Filomat 37:15 (2023), 5109–5121
https://doi.org/10.2298/FIL2315109O

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Cotton solitons on three dimensional paracontact metric manifolds

Mustafa Özkan[a], İrem Küpeli Erken[a], Cengizhan Murathan[b]

[a]*Bursa Technical University Faculty of Engineering and Natural Science Department of Mathematics Bursa, Turkey*
[b]*Bursa Uludağ University Art and Science Faculty Department of Mathematics Bursa, Turkey*

Abstract. In this paper, we study Cotton solitons on three-dimensional paracontact metric manifolds. We especially focus on three-dimensional paracontact metric manifolds with harmonic vector field ξ and characterize them for all possible types of operator h. Finally, we constructed an example which satisfies our results.

1. Introduction

The study of geometric evolution equations is one of the principal research subjects motivated by either physical or mathematical questions. Several years ago, the notion of the Yamabe flow was introduced by Richard Hamilton at the same time as the Ricci flow (see [5, 6]), as a tool for constructing metrics of constant scalar curvature in a given conformal class of Riemannian metrics on (M^n, g) $(n \geq 3)$. On a smooth semi-Riemannian manifold, the Yamabe flow can be defined as the evolution of the semi-Riemannian metric g_0 in time t to $g = g(t)$ by the equation

$$\frac{\partial}{\partial t} g = -rg, \quad g(0) = g_0,$$

where r denotes the scalar curvature which corresponds to g.

The significance of the Yamabe flow lies in the fact that it is a natural geometric deformation to metrics of constant scalar curvature. One notes that Yamabe flow corresponds to the fast diffusion case of the porous medium equation (the plasma equation) in mathematical physics. In dimension $n = 2$ the Yamabe flow is equivalent to the Ricci flow (defined by $\frac{\partial}{\partial t} g = -2S(t)$, where S stands for Ricci tensor). However in dimension $n > 2$ the Yamabe and Ricci flows do not agree, since the first one preserves the conformal class of the metric but the Ricci flow does not in general. Just as a Ricci soliton is a special soliton of the Ricci flow, a Yamabe soliton is a special solution of the Yamabe flow that moves by one parameter family of diffeomorphisms ϕ_t generated by a fixed (time-independent) vector field V on M, and homotheties, i.e. $g(.,t) = \sigma(t)\phi_t{}^* g_0$.

Weyl tensor is a significant tool in the study of manifold geometry. However, geometers need to find a distinct way in three-dimension. In general, Cotton tensor C, is a non-vanishing conformal invariant on

2020 *Mathematics Subject Classification*. Primary 53B30; Secondary 53C21, 53C25, 53D10.
Keywords. Cotton soliton; Paracontact metric manifold; Para-Sasakian manifold.
Received: 31 August 2022; Accepted: 03 November 2022
Communicated by Mića Stanković
Email addresses: mustafa.ozkan@btu.edu.tr (Mustafa Özkan), irem.erken@btu.edu.tr (İrem Küpeli Erken), cengiz@uludag.edu.tr (Cengizhan Murathan)

a three-dimensional paracontact metric manifold contrary to Weyl tensor. The $(0,2)$- Cotton tensor C is defined by

$$C_{ij} = \frac{1}{2\sqrt{g}} C_{nmi} \epsilon^{nml} g_{lj}, \tag{1}$$

where ϵ^{ijk} denotes the Levi-Civita permutation symbol ($\epsilon^{123} = 1$) and $g = |det(g_{ij})|$. It is trace-free and divergence-free tensor.

In [7], a new geometric flow based on the conformally invariant Cotton tensor was introduced. A Cotton flow is a one-parameter family $g(t)$ of three-dimensional metrics satisfying

$$\frac{\partial}{\partial t} g(t) = -\lambda C_{g(t)}, \tag{2}$$

where $C_{g(t)}$ is the $(0,2)$-Cotton tensor corresponding to the metric $g(t)$. A *Cotton soliton* is a metric defined on a three-dimensional smooth manifold which satisfies

$$L_V g + C - \sigma g = 0, \tag{3}$$

where V is a vector field, called potential vector field, σ is constant and L denotes the Lie derivative [7]. Cotton soliton is *trivial* if $C = 0$ (i.e. conformally flat). Also, Cotton soliton is said to be *shrinking, steady* and *expanding* according as σ is positive, zero and negative respectively. The potential vector field V is a gradient vector field, i.e. $V = \nabla f$ for some smooth function f, then the metric g is said to be *gradient Cotton soliton* and the following equation holds for a smooth f on M:

$$2Hess f + C = \sigma g. \tag{4}$$

As in Ricci and Yamabe soliton, Cotton soliton is a fixed point of (2) up to diffeomorphism and rescaling.

Calvino-Louzao et.al. [1] studied compact Riemannian Cotton solitons and proved that compact Riemannian Cotton solitons are locally conformally flat in Riemannian structure. Moreover, they investigated left-invariant Cotton solitons on homogeneous manifolds in [2]. Three-dimensional almost coKähler such that the characteristic vector field ξ is an eigenvector field of the Ricci operator Q (i.e. $Q\xi = \rho\xi$, where ρ is a smooth function on M) were studied by Chen in [3]. Furthermore, the same author investigated Cotton solitons on three-dimensional contact metric manifolds [4].

In the light of previous works, the fact that there are only studies about Cotton solitons on contact geometry motivate us to study Cotton solitons on 3-dimensional paracontact metric manifolds. The paper is organized in the following way. In section 2, we recall some notations needed for this paper. Section 3 deals with the computations of the components of the $(0,2)-$Cotton Tensor. In the last section, we consider three-dimensional paracontact metric manifold M with $\hbar_1$ and $\hbar_3$ types such that the characteristic vector field is harmonic (i.e. $Q\xi = \rho\xi$) and ρ is constant along the characteristic vector field ξ. Then we proved that if M admits a Cotton soliton with potential vector field being collinear with characteristic vector field ξ, then M is either para-Sasakian, or locally conformally flat. The results for three-dimensional paracontact metric manifolds with $\hbar_2$ type are different from three-dimensional contact metric manifolds and three-dimensional paracontact metric manifolds with $\hbar_1$ and $\hbar_3$ types. We consider a three-dimensional paracontact metric manifold with $\hbar_2$ type such that the characteristic vector field is harmonic and ρ is constant along the characteristic vector field ξ. Then we proved that if M admits a Cotton soliton with potential vector field being collinear with characteristic vector field ξ, then M is locally conformally flat, has scalar curvature -6 and Cotton soliton is steady. Also, we studied the three-dimensional paracontact metric manifold with $\hbar_1$ type admiting a gradient Cotton soliton. Finally, an example which satisfies our results is constructed.

2. Preliminaries

A $(2n+1)-$ dimensional manifold M is called *almost paracontact manifold* if it admits triple (ϕ, ξ, η) satisfying the followings:

- $\eta(\xi) = 1, \phi^2 = I - \eta \otimes \xi$,
- ϕ induces on almost paracomplex structure on each fiber of $\mathcal{D} = ker(\eta)$,

where ϕ, ξ and η are $(1,1)-$tensor field, vector field and $1-$form, respectively. One can easily checked that $\phi\xi = 0, \eta \circ \phi = 0$ and $rank\phi = 2n$, by the definition. Here, ξ is a unique vector field (called *Reeb* or *characteristic vector field*) dual to η and satisfying $d\eta(\xi, X) = 0$ for all X. When the tensor field $N_\phi := [\phi, \phi] - 2d\eta \otimes \xi$ vanishes identically, the almost paracontact manifold is said to be *normal*. If the structure (M, ϕ, ξ, η) admits a pseudo-Riemannian metric such that

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y),$$

then we say that (M, ϕ, ξ, η, g) is an *almost paracontact metric manifold*. Note that any pseudo-Riemannian metric with a given almost paracontact metric manifold structure is necessarily of signature $(n+1, n)$. For an almost paracontant metric manifold, one can always find an orthogonal basis $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, \xi\}$, namely $\phi-$basis, such that $g(X_i, X_j) = -g(Y_i, Y_j) = \delta_{ij}$ and $Y_i = \phi X_i$, for any $i, j \in \{1, \ldots, n\}$.

Further, an almost paracontact metric manifold is said to be *paracontact metric manifold* if the following holds for all vector fields X, Y on M:

$$d\eta(X, Y) = g(X, \phi Y).$$

In paracontact metric manifold, one defines a symmetric operator $h := \frac{1}{2}L_\xi\phi$. The operator h also satisfies the followings:

$$\begin{cases} h\xi = 0, & \phi h = -h\phi, \\ traceh = 0, & \nabla_X\xi = -\phi X + \phi hX, \end{cases} \tag{5}$$

where ∇ is the Levi-Civita connection of the pseudo-Riemannian manifold. If ξ is a Killing vector field, then the paracontact metric manifold is called a *K-paracontact manifold*. A normal paracontact metric manifold is said to be a *para-Sasakian manifold*. A para-Sasakian maniold is also K-paracontact and the converse holds only in dimension 3. Küpeli Erken and Murathan proved the following Theorem.

Theorem 2.1. *[8] Let (M, ϕ, ξ, η, g) be a three-dimensional paracontact metric manifold. ξ is a harmonic vector field if and only if the characteristic vector field ξ is an eigenvector of the Ricci operator.*

Theorem 2.2. *[9] An almost paracontact metric structure (ϕ, ξ, η, g) is para-Sasakian if and only if*

$$(\nabla_X\phi)Y = -g(X, Y)\xi + \eta(Y)X.$$

Now, we give some information about the canonical forms of h.

The tensor h the canonical form (I). Let (M, ϕ, ξ, η, g) be a three-dimensional paracontact metric manifold and let

$$U_1 = \{p \in M | h(p) \neq 0\} \subset M$$

$$U_2 = \{p \in M | h(p) = 0, in\ a\ neighborhood\ of\ p\} \subset M.$$

That h is a smooth function on M implies $U_1 \cup U_2$ is an open and dense subset of M, so any property satisfied in $U_1 \cup U_2$ is also satisfied in M. For any point $p \in U_1 \cup U_2$, there exists a local orthonormal ϕ-basis $\{e, \phi e, \xi\}$ of smooth eigenvectors of h in a neighborhood of p, where $-g(e, e) = g(\phi e, \phi e) = g(\xi, \xi) = 1$. On U_1, we put $he = \lambda e$, where λ is a non-vanishing smooth function. Since $trh = 0$, we have $h\phi e = -\lambda\phi e$. In this case, we will say the operator h is of $\mathfrak{h}_1$ type.

Lemma 2.3. *[8] Let (M, ϕ, ξ, η, g) be a three-dimensional paracontact metric manifold with h of $\mathfrak{h}_1$ type. Then for the covariant derivative on U_1, the following equations are valid*

$$
\begin{aligned}
&\text{(i)} \ \nabla_e e = a\phi e, \\
&\text{(ii)} \ \nabla_e \phi e = ae + (1-\lambda)\xi, \\
&\text{(iii)} \ \nabla_e \xi = (\lambda - 1)\phi e, \\
&\text{(iv)} \ \nabla_{\phi e} e = c\phi e - (\lambda + 1)\xi, \\
&\text{(v)} \ \nabla_{\phi e} \phi e = ce, \\
&\text{(vi)} \ \nabla_{\phi e} \xi = -(\lambda + 1)e, \\
&\text{(vii)} \ \nabla_\xi e = b\phi e, \\
&\text{(viii)} \ \nabla_\xi \phi e = be, \\
&\text{(ix)} \ \nabla_\xi \xi = 0,
\end{aligned} \tag{6}
$$

where $\omega = S(\xi, .)_{ker\eta}, b = g(\nabla_\xi e, \phi e), A = \omega(e), B = \omega(\phi e)$ and

$$a = \frac{A - \phi e(\lambda)}{2\lambda}, \tag{7}$$

$$c = -\left(\frac{B + e(\lambda)}{2\lambda}\right). \tag{8}$$

From (6), we have

$$
\begin{cases}
[e, \phi e] & = ae - c\phi e + 2\xi, \\
[e, \xi] & = (\lambda - 1 - b)\phi e, \\
[\phi e, \xi] & = (-\lambda - 1 - b)e.
\end{cases} \tag{9}
$$

The components of the Ricci operator Q for $\mathfrak{h}_1$ type are given by

$$
\begin{cases}
Qe & = (1 - \lambda^2 + \frac{1}{2}r - 2b\lambda)e - Z\phi e + A\xi, \\
Q\phi e & = Ze + (1 - \lambda^2 + \frac{1}{2}r + 2b\lambda)\phi e + B\xi, \\
Q\xi & = -Ae + B\phi e + 2(\lambda^2 - 1)\xi,
\end{cases} \tag{10}
$$

where $Z = \xi(\lambda)$.

The tensor h the canonical form (II). Let (M, ϕ, ξ, η, g) be a three-dimensional paracontact metric manifold and p is a point of M. Then there exists a local pseudo-orthonormal basis $\{e_1, e_2, \xi\}$ in a neighborhood of p, where $g(e_1, e_1) = g(e_2, e_2) = g(e_1, \xi) = g(e_2, \xi) = 0$ and $g(e_1, e_2) = 1$.

Lemma 2.4. *[8] Let U be the open subset of M, where $h \neq 0$. For every $p \in U$, there exists an open neighborhood of p such that $he_1 = e_2, he_2 = 0, h\xi = 0$ and $\phi e_1 = \pm e_1, \phi e_2 = \mp e_2$.*

In this case, we say h is of $\mathfrak{h}_2$ type.

Lemma 2.5. *[8] Let (M, ϕ, ξ, η, g) be a three-dimensional paracontact metric manifold with h of $\mathfrak{h}_2$ type. Then for the covariant derivative on U, the following equations are valid*

$$
\begin{aligned}
&\text{(i)}\ \nabla_{e_1}e_1 = -b_2e_1 + \xi,\\
&\text{(ii)}\ \nabla_{e_1}e_2 = b_2e_2 + \xi,\\
&\text{(iii)}\ \nabla_{e_1}\xi = -e_1 - e_2,\\
&\text{(iv)}\ \nabla_{e_2}e_1 = -\tilde{b_2}e_1 - \xi,\\
&\text{(v)}\ \nabla_{e_2}e_2 = \tilde{b_2}e_2,\\
&\text{(vi)}\ \nabla_{e_2}\xi = e_2,\\
&\text{(vii)}\ \nabla_{\xi}e_1 = a_2e_1,\\
&\text{(viii)}\ \nabla_{\xi}e_2 = -a_2e_2,
\end{aligned} \tag{11}
$$

where $a_2 = g(\nabla_\xi e_1, e_2), b_2 = g(\nabla_{e_1}e_2, e_1), \tilde{b_2} = -\frac{1}{2}\omega(e_1)$ *and* $\omega(e_1) = S(\xi, e_1) = A_2$.

From (11) we have

$$
\begin{cases}
[e_1, e_2] &= \tilde{b_2}e_1 + b_2e_2 + 2\xi,\\
[e_1, \xi] &= -(1 + a_2)e_1 - e_2,\\
[e_2, \xi] &= (1 + a_2)e_2.
\end{cases} \tag{12}
$$

The components of the Ricci operator Q for $\mathfrak{h}_2$ are given by

$$
\begin{cases}
Qe_1 &= (1 + \frac{1}{2}r)e_1 - 2a_2e_2 + A_2\xi,\\
Qe_2 &= (1 + \frac{1}{2}r)e_2,\\
Q\xi &= A_2e_2 - 2\xi.
\end{cases} \tag{13}
$$

The tensor h the canonical form (III). Let (M, ϕ, ξ, η, g) be a three-dimensional paracontact metric manifold and let p is a point of M. Then there exists a local orthonormal ϕ-basis $\{e, \phi e, \xi\}$ in a neighborhood of p, where $-g(e, e) = g(\phi e, \phi e) = g(\xi, \xi) = 1$. Now, let U_1 be the open subset of M where $h \neq 0$ and let U_2 be the open subset of points $p \in M$ such that $h = 0$ in a neighborhood of p. $U_1 \cup U_2$ is an open subset of M. For every $p \in U_1$ there exists an open neighborhood of p such that $he = \lambda\phi e, h\phi e = -\lambda e$ and $h\xi = 0$ where λ is a non-vanishing smooth function. In this case, we say that h is of $\mathfrak{h}_3$ type.

Lemma 2.6. *[8] Let (M, ϕ, ξ, η, g) be a three-dimensional paracontact metric manifold with h of $\mathfrak{h}_3$ type. Then for the covariant derivative on U_1, the following equations are valid*

$$
\begin{aligned}
&\text{(i)}\ \nabla_e e = a_3\phi e + \lambda\xi,\\
&\text{(ii)}\ \nabla_e\phi e = a_3e + \xi,\\
&\text{(iii)}\ \nabla_e\xi = -\phi e + \lambda e,\\
&\text{(iv)}\ \nabla_{\phi e}e = b_3\phi e - \xi,\\
&\text{(v)}\ \nabla_{\phi e}\phi e = b_3e + \lambda\xi,\\
&\text{(vi)}\ \nabla_{\phi e}\xi = -e - \lambda\phi e,\\
&\text{(vii)}\ \nabla_\xi e = \tilde{b_3}\phi e,\\
&\text{(viii)}\ \nabla_\xi\phi e = \tilde{b_3}e,
\end{aligned} \tag{14}
$$

where a_3, b_3 and $\tilde{b_3}$ are defined by

$$a_3 = -\frac{1}{2\lambda}[\omega(\phi e) + \phi e(\lambda)], \qquad A_3 = \omega(e) = S(e, \xi), \tag{15}$$

$$b_3 = \frac{1}{2\lambda}[\omega(e) - e(\lambda)], \qquad B_3 = \omega(\phi e) = S(\phi e, \xi) \tag{16}$$

$$\tilde{b_3} = g(\nabla_\xi e, \phi e),$$

respectively.

From (14) we have

$$\begin{cases} [e, \phi e] &= a_3 e - b_3 \phi e + 2\xi, \\ [e, \xi] &= \lambda e - (1 + \tilde{b_3})\phi e, \\ [\phi e, \xi] &= -(1 + \tilde{b_3})e - \lambda \phi e. \end{cases} \tag{17}$$

The components of the Ricci operator Q for $\hbar_3$ are given by

$$\begin{cases} Qe &= (1 + \lambda^2 + \frac{1}{2}r + Z)e - 2\tilde{b_3}\lambda\phi e + A_3\xi, \\ Q\phi e &= 2\tilde{b_3}\lambda e + (1 + \lambda^2 + \frac{1}{2}r + Z)\phi e + B_3\xi, \\ Q\xi &= -A_3 e + B_3\phi e - 2(1 + \lambda^2)\xi, \end{cases} \tag{18}$$

where $Z = \xi(\lambda)$.

3. Cotton Solitons

In this section, we give the components of the Cotton tensor and calculate the scalar curvature for each three-dimensional paracontact metric manifolds according to their h types.

Using the relations $S(X, Y) = \sum_{i=1}^{2n+1} \varepsilon_i g(R(e_i, X)Y, e_i)$ and $r = \sum_{i=1}^{2n+1} \varepsilon_i S(e_i, e_i)$. We derive a useful formula for the scalar curvature.

Lemma 3.1. *Let (M, ϕ, ξ, η, g) be a three-dimensional paracontact metric manifold with $\hbar_1$ type. Then the scalar curvature r is given as follows:*

$$r = trace(Q) = 2[-\phi e(a) + e(c) - a^2 + c^2 - 2b + \lambda^2 - 1]. \tag{19}$$

Proposition 3.2. *Let (M, ϕ, ξ, η, g) be a three-dimensional paracontact metric manifold with $\hbar_1$ type. If the characteristic vector field ξ is a harmonic vector field in the open subset U_1, then the following relations are valid for the components of Cotton tensor C.*

$$C_{11} = C(e, e) = (\lambda + 1)[\frac{1}{2}r + 3 - 3\lambda^2 - 2b\lambda] - \xi(Z) - 4b^2\lambda, \tag{20}$$

$$C_{12} = C(e, \phi e) = 2\lambda\xi(b) + 4bZ + Z(\lambda + 1) - \frac{1}{4}\xi(r), \tag{21}$$

$$C_{13} = C(e, \xi) = -e(Z) - 4ab\lambda - \phi e(\lambda^2 + 2b\lambda) - 2cZ + \frac{1}{4}\phi e(r), \tag{22}$$

$$C_{22} = C(\phi e, \phi e) = -\xi(Z) - 4b^2\lambda + (\lambda - 1)(\frac{r}{2} + 2b\lambda) + 3(\lambda - 1)(1 - \lambda^2), \tag{23}$$

$$C_{23} = C(\phi e, \xi) = e(-\lambda^2 + 2b\lambda) + 2aZ + \phi e(Z) + 4bc\lambda + \frac{1}{4}e(r), \tag{24}$$

$$C_{33} = C(\xi, \xi) = -4b\lambda^2 + 6(1 - \lambda^2) + r. \tag{25}$$

Proof. Well-known Cotton tensor equation is defined as

$$C(X, Y)Z = (\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) - \frac{1}{4}[X(r)g(Y, Z) - Y(r)g(X, Z)] \tag{26}$$

for all vector fields X, Y, Z, where S is the Ricci curvature tensor and r is the scalar curvature. From (1) and using the notation $C_{ijk} = C(e_i, e_j)e_k$ for all $i, j = 1, 2, 3$, we get

$$\begin{aligned} C_{11} &= \frac{1}{2}[C_{nm1}\epsilon^{nml}g_{l1}] = \frac{1}{2}[-C_{nm1}\epsilon^{nm1}] = -\frac{1}{2}[C_{2m1}\epsilon^{2m1} + C_{3m1}\epsilon^{3m1}] \\ &= -C_{231}. \end{aligned}$$

Using similar calculations we have

$$C_{12} = C_{311}, \quad C_{13} = C_{121}, \quad C_{22} = C_{312}, \quad C_{23} = C_{122}, \quad C_{33} = C_{123}.$$

From the assumption of ξ is a harmonic vector field, using Theorem 2.1 and (18), we have $A = B = 0$. By using (1) and (18) after a long but straightforward calculations we compute the components C_{ij} as follows:

$$\begin{aligned} C_{11} &= -C_{231} = -[C(\phi e, \xi)e] \\ &= -[(\nabla_{\phi e} S)(\xi, e) - (\nabla_\xi S)(\phi e, e)] \\ &= (\lambda + 1)(1 - \lambda^2 + \frac{1}{2}r - 2b\lambda) - 2(\lambda^2 - 1)(\lambda + 1) \\ &\quad - \xi(Z) + b(1 - \lambda^2 + \frac{1}{2}r - 2b\lambda) - b(1 - \lambda^2 + \frac{1}{2}r + 2b\lambda) \\ &= (\lambda + 1)[\frac{1}{2}r + 3 - 3\lambda^2 - 2b\lambda] - \xi(Z) - 4b^2\lambda, \end{aligned}$$

$$\begin{aligned} C_{12} &= C_{311} = [C(\xi, e)e] \\ &= [(\nabla_\xi S)(e, e) - (\nabla_e S)(\xi, e)] + \frac{1}{4}\xi(r) \\ &= -\xi(1 - \lambda^2 + \frac{1}{2}r - 2b\lambda) + 2bZ - (\lambda - 1)Z + \frac{1}{4}\xi(r) \\ &= 2\lambda\xi(b) + 4bZ + Z(\lambda + 1) - \frac{1}{4}\xi(r), \end{aligned}$$

$$\begin{aligned} C_{13} &= C_{121} = [C(e, \phi e)e] \\ &= [(\nabla_e S)(\phi e, e) - (\nabla_{\phi e} S)(e, e)] - \frac{1}{4}\phi e(r) \\ &= -e(Z) - 4ab\lambda + \phi e(1 - \lambda^2 + \frac{1}{2}r - 2b\lambda) - 2cZ - \frac{1}{4}\phi e(r) \\ &= -e(Z) - 4ab\lambda - \phi e(\lambda^2 + 2b\lambda) - 2cZ + \frac{1}{4}\phi e(r), \end{aligned}$$

$$\begin{aligned} C_{22} &= C_{312} = [C(\xi, e)\phi e] \\ &= [(\nabla_\xi S)(e, \phi e) - (\nabla_e S)(\xi, \phi e)] \\ &= -\xi(Z) - 4b^2\lambda + (\lambda - 1)(1 - \lambda^2 + \frac{1}{2}r + 2b\lambda) + 2(\lambda^2 - 1)(1 - \lambda) \\ &= -\xi(Z) - 4b^2\lambda + (\lambda - 1)(\frac{1}{2}r + 2b\lambda) + 3(\lambda - 1)(1 - \lambda^2), \end{aligned}$$

$$\begin{aligned} C_{23} &= C_{122} = [C(e, \phi e)\phi e] \\ &= [(\nabla_e S)(\phi e, \phi e) - (\nabla_{\phi e} S)(e, \phi e)] - \frac{1}{4}e(r) \\ &= e(1 - \lambda^2 + \frac{1}{2}r + 2b\lambda) + 2aZ + \phi e(Z) + 4bc\lambda - \frac{1}{4}e(r) \\ &= e(-\lambda^2 + 2b\lambda) + 2aZ + \phi e(Z) + 4bc\lambda + \frac{1}{4}e(r), \end{aligned}$$

$$
\begin{aligned}
C_{33} = C_{123} &= [C(e, \phi e)\xi] \\
&= [(\nabla_e S)(\phi e, \xi) - (\nabla_{\phi e} S)(e, \xi)] \\
&= (\lambda - 1)[3\lambda^2 - 3 - \frac{r}{2} - 2b\lambda] + (\lambda + 1)[-3\lambda^2 + 3 + \frac{1}{2}r - 2b\lambda] \\
&= -4b\lambda^2 + 6(1 - \lambda^2) + r.
\end{aligned}
$$

□

To calculate r for $\mathfrak{h}_2$ type we construct a new pseudo-orthonormal frame $\{\tilde{e_1}, \tilde{e_2}, \tilde{e_3}\}$ such as $\tilde{e_1} = \frac{e_1+e_2}{\sqrt{2}}, \tilde{e_2} = \frac{e_1-e_2}{\sqrt{2}}$ and $\tilde{e_3} = \xi$. So, we get $g(\tilde{e_1}, \tilde{e_1}) = 1 = -g(\tilde{e_2}, \tilde{e_2}), g(\tilde{e_1}, \tilde{e_2}) = 0$ and $h\tilde{e_1} = h\tilde{e_2} = e_2$. Then we give the following lemma.

Lemma 3.3. *Let (M, ϕ, ξ, η, g) be a three-dimensional paracontact metric manifold with $\mathfrak{h}_2$ type. Then the scalar curvature r is given as follows:*

$$r = trace(Q) = 2[-e_1(\tilde{b_2}) + e_2(b_2) + 2b_2\tilde{b_2} - 2a_2 - 1]. \tag{27}$$

Since the proof of the following proposition is quite similar to Proposition 3.2, so we don't give the proof of it.

Proposition 3.4. *Let (M, ϕ, ξ, η, g) be a three-dimensional paracontact metric manifold with $\mathfrak{h}_2$ type. If the characteristic vector field ξ is a harmonic vector field in the open subset U, then the following relations are valid for the components of Cotton tensor C.*

$$C_{11} = C(e_1, e_1) = -2\xi(a_2) + 2a_2(1 + 2a_2) - 3 - \frac{1}{2}r, \tag{28}$$

$$C_{12} = C(e_1, e_2) = -3 - \frac{1}{2}r - \frac{1}{4}\xi(r), \tag{29}$$

$$C_{13} = C(e_1, \xi) = 2e_2(a_2) + 4a_2\tilde{b_2} + \frac{1}{4}e_1(r), \tag{30}$$

$$C_{22} = C(e_2, e_2) = 0, \tag{31}$$

$$C_{23} = C(e_2, \xi) = -\frac{1}{4}e_2(r), \tag{32}$$

$$C_{33} = C(\xi, \xi) = 6 + r. \tag{33}$$

Lemma 3.5. *Let (M, ϕ, ξ, η, g) be a three-dimensional paracontact metric manifold with $\mathfrak{h}_3$ type. Then the scalar curvature r is given as follows:*

$$r = trace(Q) = 2[-\phi e(a_3) + e(b_3) - a_3^2 + b_3^2 - 2\tilde{b_3} - \lambda^2 - 1]. \tag{34}$$

Since the proof of the following proposition is quite similar to Proposition 3.2, so we don't give the proof of it.

Proposition 3.6. *Let (M, ϕ, ξ, η, g) be a three-dimensional paracontact metric manifold with $\mathfrak{h}_3$ type. If the characteristic vector field ξ is a harmonic vector field in the open subset U_1, then the following relations are valid for the* *components of Cotton tensor C.*

$$C_{11} = C(e, e) = 3(\lambda^2 + 1) + Z(1 - 2\tilde{b_3}) + \frac{1}{2}r + 2\lambda(\lambda\tilde{b_3} - \xi(\tilde{b_3})), \tag{35}$$

$$C_{12} = C(e, \phi e) = -\frac{1}{4}\xi(r) - \lambda(3Z + 3\lambda^2 + 4 + \frac{1}{2}r) + 2\tilde{b_3}\lambda(1 + 2\tilde{b_3}), \tag{36}$$

$$C_{13} = C(e, \xi) = -2e(\tilde{b_3}\lambda) + \phi e(\lambda^2 + Z) - 4b_3\tilde{b_3}\lambda + \frac{1}{4}\phi e(r), \tag{37}$$

$$C_{22} = C(\phi e, \phi e) = -2[\xi(\tilde{b_3})\lambda + \tilde{b_3}Z + \tilde{b_3}\lambda^2] - 3(\lambda^2 + 1) - \frac{1}{2}r - Z, \tag{38}$$

$$C_{23} = C(\phi e, \xi) = e(\lambda^2 + Z) + 4a_3\tilde{b_3}\lambda + 2\phi e(\tilde{b_3}\lambda) + \frac{1}{4}e(r), \tag{39}$$

$$C_{33} = C(\xi, \xi) = r + 4\lambda^2\tilde{b_3} + 2Z + 6(1 + \lambda^2). \tag{40}$$

4. 3-dimensional Paracontact metric manifolds with harmonic vector field ξ

Theorem 4.1. *Let (M, ϕ, ξ, η, g) be a three-dimensional paracontact metric manifold with $\tilde{h}_1$ type such that the characteristic vector field is harmonic (i.e. $Q\xi = \rho\xi$) and ρ is constant along the characteristic vector field ξ. If M admits a Cotton soliton with potential vector field being collinear with characteristic vector field ξ, then M is either para-Sasakian, or locally conformally flat.*

Proof. Firstly, we denote U_1 and U_2 as follows:

$$U_1 = \{p \in M : \lambda = 0 \textit{ in a neighborhood of } p\}$$

and

$$U_2 = \{p \in M : \lambda \neq 0 \textit{ in a neighborhood of } p\}.$$

If we only study on U_1, then M is para-Sasakian from Theorem 2.2. Now, assume that U_2 is a non-empty set and let $\{e, \phi e, \xi\}$ is a ϕ-basis in U_2.

From the characteristic vector field is harmonic and (18), we have $\rho = 2(\lambda^2 - 1), \xi(\rho) = \xi(\lambda) = Z = 0$ and $A = B = 0$.

If $V = 0$ (3) returns to $C = \sigma g$. It could be shown obviously that the tensor C is trace-free. So, σ is equal to zero. Hence, M is locally conformally flat.

Now, we assume that $V = f\xi$, where f is a non-vanishing smooth function. Substituting V by $f\xi$ and using (5), equation (3) becomes:

$$\sigma g(X, Y) = 2fg(\phi hX, Y) + X(f)\eta(Y) + Y(f)\eta(X) + C(X, Y) \tag{41}$$

Putting $X = Y = e$ in (41) and using (20) we obtain

$$\sigma = -(\lambda + 1)(\frac{1}{2}r - 2b\lambda) + 4b^2\lambda - 3(\lambda + 1)(1 - \lambda^2). \tag{42}$$

Similarly, letting $X = Y = \phi e$ in (41) and using (23) we get

$$\sigma = (\lambda - 1)(\frac{1}{2}r + 2b\lambda) + 3(\lambda - 1)(1 - \lambda^2) - 4b^2\lambda. \tag{43}$$

On the other hand, if we put $X = e$ and $Y = \phi e$ in (41) and use (21) we have

$$2\lambda f = -2\lambda\xi(b) + \frac{1}{4}\xi(r). \tag{44}$$

If we add (42) and (43) we have

$$2\sigma = -r + 4b\lambda^2 - 6(1 - \lambda^2). \tag{45}$$

Comparing (42) with (45) after some calculations, we get

$$\sigma = 2b(\lambda^2 - 1 - 2b).$$

Differentiating the above equation along the vector field ξ, and from the fact that σ is constant and $\xi(\lambda) = 0$ we find

$$\xi(b)[4b + 1 - \lambda^2] = 0. \tag{46}$$

Now there are two possibilities. The first one is $\xi(b) = 0$. Differentiating (45) along the vector field ξ we have $\xi(r) = 0$. From the equation (44), f must be zero since λ is non-vanishing smooth function. So, we obtain that the Cotton soliton is trivial.

The second one is $\xi(b) \neq 0$. Then we get $4b + 1 - \lambda^2 = 0$ from (46). By differentiating this along the vector field ξ, we obtain $\xi(b) = 0$, which leads to a contradiction with $\xi(b) \neq 0$.

This completes the proof of the theorem. □

Theorem 4.2. *Let (M, ϕ, ξ, η, g) be a three-dimensional paracontact metric manifold with $\mathfrak{h}_2$ type such that the characteristic vector field is harmonic (i.e. $Q\xi = \rho\xi$) and ρ is constant along the characteristic vector field ξ. If M admits a Cotton soliton with potential vector field being collinear with characteristic vector field ξ, then M is locally conformally flat, has scalar curvature -6 and Cotton soliton is steady.*

Proof. The proof of the first part is similar to the proof of the Theorem 4.1, namely, if $V = 0$, then M is locally conformally flat. Now, assume that $V = f\xi$, where f is a non-vanishing constant function. The equation (41) is also valid for $\mathfrak{h}_2$ type. Putting $X = e_1$ and $Y = e_2$ in (41) and using (29), we get

$$\sigma = -3 - \frac{1}{2}r - \frac{1}{4}\xi(r). \tag{47}$$

Letting $X = Y = \xi$ in (41) and using (33) we have

$$\sigma = 6 + r. \tag{48}$$

Comparing (47) with (48), we have $\sigma = 0$. By (48), we obtain $r = -6$. This completes the proof of the theorem. □

Theorem 4.3. *Let (M, ϕ, ξ, η, g) be a three-dimensional paracontact metric manifold with $\mathfrak{h}_3$ type such that the characteristic vector field is harmonic (i.e. $Q\xi = \rho\xi$) and ρ is constant along the characteristic vector field ξ. If M admits a Cotton soliton with potential vector field being collinear with characteristic vector field ξ, then M is either para-Sasakian, or locally conformally flat.*

Proof. The proof of the first part is similar to the proof of the Theorem 4.1. If $V = 0$, then M is locally conformally flat. Now, assume that $V = f\xi$, where f is a non-vanishing constant function. The equation (41) is also valid for $\mathfrak{h}_3$ type.

Letting $X = Y = e$ in (41) and using (35) we get

$$\sigma = 2\lambda f - 3(\lambda^2 + 1) - \frac{1}{2}r - 2\lambda^2\tilde{b_3} + 2\lambda\xi(\tilde{b_3}). \tag{49}$$

Again, putting $X = Y = \phi e$ in (41) and by the help of (38), we obtain

$$\sigma = -2f\lambda - 2[\xi(\tilde{b_3})\lambda + \tilde{b_3}\lambda^2] - 3(\lambda^2 + 1) - \frac{1}{2}r. \tag{50}$$

On the other hand, if we put $X = e$ and $Y = \phi e$ in (41) and use (36), we have

$$-\frac{1}{4}\xi(r) - \lambda(3\lambda^2 + 4 + \frac{1}{2}r) + 2\tilde{b_3}\lambda(2\tilde{b_3} + 1) = 0. \tag{51}$$

By adding (49) with (50) we get

$$\sigma = -3(\lambda^2 + 1) - \frac{1}{2}r - 2\lambda^2 \tilde{b_3}. \tag{52}$$

Comparing (49) with (52), we conclude that $f = -\xi(\tilde{b_3})$. By differentiating the equations (52) and (51) along the vector field ξ, we obtain $\xi(r) = 4\lambda^2 f$ and $f(\lambda^2 + 1 + 4\tilde{b_3}) = 0$, respectively. Since $f \neq 0$, we get $\lambda^2 + 1 + 4\tilde{b_3} = 0$. Differentiating this along the vector field ξ, we have $f = 0$, which is a contradiction with the fact that $f \neq 0$.

Hence, we complete the proof of the theorem. □

Remark 4.4. *From Theorem 4.1, Theorem 4.2 and Theorem 4.3, we proved that there do not exist any non-conformally flat three-dimensional paracontact metric manifold admitting a Cotton soliton with the characteristic vector field is harmonic (i.e. $Q\xi = \rho\xi$) and ρ is constant along the characteristic vector field ξ such that the potential vector field being collinear with characteristic vector field ξ.*

Theorem 4.5. *Let (M, ϕ, ξ, η, g) be a three-dimentional paracontact metric manifold with $\tilde{h}_1$ type such that the characteristic vector field is harmonic. If M admits a gradient Cotton soliton, then M is either para-Sasakian or locally conformally flat.*

Proof. We denote U_1 and U_2 as follows:

$$U_1 = \{p \in M : \lambda = 0 \textit{ in a neighborhood of } p\}$$

and

$$U_2 = \{p \in M : \lambda \neq 0 \textit{ in a neighborhood of } p\}.$$

If $M = U_1$, then M is para-Sasakian. Consider ϕ-basis $\{e, \phi e, \xi\}$ on non-empty set U_2. The potential vector field V equals $\nabla f = f_1 e + f_2 \phi e + f_3 \xi$, where f_1, f_2, f_3 are smooth functions. Since C is divergence-free and from [[1], Remark 3], we get $Q\nabla f = 0$. Then following three equation holds from (6)

$$\begin{cases} f_1(1 - \lambda^2 + \frac{1}{2}r - 2b\lambda) = 0 \\ f_2(1 - \lambda^2 + \frac{1}{2}r + 2b\lambda) = 0 \\ f_3(\lambda^2 - 1) = 0. \end{cases} \tag{53}$$

Obviously, if $V = 0$ then Cotton soliton is trivial. Now, we assume that at least one of the functions is different from zero and seperate the proof three parts.

Case I: ($\lambda = 1$) In this case we have $a = c = 0$ from (7) and (8). By (3.1), we derive $r = -4b$. Then from the first term of (53) we have $bf_1 = 0$. Using the well-known formula $\frac{1}{2}\nabla r = divQ$ for every semi-Riemannian manifolds, after some calculations we obtain b is constant. If $b = 0$, then the equations (20)-(25) are zero ,that is, M is locally conformally flat. If $b \neq 0$ then $f_1 = 0$. Then the components of C return to the followings:

$$C_{12} = C_{13} = C_{23} = 0,\ C_{11} = -8b - 4b^2,\ C_{22} = -4b^2,\ C_{33} = -8b. \tag{54}$$

The gradient Cotton soliton equation (4) can be written as

$$2g(\nabla_X \nabla f, Y) + C(X, Y) = \sigma g(X, Y), \tag{55}$$

where X and Y are vector fields on M. By putting $X = e$ and $Y = \phi e$ in (55) and using (54) we have $e(f_2) = 0$. Similarly, if we write $X = \xi$ and $Y = \phi e$ in (55) we get $\xi(f_2) = 0$. If we act f_2 to the second term of (9) we find $\phi e(f_2) = 0$. In (55), after putting $X = Y = e$ and $X = \phi e, Y = \phi e$ we get $\sigma = 8b + 4b^2$ and $\sigma = -4b^2$ respectively. It implies that $b = -1$. Using the similar calculations from (55), we find that $e(f_3) = 0, \phi e(f_3) = 0$

and $\xi(f_3) = -6$. After acting f_3 to the first term of (9) we obtain $\xi(f_3) = 0$. Hence, we find that b must be zero.

Case II: ($\lambda = -1$**)** Applying the same method as in Case I, we get the same results.

Case III: ($\lambda \neq \pm 1$ in some $O \subset U_2$) In this case, we observe that $f_3 = 0$ from the last term of (53). By the help of (6) and (25), after putting $X = Y = \xi$ in (55) we get

$$-4b\lambda^2 + 6(1-\lambda^2) + r = \sigma. \tag{56}$$

Taking $X = e$ and $Y = \xi$ in (55) and using (6), (7) and (22) we have

$$2f_2(1-\lambda) + 4a\lambda^2 - 2\lambda\phi e(b) + \frac{1}{4}\phi e(r) = 0. \tag{57}$$

Putting $X = \phi e$ and $Y = \xi$ and using (24) in (55), we conculude that

$$-2(\lambda+1)f_1 + 4c\lambda^2 + 2e(b)\lambda + \frac{1}{4}e(r) = 0. \tag{58}$$

Consider the following two open sets such as the union set is open and dense in the closure of O as follows:

$$O_1 = \{p \in O : 1 - \lambda^2 + \frac{1}{2}r - 2b\lambda \neq 0 \ \textit{in a neighborhood of } p\}$$

and

$$O_2 = \{p \in O : 1 - \lambda^2 + \frac{1}{2}r - 2b\lambda = 0 \ \textit{in a neighborhood of } p\}.$$

In the set O_1, we get $f_1 = 0$ from the first term of (53). It implies that $f_2 \neq 0$ from the assumption. So, from the second term of (53) we get $1 - \lambda^2 + \frac{1}{2}r + 2b\lambda = 0$. Comparing the above equation with (56), we have

$$\sigma = 4(1-\lambda^2) - 4b(\lambda + \lambda^2). \tag{59}$$

By using $d^2 f = 0$, Poincare Lemma, we have the relation

$$g(\nabla_X \nabla f, Y) = g(\nabla_Y \nabla f, X), \tag{60}$$

where X and Y are vector fields. Taking $X = \xi$ and $Y = e$ in (60), and using (6) we get

$$b = \lambda - 1.$$

We obtain that λ and b are constants after substituting the above equation in (59). Hence, $a = c = 0$ by (7) and (8). (56) gives that r is constant. From (57), we observe $f_2(1-\lambda) = 0$. Since $\lambda \neq 1$, we get $f_2 = 0$ in O which leads to a contradiction with $f_2 \neq 0$. So, it means that O_1 is empty.

In O_2, we have

$$1 - \lambda^2 + \frac{1}{2}r - 2b\lambda = 0. \tag{61}$$

Then we get $bf_2 = 0$ from the second term of (53), where λ is a non-vanishing smooth function. Let define two sets V_1 and V_2 as follows:

$$V_1 = \{p \in O_2 : b \neq 0\}$$

and

$$V_2 = \{p \in O_2 : b = 0\}.$$

The union set is open and dense in the closure of O_2. Hence, we have $f_2 = 0$ in V_1. Putting $X = \xi$ and $Y = \phi e$ in (60) and using (6), we get

$$b = 1 - \lambda,$$

since $f_1 \neq 0$ in V_1. With the similar calculations as we did before, we get $a = c = 0$ and b, λ and r are constants. By (58), we have $f_1 = 0$. It means that V_1 is empty.

From the fact that $b = 0$ in V_2, we obtain from (61) that $r = -2(1 - \lambda^2)$. Replacing the last equation into (56) we obtain $\sigma = 4(1 - \lambda^2)$. It implies that λ and r are constants. Then $a = c = 0$ from (7) and (8). On the other hand, we get $f_1 = f_2 = 0$ from (57) and (58) since $\lambda \neq \pm 1$. Hence, V_2 is empty. So, this completes the proof of the theorem. □

Now, we will give an example which satisfies Theorem 4.2.

Example 4.6. *Let us choose a local pseudo-orthonormal frame* $\{e_1, e_2, e_3 = \xi\}$ *for a three-dimensional paracontact metric manifold where* $[e_1, e_2] = 2\xi$, $[e_1, \xi] = -2e_1 - e_2$ *and* $[e_2, \xi] = 2e_2$ *for* $g(e_1, e_1) = g(e_2, e_2) = g(e_1, \xi) = g(e_2, \xi) = 0$ *and* $g(e_1, e_2) = g(e_3, e_3) = 1$. *Using the equation (3) and Propositon 3.4, we see that the manifold admits Cotton soliton for* $V = 3\xi$, $\tilde{b_2} = b_2 = 0$ *and* $a_2 = 1$. *We conclude that the scalar curvature* $r = -6$ *and steady from Lemma 3.3.*

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