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Computation of the iterated Aluthge, Duggal, and Mean transforms

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Abstract. In this paper, we consider the computation of the Aluthge, Duggal, and Mean transforms of $n \times n$ matrices using the singular valued decomposition. We provide examples to illustrate the superiority of our technique in finding iterated Aluthge transform of a matrix over the more traditional methods. From the observations using Python, we know that the iterated Aluthge and Mean transforms of the matrices become normal. However, the iterated Duggal transform of the matrices may not be normal.

1. Introduction

Let $\mathcal{H}$ be a separable complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ has the unique polar decomposition $T = U|T|$ where $|T| = (T^*T)^{\frac{1}{2}}$ and U is the appropriate partial isometry satisfying $\ker(U) = \ker(|T|) = \ker(T)$. Here, $\ker(T) = \{x \in \mathcal{H} \mid Tx = 0\}$. In 1990, A. Aluthge([2]) introduced the *Aluthge transform* $\widetilde{T}$ of $T \in \mathcal{L}(\mathcal{H})$ given by

$$\widetilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}.$$

For an operator $T \in \mathcal{L}(\mathcal{H})$, the sequence $\{\widetilde{T}^{(n)}\}$ of the *Aluthge iterates* of T is defined by $\widetilde{T}^{(0)} = T$ and $\widetilde{T}^{(n)} = \widetilde{\widetilde{T}^{(n-1)}}$ for $n \in \mathbb{N}$ where $\mathbb{N}$ denotes the set of positive integers.

The main advantages of the Aluthge transformation of operators are as follows.

(i) If T is p-hyponormal with $0 < p < \frac{1}{2}$, then $\widetilde{(\widetilde{T})}$ is hyponormal (see [2]).
(ii) If T is a quasiaffinity, i.e., it has trivial kernel and dense range, then $\mathrm{Lat}(T)$ is nontrivial if and only if $\mathrm{Lat}(\widetilde{T})$ is nontrivial (see [6]).

The *Duggal transform* $\widetilde{T}^D$ of $T \in \mathcal{L}(\mathcal{H})$ is given by $\widetilde{T}^D := |T|U$ (see [7]). The *mean transform* $\widehat{T}$ of T is defined by $\widetilde{T}^M := \frac{1}{2}(T + \widetilde{T}^D)$ (see [8]). Now, we state important concepts, that is, "pseudo inverse" and "singular valued decomposition" in linear algebra for our study.

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Definition 1.1. Let $m, n \in \mathbb{N}$. For a matrix $T \in \mathbb{C}^{m\times n}$, a *pseudo inverse* of T is defined as matrix $T^{\dagger} \in \mathbb{C}^{n\times m}$ satisfying all of the following four criteria, known as the Moore-Penrose conditions:

$$TT^{\dagger}T = T, T^{\dagger}TT^{\dagger} = T^{\dagger}, (TT^{\dagger})^* = TT^{\dagger}, \text{ and } (T^{\dagger}T)^* = T^{\dagger}T. \tag{1}$$

Since the pseudo inverse exists and is unique for any matrix T, there is precisely one matrix $T^{\dagger}$.

Theorem 1.2. The *singular value decomposition* (SVD) of an $m \times n$ complex matrix T is the factorization of the form

$$T = W\Sigma V^*$$

where Σ is an $m \times n$ rectangular diagonal matrix with non-negative real numbers on the diagonal, W is an $m \times m$ complex unitary matrix, and V is an $n \times n$ complex unitary matrix.

In 2017-2018, D. Pappas, W. N. Kataikis, and I. P. Stanimirovic ([9] and [10]) developed algorithms for symbolic computation of the Aluthge and Duggal transforms of a polynomial matrix, respectively. Like this study, we study the Aluthge, Duggal and Mean transforms with a newly developed algorithm(see Section 3) and determine whether the iterated transformations become normal.

In this paper, we study the computation of the iterated Aluthge, Duggal, and Mean transforms of $n \times n$ matrices using the singular value decomposition. We provide examples to illustrate the superiority of our technique in finding iterated Aluthge transform of a matrix over the more traditional methods. From the observations using Python, we know that the iterated Aluthge and Mean transforms of $n \times n$ matrices become normal. But, the iterated Duggal transform of $n \times n$ matrices may not be normal.

2. Main results

In this section, we focus on the computation of the (iterated) Aluthge transform of $n \times n$ matrices. In general, it is not easy to directly calculate the Aluthge transform after finding the polar decomposition of a matrix. So we use the singular value decomposition to find the Aluthge transform of a matrix. We start with the following lemma.

Lemma 2.1. *With the same matrices W, V, Σ as in Theorem 1.2, let $W\Sigma V^*$ be the singular value decomposition of $T \in \mathbb{C}^{n\times n}$. Then $T^{\dagger} = V\Sigma^{\dagger}W^*$ is the pseudo inverse of T, where*

$$\Sigma^{\dagger} = \begin{cases} {\sigma_i}^{-1}, & \text{if} \quad \sigma_i \neq 0 \\ 0, & \text{otherwise} \end{cases} \quad \text{for } i \in \{1, 2, \cdots, n\} \tag{2}$$

and σ_i is the entries of the diagonal matrix Σ.

The following theorem provides the form of the Aluthge transform of matrices T using the singular value decomposition.

Theorem 2.2. *Let T be an $n \times n$ matrix. If $W\Sigma V^*$ is the singular value decomposition of T, then the Aluthge transform of T has the following form*

$$\widetilde{T} = \hat{\Sigma}WV^*\hat{\Sigma}$$

where $\hat{\Sigma} = V\Sigma^{\frac{1}{2}}V^*$.

Proof. Let $T = U|T|$ be a polar decomposition of T and let $T = W\Sigma V^*$ be the singular value decomposition of T. Since $|T|^2 = T^*T = V\Sigma^2V^* = (V\Sigma V^*)^2$, it follows that

$$|T| = V\Sigma V^* \text{ and } |T|^{\dagger} = (V\Sigma V^*)^{\dagger} = V\Sigma^{\dagger}V^* \tag{3}$$

where V is unitary and it holds that $V^* = V^\dagger$. Consider $U = W\Sigma\Sigma^\dagger V^*$, we obtain from Definition 1.1 and (3) that

$$\begin{aligned} U|T| &= (W\Sigma\Sigma^\dagger V^*)(V\Sigma V^*) \\ &= W\Sigma\Sigma^\dagger V^* V\Sigma V^* \\ &= W\Sigma\Sigma^\dagger \Sigma V^* \\ &= W\Sigma V^* \\ &= T. \end{aligned} \tag{4}$$

Since $(\Sigma\Sigma^\dagger)^* = \Sigma\Sigma^\dagger$, we get that

$$\begin{aligned} UU^*U &= (W\Sigma\Sigma^\dagger V^*)(W\Sigma\Sigma^\dagger V^*)^*(W\Sigma\Sigma^\dagger V^*) \\ &= (W\Sigma\Sigma^\dagger V^*)(V\Sigma\Sigma^\dagger W^*)(W\Sigma\Sigma^\dagger V^*) \\ &= W\Sigma\Sigma^\dagger V^* V\Sigma\Sigma^\dagger W^* W\Sigma\Sigma^\dagger V^* \\ &= W(\Sigma\Sigma^\dagger\Sigma)(\Sigma^\dagger\Sigma\Sigma^\dagger)V^* \\ &= W\Sigma\Sigma^\dagger V^* \\ &= U. \end{aligned} \tag{5}$$

Therefore, $\Sigma^{\frac{1}{2}}(\Sigma^{\frac{1}{2}})^\dagger = (\Sigma^{\frac{1}{2}})^\dagger\Sigma^{\frac{1}{2}}$ implies that

$$\begin{aligned} \widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} &= V\Sigma^{\frac{1}{2}}V^*W\Sigma\Sigma^\dagger V^*V\Sigma^{\frac{1}{2}}V^* \;(= \hat{\Sigma}W\Sigma\Sigma^\dagger V^*\hat{\Sigma}) \\ &= V\Sigma^{\frac{1}{2}}V^*W\Sigma\Sigma^\dagger\Sigma^{\frac{1}{2}}V^* \\ &= V\Sigma^{\frac{1}{2}}V^*W\Sigma^{\frac{1}{2}}\Sigma^{\frac{1}{2}}(\Sigma^\dagger)^{\frac{1}{2}}(\Sigma^\dagger)^{\frac{1}{2}}\Sigma^{\frac{1}{2}}V^* \\ &= V\Sigma^{\frac{1}{2}}V^*W\Sigma^{\frac{1}{2}}\Sigma^{\frac{1}{2}}(\Sigma^{\frac{1}{2}})^\dagger(\Sigma^{\frac{1}{2}})^\dagger\Sigma^{\frac{1}{2}}V^* \\ &= V\Sigma^{\frac{1}{2}}V^*W\Sigma^{\frac{1}{2}}(\Sigma^{\frac{1}{2}})^\dagger\Sigma^{\frac{1}{2}}(\Sigma^{\frac{1}{2}})^\dagger\Sigma^{\frac{1}{2}}V^* \\ &= V\Sigma^{\frac{1}{2}}V^*W\Sigma^{\frac{1}{2}}(\Sigma^{\frac{1}{2}})^\dagger\Sigma^{\frac{1}{2}}V^* \\ &= V\Sigma^{\frac{1}{2}}V^*W\Sigma^{\frac{1}{2}}V^* \\ &= (V\Sigma^{\frac{1}{2}}V^*)WV^*(V\Sigma^{\frac{1}{2}}V^*) \\ &= \hat{\Sigma}WV^*\hat{\Sigma}. \end{aligned}$$

Hence the Aluthge transform of T is $\widetilde{T} = \hat{\Sigma}WV^*\hat{\Sigma}$ where $\hat{\Sigma} = V\Sigma^{\frac{1}{2}}V^*(= |T|^{\frac{1}{2}})$. □

Remark 2.3. From the proof of Theorem 2.2, we know that

$$\widetilde{T} = \hat{\Sigma}W\Sigma\Sigma^\dagger V^*\hat{\Sigma} = \hat{\Sigma}WV^*\hat{\Sigma}.$$

So, there is no need to find U or check the existence of an inverse of T to find the Aluthge transform.

Corollary 2.4. *Let T be an $n\times n$ matrix. If $W\Sigma V^*$ is the singular value decomposition of T, then $T = (W\Sigma\Sigma^\dagger V^*)(V\Sigma V^*)$ is the unique polar decomposition of T.*

Proof. Let $U = W\Sigma\Sigma^\dagger V^*$. We now want to show that

$$\ker(W\Sigma\Sigma^\dagger V^*) = \ker(V\Sigma V^*) = \ker(W\Sigma V^*).$$

If $x \in \ker(W\Sigma\Sigma^\dagger V^*)$, then $W\Sigma\Sigma^\dagger V^* x = 0$ and so $\Sigma^\dagger V^* x = \Sigma^\dagger \Sigma\Sigma^\dagger V^* x = 0$ and $\ker(\Sigma) = \ker(\Sigma^\dagger)$. Thus $\Sigma V^* x = 0$ and so $V\Sigma V^* x = 0$. Therefore $x \in \ker(V\Sigma V^*)$. The converse implication hold by a similar method. Moreover, $\ker(V\Sigma V^*) = \ker(W\Sigma V^*)$ holds clearly. Thus U is a partial isometry by(5) and $|T| = V\Sigma V^*$ holds by (3) the proof of Theorem 2.2. Hence

$$T = (W\Sigma\Sigma^\dagger V^*)(V\Sigma V^*)$$

is the unique polar decomposition of T. □

We now give the form of the iterated Aluthge transform of an matrix T using the singular value decomposition.

Theorem 2.5. *Let T be an $n \times n$ matrix in $\mathbb{C}^{n\times n}$. If $T = W\Sigma V^*$ is the singular value decomposition of T, then the n-th iterated Aluthge transform of T is*

$$\widetilde{T}^{(n)} = \hat{\Sigma}_{n-1} W_{n-1} V_{n-1}^* \hat{\Sigma}_{n-1}$$

where $\hat{\Sigma}_{n-1} = V_{n-1}\Sigma_{n-1}^{\frac{1}{2}} V_{n-1}^*$ for each $n \geq 1$, $\hat{\Sigma}_0 = \hat{\Sigma}$, $W_0 = W$, and $V_0^* = V^*$.

Proof. If $n = 1$, then it is clear by Theorem 2.2. Assume that the n-th iterated Aluthge transform of T is

$$\widetilde{T}^{(n)} = \hat{\Sigma}_{n-1} W_{n-1} V_{n-1}^* \hat{\Sigma}_{n-1}$$

where $\hat{\Sigma}_{n-1} = V_{n-1}\Sigma_{n-1}^{\frac{1}{2}} V_{n-1}^*$ for each $n \geq 1$. Then we can find the singular value decomposition of $\widetilde{T}^{(n)}$ as follows;

$$\widetilde{T}^{(n)} = W_n \Sigma_n V_n^*.$$

By the similar way of Theorem 2.2, we obtain that the $(n+1)$-th iterated Aluthge transform of T is

$$\widetilde{T}^{(n+1)} = \hat{\Sigma}_n W_n V_n^* \hat{\Sigma}_n$$

where $\hat{\Sigma}_n = V_n \Sigma_n^{\frac{1}{2}} V_n^*$. So, we complete the proof. □

We say that a matrix N in $\mathbb{C}^{n\times n}$ is *normal* if $NN^* = N^*N$ where N^* is an adjoint matrix of N.

Corollary 2.6. *If T is an $n \times n$ matrix, then the limit points of the sequence $\{\widetilde{T}^{(n)}\}_{n\in\mathbb{N}}$ are normal where $\widetilde{T}^{(n)} = \hat{\Sigma}_{n-1} W_{n-1} V_{n-1}^* \hat{\Sigma}_{n-1}$ for each $n \geq 1$.*

Proof. The proof follows from Theorem 2.5 and [5] (or[1, Proposition 1]). □

Next, we give several examples of the iterated Aluthge transform of a matrix T by Theorems 2.2 and 2.5.

Example 2.7. Let $T \in \mathbb{C}^{3\times 3}$ be defined as

$$T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix}.$$

Then $|T|$ is clearly invertible and the singular value decomposition (SVD) of T is as follows.

$$T = W\Sigma V^* = \begin{pmatrix} 0 & 0 & -1 \\ -2^{-\frac{1}{2}} & -2^{-\frac{1}{2}} & 0 \\ 2^{-\frac{1}{2}} & -2^{-\frac{1}{2}} & 0 \end{pmatrix} \begin{pmatrix} 2^{\frac{1}{2}} & 0 & 0 \\ 0 & 2^{\frac{1}{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Therefore we have

$$\begin{aligned}
\hat{\Sigma} &= V\Sigma^{\frac{1}{2}}V^* \\
&= \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 2^{\frac{1}{4}} & 0 & 0 \\ 0 & 2^{\frac{1}{4}} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 2^{\frac{1}{4}} & 0 & 0 \\ 0 & 2^{\frac{1}{4}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
\text{and } WV^* &= \begin{pmatrix} 0 & 0 & -1 \\ -2^{-\frac{1}{2}} & -2^{-\frac{1}{2}} & 0 \\ 2^{-\frac{1}{2}} & -2^{-\frac{1}{2}} & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 2^{-\frac{1}{2}} & 2^{-\frac{1}{2}} & 0 \\ 2^{-\frac{1}{2}} & -2^{-\frac{1}{2}} & 0 \end{pmatrix}.
\end{aligned}$$

Hence the Aluthge transform of T is

$$\begin{aligned}
\widetilde{T} &= \hat{\Sigma}WV^*\hat{\Sigma} \\
&= \begin{pmatrix} 2^{\frac{1}{4}} & 0 & 0 \\ 0 & 2^{\frac{1}{4}} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 2^{-\frac{1}{2}} & 2^{-\frac{1}{2}} & 0 \\ 2^{-\frac{1}{2}} & -2^{-\frac{1}{2}} & 0 \end{pmatrix}\begin{pmatrix} 2^{\frac{1}{4}} & 0 & 0 \\ 0 & 2^{\frac{1}{4}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 2^{\frac{1}{4}} \\ 1 & 1 & 0 \\ 2^{-\frac{1}{4}} & -2^{-\frac{1}{4}} & 0 \end{pmatrix}.
\end{aligned}$$

On the other hand, the singular valued decomposition of $\widetilde{T}$ is

$$\widetilde{T} = W_1\Sigma_1V_1^* = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 2^{\frac{1}{2}} & 0 & 0 \\ 0 & 2^{\frac{1}{4}} & 0 \\ 0 & 0 & 2^{\frac{1}{4}} \end{pmatrix}\begin{pmatrix} -2^{-\frac{1}{2}} & -2^{-\frac{1}{2}} & 0 \\ 2^{-\frac{1}{2}} & -2^{-\frac{1}{2}} & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Therefore we have

$$\begin{aligned}
\hat{\Sigma_1} &= V_1\Sigma_1^{\frac{1}{2}}V_1^* \\
&= \begin{pmatrix} -2^{-\frac{1}{2}} & 2^{-\frac{1}{2}} & 0 \\ -2^{-\frac{1}{2}} & -2^{-\frac{1}{2}} & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 2^{\frac{1}{4}} & 0 & 0 \\ 0 & 2^{\frac{1}{8}} & 0 \\ 0 & 0 & 2^{\frac{1}{8}} \end{pmatrix}\begin{pmatrix} -2^{-\frac{1}{2}} & -2^{-\frac{1}{2}} & 0 \\ 2^{-\frac{1}{2}} & -2^{-\frac{1}{2}} & 0 \\ 0 & 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 2^{-\frac{3}{4}} + 2^{-\frac{7}{8}} & 2^{-\frac{3}{4}} - 2^{-\frac{7}{8}} & 0 \\ 2^{-\frac{3}{4}} - 2^{-\frac{7}{8}} & 2^{-\frac{3}{4}} + 2^{-\frac{7}{8}} & 0 \\ 0 & 0 & 2^{\frac{1}{8}} \end{pmatrix} \\
\text{and } W_1V_1^* &= \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} -2^{-\frac{1}{2}} & -2^{-\frac{1}{2}} & 0 \\ 2^{-\frac{1}{2}} & -2^{-\frac{1}{2}} & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 2^{-\frac{1}{2}} & 2^{-\frac{1}{2}} & 0 \\ 2^{-\frac{1}{2}} & -2^{-\frac{1}{2}} & 0 \end{pmatrix}.
\end{aligned}$$

Hence the 2-th iterated Aluthge transform of $\widetilde{T}^{(2)}$ is

$$\begin{aligned}
\widetilde{T}^{(2)} &= \hat{\Sigma}_1 W_1 V_1^* \hat{\Sigma}_1 \\
&= \begin{pmatrix} 2^{-\frac{3}{4}} + 2^{-\frac{7}{8}} & 2^{-\frac{3}{4}} - 2^{-\frac{7}{8}} & 0 \\ 2^{-\frac{3}{4}} - 2^{-\frac{7}{8}} & 2^{-\frac{3}{4}} + 2^{-\frac{7}{8}} & 0 \\ 0 & 0 & 2^{\frac{1}{8}} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 2^{-\frac{1}{2}} & 2^{-\frac{1}{2}} & 0 \\ 2^{-\frac{1}{2}} & -2^{-\frac{1}{2}} & 0 \end{pmatrix} \begin{pmatrix} 2^{-\frac{3}{4}} + 2^{-\frac{7}{8}} & 2^{-\frac{3}{4}} - 2^{-\frac{7}{8}} & 0 \\ 2^{-\frac{3}{4}} - 2^{-\frac{7}{8}} & 2^{-\frac{3}{4}} + 2^{-\frac{7}{8}} & 0 \\ 0 & 0 & 2^{\frac{1}{8}} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} - 2^{-\frac{9}{8}} & \frac{1}{2} - 2^{-\frac{9}{8}} & 2^{-\frac{5}{8}} + 2^{-\frac{3}{4}} \\ \frac{1}{2} + 2^{-\frac{9}{8}} & \frac{1}{2} + 2^{-\frac{9}{8}} & 2^{-\frac{5}{8}} - 2^{-\frac{3}{4}} \\ 2^{-\frac{1}{4}} & -2^{-\frac{1}{4}} & 0 \end{pmatrix}.
\end{aligned}$$

Example 2.8. Let $T \in \mathbb{C}^{4\times 4}$ be defined as

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then T is obviously non-invertible matrix and the singular value decomposition of T is as follows:

$$T = W\Sigma V^* = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Thus

$$\begin{aligned}
\hat{\Sigma} &= V\Sigma^{\frac{1}{2}} V^* \\
&= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

and

$$WV^* = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Hence the Aluthge transform of T is

$$\begin{aligned}\widetilde{T} &= \hat{\Sigma}WV^*\hat{\Sigma}\\
&= \begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&2^{\frac{1}{2}}&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\-1&0&0&0\end{pmatrix}\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&2^{\frac{1}{2}}&0\\0&0&0&1\end{pmatrix}\\
&= \begin{pmatrix}0&0&0&0\\0&0&2^{\frac{1}{2}}&0\\0&0&0&2^{\frac{1}{2}}\\0&0&0&0\end{pmatrix}.\end{aligned}$$

On the other hand, the singular valued decomposition of $\widetilde{T}$ is

$$\widetilde{T} = W_1\Sigma_1V_1^* = \begin{pmatrix}0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}2^{\frac{1}{2}}&0&0&0\\0&2^{\frac{1}{2}}&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}0&0&0&1\\0&0&1&0\\1&0&0&0\\0&1&0&0\end{pmatrix}.$$

Thus we get that

$$\begin{aligned}\hat{\Sigma_1} &= V_1\Sigma_1^{\frac{1}{2}}V_1^*\\
&= \begin{pmatrix}0&0&1&0\\0&0&0&1\\0&1&0&0\\1&0&0&0\end{pmatrix}\begin{pmatrix}2^{\frac{1}{4}}&0&0&0\\0&2^{\frac{1}{4}}&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}0&0&0&1\\0&0&1&0\\1&0&0&0\\0&1&0&0\end{pmatrix}\\
&= \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&2^{\frac{1}{4}}&0\\0&0&0&2^{\frac{1}{4}}\end{pmatrix}\end{aligned}$$

and

$$W_1\Sigma_1V_1^* \quad = \quad \begin{pmatrix}0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}0&0&0&1\\0&0&1&0\\1&0&0&0\\0&1&0&0\end{pmatrix} = \begin{pmatrix}1&0&0&0\\0&0&1&0\\0&0&0&1\\0&1&0&0\end{pmatrix}.$$

Hence the 2-th iterated Aluthge transform of $\tilde{T}^{(2)}$ is

$$\begin{aligned}\widetilde{T}^{(2)} &= \hat{\Sigma_1}W_1V_1^*\hat{\Sigma_1}\\
&= \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&2^{\frac{1}{4}}&0\\0&0&0&2^{\frac{1}{4}}\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&0&0&1\\0&1&0&0\end{pmatrix}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&2^{\frac{1}{4}}&0\\0&0&0&2^{\frac{1}{4}}\end{pmatrix}\\
&= \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&2^{\frac{1}{2}}\\0&0&0&0\end{pmatrix}.\end{aligned}$$

Remark 2.9. In Example 2.7, $\Sigma\Sigma^\dagger = \begin{pmatrix} 2^{\frac{1}{2}} & 0 & 0 \\ 0 & 2^{\frac{1}{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2^{\frac{1}{2}} & 0 & 0 \\ 0 & 2^{\frac{1}{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}^\dagger$ is the identity matrix. But, in Example 2.8, $\Sigma\Sigma^\dagger = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^\dagger$ is not the identity matrix.

Remark 2.10. For a given matrix T in $\mathbb{C}^{n\times n}$, it is not always easy to find the polar decomposition of T. However, from the method in Theorem 2.2, we can obtain the polar decomposition of the given matrix T as follows.

(i) If $T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix}$, then by Corollary 2.4, $U = W\Sigma\Sigma^\dagger V^\dagger = \begin{pmatrix} 0 & 0 & 1 \\ 2^{-\frac{1}{2}} & 2^{-\frac{1}{2}} & 0 \\ 2^{-\frac{1}{2}} & -2^{-\frac{1}{2}} & 0 \end{pmatrix}$ and $|T| = \begin{pmatrix} 2^{\frac{1}{2}} & 0 & 0 \\ 0 & 2^{\frac{1}{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Hence we get that

$$U|T| = \begin{pmatrix} 0 & 0 & 1 \\ 2^{-\frac{1}{2}} & 2^{-\frac{1}{2}} & 0 \\ 2^{-\frac{1}{2}} & -2^{-\frac{1}{2}} & 0 \end{pmatrix}\begin{pmatrix} 2^{\frac{1}{2}} & 0 & 0 \\ 0 & 2^{\frac{1}{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix} = T$$

is the polar decomposition of T.

(ii) Let $T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then $U = W\Sigma\Sigma^\dagger V^\dagger = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $|T| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ from Corollary 2.4. Therefore we obtain that

$$U|T| = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is the polar decomposition of T.

Recall that the *Duggal transform* $\widetilde{T}^D$ of T is given by $\widetilde{T}^D := |T|U$. The *mean transform* $\widehat{T}^M$ of T is defined by $\widehat{T}^M := \frac{1}{2}(T + \widetilde{T}^D)$.

Proposition 2.11. *Let T be an $n \times n$ matrix. If $W\Sigma V^*$ is the singular value decomposition of T, then the following statements hold.*
(i) $\widetilde{T}^D = V\Sigma V^* W\Sigma\Sigma^\dagger V^*$ *is the Duggal transform of T.*
(ii) $\widehat{T}^M = \frac{1}{2}\left(W\Sigma V^* + V\Sigma V^* W\Sigma\Sigma^\dagger V^*\right)$ *is the mean transform of T.*

Proof. Since $U = W\Sigma\Sigma^\dagger V^*$ is a partial isometry and $|T| = V\Sigma V^*$, the proof follows from the proof of Theorem 2.2 and the definitions of Duggal and Mean transforms of T, respectively. □

Example 2.12. (i) If $T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix}$, then Remark 2.10 gives that

$$\widetilde{T}^D = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \sqrt{2} \\ 1 & 1 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

and $\widetilde{T}^M = \frac{1}{2}\left(\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \sqrt{2} \\ 1 & 1 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & \frac{1+\sqrt{2}}{2} \\ 1 & 1 & 0 \\ \frac{2+\sqrt{2}}{4} & -\frac{2+\sqrt{2}}{4} & 0 \end{pmatrix}$. Moreover, we have

$$(\widetilde{T}^D)^{(2)} = \begin{pmatrix} \frac{1+\sqrt{2}}{2} & \frac{-1+\sqrt{2}}{2} & 0 \\ \frac{-1+\sqrt{2}}{2} & \frac{1+\sqrt{2}}{2} & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \end{pmatrix} = \begin{pmatrix} \frac{2-\sqrt{2}}{4} & \frac{2-\sqrt{2}}{4} & \frac{1+\sqrt{2}}{2} \\ \frac{2+\sqrt{2}}{4} & \frac{2+\sqrt{2}}{4} & \frac{-1+\sqrt{2}}{2} \\ 1 & -1 & 0 \end{pmatrix}$$

and

$$\begin{aligned}
(\widetilde{T}^M)^{(2)} &= \frac{1}{2}(\widetilde{T}^M + \widetilde{(\widetilde{T}^M)}^D) \\
&= \frac{1}{2}\left(\begin{pmatrix} 0 & 0 & \frac{1+\sqrt{2}}{2} \\ 1 & 1 & 0 \\ \frac{2+\sqrt{2}}{4} & -\frac{2+\sqrt{2}}{4} & 0 \end{pmatrix} + \begin{pmatrix} \frac{2-\sqrt{2}}{8} & \frac{2-\sqrt{2}}{8} & \frac{1+3\sqrt{2}}{4} \\ \frac{6+\sqrt{2}}{8} & \frac{6+\sqrt{2}}{8} & \frac{-1+\sqrt{2}}{4} \\ \frac{2+\sqrt{2}}{4} & -\frac{2+\sqrt{2}}{4} & 0 \end{pmatrix}\right) \\
&= \begin{pmatrix} \frac{2-\sqrt{2}}{16} & \frac{2-\sqrt{2}}{16} & \frac{3+5\sqrt{2}}{8} \\ \frac{14+\sqrt{2}}{16} & \frac{14+\sqrt{2}}{16} & \frac{-1+\sqrt{2}}{8} \\ \frac{2+\sqrt{2}}{4} & -\frac{2+\sqrt{2}}{4} & 0 \end{pmatrix}.
\end{aligned}$$

(ii) Let $T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then Remark 2.10 implies that

$$\widetilde{T}^D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $\widetilde{T}^M = \frac{1}{2}\left(\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Moreover, by a similar way, we have $\widetilde{(\widetilde{T}^M)}^D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and

$$(\widetilde{T}^M)^{(2)} = \frac{1}{2}(\widetilde{T}^M + \widetilde{(\widetilde{T}^M)}^D) = \begin{pmatrix} 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Theorem 2.13. *Let T be an $n \times n$ matrix. If $T = W\Sigma V^*$ is the singular value decomposition of T, then the following statements hold.*
(i) *The n-th iterated Duggal transform of T is*

$$\widetilde{T}^{D^{(n)}} = V_{n-1}\Sigma_{n-1}V^*_{n-1}W_{n-1}\Sigma_{n-1}\Sigma_{n-1}{}^{\dagger}V^*_{n-1}$$

for $n \geq 1$ *where* $\hat{\Sigma}_0 = \hat{\Sigma}$, $W_0 = W$, *and* $V_0^* = V^*$.
(ii) *The n-th iterated mean transform of T is*

$$\widetilde{T^M}^{(n)} = \frac{1}{2}\Big(W_{n-1}\Sigma_{n-1}V_{n-1}^* + V_{n-1}\Sigma_{n-1}V_{n-1}^*W_{n-1}\Sigma_{n-1}\Sigma_{n-1}^{\dagger}V_{n-1}^*\Big)$$

where $\hat{\Sigma}_0 = \hat{\Sigma}$, $W_0 = W$, *and* $V_0^* = V^*$.

Proof. (i) If $n = 1$, then it is clear by Proposition 2.11. Assume that the n-th iterated Duggal transform of T is

$$V_{n-1}\Sigma_{n-1}V_{n-1}^*W_{n-1}\Sigma_{n-1}{\Sigma_{n-1}}^{\dagger}V_{n-1}^*$$

for each $n \geq 1$. Then we can find the singular value decomposition of $\widetilde{T^D}^{(n)}$ as follows;

$$\widetilde{T^D}^{(n)} = W_n\Sigma_nV_n^*.$$

By the similar way of Proposition 2.11, we get that the $(n+1)$-th iterated Duggal transform of T is

$$\widetilde{T^D}^{(n+1)} = V_n\Sigma_nV_n^*W_n\Sigma_n{\Sigma_n}^{\dagger}V_n^*.$$

(ii) If $n = 1$, then it is clear by Proposition 2.11. Assume that the n-th iterated mean transform of T is

$$\widetilde{T^M}^{(n)} = \frac{1}{2}\Big(W_{n-1}\Sigma_{n-1}V_{n-1}^* + V_{n-1}\Sigma_{n-1}V_{n-1}^*W_{n-1}\Sigma_{n-1}\Sigma_{n-1}^{\dagger}V_{n-1}^*\Big)$$

for each $n \geq 1$. Then we can find the singular value decomposition of $\widetilde{T^M}^{(n)}$ as follows;

$$\widetilde{T^M}^{(n)} = W_n\Sigma_nV_n^*.$$

By the similar way of Proposition 2.11, we obtain that the $(n+1)$-th iterated mean transform of T is

$$\frac{1}{2}\Big(W_n\Sigma_nV_n^* + V_n\Sigma_nV_n^*W_n\Sigma_n\Sigma_n^{\dagger}V_n^*\Big).$$

So, we complete the proof. □

Remark 2.14. Let $T \in \mathbb{C}^{n\times n}$ be a polar decomposition of T. If T is normal, then $U|T| = |T|U$ holds from [4]. Hence we have

$$\widetilde{T} = \widetilde{T}^D = \widetilde{T}^M.$$

For the matrix T in Examples 2.7 and 2.8, we know that T is not normal. There exists a $k \in \mathbb{N}$ such that $\widetilde{T}^{(k)}$, $(\widetilde{T}^D)^{(k)}$, and $(\widetilde{T}^M)^{(k)}$ are not normal. But, the limit points of the sequences $\{\widetilde{T}^{(k)}\}$ and $\{(\widetilde{T}^M)^{(k)}\}$ become normal.

3. Algorithm: Python code implement

3.1. Directionality

We aim to create a module that can perform a lot of iterations with a small amount of computation. We want to check when each iterated transforms of a matrix becomes a normal matrix. We iterate transformations of each matrix defined in Examples 2.7 and 2.8 via Python and observe the results.

3.2. Process

We decompose the input matrix into three matrices through the singular value decomposition to make the necessary materials for transformation. Then we reconstruct them to perform the transformation we want. After repeating this process several times and saving the series of processes, we visually show that the result of the transformation gradually satisfies the normality through the Frobenius norm. The table below briefly shows what each method does.

Table 1: Methods

Class : `Transformation`	
Method Name	description

`Transform`	The three transformations (Aluthge, Duggal and Mean) are executed as many times as the number of iterations received and the outputs are saved in the form of `list` (`Transformation.total_info`).
	
	`Transformation.total_info=`
	`[ Transformation.Aluthge_info,`
	`Transformation.Duggal_info,`
	`Transformation.Mean_info ]`

`Normal_calculator`	Computing the Frobenius norm and normal characteristic - $\|\widetilde{T}\|$ and $\|\widetilde{T}\widetilde{T}^* - \widetilde{T}^*\widetilde{T}\|$ - of matrices in the `Transformation.total_info`. Then each result values are stored in `Transformation.norm_info` and `Transformation.normal_info` in the form of `list`, respectively.

Computing Aluthge transformation does not use the pseudo-inverse as the result is the same whether $\Sigma\Sigma^\dagger$ is invertible or not. However, since our purpose is to look at the flow within the same number of iterations for not only Aluthge transformation but also Duggal transformation and Mean transformation, we compute it by applying $U = W\Sigma\Sigma^\dagger V^*$ uniformly.

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Algorithm 1 Algorithm of only one ciycle

Input: $T \in \mathbb{C}^{n\times n}$
Output: $\widetilde{T}, \widetilde{T}^D, \widetilde{T}^M$

$W, \Sigma, V^* \leftarrow$ singular value decomposition of T
$|T| \leftarrow V\Sigma V^*$
$\Sigma^{\frac{1}{2}} \leftarrow$ element-wise square root of Σ
$U \leftarrow W\Sigma\Sigma^\dagger V^*$
$\hat{\Sigma} \leftarrow V\Sigma^{\frac{1}{2}}V^*$

$\widetilde{T} \leftarrow \hat{\Sigma}U\hat{\Sigma}; \quad \widetilde{T}^D \leftarrow |T|U; \quad \widetilde{T}^M \leftarrow \frac{1}{2}(T + \widetilde{T}^D)$

3.3. Example

Example 3.1. Let T define the same as in the Example 2.7. Then the graph of the 10-th iterated Aluthge, Duggal, and Mean transform of T are as follows. Each of the three transformations is iterated 10 times using the Python code, and the results of the two iterations are summarized in the table below (display up to 4 decimal places).

Table 2: Example 3.1

Matrix = [[0 , 0 , 1] , [1 , 1 , 0] , [1 , -1 , 0]]	
Transformation	Output
Aluthge Transform	[[0 , 0 , 1.1892] , [1 , 1 , 0] , [0.8409 , -0.8409 , 0]] [[0.0415 , 0.0415 , 1.243] , [0.9585 , 0.9585 , 0.0538] , [0.8409 , -0.8409 , 0]]
Duggal Transform	[[0 , 0 , 1.4142] , [1 , 1 , 0] , [0.7071 , -0.7071 , 0]] [[0.1464 , 0.1464 , 1.2071] , [0.8536 , 0.8536 , 0.2071] , [1 , -1 , 0]]
Mean Transform	[[0 , 0 , 1.2071] , [1 , 1 , 0] , [0.8536 , -0.8536 , 0]] [[0.0366 , 0.0366 , 1.2589] , [0.9634 , 0.9634 , 0.0518] , [0.8536 , -0.8536 , 0]]

The following is a plot of the normal characteristic of each transformation.

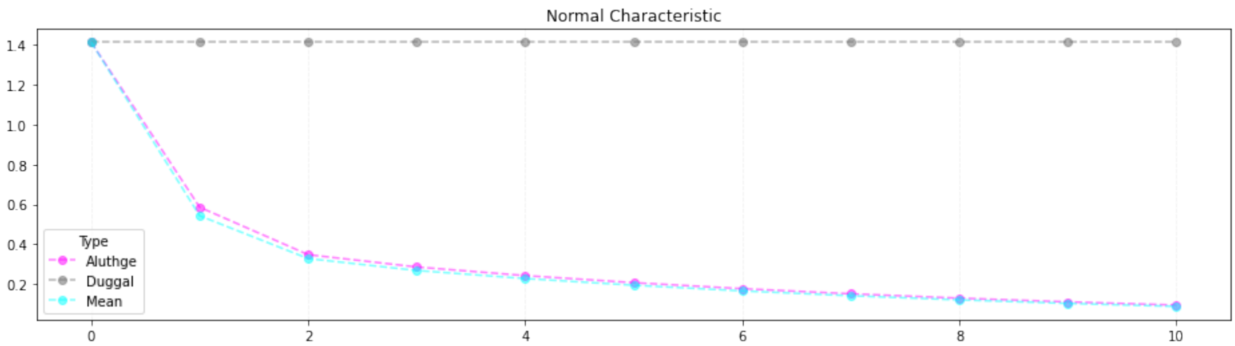

Figure 1: Example 3.1

It is confirmed that the Mean transformation converges to the normal matrix slightly faster than the Aluthge transformation, but Duggal Transformation does not converge to the normal matrix.

Example 3.2. Let T define the same as in the Example 2.8. All the steps are same as the previous example.

Table 3: Example 3.2

Matrix = [[0 , 1 , 0 , 0] , [0 , 0 , 2 , 0] , [0 , 0 , 0 , 1] , [0 , 0 , 0 , 0]]	
Transformation	Output
Aluthge Transform	[[0 , 0 , 0 , 0] , [0 , 0 , 1.4142 , 0] , [0 , 0 , 0 , 1.4142] , [0 , 0 , 0 , 0]] [[0 , 0 , 0 , 0] , [0 , 0 , 0 , 0] , [0 , 0 , 0 , 1.4142] , [0 , 0 , 0 , 0]]
Duggal Transform	[[0 , 0 , 0 , 0] , [0 , 0 , 1 , 0] , [0 , 0 , 0 , 2] , [0 , 0 , 0 , 0]] [[0 , 0 , 0 , 0] , [0 , 0 , 0 , 0] , [0 , 0 , 0 , 1] , [0 , 0 , 0 , 0]]
Mean Transform	[[0 , 0.5 , 0 , 0] , [0 , 0 , 1.5 , 0] , [0 , 0 , 0 , 1.5] , [0 , 0 , 0 , 0]] [[0 , 0.25 , 0 , 0] , [0 , 0 , 1 , 0] , [0 , 0 , 0 , 1.5] , [0 , 0 , 0 , 0]]

The following is a plot of the normal characteristic of each transformation.

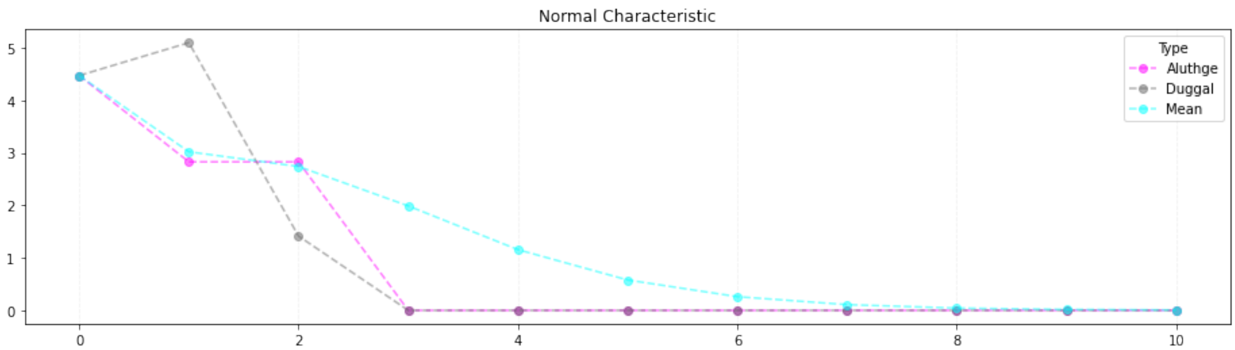

Figure 2: Example 3.2

Unlike the previous Example 3.1, it is confirmed that Aluthge transformation and Duggal transformation converge to the normal matrix after the third iteration, and the Mean transformation converges relatively slowly.

If you would like to use our open source code and additional examples, please follow the address.
`https://github.com/MinwooPark96/Aluthge-Duggal-and-Mean-Transformation.git`

4. Conclusion

In general, it is not easy to find the Aluthge transform, the Duggal transform, and the Mean transform using the polar decomposition of an $n \times n$ matrix. By applying SVD, we can more easily obtain Aluthge transform, Duggal transform, and Mean transform of an $n \times n$ matrix (see Theorem 2.2 and Proposition 2.11). Through an algorithm with Python, we verify that the n-th iterated Aluthge transform and n-th iterated Mean transform of a matrix converge to a normal matrix. However, in the case of the Duggal transform, it is confirmed through several examples that the matrix obtained by the n-th iterated Duggal transform does not necessarily become a normal operator (see Figures 1 and 2).

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