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An investigation into inequalities obtained through the new lemma for exponential type convex functions

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Abstract. In this paper, we obtained a new general Lemma (Lemma 2.1) and proved some new integral inequalities for exponential type convex functions. To prove these inequalities, we used the Hölder, Hölder-İşcan, Power Mean and Improved Power-Mean integral inequalities. Finally, some applications for special means were also given.

1. Introduction

The theory of inequalities has been the focus of researchers in the last decade. One of the main reasons for this interest is the definition of convexity. This definition firstly has appeared in Jensen's (the celebrated Danish mathematician) papers in 1905. The convex functions have experienced a rapid development because the convex functions are closely related to the theory of inequalities and many important inequalities (such as Hermite-Hadamard inequality, Minkowski inequality and Jensen inequality) are consequences of the applications of convex functions.([1]-[6])

The definition of the convex function can be represented as follows:

Definition 1.1. *A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex if*

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b)$$

holds for all $a, b \in I$ and $t \in [0, 1]$.

Theorem 1.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the invertal I of real numbers and $a, b \in I$ with $a < b$. The following double inequality:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

is known in the literature as the Hermite-Hadamard inequality for convex mapping.

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Many different definitions and identities have been obtained by using convex functions. Even the relationships between these definitions have been proven.([7]) Some of these definitions are $s-$convex, $m-$convex, $h-$convex, quasi-convex, harmonically convex. ([8]-[12])

In reference [14], Kadakal and İşcan provided a new definition which is called exponential type convex function and showed that this definition satisfied some properties given below. Also, using the definition of the exponential type convex function, they proved the Hermite-Hadamard inequalities again in [14].

Definition 1.3. *A nonnegative function $f : I \to \mathbb{R}$ is called exponential type convex function if, for every $a, b \in I$ and $t \in [0, 1]$,*

$$f(ta + (1-t)b) \leq (e^t - 1)f(a) + (e^{1-t} - 1)f(b).$$

The class of all exponential type convex functions on invertal I is indicated by $EXPC(I)$.

A new definition associated with the convex function is exponential type convex function. This relationship is given as follows:

Proposition 1.4. *For all $t \in [0, 1]$, the inequalities $e^t - 1 \geq t$ and $e^{1-t} - 1 \geq 1 - t$ hold.*

Proposition 1.5. *Every nonnegative convex function is exponential type convex function.*

Theorem 1.6. *[[14]] Let $f : [a, b] \to \mathbb{R}$ be an exponential type convex function. If $a < b$ and $f \in L[a, b]$, then the following Hermite-Hadamard type inequalities holds:*

$$\frac{1}{2(\sqrt{e}-1)} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq (e-2)[f(a) + f(b)].$$

Hölder's inequality is a fundamental inequality between integrals and an indispensable tool for the study of L^p spaces. Many new generalizations and refinements have been obtained in the theory of inequalities using different convex functions and this inequality. However, in [19], İşcan proved a new form of the Hölder inequality using a simple method. Using the Hölder-Işcan inequality, better upper bounds are obtained than in previous studies. This new inequality is as follows:

Theorem 1.7. *(Hölder-İşcan integral inequality) Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are real functions defined on $[a, b]$ and if $|f|^p$ and $|g|^q$ are integrable on $[a, b]$, then*

$$\begin{aligned} \int_a^b |f(x)g(x)|\,dx \leq \frac{1}{b-a}\Bigg\{ & \left(\int_a^b (b-x)|f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_a^b (b-x)|g(x)|^q dx\right)^{\frac{1}{q}} \\ & + \left(\int_a^b (x-a)|f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_a^b (x-a)|g(x)|^q dx\right)^{\frac{1}{q}} \Bigg\} \end{aligned}$$

The power-mean integral inequality, which is a different version of the Hölder integral inequality, is well known for its elementary role in many branches of mathematical analysis.Therefore, in [20], Kadakal *et al.* showed and confirmed the improved power-mean integral inequality, which gives better results than the power-mean integral inequality. This new generalized expression is as follows:

Theorem 1.8. *(Improved Power-Mean integral inequality) Let $q \geq 1$. If f and g are real functions defined on $[a, b]$ and if $|f|$, $|f|\,|g|^q$ are integrable on $[a, b]$, then*

$$\begin{aligned} \int_a^b |f(x)g(x)|\,dx \leq \frac{1}{b-a}\Bigg\{ & \left(\int_a^b (b-x)|f(x)|\,dx\right)^{1-\frac{1}{q}} \left(\int_a^b (b-x)|f(x)|\,|g(x)|^q dx\right)^{\frac{1}{q}} \\ & + \left(\int_a^b (x-a)|f(x)|\,dx\right)^{1-\frac{1}{q}} \left(\int_a^b (x-a)|f(x)|\,|g(x)|^q dx\right)^{\frac{1}{q}} \Bigg\} \end{aligned}$$

Definition 1.9. *(Beta Function) The Beta function denoted by* $\beta(a,b)$ *is defined by*

$$\beta(a,b)=\int_0^1 t^{\alpha-1}(1-t)^{b-1}\,dt,\ a,b>0.$$

Corollary 1.10. *Beta function provides the following properties:*

1. $\beta(a,b)=\beta(b,a)$
2. $\beta(a+1,b)=\frac{a}{a+b}\beta(a,b)$

The main purpose of this article is to obtain some new generalizations of Hermite-Hadamard's integral inequality. In line with this purpose, we achieve a new integral identity for continuously differentiable functions. This identity will help as an auxiliary result to obtain main results of the article. Also, we discuss several new special cases in detail. Finally, we present applications for some of the special means of real numbers.

2. Auxiliary Results

First of all, we need to establish the following new lemma which will play an important role in obtaining main results of the article:

Lemma 2.1. *Let* $f: I^\circ\subset\mathbb{R}\to\mathbb{R}$ *be a continuously differentiable function on* I°*, where* $a,b\in I^\circ$*, with* $a<b$ *and* $n\in\mathbb{N}$*. If* $f''\in L[a,b]$*, then the following equality holds:*

$$\begin{aligned}&\Xi(f_n,[a,b])\\ =\ &\sum_{i=0}^{n-1}\frac{1}{2n}\left[f\left(\frac{(n-i)a+ib}{n}\right)+f\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\\ =\ &\sum_{i=0}^{n-1}\frac{(b-a)^2}{2n^3}\left[\int_0^1 t(1-t)f''\left(t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)dt\right].\end{aligned}\tag{1}$$

Proof. Let $n\in\mathbb{N}$ arbitrarily and $i\in\{1,2,...,n-1\}$. Then by integration by parts, we have the following identity:

$$\begin{aligned}\Lambda_i &= \int_0^1 t(1-t)f''\left(t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)dt\\ &= \frac{n}{a-b}(t-t^2)f'\left(t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)\Bigg|_0^1\\ &\quad -\frac{n}{a-b}\int_0^1(1-2t)f'\left(t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)dt\\ &= \frac{n}{b-a}\int_0^1(1-2t)f'\left(t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)dt.\end{aligned}$$

Again using integration by parts, we get

$$\begin{aligned}\Lambda_i &= \frac{n}{b-a}\int_0^1 (1-2t) f'\left(t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)dt \\ &= \frac{n}{b-a}\left[\frac{n}{a-b}(1-2t) f\left(t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)\Bigg|_0^1\right. \\ &\quad \left.+\frac{2n}{a-b}\int_0^1 f\left(t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)dt\right] \\ &= \frac{n^2}{(b-a)^2}\left\{f\left(\frac{(n-i)a+ib}{n}\right)+f\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right\} \\ &\quad -\frac{2n^2}{(b-a)^2}\int_0^1 f\left(t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)dt.\end{aligned} \tag{2}$$

Substituting $x = t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}$ in (2), then we have

$$\begin{aligned}\Lambda_i &= \frac{n^2}{(b-a)^2}\left\{f\left(\frac{(n-i)a+ib}{n}\right)+f\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right\} \\ &\quad -\frac{2n^3}{(b-a)^3}\int_{\frac{(n-i)a+ib}{n}}^{\frac{(n-i-1)a+(i+1)b}{n}} f(x)dx.\end{aligned}$$

Multiplying the both sides by $\frac{(b-a)^2}{2n^3}$ in the above identity, we obtain

$$\begin{aligned}\frac{(b-a)^2}{2n^3}\Lambda_i &= \frac{1}{2n}\left[f\left(\frac{(n-i)a+ib}{n}\right)+f\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right] \\ &\quad -\frac{1}{b-a}\int_{\frac{(n-i)a+ib}{n}}^{\frac{(n-i-1)a+(i+1)b}{n}} f(x)dx.\end{aligned}$$

Consequently, we have

$$\begin{aligned}\sum_{i=0}^{n-1}\frac{(b-a)^2}{2n^3}\Lambda_i &= \sum_{i=0}^{n-1}\frac{1}{2n}\left[f\left(\frac{(n-i)a+ib}{n}\right)+f\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right] \\ &\quad -\frac{1}{b-a}\sum_{i=0}^{n-1}\int_{\frac{(n-i)a+ib}{n}}^{\frac{(n-i-1)a+(i+1)b}{n}} f(x)dx \\ &= \sum_{i=0}^{n-1}\frac{1}{2n}\left[f\left(\frac{(n-i)a+ib}{n}\right)+f\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right] \\ &\quad -\frac{1}{b-a}\int_a^b f(x)dx.\end{aligned}$$

That completes the proof. □

Remark 2.2. *In Lemma 2.1, we get the following conditions:*

1. If we take $n = 1$ in equation (1), then we have the following equation in [17]:

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx = \frac{(b-a)^2}{2}\int_0^1 t(1-t) f''(ta+(1-t)b)\,dt.$$

2. If we take $n = 2$ in equation (1), then

$$\begin{aligned}&\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\\ =&\frac{(b-a)^2}{16}\left\{\int_0^1 t(1-t)f''\left(ta+(1-t)\frac{a+b}{2}\right)dt\right.\\ &\left.+\int_0^1 t(1-t)f''\left(t\frac{a+b}{2}+(1-t)b\right)dt\right\}.\end{aligned}$$

Above result can be obtained if we choose $\lambda = \frac{1}{2}$ in Lemma 2.1 ([18]).

3. Main Results

Theorem 3.1. *Let $f : I \to \mathbb{R}$ be a continuously differentiable function on I°, $a, b \in I^\circ$ with $a < b$, and assume that $f'' \in L[a, b]$. If $|f''|$ is an exponential type convex function on $[a, b]$, then the following integral inequality holds:*

$$\begin{aligned}&\left|\Xi(f_n,[a,b])\right|\\ \le&\sum_{i=0}^{n-1}\frac{(b-a)^2}{n^3}\left(\frac{17}{6}-e\right)A\left(\left|f''\left(\frac{(n-i)a+ib}{n}\right)\right|,\left|f''\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right|\right)\end{aligned}$$

for $t \in [0, 1]$, where $A(u, v)$ is the arithmetic mean of u and v.

Proof. From Lemma 2.1 and the exponential type convex function of $|f''|$, we get

$$\begin{aligned}&\left|\Xi(f_n,[a,b])\right|\\ \le&\sum_{i=0}^{n-1}\frac{(b-a)^2}{2n^3}\left[\int_0^1\left|t-t^2\right|\left|f''\left(t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)\right|dt\right]\\ \le&\sum_{i=0}^{n-1}\frac{(b-a)^2}{2n^3}\int_0^1\left|t-t^2\right|\left\{\left(e^t-1\right)\left|f''\left(\frac{(n-i)a+ib}{n}\right)\right|\right.\\ &\left.+\left(e^{1-t}-1\right)\left|f''\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right|\right\}dt\\ =&\sum_{i=0}^{n-1}\frac{(b-a)^2}{2n^3}\left[\left|f''\left(\frac{(n-i)a+ib}{n}\right)\right|\int_0^1\left(t-t^2\right)\left(e^t-1\right)dt\right.\\ &\left.+\left|f''\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right|\int_0^1\left(t-t^2\right)\left(e^{1-t}-1\right)dt\right]\\ =&\sum_{i=0}^{n-1}\frac{(b-a)^2}{n^3}\left(\frac{17}{6}-e\right)A\left(\left|f''\left(\frac{(n-i)a+ib}{n}\right)\right|,\left|f''\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right|\right),\end{aligned}$$

where

$$\int_0^1\left(t-t^2\right)\left(e^t-1\right)dt=\int_0^1\left(t-t^2\right)\left(e^{1-t}-1\right)dt=\frac{17}{6}-e.$$

□

Corollary 3.2. *If we choose $n = 1$ in Theorem 3.1, then we obtain*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\le(b-a)^2\left(\frac{17}{6}-e\right)\frac{\left|f''(a)\right|+\left|f''(b)\right|}{2}.$$

Corollary 3.3. *If we choose $n = 2$ in Theorem 3.1, then we obtain*

$$\begin{aligned}
&\left|\frac{1}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{(b-a)^2}{8}\left(\frac{17}{6}-e\right)\left[\frac{\left|f''(a)\right|+\left|f''(b)\right|}{2}+\left|f''\left(\frac{a+b}{2}\right)\right|\right].
\end{aligned}$$

Theorem 3.4. *Let $f : I \to \mathbb{R}$ be a continuously differentiable function on I°, $a, b \in I^\circ$ with $a < b$ and assume that $f'' \in L[a, b]$. If $\left|f''\right|^q$ is an exponential type convex function on $[a, b]$, then*

$$\begin{aligned}
&\left|\Xi(f_n,[a,b])\right| \\
\leq\ & \sum_{i=0}^{n-1}\frac{(b-a)^2}{2^{1-\frac{1}{q}}n^3}\beta^{\frac{1}{p}}(p+1,p+1)(e-2)^{\frac{1}{q}} \\
&\times A^{\frac{1}{q}}\left(\left|f''\left(\frac{(n-i)a+ib}{n}\right)\right|^q,\left|f''\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right|^q\right)
\end{aligned} \tag{3}$$

holds for $t \in [0, 1]$, where $\frac{1}{p} + \frac{1}{q} = 1$ and A is the arithmetic mean.

Proof. From Lemma 2.1 and Hölder integral inequality, we get

$$\begin{aligned}
&\left|\Xi(f_n,[a,b])\right| \\
\leq\ & \sum_{i=0}^{n-1}\frac{(b-a)^2}{2n^3}\left[\int_0^1\left|t-t^2\right|\left|f''\left(t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)\right|dt\right] \\
\leq\ & \sum_{i=0}^{n-1}\frac{(b-a)^2}{2n^3}\left\{\int_0^1\left|t-t^2\right|^p dt\right\}^{\frac{1}{p}} \\
&\times\left\{\int_0^1\left|f''\left(t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)\right|^q dt\right\}^{\frac{1}{q}}.
\end{aligned}$$

Using the exponential type convex function of $\left|f''\right|^q$, we have

$$\begin{aligned}
&\left|\Xi(f_n,[a,b])\right| \\
\leq\ & \sum_{i=0}^{n-1}\frac{(b-a)^2}{2n^3}\left\{\int_0^1 t^p(1-t)^p dt\right\}^{\frac{1}{p}} \\
&\times\left\{\int_0^1\left[\left(e^t-1\right)\left|f''\left(\frac{(n-i)a+ib}{n}\right)\right|^q+\left(e^{1-t}-1\right)\left|f''\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right|^q\right]dt\right\}^{\frac{1}{q}}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{n-1} \frac{(b-a)^2}{2n^3} \beta^{\frac{1}{p}} (p+1, p+1) \\
&\quad \times \left\{ \left| f'' \left(\frac{(n-i)a+ib}{n} \right) \right|^q \int_0^1 \left(e^t - 1 \right) dt + \left| f'' \left(\frac{(n-i-1)a+(i+1)b}{n} \right) \right|^q \int_0^1 \left(e^{1-t} - 1 \right) dt \right\}^{\frac{1}{q}} \\
&= \sum_{i=0}^{n-1} \frac{(b-a)^2}{2^{1-\frac{1}{q}} n^3} \beta^{\frac{1}{p}} (p+1, p+1) (e-2)^{\frac{1}{q}} \\
&\quad \times A^{\frac{1}{q}} \left(\left| f'' \left(\frac{(n-i)a+ib}{n} \right) \right|^q, \left| f'' \left(\frac{(n-i-1)a+(i+1)b}{n} \right) \right|^q \right).
\end{aligned}
$$

Thus, the proof is completed. □

Corollary 3.5. *In inequality (3), if we choose $n = 1$, then we obtain*

$$
\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \le \frac{(b-a)^2}{2^{1-\frac{1}{q}}} \beta^{\frac{1}{p}} (p+1, p+1) (e-2)^{\frac{1}{q}} \left(\frac{\left| f''(a) \right|^q + \left| f''(b) \right|^q}{2} \right)^{\frac{1}{q}}.
$$

Corollary 3.6. *In inequality (3), if we choose $n = 2$, then we obtain*

$$
\begin{aligned}
&\left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| \\
&\le \frac{(b-a)^2}{2^{4-\frac{1}{q}}} \beta^{\frac{1}{p}} (p+1, p+1) (e-2)^{\frac{1}{q}} \left[\left(\frac{\left| f''(a) \right|^q + \left| f''(\frac{a+b}{2}) \right|^q}{2} \right)^{\frac{1}{q}} + \left(\frac{\left| f''(\frac{a+b}{2}) \right|^q + \left| f''(b) \right|^q}{2} \right)^{\frac{1}{q}} \right].
\end{aligned}
$$

Theorem 3.7. *Let $f : I \to \mathbb{R}$ be a continuously differentiable function on I°, $a, b \in I^\circ$ with $a < b$, $q \ge 1$ and assume that $f'' \in L[a, b]$. If $\left| f'' \right|^q$ is an exponential type convex function on $[a, b]$, then the following inequality holds:*

$$
\begin{aligned}
&\left| \Xi(f_n, [a, b]) \right| \\
&\le \sum_{i=0}^{n-1} \frac{(b-a)^2}{12^{1-\frac{1}{q}} n^3} \left(\frac{17}{6} - e \right)^{\frac{1}{q}} \\
&\quad \times A^{\frac{1}{q}} \left(\left| f'' \left(\frac{(n-i)a+ib}{n} \right) \right|^q, \left| f'' \left(\frac{(n-i-1)a+(i+1)b}{n} \right) \right|^q \right).
\end{aligned}
\tag{4}
$$

Proof. By using Lemma 2.1, Power-mean integral inequality and the property of the exponential type convex function of $\left| f'' \right|^q$, we obtain

$$
\begin{aligned}
&\left| \Xi(f_n, [a, b]) \right| \\
&\le \sum_{i=0}^{n-1} \frac{(b-a)^2}{2n^3} \left[\int_0^1 \left| t - t^2 \right| \left| f'' \left(t \frac{(n-i)a+ib}{n} + (1-t) \frac{(n-i-1)a+(i+1)b}{n} \right) \right| dt \right]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum_{i=0}^{n-1}\frac{(b-a)^2}{2n^3}\left\{\int_0^1\left|t-t^2\right|dt\right\}^{1-\frac{1}{q}}\\
&\quad\times\left\{\int_0^1\left|t-t^2\right|\left|f''\left(t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)\right|^q dt\right\}^{\frac{1}{q}}\\
&\leq \sum_{i=0}^{n-1}\frac{(b-a)^2}{2n^3}\left\{\int_0^1\left(t-t^2\right)dt\right\}^{1-\frac{1}{q}}\\
&\quad\times\left\{\int_0^1\left(t-t^2\right)\left[\left(e^t-1\right)\left|f''\left(\frac{(n-i)a+ib}{n}\right)\right|^q\right.\right.\\
&\quad\left.\left.+\left(e^{1-t}-1\right)\left|f''\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right|^q\right]dt\right\}^{\frac{1}{q}}\\
&= \sum_{i=0}^{n-1}\frac{(b-a)^2}{2n^3}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(\left|f''\left(\frac{(n-i)a+ib}{n}\right)\right|^q\int_0^1\left(t-t^2\right)\left(e^t-1\right)dt\right.\\
&\quad\left.+\left|f''\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right|^q\int_0^1\left(t-t^2\right)\left(e^{1-t}-1\right)dt\right)^{\frac{1}{q}}\\
&= \sum_{i=0}^{n-1}\frac{(b-a)^2}{12^{1-\frac{1}{q}}n^3}\left(\frac{17}{6}-e\right)^{\frac{1}{q}}A^{\frac{1}{q}}\left(\left|f''\left(\frac{(n-i)a+ib}{n}\right)\right|^q,\left|f''\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right|^q\right).
\end{aligned}
$$

This completes the proof. □

Remark 3.8. *Under the assumptions of Theorem 3.7 with $q = 1$, we get the conclusion of Theorem 3.1.*

Corollary 3.9. *If we choose $n = 1$ in Theorem 3.7, then we obtain*

$$
\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{(b-a)^2}{12^{1-\frac{1}{q}}}\left(\frac{17}{6}-e\right)^{\frac{1}{q}}\left(\frac{\left|f''(a)\right|^q+\left|f''(b)\right|^q}{2}\right)^{\frac{1}{q}}.
$$

Corollary 3.10. *If we choose $n = 2$ in Theorem 3.7, then we obtain*

$$
\begin{aligned}
&\left|\frac{1}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
&\leq \frac{(b-a)^2}{8\cdot 12^{1-\frac{1}{q}}}\left(\frac{17}{6}-e\right)^{\frac{1}{q}}\left[\left(\frac{\left|f''(a)\right|^q+\left|f''(\frac{a+b}{2})\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f''(\frac{a+b}{2})\right|^q+\left|f''(b)\right|^q}{2}\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

Theorem 3.11. *Let $f : I \to \mathbb{R}$ be a continuously differentiable function on I°, $a, b \in I^\circ$ with $a < b$ and assume that*

$f'' \in L[a,b]$. If $\left|f''\right|^q$ is an exponential type convex function on $[a,b]$, then we have the following inequality;

$$\begin{aligned} & \left|\Xi(f_n,[a,b])\right| \\ \leq & \sum_{i=0}^{n-1} \frac{(b-a)^2}{2n^3} \\ & \times \left\{ \beta^{\frac{1}{p}}(p+1,p+2) \left[\left(e-\frac{5}{2}\right) \left| f''\left(\frac{(n-i)a+ib}{n}\right)\right|^q + \frac{1}{2}\left| f''\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right|^q \right]^{\frac{1}{q}} \right. \\ & \left. +\beta^{\frac{1}{p}}(p+2,p+1) \left[\frac{1}{2}\left| f''\left(\frac{(n-i)a+ib}{n}\right)\right|^q + \left(e-\frac{5}{2}\right)\left| f''\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right|^q \right]^{\frac{1}{q}} \right\} \end{aligned} \tag{5}$$

for $t \in [0,1]$, where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using the Hölder-İşcan integral inequality and Lemma 2.1, we get

$$\begin{aligned} & \left|\Xi(f_n,[a,b])\right| \\ \leq & \sum_{i=0}^{n-1} \frac{(b-a)^2}{2n^3} \left[\int_0^1 \left|t-t^2\right| \left| f''\left(t\frac{(n-i)a+ib}{n} + (1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)\right| dt \right] \\ \leq & \sum_{i=0}^{n-1} \frac{(b-a)^2}{2n^3} \left\{ \left(\int_0^1 (1-t)\left|t-t^2\right|^p dt \right)^{\frac{1}{p}} \right. \\ & \times \left(\int_0^1 (1-t) \left| f''\left(t\frac{(n-i)a+ib}{n} + (1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)\right|^q dt \right)^{\frac{1}{q}} \\ & \left. + \left(\int_0^1 t\left|t-t^2\right|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 t \left| f''\left(t\frac{(n-i)a+ib}{n} + (1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)\right|^q dt \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

By the definition of the exponential type convex function, we obtain

$$\begin{aligned} & \left|\Xi(f_n,[a,b])\right| \\ \leq & \sum_{i=0}^{n-1} \frac{(b-a)^2}{2n^3} \left\{ \left(\int_0^1 t^p(1-t)^{p+1} dt \right)^{\frac{1}{p}} \right. \\ & \times \left(\int_0^1 (1-t) \left[\left(e^t-1\right)\left| f''\left(\frac{(n-i)a+ib}{n}\right)\right|^q + \left(e^{1-t}-1\right)\left| f''\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right|^q \right] dt \right)^{\frac{1}{q}} \\ & + \left(\int_0^1 t^{p+1}(1-t)^p dt \right)^{\frac{1}{p}} \\ & \left. \times \left(\int_0^1 t \left[\left(e^t-1\right)\left| f''\left(\frac{(n-i)a+ib}{n}\right)\right|^q + \left(e^{1-t}-1\right)\left| f''\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right|^q \right] dt \right)^{\frac{1}{q}} \right\} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{n-1} \frac{(b-a)^2}{2n^3} \left\{ \beta^{\frac{1}{p}}(p+1,p+2) \left[\left| f''\left(\frac{(n-i)a+ib}{n}\right) \right|^q \int_0^1 (1-t)\left(e^t-1\right) dt \right. \right. \\
&\quad \left. + \left| f''\left(\frac{(n-i-1)a+(i+1)b}{n}\right) \right|^q \int_0^1 (1-t)\left(e^{1-t}-1\right) dt \right]^{\frac{1}{q}} \\
&\quad + \beta^{\frac{1}{p}}(p+2,p+1) \\
&\quad \left. \times \left[\left| f''\left(\frac{(n-i)a+ib}{n}\right) \right|^q \int_0^1 t\left(e^t-1\right) dt + \left| f''\left(\frac{(n-i-1)a+(i+1)b}{n}\right) \right|^q \int_0^1 t\left(e^{1-t}-1\right) dt \right]^{\frac{1}{q}} \right\} \\
&= \sum_{i=0}^{n-1} \frac{(b-a)^2}{2n^3} \\
&\quad \times \left\{ \beta^{\frac{1}{p}}(p+1,p+2) \left[\left(e-\frac{5}{2}\right) \left| f''\left(\frac{(n-i)a+ib}{n}\right) \right|^q + \frac{1}{2} \left| f''\left(\frac{(n-i-1)a+(i+1)b}{n}\right) \right|^q \right]^{\frac{1}{q}} \right. \\
&\quad \left. + \beta^{\frac{1}{p}}(p+2,p+1) \left[\frac{1}{2} \left| f''\left(\frac{(n-i)a+ib}{n}\right) \right|^q + \left(e-\frac{5}{2}\right) \left| f''\left(\frac{(n-i-1)a+(i+1)b}{n}\right) \right|^q \right]^{\frac{1}{q}} \right\},
\end{aligned}
$$

where

$$
\begin{aligned}
\int_0^1 t^p (1-t)^{p+1} dt &= \beta(p+1,p+2), \\
\int_0^1 t^{p+1} (1-t)^p dt &= \beta(p+2,p+1), \\
\int_0^1 (1-t)\left(e^t-1\right) dt &= \int_0^1 t\left(e^{1-t}-1\right) dt = e-\frac{5}{2}, \\
\int_0^1 (1-t)\left(e^{1-t}-1\right) dt &= \int_0^1 t\left(e^t-1\right) dt = \frac{1}{2}.
\end{aligned}
$$

□

Remark 3.12. *The inequality (5) is better than the inequality (3).*

Proof. By using the properties

$$\beta(p+1,p+2) = \beta(p+2,p+1) = \frac{p+1}{2(p+1)}\beta(p+1,p+1)$$

and the concavity of the function $k:[0,\infty)\to\mathbb{R}$, $k(x)=x^s$, $0<s\le 1$, that is, if we use the property

$$\frac{u^s+v^s}{2} \le \left(\frac{u+v}{2}\right)^s$$

we can write the right hand-side of the inequality (5) as follow:

$$
\begin{aligned}
&\sum_{i=0}^{n-1}\frac{(b-a)^2}{2n^3}\\
&\times\Bigg\{\beta^{\frac{1}{p}}(p+1,p+2)\left[\left(e-\frac{5}{2}\right)\left|f''\left(\frac{(n-i)a+ib}{n}\right)\right|^q+\frac{1}{2}\left|f''\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right|^q\right]^{\frac{1}{q}}\\
&+\beta^{\frac{1}{p}}(p+2,p+1)\left[\frac{1}{2}\left|f''\left(\frac{(n-i)a+ib}{n}\right)\right|^q+\left(e-\frac{5}{2}\right)\left|f''\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right|^q\right]^{\frac{1}{q}}\Bigg\}\\
\leq\;& 2\sum_{i=0}^{n-1}\frac{(b-a)^2}{2n^3}\beta^{\frac{1}{p}}(p+1,p+2)\left[\frac{(e-2)\left|f''\left(\frac{(n-i)a+ib}{n}\right)\right|^q+(e-2)\left|f''\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right|^q}{2}\right]^{\frac{1}{q}}\\
=\;& 2\sum_{i=0}^{n-1}\frac{(b-a)^2}{2n^3}\\
&\times\left[\frac{p+1}{2(p+1)}\beta(p+1,p+1)\right]^{\frac{1}{p}}\left[\frac{(e-2)\left|f''\left(\frac{(n-i)a+ib}{n}\right)\right|^q+(e-2)\left|f''\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right|^q}{2}\right]^{\frac{1}{q}}\\
=\;& \sum_{i=0}^{n-1}\frac{(b-a)^2}{2^{1-\frac{1}{q}}n^3}\beta^{\frac{1}{p}}(p+1,p+1)(e-2)^{\frac{1}{q}}\\
&\times A^{\frac{1}{q}}\left(\left|f''\left(\frac{(n-i)a+ib}{n}\right)\right|^q,\left|f''\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right|^q\right),
\end{aligned}
$$

which is the required result. This completes the proof of the Remark. □

Corollary 3.13. *If we choose $n=1$ in inequality (5), then we obtain*

$$
\begin{aligned}
\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq\;& \frac{(b-a)^2}{2}\Bigg[\beta^{\frac{1}{p}}(p+1,p+2)\left(\left(e-\frac{5}{2}\right)\left|f''(a)\right|^q+\frac{1}{2}\left|f''(b)\right|^q\right)^{\frac{1}{q}}\\
&+\beta^{\frac{1}{p}}(p+2,p+1)\left(\frac{1}{2}\left|f''(a)\right|^q+\left(e-\frac{5}{2}\right)\left|f''(b)\right|^q\right)^{\frac{1}{q}}\Bigg].
\end{aligned}
$$

Theorem 3.14. *Let $f: I\to\mathbb{R}$ be a continuously differentiable function on I°, $a,b\in I^\circ$ with $a<b$, $q\geq 1$ and assume that $f''\in L[a,b]$. If $\left|f''\right|^q$ is an exponential type convex function on $[a,b]$, then the following inequality holds:*

$$
\begin{aligned}
&\left|\Xi(f_n,[a,b])\right|\\
\leq\;& \sum_{i=0}^{n-1}\frac{(b-a)^2}{2n^3}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}\\
&\times\Bigg\{\left[\left(\frac{131}{12}-4e\right)\left|f''\left(\frac{(n-i)a+ib}{n}\right)\right|^q+\left(3e-\frac{97}{12}\right)\left|f''\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right|^q\right]^{\frac{1}{q}}\\
&+\left[\left(3e-\frac{97}{12}\right)\left|f''\left(\frac{(n-i)a+ib}{n}\right)\right|^q+\left(\frac{131}{12}-4e\right)\left|f''\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right|^q\right]^{\frac{1}{q}}\Bigg\}.
\end{aligned}
\tag{6}
$$

Proof. By using Lemma 2.1, improved power-mean integral inequality and the property of the exponential type convex function of $\left|f''\right|^q$, we obtain

$$
\begin{aligned}
&\left|\Xi(f_n,[a,b])\right| \\
\leq\ & \sum_{i=0}^{n-1}\frac{(b-a)^2}{2n^3}\left[\int_0^1\left|t-t^2\right|\left|f''\left(t\frac{(n-i)\,a+ib}{n}+(1-t)\,\frac{(n-i-1)\,a+(i+1)\,b}{n}\right)\right|dt\right] \\
\leq\ & \sum_{i=0}^{n-1}\frac{(b-a)^2}{2n^3}\left[\left(\int_0^1(1-t)\left|t-t^2\right|dt\right)^{1-\frac{1}{q}}\right. \\
&\times\left(\int_0^1(1-t)\left|t-t^2\right|\left|f''\left(t\frac{(n-i)\,a+ib}{n}+(1-t)\,\frac{(n-i-1)\,a+(i+1)\,b}{n}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\left.+\left(\int_0^1 t\left|t-t^2\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|t-t^2\right|\left|f''\left(t\frac{(n-i)\,a+ib}{n}+(1-t)\,\frac{(n-i-1)\,a+(i+1)\,b}{n}\right)\right|^q dt\right)^{\frac{1}{q}}\right] \\
\leq\ & \sum_{i=0}^{n-1}\frac{(b-a)^2}{2n^3}\left[\left(\frac{1}{12}\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)\left(t-t^2\right)\right.\right. \\
&\times\left.\left[\left(e^t-1\right)\left|f''\left(\frac{(n-i)\,a+ib}{n}\right)\right|^q+\left(e^{1-t}-1\right)\left|f''\left(\frac{(n-i-1)\,a+(i+1)\,b}{n}\right)\right|^q\right]dt\right)^{\frac{1}{q}} \\
&+\left(\frac{1}{12}\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left(t-t^2\right)\right. \\
&\times\left.\left.\left[\left(e^t-1\right)\left|f''\left(\frac{(n-i)\,a+ib}{n}\right)\right|^q+\left(e^{1-t}-1\right)\left|f''\left(\frac{(n-i-1)\,a+(i+1)\,b}{n}\right)\right|^q\right]dt\right)^{\frac{1}{q}}\right]
\end{aligned}
$$

$$
\begin{aligned}
=\ & \sum_{i=0}^{n-1}\frac{(b-a)^2}{2n^3}\left[\left(\frac{1}{12}\right)^{1-\frac{1}{q}}\left(\left|f''\left(\frac{(n-i)\,a+ib}{n}\right)\right|^q\int_0^1(1-t)\left(t-t^2\right)\left(e^t-1\right)dt\right.\right. \\
&\qquad\left.+\left|f''\left(\frac{(n-i-1)\,a+(i+1)\,b}{n}\right)\right|^q\int_0^1(1-t)\left(t-t^2\right)\left(e^{1-t}-1\right)dt\right)^{\frac{1}{q}} \\
&\qquad+\left(\frac{1}{12}\right)^{1-\frac{1}{q}}\left(\left|f''\left(\frac{(n-i)\,a+ib}{n}\right)\right|^q\int_0^1 t\left(t-t^2\right)\left(e^t-1\right)dt\right. \\
&\qquad\left.\left.+\left|f''\left(\frac{(n-i-1)\,a+(i+1)\,b}{n}\right)\right|^q\int_0^1 t\left(t-t^2\right)\left(e^{1-t}-1\right)dt\right)^{\frac{1}{q}}\right] \\
=\ & \sum_{i=0}^{n-1}\frac{(b-a)^2}{2n^3}\left(\frac{1}{12}\right)^{1-\frac{1}{q}} \\
&\times\left\{\left[\left(\frac{131}{12}-4e\right)\left|f''\left(\frac{(n-i)\,a+ib}{n}\right)\right|^q+\left(3e-\frac{97}{12}\right)\left|f''\left(\frac{(n-i-1)\,a+(i+1)\,b}{n}\right)\right|^q\right]^{\frac{1}{q}}\right. \\
&\times\left\{\left[\left(\frac{131}{12}-4e\right)\left|f''\left(\frac{(n-i)\,a+ib}{n}\right)\right|^q+\left(3e-\frac{97}{12}\right)\left|f''\left(\frac{(n-i-1)\,a+(i+1)\,b}{n}\right)\right|^q\right]^{\frac{1}{q}}\right.
\end{aligned}
$$

In the above inequality, we can do the following basic computations;

$$\begin{aligned}\int_0^1 (1-t)\left|t-t^2\right| dt &= \int_0^1 t\left|t-t^2\right| dt = \frac{1}{12},\\ \int_0^1 (1-t)\left(t-t^2\right)\left(e^t-1\right) dt &= \int_0^1 t\left(t-t^2\right)\left(e^{1-t}-1\right) dt = \frac{131}{12}-4e,\\ \int_0^1 (1-t)\left(t-t^2\right)\left(e^{1-t}-1\right) dt &= \int_0^1 t\left(t-t^2\right)\left(e^t-1\right) dt = 3e-\frac{97}{12}.\end{aligned}$$

□

Remark 3.15. *The inequality (6) is better than the inequality (4).*

Proof. By using concavity of the function $k:[0,\infty)\to\mathbb{R}, k(x)=x^s, 0<s\leq 1$, we can write the right hand-side of the inequality (6) as follow:

$$\begin{aligned}&\sum_{i=0}^{n-1}\frac{(b-a)^2}{2n^3}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}\\ &\times\left\{\left[\left(\frac{131}{12}-4e\right)\left|f''\left(\frac{(n-i)a+ib}{n}\right)\right|^q+\left(3e-\frac{97}{12}\right)\left|f''\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right|^q\right]^{\frac{1}{q}}\right.\\ &\left.+\left[\left(3e-\frac{97}{12}\right)\left|f''\left(\frac{(n-i)a+ib}{n}\right)\right|^q+\left(\frac{131}{12}-4e\right)\left|f''\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right|^q\right]^{\frac{1}{q}}\right\}\\ \leq\ & 2\sum_{i=0}^{n-1}\frac{(b-a)^2}{2n^3}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}\left[\frac{\left(\frac{17}{6}-e\right)\left|f''\left(\frac{(n-i)a+ib}{n}\right)\right|^q+\left(\frac{17}{6}-e\right)\left|f''\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right|^q}{2}\right]^{\frac{1}{q}}\\ =\ & \sum_{i=0}^{n-1}\frac{(b-a)^2}{12^{1-\frac{1}{q}}n^3}\left(\frac{17}{6}-e\right)^{\frac{1}{q}}A^{\frac{1}{q}}\left(\left|f''\left(\frac{(n-i)a+ib}{n}\right)\right|^q,\left|f''\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right|^q\right),\end{aligned}$$

which is the required result. This completes the proof of the Remark. □

Remark 3.16. *Under the assumption of Theorem 3.14 with $q=1$, we get the following the inequality*

$$\left|\Xi(f_n,[a,b])\right|\leq\sum_{i=0}^{n-1}\frac{(b-a)^2}{n^3}\left(\frac{17}{6}-e\right)A\left[\left|f''\left(\frac{(n-i)a+ib}{n}\right)\right|,\left|f''\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right|\right].$$

Corollary 3.17. *If we choose $n=1$ in Theorem 3.14, then we obtain*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq\ & \frac{(b-a)^2}{2}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}\\ &\times\left[\left(\left(\frac{131}{12}-4e\right)\left|f''(a)\right|^q+\left(3e-\frac{97}{12}\right)\left|f''(b)\right|^q\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\left(3e-\frac{97}{12}\right)\left|f''(a)\right|^q+\left(\frac{131}{12}-4e\right)\left|f''(b)\right|^q\right)^{\frac{1}{q}}\right].\end{aligned}$$

4. Applications to Special Means

In this section, we will apply some special means to the results we have achieved. Let $a, b \in \mathbb{R}$,

1. The arithmetic mean:

$$A = A(a,b) := \frac{a+b}{2}, \ \ a,b \geq 0.$$

2. The harmonic mean:

$$H = H(a,b) := \frac{2ab}{a+b}, \ \ a,b > 0.$$

3. The logarithmic mean:

$$L = L(a,b) := \begin{cases} a, & if\ a = b \\ \frac{b-a}{\ln b - \ln a}, & if\ \ a \neq b, \end{cases} \qquad a,b > 0.$$

4. The p-logarithmic mean:

$$L_p = L_p(a,b) := \begin{cases} a, & if\ \ a = b \\ \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & if\ \ a \neq b, \end{cases} \quad p \in \mathbb{R} \setminus \{-1, 0\},\ a,b > 0.$$

Proposition 4.1. *Let $a, b \in \mathbb{R}$, $0 < a < b$ and $t \in \mathbb{N}$, $t \geq 3$. Then the following inequality holds;*

$$\left|\sum_{i=0}^{n-1} \frac{1}{n} A\left(\left(\frac{(n-i)\,a + ib}{n}\right)^t, \left(\frac{(n-i-1)\,a + (i+1)\,b}{n}\right)^t\right) - L_t^t(a,b)\right|$$
$$\leq \sum_{i=0}^{n-1} \frac{(b-a)^2\, t(t-1)}{n^3}\left(\frac{17}{6} - e\right) A\left(\left|\frac{(n-i)\,a + ib}{n}\right|^{t-2}, \left|\frac{(n-i-1)\,a + (i+1)\,b}{n}\right|^{t-2}\right).$$

Proof. The assertion follows from Theorem 3.1 applied for $f(x) = x^t$, $x \in [a,b]$, $t \in \mathbb{N}$, $t \geq 3$. □

Proposition 4.2. *Let $a, b \in \mathbb{R}$, $0 < a < b$ and $t \in \mathbb{N}$, $t \geq 3$. Then, for all $q > 1$, the following inequality holds;*

$$\left|\sum_{i=0}^{n-1} \frac{1}{n} A\left(\left(\frac{(n-i)\,a + ib}{n}\right)^t, \left(\frac{(n-i-1)\,a + (i+1)\,b}{n}\right)^t\right) - L_t^t(a,b)\right|$$
$$\leq \sum_{i=0}^{n-1} \frac{(b-a)^2\, t(t-1)}{2^{1-\frac{1}{q}} n^3} \beta^{\frac{1}{p}}(p+1, p+1)\,(e-2)^{\frac{1}{q}}$$
$$\times A^{\frac{1}{q}}\left(\left|\frac{(n-i)\,a + ib}{n}\right|^{(t-2)q}, \left|\frac{(n-i-1)\,a + (i+1)\,b}{n}\right|^{(t-2)q}\right).$$

Proof. The assertion follows from the inequality (3) applied for $f(x) = x^t$, $x \in [a,b]$, $t \in \mathbb{N}$, $t \geq 3$. □

Proposition 4.3. *Let $a, b \in \mathbb{R}$, $0 < a < b$ and $t \in \mathbb{N}$, $t \geq 3$. Then, for all $q \geq 1$, the following inequality holds;*

$$\left|\sum_{i=0}^{n-1} \frac{1}{n} A\left(\left(\frac{(n-i)\,a + ib}{n}\right)^t, \left(\frac{(n-i-1)\,a + (i+1)\,b}{n}\right)^t\right) - L_t^t(a,b)\right|$$
$$\leq \sum_{i=0}^{n-1} \frac{(b-a)^2\, t(t-1)}{12^{1-\frac{1}{q}} n^3}\left(\frac{17}{6} - e\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(\left|\frac{(n-i)\,a + ib}{n}\right|^{(t-2)q}, \left|\frac{(n-i-1)\,a + (i+1)\,b}{n}\right|^{(t-2)q}\right)$$

Proof. The assertion follows from Theorem 3.7 applied for $f(x) = x^t$, $x \in [a,b]$, $t \in \mathbb{N}$, $t \geq 3$. □

5. Conclusion

In this study, a new Lemma (Lemma 2.1) was proved and new generalizations were made for exponential type convex function which is a new definition. Also using the Lemma 2.1, different types of integral inequalities were obtained (different results were found for various values of n, $n \in \mathbb{N}$) and comparisons were made between these inequalities. In these comparisons, the inequality (5) yields a better result compared to the inequality (3). Similarly, Theorem 3.14 yields a better result compared to the Theorem 3.7. In addition to this work, researchers interested in this field can obtain new inequalities for concave functions and different convex functions. We hope that this study will inspire new interesting results for mathematicians working in the theory of inequalities.

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