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On topological gyrogroups

Piyu Li[a], Rongxin Shen[b]

[a] *School of Mathematics and Statistics, Xuzhou Institute of Technology, Xuzhou 221018, China*
[b] *School of Mathematics and Physics, Taizhou University, Taizhou 225300, China*

Abstract. The concept of gyrogroups is a generalization of groups which do not explicitly have associativity. In this paper, we show that every first-countable strongly topological gyrogroup admits a left-invariant metric generating the original topology of it and every T_0 compact paratopological gyrogroup is a Hausdorff compact topological gyrogroup. Also, some basic properties on topological gyrogroups and paratopological gyrogroups are discussed.

1. Introduction

The concept of gyrogroup was firstly posed by A. A. Ungar in the study of c-ball of relativistically admissible velocities with Einstein velocity addition [21]. The Einstein velocity addition $\oplus_E$ in the c-ball is given by the following equation

$$\mathbf{u} \oplus_E \mathbf{v} = \frac{1}{1 + \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{c^2}} \{\mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v} + \frac{1}{c^2}\frac{\gamma_{\mathbf{u}}}{1 + \gamma_{\mathbf{u}}}\langle \mathbf{u}, \mathbf{v} \rangle \mathbf{u}\},$$

where $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3_c = \{\mathbf{v} \in \mathbb{R}^3 : ||\mathbf{v}|| < c\}$ and $\gamma_{\mathbf{u}}$ is the Lorentz factor given by

$$\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1 - \frac{||\mathbf{u}||^2}{c^2}}}.$$

The system $(\mathbb{R}^3_c, \oplus_E)$ does not form a group since $\oplus_E$ is not associative. Loosely speaking, a gyrogroup (see Definition 2.1) is a nonassociative group-like structure that shares many properties with groups and, in fact, every group may be viewed as a gyrogroup with trivial gyroautomorphisms. It turns out that gyrogroups share remarkable analogies with groups. Several well-known results in group theory can be naturally extended to the case of gyrogroups such as the Lagrange theorem [18], the fundamental isomorphism theorems, the Cayley theorem [19], the orbit-stabilizer theorem, the class equation, and the Burnside lemma [17].

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Email addresses: lpy91132006@aliyun.com (Piyu Li), srx20212021@163.com (Rongxin Shen)

From the topological aspect, Atiponrat [1] firstly introduced the notion of topological gyrogroups and the separation axioms and some basic properties of the topological gyrogroups were studied. In particular, Cai et al. [9] extended the famous Birkhoff-Kakutani theorem by proving that every first-countable Hausdorff topological gyrogroup is metrizable [9, Theorem 2.3]. In 2019, M. Bao and F. Lin [5] defined the strongly topological gyrogroups and proved that every feathered strongly topological gyrogroup is paracompact. In fact, this kind of spaces has been studied for many years, see [3, 7, 13, 14, 16, 22, 23]. In [2, 12], the authors studied some separation axioms of paratopological gyrogroups. However, the conditions under which a paratopolgical gyrogroup turns to be a topological gyrogroup was not considered. As studied in paratoplogical groups, it happens quite frequently that a paratopological group satisfying a natural compactness-type condition turns out to be a topological group [20].

In this paper, we study some properties of topological gyrogroups. We mainly show that every first-countable strongly topological gyrogroup admits a left-invariant metric generating its topology and every T_0 compact paratopological gyrogroup is a topological gyrogroup.

2. Preliminaries

In this section, we introduce the necessary notations, terminologies and some facts about topological gyrogroups.

Let G be a non-empty set and $\oplus : G \times G \to G$ a binary operation on G. Then the pair $(G, \oplus)$ is called a *groupoid* or a *magma*. An automorphism φ of a groupoid $(G, \oplus)$ is a bijective self-mapping of G, $\varphi : G \to G$, which preserves its groupoid operation, that is, $\varphi(a \oplus b) = \varphi(a) \oplus \varphi(b)$ for every $a, b \in G$. The symbol $Aut(G, \oplus)$ denotes the set of all automorphisms of a groupoid $(G, \oplus)$.

Definition 2.1. ([22]) Let $(G, \oplus)$ be a groupoid. The system $(G, \oplus)$ is called a gyrogroup, if its binary operation satisfies the following conditions:

(G1) There exists a unique identity element $e \in G$ such that $e \oplus a = a = a \oplus e$ for all $a \in G$.

(G2) For each $x \in G$, there exists a unique inverse element $\ominus x \in G$ such that $\ominus x \oplus x = e = x \oplus (\ominus x)$.

(G3) For all $x, y \in G$, there exists $gyr[x, y] \in Aut(G, \oplus)$ with the property that $x \oplus (y \oplus z) = (x \oplus y) \oplus gyr[x, y](z)$ for all $z \in G$.

(G4) For any $x, y \in G$, $gyr[x \oplus y, y] = gyr[x, y]$.

Notice that a group is a gyrogroup $(G, \oplus)$ such that $gyr[x, y]$ is the identity function for all $x, y \in G$.

Proposition 2.2. ([22]) *Let $(G, \oplus)$ be a gyrogroup and $a, b, c \in G$. Then:*

(1) $\ominus(\ominus a) = a$;

(2) $\ominus a \oplus (a \oplus b) = b$;

(3) $\ominus(a \oplus b) = gyr[a, b](\ominus b \ominus a)$;

(4) $gyr[a, b] = gyr^{-1}[b, a]$, *the inverse of* $gyr[b, a]$.

Definition 2.3. ([1]) A triple $(G, \tau, \oplus)$ is called a topological gyrogroup if and only if

(1) (G, τ) is a topological space;

(2) $(G, \oplus)$ is a gyrogroup;

(3) The binary operation $\oplus : G \times G \to G$ is continuous where $G \times G$ is endowed with the product topology and the operation of taking the inverse $\ominus(\cdot) : G \to G$, i.e. $x \to \ominus x$, is continuous.

If a triple $(G, \tau, \oplus)$ satisfies the first two conditions and its binary operation is continuous, we call such triple a paratopological gyrogroup [2]. Sometimes we will just say that G is a topological gyrogroup (paratopological gyrogroup) if the binary operation and the topology are clear from the context.

Definition 2.4. ([5]) Let G be a topological gyrogroup. We say that G is a strongly topological gyrogroup if there exists a neighborhood base U of the identity element e such that, for every $U \in \mathcal{U}$, $gyr[x, y](U) = U$ for any $x, y \in G$. For convenience, we say that G is a strongly topological gyrogroup with neighborhood base $\mathcal{U}$ of e. Clearly, we may assume that U is symmetric for each $U \in \mathcal{U}$.

Throughout this paper, all topological spaces are assumed to be T_1, unless otherwise is explicitly stated. The readers may consult [4, 10] for notations and terminologies not explicitly given here.

3. First-countable strongly topological gyrogroups

In this section we consider the left-invariant metric on first-countable strongly topological gyrogroups.

We first consider a continuous prenorm on a topological gyrogroup. Let G be a gyrogroup with an identity element e and N a real-valued function on G. At first, we assume that there is no any topology on G. We call N a *prenorm* on G if the following conditions are satisfied for all $x, y \in G$:

(1) $N(e) = 0$;

(2) $N(x \oplus y) \le N(x) + N(y)$;

(3) $N(\ominus x) = N(x)$.

Lemma 3.1. ([5]) *Let G be a strongly topological gyrogroup with the symmetric neighborhood base $\mathcal{U}$ at the identity element e, and let $\{U_n : n \in \mathbb{N}\}$ and $\{V(m/2^n) : n, m \in \mathbb{N}\}$ be two sequences of open neighborhoods satisfying the following conditions* (1)-(5)*:*

(1) *$U_n \in \mathcal{U}$ for each $n \in \mathbb{N}$;*

(2) *$U_{n+1} \oplus U_{n+1} \subseteq U_n$, $n \in \mathbb{N}$;*

(3) *$V(1) = U_0$;*

(4) *For any $n \ge 1$, put $V(1/2^n) = U_n, V(2m/2^n) = V(m/2^{n-1})$ for $m = 1, \cdots, 2^{n-1}$, and $V((2m+1)/2^n) = U_n \oplus V(m/2^{n-1}) = V(1/2^n) \oplus V(m/2^{n-1})$ for each $m = 1, \cdots, 2^{n-1} - 1$;*

(5) *$V(m/2^n) = G$ when $m > 2^n$.*

Then there exists a prenorm N on G satisfies the following conditions:

(a) *for any fixed $x, y \in G$, we have $N(\mathrm{gyr}[x, y](z)) = N(z)$ for any $z \in G$;*

(b) *for any $n \in \mathbb{N}$, $\{x \in G : N(x) < 1/2^n\} \subseteq U_n \subseteq \{x \in G : N(x) \le 2/2^n\}$.*

In the proof of the above lemma, the authors defined a real-valued function f on G as follows $f(x) = inf\{r > 0 : x \in V(r)\}$ for each $x \in G$ and $N(x) = sup_{g \in G}|f(x \oplus g) - f(g)|$. One can easily show that $gyr(V(\frac{m}{2^n})) = V(\frac{m}{2^n})$ for each $n \in \mathbb{N}$ and $m \in \mathbb{N} \setminus \{0\}$.

Theorem 3.2. *Every first-countable strongly topological gyrogroup G admits a left-invariant metric ϱ generating the original topology of G.*

Proof. Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be a countable symmetric neighborhood base at the identity element e such that $gyr[x, y](U) = U$ for any $x, y \in G$ and $U_{n+1} \oplus U_{n+1} \subseteq U_n$ for each $n \in \mathbb{N}$. By Lemma 3.1, choose a continuous prenorm N on G which satisfies $N(gyr[x, y](z)) = N(z)$ for any $x, y, z \in G$ and

$$\{x \in G : N(x) < 1/2^n\} \subseteq U_n \subseteq \{x \in G : N(x) \le 2/2^n\},$$

for each integer $n \ge 0$. Put $B_N(1/2^n) = \{x \in G : N(x) < 1/2^n\}$. It is clear that the open sets $B_N(1/2^n)$ also form a base of G at e.

Now, for arbitrary x and y in G, put $\varrho_N(x, y) = N(\ominus x \oplus y)$. Let us show that ϱ is a metric on G. Let us show that ϱ_N is a metric on G generating the original topology on G.

(1) Clearly, $\varrho_N(x, y) = N(\ominus x \oplus y) \ge 0$, for every $x, y \in G$. At the same time, $\varrho_N(x, x) = N(e) = 0$, for each $x \in G$. Assume that $\varrho_N(x, y) = N(\ominus x \oplus y) = 0$, that is, $N(\ominus x \oplus y) = 0$. Then, for each $n \in \mathbb{N}$, $\ominus x \oplus y \in \{x \in G : N(x) < 1/2^n\} \subseteq U_n$. Since $\{e\} = \bigcap_{n \in \mathbb{N}} U_n$, it follows that $\ominus x \oplus y = e$, that is, $x = y$.

(2) For every $x, y \in G$, by [22, Theorem 2.11], we have

$$\begin{aligned}\varrho_N(x, y) &= N(\ominus x \oplus y) = N(\ominus(\ominus x \oplus y)) \\ &= N(gyr[\ominus x, y](\ominus y \oplus x)) = N(\ominus y \oplus x) = \varrho_N(y, x).\end{aligned}$$

(3) For every $x, y, z \in G$, it follows from [22, Theorem 2.11] that

$$\begin{aligned}\varrho_N(x, y) &= N(\ominus x \oplus y) \\ &= N((\ominus x \oplus z) \oplus gyr[\ominus x, z](\ominus z \oplus y)) \\ &\le N(\ominus x \oplus z) + N(gyr[\ominus x, z](\ominus z \oplus y)) \\ &= \varrho_N(x, z) + \varrho_N(z, y)\end{aligned}$$

Thus, ϱ_N is a metric on G. In the following, we show ϱ_N is left-invariant, that is, $\varrho_N(x, y) = \varrho_N(z \oplus x, z \oplus y)$ for all $x, y, z \in G$. Indeed,

$$
\begin{aligned}
&(\ominus x \oplus y) \oplus g \in V(r) \\
\Leftrightarrow\ & \ominus x \oplus (y \oplus \mathrm{gyr}[y, \ominus x](g)) \in V(r) \\
\Leftrightarrow\ & y \oplus \mathrm{gyr}[y, \ominus x](g) \in x \oplus V(r) \\
\Leftrightarrow\ & z \oplus (y \oplus \mathrm{gyr}[y, \ominus x](g)) \in z \oplus (x \oplus V(r)) = (z \oplus x) \oplus V(r) \\
\Leftrightarrow\ & (z \oplus y) \oplus \mathrm{gyr}[z, y](\mathrm{gyr}[y, \ominus x](g)) \in (z \oplus x) \oplus V(r) \\
\Leftrightarrow\ & \ominus(z \oplus x) \oplus ((z \oplus y) \oplus \mathrm{gyr}[z, y]\mathrm{gyr}[y, \ominus x](g)) \in V(r) \\
\Leftrightarrow\ & (\ominus(z \oplus x) \oplus (z \oplus y)) \oplus \mathrm{gyr}[\ominus(z \oplus x), z \oplus y]\mathrm{gyr}[z, y]\mathrm{gyr}[y, \ominus x](g) \in V(r) \\
\Leftrightarrow\ & \mathrm{gyr}[\ominus x, y]\mathrm{gyr}[y, z]\mathrm{gyr}[z \oplus y, \ominus(z \oplus x)](\ominus(z \oplus x) \oplus (z \oplus y)) \oplus g \in V(r)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
N(\ominus x \oplus y) &= N(\mathrm{gyr}[\ominus x, y]\mathrm{gyr}[y, z]\mathrm{gyr}[z \oplus y, \ominus(z \oplus x)](\ominus(z \oplus x) \oplus (z \oplus y))) \\
&= N(\ominus(z \oplus x) \oplus (z \oplus y)).
\end{aligned}
$$

This implies that $\varrho_N(x, y) = \varrho_N(z \oplus x, z \oplus y)$ for all $x, y, z \in G$ and ϱ_N is left-invariant.

Since $B_N(\varepsilon)$ is the spherical ϱ_N-neighborhood of the identity element e of radius ε, it follows that the the spherical ϱ_N-neighborhood of x is precisely the set $x \oplus B_N(\varepsilon)$. Take any point $x \in G$. Since the sets $B_N(1/2^n)$ form a base of e, and G is a strongly topological gyrogroup, the sets $x \oplus B_N(1/2^n)$ constitute a base of G at x. Thus the metric ϱ_N generating the original topology of the space G, that is, G is metrizable by a left-invariant metric. □

The authors in [8, Corollary 3.14] proved every first-countable left ω-narrow strongly topological gyrogroup is separable. Since, by Proposition 3.2, every first-countable strongly topological gyrogroup admits a left-invariant metric generating the original topology. Hence, the following result improves [8, Corollary 3.14].

Proposition 3.3. *The left ω-narrow topological gyrogroup G admits a left-invariant metric ϱ generating the original topology of G. Then G is separable.*

Proof. Let G be a first-countable left ω-narrow topological gyrogroup and $\{U_n : n \in \mathbb{N}\}$ a local base at the identity element $e \in G$.

First we construct a countable subset C of G such that $C \oplus U_n = G$ for each n. Indeed, for each $n \in \mathbb{N}$, take a countable subset $A_n \subseteq G$ such that $A_n \oplus U_n = G$. Put $C_n = A_n \cup (\ominus A_n) \cup \{e\}$ and $C = \bigcup_{n=1}^{\infty} C_n$. Then C_n and C are countable subsets of G.

We claim that C is dense in G. To see this, let U be an arbitrary non-empty open subset of G. Fix a point $x \in U$. There exist $n \in \mathbb{N}$ such that $B(x, \frac{1}{n}) = \{y \in G : \varrho(x, y) < \frac{1}{n}\} \subseteq U$. Take $m > n$. Then $C \oplus B(e, \frac{1}{m}) = G$. It follows that there exist $h \in C$ and $u_m \in B(e, \frac{1}{m})$ such that $x = h \oplus u_m$. Hence $u_m = \ominus h \oplus x \in B(e, \frac{1}{m})$. This implies that $\varrho(\ominus h \oplus x, e) < \frac{1}{m}$. Since ϱ is a left-invariant metric on G, $\varrho(x, h) = \varrho(\ominus h \oplus x, e) < \frac{1}{m} < \frac{1}{n}$. It follows that $h \in B(x, \frac{1}{n}) \subseteq U$, which completes the proof. □

At the end of this section, we give a result about first-countable right ω-narrow topological gyrogroups.

Proposition 3.4. *Every first-countable right ω-narrow topological gyrogroup is separable.*

Proof. Let G be a first-countable right ω-narrow topological gyrogroup and $\{U_n : n \in \mathbb{N}\}$ a local base at the identity element $e \in G$ consisting of symmetric open subsets.

For each $n \in \mathbb{N}$, take a countable subset $A_n \subseteq G$ such that $U_n \oplus A_n = G$. Put $A = \bigcup_{n=1}^{\infty} A_n$. Then A is a countable subsets of G and $U_n \oplus A = G$ for each $n \in \mathbb{N}$.

We claim that A is dense in G. To see this, let U be an arbitrary non-empty open subset of G. Fix a point $x \in U$. Then there exists an open neighborhood U_n of e such that $U_n \oplus x \subseteq U$. Since $U_n \oplus A = G$, there exists $u_n \in U_n$ and $a \in A$ such that $u_n \oplus a = x$. It follows that

$$a = \ominus u_n \oplus x \in U_n \oplus x \subseteq U.$$

This implies that A is dense in G and G is separable. □

4. Paratoplogical gyrogroups

In this section, we consider some conditions under which a paratopological gyrogroup is a topological gyrogroup.

Proposition 4.1. *Let X be a compact Hausdorff paratopological gyrogroup. Then the inverse operation in X is continuous and, therefore, X is a topological gyrogroup.*

Proof. Let e be the identity element of X. Since X is Hausdorff, the set $M = \{(x, y) \in X \times X : x \oplus y = e\}$ is closed in $X \times X$.

Now, take any closed subset F of X, and put $P = (X \times F) \cap M$. Then F and $X \times F$ are compact, P closed in $X \times F$, since M is closed. It follows that P is compact. Now, $(x, y) \in P$ if and only if $y \in F$ and $x \oplus y = e$, that is, $x = \ominus y$. It follows that the image of P under the natural projection of $X \times X$ onto the first factor X is precisely $\ominus F$. Since F is compact and the projection mappings is continuous, we conclude that $\ominus F$ is compact, and therefore, closed in X. Thus, the inverse operation in X is continuous, which completes the proof. □

In the following, we establish that a T_0 compact paratopological gyrogroup is a topological gyrogroup. To begin with, we give a property of the Alexandroff specialization order in compact spaces. Given a topological space (X, τ), the Alexandroff specialization order is the partial order defined as

$$x \leq_\tau y \text{ if and only if} x \in cl_\tau\{y\}.$$

For the Alexandroff specialization order, the following result was established.

Theorem 4.2. ([15]) *Let (X, τ) be a T_0 compact topological space. For each $x \in X$, the set*

$$P(x) = \{y \in X : y \leq_\tau x\}$$

has a minimal element.

With the method in [15, Theorem 2.4], we give the following result.

Theorem 4.3. *Every T_0 compact paratopological gyrogroup (G, τ) is a Hausdorff compact topological gyrogroup.*

Proof. In the following proof, we will show that (G, τ) is Hausdorff. Thus, together with Proposition 4.1, one can get the result. First we give the following claim.

Claim 1. (G, τ) is T_1.

Indeed, take $x \in G$. By Theorem 4.2, there is a minimal element, say t, of the set $P(x)$. It is clear that $\{t\}$ is τ-closed. Now for each $y \in S$, since the right translation $x \mapsto x \oplus y$ is a homeomorphism, $\{t \oplus y\}$ is also τ-closed. Since (G, τ) is a paratopological gyrogroup, if $x \in cl_\tau\{y\}$, we have $t \oplus x \in cl_\tau\{t \oplus y\}$ which implies that $t \oplus x = t \oplus y$. It follows that $x = y$. Thus, (G, τ) is a T_1 space.

Claim 2. (G, τ) is Hausdorff.

Indeed, let (G, τ) be a T_1 compact paratopological gyrogroup. Let $(x_\delta)_{\delta \in D}$ be a net in G which τ-converges to points $x, y \in G$. Put

$$\ominus\tau = \{\ominus U | U \in \tau\}.$$

It is clear that $(G, \ominus\tau)$ is also compact. We may assume without loss of generality that $(x_\delta)_{\delta\in D}$ $\ominus\tau$-converges to some point $z \in G$. Hence,the net $(\ominus x_\delta)_{\delta\in D}$ τ-converges to $\ominus z$, and thus, $\ominus z \oplus x \in cl_\tau\{e\}$ and $\ominus z \oplus y \in cl_\tau\{e\}$. Since (G, τ) is a T_1 space, $x = z$ and $y = z$, i.e., $x = y$. We conclude that (G, τ) is a Hausdorff space. □

We do not know whether a regular locally compact paratopological gyrogroup is a topological gyrogroup. We give the following question.

Question 4.4. *Is a regular locally compact paratopological gyrogroup a topological gyrogroup?*

When consider the separation property of paratopological gyrogroup, the micro-associative paratopological gyrogroup and locally gyroscopic invariant paratopological gyrogroup were proposed in [2] and [12], respectively.

Definition 4.5. ([2]) A paratopological gyrogroup G is micro-associative if for any neighborhood $U \subseteq G$ of the identity e, there are neighborhoods $W \subseteq V \subseteq U$ of e such that $a \oplus (b \oplus V) = (a \oplus b) \oplus V$ for any $a, b \in W$.

Definition 4.6. ([12]) Let G be a paratopological gyrogroup and $\mathcal{B}$ a local base at the identity e of G. We say that $\mathcal{B}$ is locally gyroscopic invariant if there is an open neighborhood U of e such that $gyr[a, b]V \subseteq V$ for each $V \in \mathcal{B}$ and $a, b \in U$. A paratopological gyrogroup G is called locally gyroscopic invariant if G has a locally gyroscopic invariant base at the identity.

In [12, Remark 2.9], the authors said that they do not know what are different for micro-associative paratopological gyrogroups and locally gyroscopic invariant paratopological gyrogroups. For this question, we give the following proposition.

Proposition 4.7. *A locally gyroscopic invariant paratopological gyrogroup G is micro-associative.*

Proof. Let G be a locally gyroscopic invariant paratopological gyrogroup and $\mathcal{B}$ a locally gyroscopic invariant base at the identity e of G. Then there exists an open neighborhood U_0 of e such that $gyr[a, b]V \subseteq V$ for each $V \in \mathcal{B}$ and $a, b \in U_0$. Without loss of generality, assume that $V \subseteq U_0$. Thus we have $gyr[a, b]V \subseteq V$ for each $V \in \mathcal{B}$ and $a, b \in V$. It follows that $gyr[a, b]V = V$ for each $V \in \mathcal{B}$ and $a, b \in V$. For any neighborhood $U \subseteq G$ of the identity e, take $W = V \in \mathcal{B}$ such that $W = V \subseteq U$ and $gyr[a, b]V = V$ for all $a, b \in W = V$. This implies that $a \oplus (b \oplus V) = (a \oplus b) \oplus gyr[a, b]V = (a \oplus b) \oplus V$ for all $a, b \in W$, which completes the proof. □

But we do not know whether a micro-associative paratopological gyrogroup is a locally gyroscopic invariant paratopological gyrogroup. For micro-associative paratopological gyrogroup and locally gyroscopic invariant paratopological gyrogroup, we give the following question.

Question 4.8. *Is a regular locally compact locally gyroscopic invariant (respectively, micro-associative) paratopological gyrogroup a topological gyrogroup?*

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