disconnected Hausdorff S-paracompact space.

It may be noted that the Stone-Čech compactification of a discrete space is extremally disconnected compact Hausdorff space. Then the class of Stone-Čech compactifications of discrete spaces is a subclass of extremally disconnected S-paracompact Hausdorff spaces which is in turn the subclass of extremally disconnected semi-regular spaces.

Theorem 2.16. *For a space X, the following statements are equivalent:*

1. *X is θ-Menger;*
2. *For each non-empty subset A of X and for each sequence $(\mathcal{U}_k : k \in \mathbb{N})$ of collections of θ-open sets in X such that $Cl_\theta(A) \subset \cup\mathcal{U}_k$, $k \in \mathbb{N}$, there is a sequence $(\mathcal{V}_k : k \in \mathbb{N})$, for each $k \in \mathbb{N}$, $\mathcal{V}_k$ is a finite subset of $\mathcal{U}_k$ such that $A \subset \bigcup_{k\in\mathbb{N}} \cup\mathcal{V}_k$.*

Proof. 2 ⇒ 1 obvious.

1 ⇒ 2 Let A be a non-empty subset of X and $(\mathcal{U}_k : k \in \mathbb{N})$ is a sequence of collections of θ-open sets in X such that $Cl_\theta(A) \subset \cup\mathcal{U}_k$, $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, put $\mathcal{V}_k = \mathcal{U}_k \cup \{X \setminus Cl_\theta(A)\}$. Then $(\mathcal{V}_k : k \in \mathbb{N})$ is a sequence of θ-open covers of X. By assumption X is θ-Menger, there exists a sequence $(\mathcal{V}'_k : k \in \mathbb{N})$ such that $\mathcal{V}'_k$ is a finite subset of $\mathcal{V}_k$ for each $k \in \mathbb{N}$ and $\bigcup_{k\in\mathbb{N}} \cup\mathcal{V}'_k = X$. Consider $\mathcal{U}'_k = \mathcal{V}'_k \setminus \{X \setminus Cl_\theta(A)\}$. Then $(\mathcal{U}'_k : k \in \mathbb{N})$ is a sequence of θ-open sets in X, where for each $k \in \mathbb{N}$, $\mathcal{U}'_k$ is finite subset of $\mathcal{U}_k$ such that with $A \subset \bigcup_{k\in\mathbb{N}} \cup\mathcal{U}'_k$. □

In the next theorem we provide a sufficient condition for a space to be θ-Menger.

Theorem 2.17. *A space X is θ-Menger if for each sequence $(\mathcal{A}_k : k \in \mathbb{N})$ of closed covers of X there exists a sequence $(\mathcal{B}_k : k \in \mathbb{N})$, where for each k, $\mathcal{B}_k$ is a finite subset of $\mathcal{A}_k$, such that $X = \bigcup_{k\in\mathbb{N}} \cup\mathcal{B}_k$.*

Proof. Let $(\mathcal{A}_k : k \in \mathbb{N})$ be a sequence of θ-open covers of X. For each $x \in X$ and each $k \in \mathbb{N}$ there exists a $A_{x,k} \in \mathcal{A}_k$ and an open set $B_{x,k}$ such that $x \in B_{x,k} \subset \overline{B_{x,k}} \subset A_{x,k}$. For each k, put $\mathcal{B}_k = \{\overline{B_{x,k}} : x \in X\}$. Then $(\mathcal{B}_k : k \in \mathbb{N})$ is a sequence of closed covers of X. From the assumption, there is a sequence $(\mathcal{C}_k : k \in \mathbb{N})$, where for each k, $\mathcal{C}_k$ is a finite subset of $\mathcal{B}_k$, such that $X = \bigcup_{k\in\mathbb{N}} \cup\mathcal{C}_k$. Since for each $C_k \in \mathcal{C}_k$, there is a $A'_{C_k} \in \mathcal{A}_k$ such that $C_k \subset A'_{C_k}$. Let for $k \in \mathbb{N}$, $\mathcal{A}'_k = \{A'_{C_k} : C_k \in \mathcal{C}_k\}$. Thus for each k, $\mathcal{A}'_k$ is a finite subset of $\mathcal{A}_k$ such that $\bigcup \mathcal{C}_k \subset \bigcup \mathcal{A}'_k$. Hence X is θ-Menger space. □

A space X is called almost Menger [11], if for each sequence $(\mathcal{A}_k : k \in \mathbb{N})$ of open covers of X there exists a sequence $(\mathcal{B}_k : k \in \mathbb{N})$, where $\mathcal{B}_k$ is a finite subset of $\mathcal{A}_k$ for each $k \in \mathbb{N}$, such that $\bigcup_{k\in\mathbb{N}} \cup\{\overline{B} : B \in \mathcal{B}_k\} = X$.

We prove that for the class of an extremally disconnected spaces the θ-Menger property is equivalent to the almost-Menger property:

Theorem 2.18. *For an extremally disconnected space X, the following statements are equivalent:*

1. *X is θ-Menger; item For each sequence $(\mathcal{A}_k : k \in \mathbb{N})$ of θ-open covers of X there exists a sequence $(\mathcal{B}_k : k \in \mathbb{N})$, where for each $k \in \mathbb{N}$, $\mathcal{B}_k$ is a finite subset of $\mathcal{A}_k$, such that $\bigcup_{k\in\mathbb{N}} \cup\{\overline{B} : B \in \mathcal{B}_k\} = X$;*
2. *X is almost Menger.*

Proof. (1) ⇒ (2) Obvious.

(2)⇒ (3) Let $(\mathcal{A}_k : k \in \mathbb{N})$ be a sequence of open covers of X. Since X is extremally disconnected, for each k, $\mathcal{A}'_k = \{\overline{A} : A \in \mathcal{A}_k\}$ is a θ-open cover of X. Thus $(\mathcal{A}'_k : k \in \mathbb{N})$ is a sequence of θ-open covers of X. From the assumption, there exists a sequence $(\mathcal{B}_k : k \in \mathbb{N})$, where for each k, $\mathcal{B}_k$ is a finite subset of $\mathcal{A}_k$, such that $\bigcup_{k\in\mathbb{N}} \cup\{\overline{B} : B \in \mathcal{B}_k\} = X$. Thus X is almost Menger.

(3)⇒ (1) It is proved in [13, Theorem 3.5]. □

3. θ-almost Menger and θ-weakly Menger spaces

In this section we studied θ-almost Menger and θ-weakly Menger spaces. First we recall some definitions.

A space X is said to be weakly Menger [2] if for each sequence $(\mathcal{A}_k : k \in \mathbb{N})$ of open covers of X there is a sequence $(\mathcal{B}_k : k \in \mathbb{N})$ such that for each k, $\mathcal{B}_k$ is a finite subset of $\mathcal{A}_k$ and $X = Cl(\bigcup_{k\in\mathbb{N}} \cup\mathcal{B}_k)$.

Definition 3.1. ([13, Remark 3.6]) A space X is said to be θ-almost Menger (resp., θ-weakly Menger) if for each sequence $(\mathcal{A}_k : k \in \mathbb{N})$ of open covers of X there is a sequence $(\mathcal{B}_k : k \in \mathbb{N})$, where for each k, $\mathcal{B}_k$ is a finite subset of $\mathcal{A}_k$, such that $X = \bigcup_{k\in\mathbb{N}} \cup\{Cl_\theta(Cl(B)) : B \in \mathcal{B}_k\}$, (resp., $X = Cl_\theta(Cl(\bigcup_{k\in\mathbb{N}} \cup\mathcal{B}_k))$).

From the definitions it is clear that each θ-almost Menger space is θ-weakly Menger. In the following example, we show that weakly Menger (hence θ-weakly Menger) space need not be θ-almost Menger.

Example 3.2. There is a Tychonoff weakly Menger (hence, θ-weakly Menger) space which is not θ-almost Menger.

Proof. Let D be a discrete space with cardinality ω_1, and $D^* = D \cup \{d^*\}$ is an one-point compactification of D, where $d^* \notin D$, let

$$X = (D^* \times [0, \omega]) \setminus \{< d^*, \omega >\}$$

be the subspace of the product space $D^* \times [0, \omega]$.

Note that, $D^* \times \omega$ is a σ-compact dense subset of X, that means the space X is weakly Menger (hence θ-weakly Menger).

Now, we prove that the space X is not an θ-almost Menger. We enumerate D as $\{d_\alpha : \alpha < \omega_1\}$ because cardinality of D is ω_1. Let $A_\alpha = \{d_\alpha\} \times [0, \omega]$, for each $\alpha < \omega_1$. Then

$$A_\alpha \cap A_{\alpha'} = \phi \text{ if } \alpha \neq \alpha', \text{ and } Cl_\theta(Cl(A_\alpha)) = A_\alpha \text{ for each } \alpha < \omega_1.$$

Let $B_k = D^* \times \{k\}$, for each $k \in \omega$. Then

$$Cl_\theta(Cl(B_k)) = B_k \text{ for each } k \in \omega.$$

Let

$$\mathcal{A}_k = \{A_\alpha : \alpha < \omega_1\} \cup \{B_k : k \in \omega\}, \text{ for each } k \in \mathbb{N}.$$

Observe that for each $k \in \mathbb{N}$, $\mathcal{A}_k$ is an open cover of the space X. Let us consider the sequence $(\mathcal{A}_k : k \in \mathbb{N})$ of open covers of X.

Claim: For any sequence $(\mathcal{B}_k : k \in \mathbb{N})$, where $\mathcal{B}_k$ is a finite subset of $\mathcal{A}_k$ for each $k \in \mathbb{N}$, $\bigcup_{k\in\mathbb{N}} \cup\{Cl_\theta(Cl(B)) : B \in \mathcal{B}_k\} \neq X$.

Let $(\mathcal{B}_k : k \in \mathbb{N})$ be any sequence, where for each k, $\mathcal{B}_k$ is a finite subset of $\mathcal{A}_k$. Since $\mathcal{B}_k$ is a finite subset of $\mathcal{A}_k$ for each $k \in \mathbb{N}$, then there is a $\alpha_k < \omega_1$ such that $A_\alpha \notin \mathcal{B}_k$ for each $\alpha > \alpha_k$. Put $\alpha' = sup\{\alpha_k : k \in \mathbb{N}\}$. Then $\alpha' < \omega_1$. We can choose $\alpha_0 > \alpha'$. Then

$$< d_{\alpha_0}, \omega > \notin \bigcup_{k\in\mathbb{N}} \cup\{Cl_\theta(Cl(B)) : B \in \mathcal{B}_k\}.$$

Note that, A_{α_0} is the only element of $\mathcal{A}_k$ which contain $< d_{\alpha_0}, \omega >$ for each $k \in \mathbb{N}$. It is easy to see that $\bigcup_{k\in\mathbb{N}} \cup\{B : B \in \mathcal{B}_k\} = \bigcup_{k\in\mathbb{N}} \cup\{Cl_\theta(Cl(B)) : B \in \mathcal{B}_k\}$ from the construction of the sequence $(\mathcal{A}_k : k \in \mathbb{N})$ of open covers of X. Thus $< d_{\alpha_0}, \omega > \notin \bigcup_{k\in\mathbb{N}} \cup\{Cl_\theta(Cl(B)) : B \in \mathcal{B}_k\}$ that means the space X is not an θ-almost Menger. □

Recall that a topological space X is said to be P-space [10] if every intersection of countably many open sets of X is open. For P-spaces, we prove the following result:

Theorem 3.3. *Let X be a θ-weakly Menger P-space, then X is θ-almost Menger.*

Proof. Let $(\mathcal{A}_k : k \in \mathbb{N})$ be a sequence of open covers of X. Since the space X is θ-weakly Menger, then for each k, there exists a finite subset $\mathcal{B}_k$ of $\mathcal{A}_k$ such that $X = Cl_\theta(Cl(\bigcup_{k\in\mathbb{N}} \cup\mathcal{B}_k))$. Since X is a P-space, $\bigcup_{k\in\mathbb{N}} \cup\{Cl(B) : B \in \mathcal{B}_k\}$ is a closed subset of X and $Cl(\bigcup_{k\in\mathbb{N}} \cup\mathcal{B}_k)$ is the least closed set contains $\bigcup_{k\in\mathbb{N}} \cup\mathcal{B}_k$, hence $Cl(\bigcup_{k\in\mathbb{N}} \cup\mathcal{B}_k) \subseteq \bigcup_{k\in\mathbb{N}} \cup\{Cl(B) : B \in \mathcal{B}_k\}$. Also observe that if $x \notin \bigcup_{k\in\mathbb{N}} \cup\{Cl_\theta(Cl(B)) : B \in \mathcal{B}_k\}$, then $x \notin Cl_\theta(\bigcup_{k\in\mathbb{N}} \cup\{Cl(B) : B \in \mathcal{B}_k\})$. Thus, we have $X = Cl_\theta(Cl(\bigcup_{k\in\mathbb{N}} \cup\mathcal{B}_k)) \subseteq Cl_\theta(\bigcup_{k\in\mathbb{N}} \cup\{Cl(B) : B \in \mathcal{B}_k\}) = \bigcup_{k\in\mathbb{N}} \cup\{Cl_\theta(Cl(B)) : B \in \mathcal{B}_k\}$. Hence X is θ-almost Menger. □

However in general, it remains open question whether a θ-almost Menger space is almost Menger or not. But in the following results, we provide a class of spaces in which θ-almost Menger property is equivalent to almost Menger property.

Theorem 3.4. *X is an extremally disconnected almost Menger space if and only if X is θ-almost Menger.*

Proof. The forward part is obvious. Conversely, let $(\mathcal{A}_k : k \in \mathbb{N})$ be a sequence of open covers of X. Since X is θ-almost Menger, there exists a sequence $(\mathcal{B}_k : k \in \mathbb{N})$, where $\mathcal{B}_k$ is a finite subset of $\mathcal{A}_k$ for each $k \in \mathbb{N}$, such that $X = \bigcup_{k\in\mathbb{N}} \cup\{Cl_\theta(Cl(B)) : B \in \mathcal{B}_k\}$. Also by the extremally disconnectedness of X, $Cl_\theta(Cl(B)) = Cl(B)$ for each open set B of X. Hence $X = \bigcup_{k\in\mathbb{N}} \cup\{Cl(B) : B \in \mathcal{B}_k\}$. □

Theorem 3.5. *Let X be a regular θ-almost Menger space, then X is Menger.*

Proof. Let $(\mathcal{A}_k : k \in \mathbb{N})$ be a sequence of open covers of X. Since X is regular, for each k and for each $x \in X$ there exist open sets $B_{x,k}, C_{x,k}$ such that $x \in C_{x,k} \subset Cl(C_{x,k}) \subset B_{x,k} \subset Cl(B_{x,k}) \subset A$ for some $A \in \mathcal{A}_k$. Thus for $k \in \mathbb{N}$, $\mathcal{C}_k = \{C_{x,k} : x \in X\}$ is an open cover of X. Also X is θ-almost Menger, there is a sequence $(\mathcal{C}'_k : k \in \mathbb{N})$, where for each k, $\mathcal{C}'_k$ is a finite subset of $\mathcal{C}_k$ such that $\bigcup_{k\in\mathbb{N}} \cup\{Cl_\theta(Cl(C')) : C' \in \mathcal{C}'_k\} = X$. For each k, for each $C'_{k,x} \in \mathcal{C}'_k$, there is an open set $B_{k,x}$ and $A_{C'_{x,k}} \in \mathcal{A}_k$ such that $Cl(C'_{x,k}) \subset B_{x,k} \subset Cl(B_{x,k}) \subset A_{C_{x,k}}$. Thus $Cl_\theta(Cl(C'_{x,k})) \subseteq Cl(B_{x,k}) \subset A_{C'_{x,k}}$. Let $\mathcal{A}'_k = \{A_{C'_{x,k}} : C'_{x,k} \in \mathcal{C}'_k\}$. Then the sequence $(\mathcal{A}'_k : k \in \mathbb{N})$ witnesses that the space X is Menger. □

From Theorem 3.3 and Theorem 3.5, we have the following corollary:

Corollary 3.6. *Let X be a regular P-space. Then the following statements are equivalent:*

1. *X is θ-Menger;*
2. *X is Menger;*
3. *X is almost Menger;*
4. *X is weakly Menger;*
5. *X is θ-weakly Menger;*
6. *X is θ-almost Menger.*

From the following examples, it is clear that θ-almost Menger (θ-weakly Menger) properties are not hereditary.

Example 3.7. A closed subset of an θ-weakly Menger space need not be θ-weakly Menger.

Proof. Let $\mathbb{R}$ be the set of real numbers, $\mathbb{Q}$ and $\mathbb{I}$ denotes the set of rational and irrational numbers respetively. For each $a \in \mathbb{I}$, we choose a sequence $\{a_i : i \in \omega\}$ of rational numbers which converge to a in the Euclidean topology. The rational sequence topology τ (see [30], Example 65) is defined as declaring each rational open and the sets $A_k(a) = \{a_{k,i} : i \in \omega\} \cup \{a\}$ as a basis for the irrational point a. Then the set $\mathbb{I}$ is a closed subset of $(\mathbb{R}, \tau)$ and as a subspace $\mathbb{I}$ is not θ-weakly Menger, $\mathbb{I}$ being an uncountable discrete subspace. On the other hand, $(\mathbb{R}, \tau)$ is θ-weakly Menger, because $\mathbb{Q}$ is a countable dense subset of $(\mathbb{R}, \tau)$. □

Example 3.8. Let X be the same space as in [28, Example 2.1]: Consider $U = \{u_\alpha : \alpha < \omega_1\}$, $V = \{v_i : i \in \omega\}$ and $W = \{< u_\alpha, v_i >: \alpha < \omega_1, i \in \omega\}$, where ω, ω_1 are the first infinite cardinal and the first uncountable cardinal respectively. Let $X = \{x\} \cup W \cup U$, where x does not belongs to $W \cup U$. We topologize X as follows: the basic neighborhood of x is of the form $A_x(\alpha) = \{x\} \cup \bigcup\{< u_\beta, v_i >: \beta > \alpha, i \in \omega\}$, $\alpha < \omega_1$, for $u_\alpha \in U$ for each $\alpha < \omega_1$, the basic neighborhood of u_α is of the form $A_{u_\alpha}(i) = \{u_\alpha\} \cup \{< u_\alpha, v_j >: j \geq i\}$, $i \in \omega$, and each member of W are isolated. In [28] Song showed that X is an almost Menger space having uncountable discrete closed subset, $U = \{u_\alpha : \alpha < \omega_1\}$. Thus X is a θ-almost Menger space with uncountable discrete closed subset. Hence we can conclude that closed subset of θ-almost Menger space need not be θ-almost Menger.

Proposition 3.9. *The closed and open subspace of θ-almost Menger space is θ-almost Menger.*

Proof. Let Y be a closed and open subspace of the space X, let $(\mathcal{A}_k : k \in \mathbb{N})$ be a sequence of open covers of Y. Then for each k, $\mathcal{B}_k = \mathcal{A}_k \cup \{X \setminus Y\}$ is an open cover of X. Since X is θ-almost Menger, for each k there is a finite subset $\mathcal{B}'_k$ of $\mathcal{B}_k$ such that $X = \bigcup_{k\in\mathbb{N}} \cup\{Cl_\theta(Cl(B')) : B' \in \mathcal{B}_k\}$. Since Y is a closed and open subspace X, $Cl_\theta(Cl(X \setminus Y)) = X \setminus Y$ and $Cl(U) \subset Y$ for each open subset U of Y, which implies that $Cl_\theta(Cl(B)) \subseteq Cl_{\theta_Y}(Cl_Y(B))$ for each open set B of Y. Thus, $Y \subseteq \bigcup_{k\in\mathbb{N}} \cup\{Cl_\theta(Cl(B')) : B' \in \mathcal{B}'_k \setminus \{X \setminus Y\}\} \subseteq \bigcup_{k\in\mathbb{N}} \cup\{Cl_{\theta_Y}(Cl_Y(B')) : B' \in \mathcal{B}'_k \setminus \{X \setminus Y\}\}$. Hence Y is θ-almost Menger. □

Similarly, we can prove that closed and open subset of a θ-weakly Menger space is θ-weakly Menger.

Note that, from Theorem 3.5 and Example 2.3, it is clear that the product space X^2 of a θ-almost Menger space X need not be θ-almost Menger. In the following theorem we give the necessary and sufficient conditions for the product space X^k to be θ-almost Menger for each $k \in \mathbb{N}$.

Theorem 3.10. *Let X be a topological space. Then the product space X^n is θ-almost Menger for each $n \in \mathbb{N}$ if and only if for each sequence $(\mathcal{A}_k : k \in \mathbb{N})$ of ω-covers of X there exists a sequence $(\mathcal{B}_k : k \in \mathbb{N})$, where for each k, $\mathcal{B}_k$ is a finite subset of $\mathcal{A}_k$, such that for every finite set $F \subset X$, there exists $k \in \mathbb{N}$, , such that $F \subset Cl_\theta(Cl(B))$ for some $B \in \mathcal{B}_k$.*

Proof. Let for each $n \in \mathbb{N}$, X^n be an θ-almost Menger space. Let $(\mathcal{A}_k : k \in \mathbb{N})$ be a sequence of ω-covers of X. Let $\mathbb{N} = N_1 \cup N_2 \cup .... \cup N_n \cup \cdots$ be a partition of $\mathbb{N}$ into countably many pairwise disjoint infinite subsets. For each $n \in \mathbb{N}$ and each $j \in N_n$, let $\mathcal{B}_j = \{A^n : A \in \mathcal{A}_j\}$. Then $(\mathcal{B}_j : j \in N_n)$ is a sequence of open covers of X^n. Since for $n \in \mathbb{N}$, X^n is θ-almost Menger, we can find a sequence $(C_j : j \in N_n)$ such that for each j, $C_j = \{A^n_{j_1}, A^n_{j_2}, ..., A^n_{j_{k(j)}}\}$ is a finite subset of $\mathcal{B}_j$ and $X^n = \bigcup_{j \in N_n}\{Cl_\theta(Cl(C)) : C \in C_j\}$. Let $F = \{x_1, x_2, ..., x_q\}$ be a finite subset of X. Then $(x_1, x_2, ..., x_q) \in X^q$, there is a $r \in N_q$ and $1 \leq l \leq k(r)$ such that $(x_1, x_2, ..., x_q) \in Cl_\theta(Cl(A^q_{r_l})) = (Cl_\theta(Cl(A_{r_l})))^q$. Hence $F \subset Cl_\theta(Cl(A_{r_l}))$.

Conversely, let $n \in \mathbb{N}$ be fixed and $(\mathcal{A}_k : k \in \mathbb{N})$ be a sequence of open covers of X^n, where $\mathcal{A}_k = \{A_{k,j} : j \in J_k\}$, J_k is an indexing set. Let $F \subset X$ be a finite set. Then F^n is a finite subset of X^n, thus compact subset of X^n. Then for each k, there exists a finite subset J^F_k of J_k such that $F^n \subset \bigcup_{j \in J^F_k} A_{k,j}$. By the Wallace theorem (see 3.2.10. [5]), there is an open set B_F in X such that $F \subset B_F$ and $B^n_F \subset \bigcup_{j \in J^F_k} A_{k,j}$. For each k, put $\mathcal{B}_k = \{B_F$: F is a finite subset of X$\}$. Thus $(\mathcal{B}_k : k \in \mathbb{N})$ is a sequence of ω-covers of X. From the assumption, there exists a sequence $(C_k : k \in \mathbb{N})$, where for each k, C_k is a finite subset of $\mathcal{B}_k$ such that for every finite subset F of X, there is $k \in \mathbb{N}$, such that $F \subset Cl_\theta(Cl(C))$ for some $C \in C_k$. Let $\mathcal{H}_k = \{A_{k,j} : j \in J^F_k, F \subset B_F \in C_k\}$. Then for each k, $\mathcal{H}_k$ is a finite subset of $\mathcal{A}_k$. Let $x = (x_1, ..., x_n) \in X^n$. Thus $F = \{x_1, ..., x_n\}$ is a finite subset of X, there exists a $k \in \mathbb{N}$ and $C \in C_k$ such that $F \subset Cl_\theta(Cl(C))$. Since $C \in C_k$, then $C = B_{F'}$, for some finite subset F' of X such that $B^n_{F'} \subset \bigcup_{j \in J^{F'}_k} A_{k,j}$. We have $F^n \subset Cl_\theta(Cl(B_{F'}))^n = Cl_\theta(Cl(B^n_{F'})) \subset Cl_\theta(Cl(\bigcup_{j \in J^{F'}_k} A_{k,j})) = \bigcup_{j \in J^{F'}_k} Cl_\theta(Cl(A_{k,j}))$, hence $X^n \subseteq \bigcup_{k \in \mathbb{N}} \cup\{Cl_\theta(Cl(H)) : H \in \mathcal{H}_k\}$. That means X^n is almost Menger. □

4. Preservation properties

In this section, we study the preservation of θ-almost Menger property under varies type of mappings.

Theorem 4.1. *The continuous image of θ-almost Menger space is θ-almost Menger.*

Proof. Let $f : X \to Y$ be a continuous map from an θ-almost Menger space onto a space Y. Let $(\mathcal{A}_k : k \in \mathbb{N})$ be a sequence of open covers of Y. Then for each k, $\{f^{-1}(A) : A \in \mathcal{A}_k\}$ is an open cover of X. Since X is an almost Menger, for each k, there exists a finite subset $\mathcal{B}_k$ of $\mathcal{A}_k$ such that $X = \bigcup_{k \in \mathbb{N}} \cup\{Cl_\theta(Cl(f^{-1}(B)) : B \in \mathcal{B}_k\}$. Thus we have, $Y = f(X) = f(\bigcup_{k \in \mathbb{N}} \cup\{Cl_\theta(Cl(f^{-1}(B)) : B \in \mathcal{B}_k\}) = \bigcup_{k \in \mathbb{N}} \cup\{f(Cl_\theta(Cl(f^{-1}(B)))) : B \in \mathcal{B}_k\}$. From the continuity of f it follows that for each $y \in f(Cl_\theta(Cl(f^{-1}(B))))$ and each neighbourhood U of y, $Cl(U) \cap Cl(B) \neq \emptyset$ that means $y \in Cl_\theta(Cl(B))$. Then $\bigcup_{k \in \mathbb{N}} \cup\{f(Cl_\theta(Cl(f^{-1}(B)))) : B \in \mathcal{B}_k\} \subseteq \bigcup_{k \in \mathbb{N}} \cup\{Cl_\theta(Cl((B))) : B \in \mathcal{B}_k\}$. Hence Y is θ-almost Menger. □

We show that the preimage of an θ-almost Menger space under a closed continuous map need not be θ-almost Menger. Recall the Alexandroff duplicate $A(X)$ of a space X: The underlying set of $A(X)$ is $X \times \{0, 1\}$ which is topologized as follows: let U be a neighborhood of x in X then a basic neighborhood of a point $\langle x, 0\rangle \in X \times \{0\}$ is of the form $(U \times \{0\}) \cup ((U \times \{1\}) \setminus \{\langle x, 1\rangle\})$ and each points of $X \times \{1\}$ is isolated.

Example 4.2. The preimage of a θ-almost Menger space under a closed continuous map need not be θ-almost Menger.

Proof. Let X be the same space as Example 3.8. Since $U = \{u_\alpha : \alpha < \omega_1\}$ is an uncountable discrete closed subset of X. Then the Alexandroff duplicate $A(X)$ of X is not θ-almost Menger, since $U \times \{1\}$ is an uncountable infinite discrete closed and open set in $A(X)$ and every open and closed subset of an θ-almost Menger space is θ-almost Menger. Let us consider the projection map $f : A(X) \to X$. Then f is a required closed continuous map. □

Definition 4.3. A map $f : X \to Y$ said to be θ-almost open if for each open subset A of Y, $f^{-1}(Cl_\theta(Cl(A))) \subseteq Cl_\theta(Cl(f^{-1}(A)))$.

A map f is called θ-open [27] if the image of every open set is θ-open. It may be noted that injective θ-open maps are θ-almost open.

Theorem 4.4. *Let $f : X \to Y$ be an θ-almost open, perfect continuous map and Y is an θ-almost Menger space, then X is θ-almost Menger.*

Proof. Let $(\mathcal{A}_k : k \in \mathbb{N})$ be a sequence of open covers of X. Let $y \in Y$, since $f^{-1}(y)$ is compact, for each $k \in \mathbb{N}$ there is a finite sub-collection $\mathcal{A}_{k_y}$ of $\mathcal{A}_k$ such that $f^{-1}(y) \subset \cup\mathcal{A}_{k_y}$ and for each $A \in \mathcal{A}_{k_y}$, $A \cap f^{-1}(y) \neq \phi$. Let $B_{k_y} = Y \setminus f(X \setminus \cup\mathcal{A}_{k_y})$. Since f is closed, B_{k_y} is an open neighbourhood of y in Y such that $f^{-1}(B_{k_y}) \subseteq \cup\{A : A \in \mathcal{A}_{k_y}\}$. For each $k \in \mathbb{N}$, put $\mathcal{B}_k = \{B_{k_y} : y \in Y\}$. Thus $(\mathcal{B}_k : k \in \mathbb{N})$ is a sequence of open covers of Y. Since Y is an θ-almost Menger space, there is a sequence $(\mathcal{B}'_k : k \in \mathbb{N})$, where for each k, $\mathcal{B}'_k$ is a finite subset of $\mathcal{B}_k$ such that $\bigcup_{k\in\mathbb{N}} \cup\{Cl_\theta(Cl(B')) : B' \in \mathcal{B}'_k\} = Y$. By the above construction there is a sequence $(\mathcal{A}'_k : k \in \mathbb{N})$, where for each k, $\mathcal{A}'_k$ is finite subset of $\mathcal{A}_k$, such that $\cup\{f^{-1}(B') : B' \in \mathcal{B}'_k\} \subseteq \cup\{A' : A' \in \mathcal{A}'_k\}$. Then we have,

$$X = f^{-1}(Y) = f^{-1}(\textstyle\bigcup_{k\in\mathbb{N}} \cup\{Cl_\theta(Cl(B')) : B' \in \mathcal{B}'_k\})$$

$$= \textstyle\bigcup_{k\in\mathbb{N}} \cup\{f^{-1}(Cl_\theta(Cl(B'))) : B' \in \mathcal{B}'_k\}$$

$$\subseteq \textstyle\bigcup_{k\in\mathbb{N}} \cup\{Cl_\theta(Cl(f^{-1}(B'))) : B' \in \mathcal{B}'_k\}$$

$$\subseteq \textstyle\bigcup_{k\in\mathbb{N}} \cup\{Cl_\theta(Cl(A')) : A' \in \mathcal{A}'_k\}.$$

Hence X is θ-almost Menger. □

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