Filomat 37:15 (2023), 4881–4902
https://doi.org/10.2298/FIL2315881G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A class of Yosida inclusion and graph convergence on Yosida approximation mapping with an application

Sanjeev Gupta[a,*], **Faizan Ahmad Khan**[b]

[a]*Department of Mathematics, Institute of Applied Sciences and Humanities, GLA University Mathura-281406, India*
[b]*Department of Mathematics, Faculty of Science, University of Tabuk, Tabuk-71491, Saudi Arabia*

Abstract. The proposed work is presented in two folds. The first aim is to deals with the new notion called generalized $\alpha_i\beta_j$-$H^p(.,.,...)$-accretive mappings that are the sum of two symmetric accretive mappings. It is an extension of $\alpha\beta$-$H(.,.)$-accretive mapping, studied and analyzed by Kazmi [18]. We define the proximal-point mapping associated with generalized $\alpha_i\beta_j$-$H^p(.,.,...)$-accretive mapping and demonstrate aspects on single-valued property and Lipschitz continuity. The graph convergence of generalized $\alpha_i\beta_j$-$H^p(.,.,...)$-accretive mapping is discussed.

Second aim is to introduce and study the generalized Yosida approximation mapping and Yosida inclusion problem. Next, we obtain the convergence on generalized Yosida approximation mappings by using the graph convergence of generalized $\alpha_i\beta_j$-$H^p(.,.,...)$-accretive mappings without using the convergence of its proximal-point mapping. As an application, we consider the Yosida inclusion problem in q-uniformly smooth Banach spaces and propose an iterative scheme connected with generalized Yosida approximation mapping of generalized $\alpha_i\beta_j$-$H^p(.,.,...)$-accretive mapping to find a solution of Yosida inclusion problem and discuss its convergence criteria under appropriate assumptions. Some examples are constructed and demonstrate few graphics for the convergence of proximal-point mapping as well as generalized Yosida approximation mapping linked with generalized $\alpha_i\beta_j$-$H^p(.,.,...)$-accretive mappings.

1. Introduction and preliminaries

Variational inequalities are very powerful tool to study a large variety of problems that appear in electricity, mechanics, operations research, optimal control, etc. Due to its extensive applications, variational inequality has been well researched and generalized in various directions. A wide range of issues we face in electricity, mechanics, operation research and optimal control can be organize as an inclusion problem

$$\Theta \in \mathcal{N}(x), \tag{1}$$

where $\mathcal{N} : \mathcal{H} \multimap \mathcal{H}$ is a multi-valued mapping on Hilbert spaces $\mathcal{H}$. Therefore the problem of finding a zero $\Theta \in \mathcal{N}(x)$, that is a point $x \in \mathcal{H}$ such that $\Theta \in \mathcal{N}(x)$ is a elementary problem in many fields of applied sciences.

2020 *Mathematics Subject Classification*. Primary 49J40; Secondary 47H19.
Keywords. Generalized $\alpha_i\beta_j$-$H^p(.,.,...)$-accretive mappings; Yosida approximation mapping; Proximal-point mapping; Graph convergence; Iterative algorithm; Variational inclusion.
Received: 16 August 2022; Accepted: 29 September 2022
Communicated by Dragan S. Djordjević
* Corresponding author: Sanjeev Gupta
Email addresses: guptasanmp@gmail.com (Sanjeev Gupta), fkhan@ut.edu.sa (Faizan Ahmad Khan)

Many mathematicians have worked on the well-known fact that regularization of monotone operators on Hilbert spaces into single-valued Lipschitzian monotone operators through the procedure known as the Yosida approximation. These Yosida approximation are significant in approaching solutions to general variational inclusion problems utilizing non-expansive proximal-point mapping. Many heuristics [2, 6, 7, 19–21, 27] have utilized the Yosida approximation mappings and their generalized versions to find out the solutions of variational inclusion problems.

In this connection, the accretive (monotone) property of the underlying proximal-point mappings (resolvent operators) have a significant role in the field of variational inequalities and their generalizations. Huang and Fang [15] were the first who considered and studied m-accretive mappings and its proximal-point mapping in Banach spaces. After that many mathematicians studied different kinds of generalized m-accretive mappings, we refer to [8, 22]. Sun et al. [31] proposed and analyzed M-monotone mappings in Hilbert spaces. A few research works linked with M-monotone (accretive) and their extension are given in [3, 10–14, 16–18, 25, 26, 28–30, 32, 34, 35].

In Recent years, $H(.,.)$-accretive mappings and generalized $\alpha\beta$-$H(.,.)$-accretive mappings were investigated and studied in Banach spaces, an natural extension of M-monotone mapping, see [18, 34, 35]. They studied the variational inclusions involving these underlying mappings.

Demonstration of graphical convergence related to $H(.,.)$-accretive mappings and equivalence between proximal-point mappings and graphical convergence of a sequence of $H(.,.)$-accretive mappings studied and analyzed by Li and Huang [23]. Recently, graphical convergence on A-maximal relaxed monotone, $\hat{A}$-maximal m-relaxed η-accretive studied and analyzed by Verma [32] and Balooee et al. [4]. For detailed study in this direction, see [1, 2, 5, 16, 23].

Motivated and inspired by the research work discussed above. Our research work is presented in two folds. Firstly, we consider $\alpha_i\beta_j$-$H^p(.,.,...)$-accretive mappings defined on a product set which are the sum of two symmetric accretive mappings. This notion is the generalized form of $\alpha\beta$-$H(.,.)$-accretive mappings studied and analyzed by Kazmi et. al [18]. We define the proximal-point mapping and discuss its some properties. Further, we focus on graph convergence connected with generalized $\alpha_i\beta_j$-$H^p(.,.,...)$-accretive mappings.

In the second phase, we consider and study generalized Yosida approximation mapping with few nice properties. Next, We establish the equivalence between convergence of the proximal point mappings, Yosida approximation mappings, and graph convergence of a sequence of generalized $\alpha_i\beta_j$-$H^p(.,.,...)$-accretive mappings. Further, an iterative algorithm involving generalized Yosida approximation mappings linked with generalized $\alpha_i\beta_j$-$H^p(.,.,...)$-accretive mappings is constructed and then, convergence analysis for this algorithm in the context to find uniqueness and existence of a solution to class of Yosida inclusion along some suitable assumptions is examined in the setting of q-uniformly smooth Banach space. A few examples are constructed and shown some graphics for the convergence of proximal-point mappings and generalized Yosida approximation mappings linked with the generalized $\alpha_i\beta_j$-$H^p(.,.,...)$-accretive mappings. Our work is the extension and refinement of some results available in the literature, see [2, 9, 18, 34].

Let Y be a real Banach space endowed with a norm $||.||$ and inner product $\langle .,. \rangle$ which presents the duality pairing between Y and Y^*. Let $CB(Y)$ be the family of all nonempty closed and bounded subsets of Y and 2^Y be the power set of of Y. Set $Y^p = \underbrace{Y \times Y \times ... \times Y}_{p\ times}$.

Definition 1.1. *"A multi-valued mapping $J_q : Y \multimap Y^*$, $q > 1$ is said to be the generalized duality mapping, if*

$$J_q(t) \ = \ \{\tilde{t} \in Y^* : \ \langle t, \tilde{t} \rangle = ||t||^q, \ ||\tilde{t}|| = ||t||^{q-1}\}, \quad \forall\ t \in Y.$$

It is well known that $J_q(t) = ||t||^{q-1} J_2(t)\ \forall\ t(\neq 0) \ \in Y$, where J_2 is usual normalized duality mapping on Y. If Y is equivalent to real Hilbert space X, then, J_2 become identity mapping on X", [33].

Definition 1.2. *"A Banach space Y is smooth if for every $t \in Y$ with $||t|| = 1$, there exists a unique $l \in Y^*$ such that $||l|| = l(t) = 1$", [33].*

Definition 1.3. *"Let $\Omega_Y : [0,\infty) \to [0,\infty)$, then, modulus of smoothness of Y is given as*

$$\Omega_Y(\mu) = \sup\left\{\frac{\|t+\tilde{t}\| + \|t-\tilde{t}\|}{2} - 1 : \|t\| \le 1, \|\tilde{t}\| \le \mu\right\}'', [33]. \tag{2}$$

Definition 1.4. *"A Banach space Y is said to be*
(i): *uniformly smooth if* $\lim_{\mu\to 0} \Omega_Y(\mu)/\mu = 0$;
(ii): *q-uniformly smooth ($q > 1$), if there exists $k > 0$ with $\Omega_Y(\mu) \le k\,\mu^q$, $\mu \in [0,\infty)$.*

It is observe that J become single-valued if Y is uniformly smooth", [33].

Lemma 1.5. *"A real uniformly smooth Banach space Y is q-uniformly smooth iff there exists a constant $c_q > 0$ such that, for every $t, t' \in Y$,*

$$\|t+\tilde{t}\|^q \le \|t\|^q + q\langle \tilde{t}, J_q(t)\rangle + c_q\|\tilde{t}\|^q{}'', [33].$$

Lemma 1.6. *"Let $\{b_n\}$ and $\{c_n\}$ be the two non-negative real sequences, which are satisfying the inequality $b_{n+1} \le lb_n + c_n$ with $c_n \to 0$ and $0 < l < 1$. Then, $\lim_{n\to\infty} b_n = 0$", [24].*

Definition 1.7. *Let $\mathcal{T} : Y \to Y$ be a single-valued mapping, then, $\mathcal{T}$ is said to be*
(i) accretive if

$$\langle \mathcal{T}(w^*) - \mathcal{T}(u^*),\ J_q(w^* - u^*)\rangle \ge 0 \ \ \forall w^*,\ u^* \in Y;$$

(ii) ξ-strongly accretive if there exists $\xi > 0$ with

$$\langle \mathcal{T}(w^*) - \mathcal{T}(u^*),\ J_q(w^* - u^*)\rangle \ge \xi\,\|w^* - u^*\|^q \ \ \forall w^*,\ v^* \in Y;$$

(iii) λ_T-Lipschitz continuous if there exists $\lambda_{\mathcal{T}} > 0$ with

$$\|\mathcal{T}(w^*) - \mathcal{T}(u^*)\| \le \lambda_T\,\|w^* - u^*\|, \ \ \forall w^*,\ u^* \in Y;$$

(iv) λ-expansive if there exists $\lambda > 0$ with

$$\|\mathcal{T}(w^*) - \mathcal{T}(u^*)\| \ge \lambda\,\|w^* - u^*\|, \ \ \forall w^*,\ u^* \in Y;$$

(v) $\mathcal{T}$ becomes expansive if $\lambda = 1$.

Following new notions are needed to continue subsequent sections.

Definition 1.8. *Let $i \in \{1, 2, ..., p\}, p \ge 3$ and $H^p : Y^p \to Y$ and $A_i : Y \to Y$ be the single-valued mappings. Then, H^p is said to be*
(i) α_i-strongly accretive with A_i if there exists $\alpha_i > 0$ such that

$$\left\langle H^p(v_1, ..., v_{i-1}, A_i\tilde{w}, v_{i+1}, ..., v_n) - H^p(v_1, ..., v_{i-1}, A_i\tilde{v}, v_{i+1}, ..., v_n),\ J_q(\tilde{w} - \tilde{v})\right\rangle \ge \alpha_i\,\|\tilde{w} - \tilde{v}\|^q,$$
$$\forall \tilde{w}, \tilde{v}, v_1, ..., v_{i-1}, v_{i+1}, ..., v_n \in Y;$$

(ii) β_i-relaxed accretive with A_i if there exists $\beta_i > 0$ such that

$$\left\langle H^p(v_1, ..., v_{i-1}, A_i\tilde{w}, v_{i+1}, ..., v_n) - H^p(v_1, ..., v_{i-1}, A_i\tilde{v}, v_{i+1}, ..., v_n),\ J_q(\tilde{w} - \tilde{v})\right\rangle \ge -\beta_i\,\|\tilde{w} - \tilde{v}\|^q,$$
$$\forall \tilde{w}, \tilde{v}, v_1, ..., v_{i-1}, v_{i+1}, ..., v_n \in Y;$$

(iii) s_i-Lipschitz continuous with A_i if there exists $s_i > 0$ such that

$$\left\| H^p(v_1, ..., v_{i-1}, A_i\tilde{w}, v_{i+1}, ..., v_n) - H^p(v_1, ..., v_{i-1}, A_i\tilde{v}, v_{i+1}, ..., v_n)\right\| \le s_i\,\|\tilde{w} - \tilde{v}\|,$$
$$\forall \tilde{w}, \tilde{v}, v_1, ..., v_{i-1}, v_{i+1}, ..., v_n \in Y;$$

(iv) $\alpha_1\beta_2\alpha_3\beta_4...\alpha_{p-1}\beta_p$*-symmetric accretive with* $A_1, A_2, ..., A_p$ *iff for* $i \in \{1, 3, ..., p-1\}$, $H^p(..., A_i, ...)$ *is* α_i*-strongly accretive with* A_i *and for* $j \in \{2, 4, ..., p\}$, $H^p(..., A_j, ...)$ *is* β_j*-relaxed accretive with* A_j*, where p is even, satisfying*

$$\beta_2 + \beta_4 + ... + \beta_p \le \alpha_1 + \alpha_3 + ... + \alpha_{p-1}$$

and $\beta_2 + \beta_4 + ... + \beta_p = \alpha_1 + \alpha_3 + ... + \alpha_{p-1}$ *iff* $\tilde{w} = \tilde{v}$*;*
(v) $\alpha_1\beta_2\alpha_3\beta_4, ...\beta_{p-1}, \alpha_p$*-symmetric accretive with* $A_1, A_2, ..., A_p$ *iff for* $i \in \{1, 3, ..., p\}$, $H^p(..., A_i, ...)$ *is* α_i*-strongly accretive with* A_i *and for* $j \in \{2, 4, ..., p-1\}$, $H^p(..., A_j, ...)$ *is* β_j*-relaxed accretive* A_j *where p is odd, satisfying*

$$\beta_2 + \beta_4 + ... + \beta_{p-1} \le \alpha_1 + \alpha_3 + ... + \alpha_p$$

and $\beta_2 + \beta_4 + ... + \beta_{p-1} = \alpha_1 + \alpha_3 + ... + \alpha_p$ *iff* $\tilde{w} = \tilde{v}$*.*

Definition 1.9. *For* $i \in \{1, 2, ..., p\}, p \ge 3$, *let* $\mathcal{N} : Y^p \multimap Y$*be a multi-valued mapping and* $f_i : Y \to Y$ *be a single-valued mapping . Then,* $\mathcal{N}$ *is said to be*
(i) $\bar{\mu}_i$*-strongly accretive with* f_i *if there exists* $\bar{\mu}_i > 0$ *such that*

$$\langle \tilde{w}_i - \tilde{v}_i,\ J_q(\tilde{w} - \tilde{v})\rangle \ \ge\ \bar{\mu}_i \,\|\tilde{w} - \tilde{v}\|^q, \ \ \forall \tilde{w}, \tilde{v}, v_1, ..., v_{i-1}, v_{i+1}, ..., v_p \in Y,$$
$$\tilde{w}_i \in \mathcal{N}(v_1, ..., v_{i-1}, f_i(\tilde{w}), v_{i+1}, ..., v_n), \tilde{v}_i \in \mathcal{N}(v_1, ..., v_{i-1}, f_i(\tilde{v}), v_{i+1}, ..., v_n);$$

(ii) $\bar{\gamma}_i$*-relaxed accretive with* f_i *if there exists* $\bar{\gamma}_i > 0$ *such that*

$$\langle \tilde{w}_i - \tilde{v}_i,\ J_q(\tilde{w} - \tilde{v})\rangle \ \ge\ -\bar{\gamma}_i \,\|\tilde{w} - \tilde{v}\|^q, \ \ \forall \tilde{w}, \tilde{v}, v_1, ..., v_{i-1}, v_{i+1}, ..., v_p \in Y,$$
$$\tilde{w}_i \in \mathcal{N}(v_1, ..., v_{i-1}, f_i(\tilde{w}), v_{i+1}, ..., v_n), \tilde{v}_i \in \mathcal{N}(v_1, ..., v_{i-1}, f_i(\tilde{v}), v_{i+1}, ...v_n);$$

(iii) $\bar{\mu}_1\bar{\gamma}_2\bar{\mu}_3\bar{\gamma}_4...\bar{\mu}_{p-1}\bar{\gamma}_p$*-symmetric accretive with* $f_1, f_2, ..., f_p$ *iff for* $i \in \{1, 3, ..., p-1\}$, $\mathcal{N}(..., f_i, ...)$ *is* $\bar{\mu}_i$*-strongly accretive with* f_i *and for* $j \in \{2, 4, ..., p\}$, $\mathcal{N}(..., f_j, ...)$ *is* $\bar{\gamma}_j$*-relaxed accretive with* f_j*, where p is even, satisfying*

$$\bar{\gamma}_2 + \bar{\gamma}_4 + ... + \bar{\gamma}_p \le \bar{\mu}_1 + \bar{\mu}_3 + ... + \bar{\mu}_{p-1}$$

and $\bar{\gamma}_2 + \bar{\gamma}_4 + ... + \bar{\gamma}_p = \bar{\mu}_1 + \bar{\mu}_3 + ... + \bar{\mu}_{p-1}$ *iff* $\tilde{w} = \tilde{v}$*;*
(iv) $\bar{\mu}_1\bar{\gamma}_2\bar{\mu}_3\bar{\gamma}_4, ...\bar{\mu}_p, \bar{\gamma}_{p-1}$*-symmetric accretive with* $f_1, f_2, ..., f_p$ *iff for* $i \in \{1, 3, ..., p\}$, $\mathcal{N}(..., f_i, ...)$ *is* $\bar{\mu}_i$*-strongly accretive with* f_i *and for* $j \in \{2, 4, ..., p-1\}$, $\mathcal{N}(..., f_j, ...)$ *is* $\bar{\gamma}_j$*-relaxed accretive with* f_j*, where p is odd, satisfying*

$$\bar{\gamma}_2 + \bar{\gamma}_4 + ... + \bar{\gamma}_{p-1} \le \bar{\mu}_1 + \bar{\mu}_3 + ... + \bar{\mu}_p$$

and $\bar{\gamma}_2 + \bar{\gamma}_4 + ... + \bar{\gamma}_{p-1} = \bar{\mu}_1 + \bar{\mu}_3 + ... + \bar{\mu}_p$ *iff* $\tilde{w} = \tilde{v}$*.*

2. Generalized $\alpha_i\beta_j$-$H^p(.,.,...,.)$-accretive mappings

At first, we consider some assumptions (M_1-M_4) to introduce and study the new notion $\alpha_i\beta_j$-generalized $H^p(.,.,....)$-accretive.

Let for each $i \in \{1, 2, ..., p\}$, $p \ge 3$ and $\mathcal{N} : Y^p \multimap Y$ be multi-valued mapping and $H^p : Y^p \to Y$, $A_i : Y \to Y$ and $f_i : Y \to Y$ be the single-valued mappings.
$\mathbf{M_1}$**:** If p is even, H^p is $\alpha_1\beta_2\alpha_3\beta_4...\alpha_{p-1}\beta_p$-symmetric accretive with $A_1, A_2, ..., A_p$.
$\mathbf{M_2}$**:** If p is odd, H^p is $\alpha_1\beta_2\alpha_3\beta_4...\beta_{p-1}\alpha_p$-symmetric accretive with $A_1, A_2, ..., A_p$.
$\mathbf{M_3}$**:** If p is even, N is $\bar{\mu}_1\bar{\gamma}_2\bar{\mu}_3\bar{\gamma}_4...\bar{\mu}_{p-1}\bar{\gamma}_p$-symmetric accretive with $f_1, f_2, ..., f_p$.
$\mathbf{M_4}$**:** If p is odd, N is $\bar{\mu}_1\bar{\gamma}_2\bar{\mu}_3\bar{\gamma}_4...\bar{\gamma}_{p-1}\bar{\mu}_p$-symmetric accretive with $f_1, f_2, ..., f_p$.

Definition 2.1. *Let* $p \ge 3$*, then,* $\mathcal{N}$ *is said to be a generalized* $\alpha_i\beta_j$*-*$H^p(.,.,...,.)$*-accretive mapping with mappings* $(A_1, A_2, ..., A_p)$ *and* $(f_1, f_2, ..., f_p)$

(i) iff $\mathcal{N}$ *is* $\bar{\mu}_1\bar{\gamma}_2\bar{\mu}_3\bar{\gamma}_4...\bar{\mu}_{p-1}\bar{\gamma}_p$*-symmetric accretive with* $f_1, f_2, ..., f_p$ *and* $\Big(H^p(A_1, A_2, ..., A_p) + \rho\mathcal{N}(f_1, f_2, ..., f_p)\Big)(Y) = Y$*, for all* $\rho > 0$ *if p is an even number;*

(ii) iff $\mathcal{N}$ *is* $\bar{\mu}_1\bar{\gamma}_2\bar{\mu}_3\bar{\gamma}_4...\bar{\gamma}_{p-1}\bar{\mu}_p$*-symmetric accretive with* $f_1, f_2, ..., f_n$ *and* $\Big(H^p(A_1, A_2, ..., A_p) + \rho\mathcal{N}(f_1, f_2, ..., f_p)\Big)(Y) = Y$*, for all* $\rho > 0$ *if p is an odd number.*

Proposition 2.2. *Let assumptions M_1 and M_2 hold for every $i \in \{1, 2, ...p\}$, $p \geq 3$ and $\mathcal{N} : Y^p \multimap Y$ be a generalized $\alpha_i\beta_j$-$H^p(A_1, A_2, ..., A_p)$-accretive mapping with mappings $(A_1, A_2, ..., A_p)$ and $(f_1, f_2, ..., f_p)$ and $\sum \bar{\mu}_i > \sum \bar{\gamma}_j$, $\sum \alpha_i > \sum \beta_j$, if $\langle \tilde{x} - \tilde{y},\ J_q(u' - v')\rangle \geq 0$ is satisfied for each $(v',\ \tilde{y}) \in \mathrm{Gr}\Big(\mathcal{N}(f_1, f_2, ..., f_p)\Big)$, $\tilde{x} \in \mathcal{N}\Big(f_1, f_2, ..., f_p\Big)(u')$, where $\mathrm{Gr}(\mathcal{N}\Big(f_1, f_2, ..., f_p)\Big) = \Big\{(u', \tilde{x}) : \tilde{x} \in \mathcal{N}\Big(f_1, f_2, ..., f_p\Big)(u')\Big\}$.*

Proof. Assume that there exists $(w_0,\ z_0) \notin \mathrm{Gr}\ \Big(\mathcal{N}(f_1, f_2, ..., f_p)\Big)$ such that

$$\Big\langle z_0 - x,\ J_q(w_0 - u)\Big\rangle \ \geq\ 0,\ \forall\ (u,\ x) \in\ \mathrm{Gr}\Big(\mathcal{N}(f_1, f_2, ..., f_p)\Big). \tag{3}$$

If p is even: Since, $\mathcal{N}$ is a generalized $\alpha_i\beta_j$-$H^p\Big(A_1, A_2, ..., A_p\Big)$-accretive mapping with mappings $(A_1, A_2, ..., A_p)$ and $(f_1, f_2, ..., f_p)$, then, $\mathcal{N}$ is $bar\mu_1\bar{\gamma}_2\bar{\mu}_3\bar{\gamma}_4...\bar{\mu}_{p-1}\bar{\gamma}_p$-symmetric accretive with mappings $f_1, f_2, ..., f_p$ and $\Big(H^p(A_1, A_2, ..., A_p)+\rho\mathcal{N}(f_1, f_2, ..., f_p)\Big)(Y) = Y$ holds for each $\rho > 0$, then, there exists $(w_1, z_1) \in \mathrm{Gr}\Big(\mathcal{N}(f_1, f_2, ..., f_p)\Big)$ such that

$$H^p\Big(A_1w_0, A_2w_0, ..., A_pw_0\Big) + \rho z_0 = H^p\Big(A_1w_1, A_2w_1, ..., A_pw_1\Big) + \rho z_1 \in\ Y. \tag{4}$$

From (3) and (4), we have

$$\rho z_0 - \rho z_1 = H^p\Big(A_1w_1, A_2w_1, ..., A_pw_1\Big) - H^p\Big(A_1w_0, A_2w_0, ..., A_pw_0\Big) \in\ Y,$$
$$\langle \rho z_0 - \rho z_1, J_q(u_0 - w_1)\rangle = \Big\langle H^p(A_1w_1, A_2w_1, ..., A_pw_1) - H^p(A_1w_0, A_2w_0, ..., A_pw_0), J_q(u_0 - w_1)\Big\rangle.$$

Setting $(u, x) = (w_1, z_1)$ in (3) and using M_3 in (4), we obtain

$$\begin{aligned}
[(\bar{\mu}_1 + \bar{\mu}_3 + ... + \bar{\mu}_{p-1}) - (\bar{\gamma}_2 + \bar{\gamma}_4 - ... + \bar{\gamma}_p)]\|w_0 - w_1\|^q &\leq \rho\langle z_0 - z_1, J_q(w_0 - w_1)\rangle \\
&\leq -\Big\langle H^p(A_1w_0, A_2w_0, ..., A_pw_0) - H^p(A_1w_1, A_2w_1, ..., A_pw_1),\ J_q(w_0 - w_1)\Big\rangle \\
&= -\Big\langle H^p(A_1w_0, A_2w_0, ..., A_pw_0) - H^p(A_1w_1, A_2w_0, ..., A_pw_0), J_q(w_0 - w_1)\Big\rangle \\
&\quad - \Big\langle H^p(A_1w_0, A_2w_0, ..., A_pw_0) - H^p(A_1w_0, A_2w_1, ..., A_pw_0), J_q(w_0 - w_1)\Big\rangle \\
&\quad \vdots \\
&\quad \vdots \\
&\quad - \Big\langle H^p(A_1w_0, A_2w_0, ..., A_pw_0) - H^p(A_1w_0, A_2w_0, ..., A_pw_1), J_q(w_0 - w_1)\Big\rangle \\
&\leq -\Big[(\alpha_1 + \alpha_3 + ... + \alpha_{p-1}) - (\beta_2 + \beta_4 + ... + \beta_p)\Big]\|w_0 - w_1\|^q
\end{aligned}$$

Let

$$\sum \alpha_i = \alpha_1 + \alpha_3 + ... + \alpha_{p-1},\ \sum \beta_j = \beta_2 + \beta_4 - ... + \beta_p,$$
$$\sum \bar{\mu}_i = \bar{\mu}_1 + \bar{\mu}_3 + ... + \bar{\mu}_{p-1},\ \sum \bar{\gamma}_j = \bar{\gamma}_2 + \bar{\gamma}_4 - ... + \bar{\gamma}_p.$$

Then, we have

$$\Big[\sum \alpha_i - \sum \beta_j + \rho\Big(\sum \bar{\mu}_i - \sum \bar{\gamma}_j\Big)\Big]\ \|w_0 - w_1\|^q \leq 0. \tag{5}$$

Since, $\sum \bar{\mu}_i > \sum \bar{\gamma}_j$, $\sum \alpha_i > \sum \beta_j$ and $\rho > 0$, it implies that $w_0 = w_1$. By (3), we have $z_0 = z_1$. Thus $(w_1, z_1) = (w_o, z_o) \in \mathrm{Gr}\Big(\mathcal{N}(\mathrm{f_1, f_2, ..., f_p})\Big)$. By the same procedure, we can obtain the result when p is odd.

Theorem 2.3. *Let assumptions M_1 and M_2 hold for every $i \in \{1, 2, ...p\}$, $p \geq 3$ and $\mathcal{N} : Y^p \multimap Y$ be a generalized $\alpha_i\beta_j$-$H^p(A_1, A_2, ..., A_p)$-accretive mapping with mappings $(A_1, A_2, ..., A_p)$ and $(f_1, f_2, ..., f_p)$ and $\sum \bar{\mu}_i > \sum \bar{\gamma}_j$, $\sum \alpha_i > \sum \beta_j$, then $\Big(H^p(A_1, A_2, ..., A_p) + \rho\mathcal{N}(f_1, f_2, ..., f_p)\Big)^{-1}$ is single-valued.*

Proof. For any given $u \in Y$, let $x, y \in \Big(H^p(A_1, A_2, ..., A_p) + \rho\mathcal{N}(f_1, f_2, ..., f_p)\Big)^{-1}(u)$. It follows that

$$-H^p\Big(A_1x, A_2x, ..., A_px\Big) + u \in \rho\mathcal{N}\Big(f_1, f_2, ..., f_p\Big)x,$$
$$-H^p\Big(A_1y, A_2y, ..., A_py\Big) + u \in \rho\mathcal{N}\Big(f_1, f_2, ..., f_p\Big)y.$$

Let p is even. Since, $\mathcal{N}$ is $\bar{\mu}_1\bar{\gamma}_2\bar{\mu}_3\bar{\gamma}_4...\bar{\mu}_{p-1}\bar{\gamma}_p$-symmetric accretive with $f_1, f_2, ..., f_p$, we have

$$\begin{aligned}
&(\bar{\mu}_1 + \bar{\mu}_3 + ... + \bar{\mu}_{p-1} - \bar{\gamma}_2 - \bar{\gamma}_4 - ... - \bar{\gamma}_p)\|x-y\|^q \\
&\quad \leq \frac{1}{\rho}\Big\langle -H^p(A_1x, A_2x, ..., A_px) + u - (-H^p(A_1y, A_2y, ..., A_py) + u), J_q(x-y)\Big\rangle \\
\Rightarrow\ & \rho(\bar{\mu}_1 + \bar{\mu}_3 + ... + \bar{\mu}_{p-1} - \bar{\gamma}_2 - \bar{\gamma}_4 - ... - \bar{\gamma}_p)\|x-y\|^q \\
&\quad \leq -\Big\langle H^p(A_1x, A_2x, ..., A_px) - H^p(A_1y, A_2y, ..., A_py),\ J_q(x-y)\Big\rangle \\
&\quad = -\Big\langle H^p(A_1x, A_2x, ..., A_px) - H^p(A_1y, A_2x, ..., A_px),\ J_q(x-y)\Big\rangle \\
&\qquad -\Big\langle H^p(A_1y, A_2x, ..., A_px) - H^p(A_1y, A_2y, ..., A_px),\ J_q(x-y)\Big\rangle \\
&\qquad \vdots \\
&\qquad \vdots \\
&\qquad -\Big\langle H^p(A_1y, A_2y, ..., A_px) - H^p(A_1y, A_2y, ..., A_py),\ J_q(x-y)\Big\rangle.
\end{aligned}$$

Proceed the same as to obtain (5), we have

$$\Big[\sum \alpha_i - \sum \beta_j + \rho\Big(\sum \bar{\mu}_i - \sum \bar{\gamma}_j\Big)\Big]\,\|x-y\|^q \leq 0. \tag{6}$$

Since, $\sum \bar{\mu}_i > \sum \bar{\gamma}_j$, $\sum \alpha_i > \sum \beta_j$ and $\rho > 0$, we have $\|x-y\| \leq 0$. It implies that $x = y$. Thus $(H^p(A_1, A_2, ..., A_p) + \rho\mathcal{N}(f_1, f_2, ..., f_p))^{-1}$ is single-valued. By the same procedure, we can obtain the result when p is odd.

Definition 2.4. *Let assumptions M_1 and M_2 holds for $p \geq 3$ and $\mathcal{N} : Y^p \multimap Y$ be a generalized $\alpha_i\beta_j$-$H^p(.,.,...)$-accretive mapping with mappings $(A_1, A_2, ..., A_p)$ and $(f_1, f_2, ..., f_p)$ and $\sum \bar{\mu}_i > \sum \bar{\gamma}_j$, $\sum \alpha_i > \sum \beta_j$. A proximal-point mapping $R^{H^p(.,.,..,.)}_{\rho,\mathcal{N}(.,.,..,.)} : Y \to Y$ is define as*

$$R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}(x) \ = \ \Big[H^p(A_1, A_2, ..., A_p) + \rho\mathcal{N}(f_1, f_2, ..., f_p)\Big]^{-1}(x),\ \forall\, x \in Y, \tag{7}$$

where ρ is non-negative constant.

Now, we prove the Lipschitz continuity of proximal-point mapping.

Theorem 2.5. *Let assumptions M_1 and M_2 holds for $p \geq 3$ and $\mathcal{N} : Y^p \multimap Y$ be a generalized $\alpha_i\beta_j$-$H^p(.,.,...)$-accretive mapping with mappings $(A_1, A_2, ..., A_p)$ and $(f_1, f_2, ..., f_p)$ and $\sum \bar{\mu}_i > \sum \bar{\gamma}_j$, $\sum \alpha_i > \sum \beta_j$. Then, the proximal-point mapping $R^{H^p(.,.,..,.)}_{\rho,\mathcal{N}(.,.,..,.)} : Y \to Y$ is Δ-Lipschitz continuous, where*

$$\Delta = \Big[\sum \alpha_i - \sum \beta_j + \rho\Big(\sum \bar{\mu}_i - \sum \bar{\gamma}_j\Big)\Big]^{-1}.$$

Proof. Let $x, y \in Y$, and from (7), we have

$$\begin{cases} R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}(x) \ = \ \Big(H^p(A_1, A_2, ..., A_p) + \rho\mathcal{N}(f_1, f_2, ..., f_p)\Big)^{-1}(x), \\ R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}(y) \ = \ \Big(H^p(A_1, A_2, ..., A_p) + \rho\mathcal{N}(f_1, f_2, ..., f_p)\Big)^{-1}(y). \end{cases}$$

It follows that

$$\frac{1}{\rho}\Big(x - H^p\Big(A_1(R^{H^p(.,.,...)}_{\rho,\mathcal{N}(.,.,...)}(x)), A_2(R^{H^p(.,.,...)}_{\rho,\mathcal{N}(.,.,...)}(x)), ..., A_p(R^{H^p(.,.,...)}_{\rho,\mathcal{N}(.,.,...)}(x))\Big)\Big) \in \mathcal{N}\Big(R^{H^p(.,.,...)}_{\rho,\mathcal{N}(.,.,...)}(x)\Big),$$
$$\frac{1}{\rho}\Big(y - H^p(A_1(R^{H^p(.,.,...)}_{\rho,\mathcal{N}(.,.,...)}(y)), A_2(R^{H^p(.,.,...)}_{\rho,\mathcal{N}(.,.,...)}(y)), ..., A_p(R^{H^p(.,.,...)}_{\rho,\mathcal{N}(.,.,...)}(y))\Big)\Big) \in \mathcal{N}\Big(R^{H^p(.,.,...)}_{\rho,\mathcal{N}(.,.,...)}(y)\Big).$$

Let $x^1 = R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}(x)$ and $y^1 = R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}(y)$.
If p is even: Since, $\mathcal{N}$ is $\bar{\mu}_1\bar{\gamma}_2...\bar{\mu}_{p-1}\gamma_p$-symmetric accretive with $f_1, f_2, ..., f_p$, we have

$$\Big\langle\Big(x - H^p(A_1(x^1), A_2(x^1), ..., A_p(x^1))\Big) - \Big(y - H^p(A_1(y^1), A_2(y^1), ..., A_p(y^1))\Big), J_q\Big(x^1 - y^1\Big)\Big\rangle$$
$$\geq \rho(\bar{\mu}_1 - \bar{\gamma}_2 + \bar{\mu}_3 - \bar{\gamma}_4 + ... + \bar{\mu}_{p-1} - \bar{\gamma}_p)\Big\|x^1 - y^1\Big\|^q,$$
$$\Big\langle x - y - \Big(H^p(A_1(x^1), A_2(x^1), ..., A_p(x^1)\Big) - H^p\Big(A_1(y^1), A_2(y^1), ..., A_p(y^1)\Big)\Big), J_q\Big(x^1 - y^1\Big)\Big\rangle$$
$$\geq \rho((\bar{\mu}_1 + \bar{\mu}_3 + ... + \bar{\mu}_{p-1} - (\bar{\gamma}_2 + \bar{\gamma}_4 - ... + \bar{\gamma}_p))\Big\|x^1 - y^1\Big\|^q.$$

Let

$$\sum \alpha_i = \alpha_1 + \alpha_3 + ... + \alpha_{p-1},\ \sum \beta_j = \beta_2 + \beta_4 - ... + \beta_p,$$
$$\sum \bar{\mu}_i = \bar{\mu}_1 + \bar{\mu}_3 + ... + \bar{\mu}_{p-1},\ \sum \bar{\gamma}_j = \bar{\gamma}_2 + \bar{\gamma}_4 - ... + \bar{\gamma}_p.$$

We have

$$\begin{aligned}\Big\|x - y\Big\|\ \Big\|x^1 - y^1\Big\|^{q-1} &\geq \Big\langle x - y,\ J_q\Big(x^1 - y^1\Big)\Big\rangle \\ &\geq \Big\langle H^p\Big(A_1(x^1), A_2(x^1), ..., A_p(x^1)\Big) - H^p\Big(A_1(y^1), A_2(y^1), ..., A_p(y^1)\Big), \\ &\qquad J_q\Big(x^1 - y^1\Big)\Big\rangle + \rho\left(\sum \bar{\mu}_i - \sum \bar{\gamma}_j\right)\Big\|x^1 - y^1\Big\|^q \\ &\geq \alpha_1\Big\|x^1 - y^1\Big\|^q - \beta_2\Big\|x^1 - y^1\Big\|^q + \alpha_3\Big\|x^1 - y^1\Big\|^q - ...\beta_p\|x^1 - y^1\|^q \\ &\qquad + \rho\left(\sum \bar{\mu}_i - \sum \bar{\gamma}_j\right)\Big\|x^1 - y^1\Big\|^q \\ &= \left[\sum \alpha_i - \sum \beta_j + \rho\left(\sum \bar{\mu}_i - \sum \bar{\gamma}_j\right)\right]\ \Big\|x^1 - y^1\Big\|^q.\end{aligned}$$

Hence

$$\Big\|x - y\Big\|\ \Big\|x^1 - y^1\Big\|^{q-1} \geq \left[\sum \alpha_i - \sum \beta_j + \rho\left(\sum \bar{\mu}_i - \sum \bar{\gamma}_j\right)\right]\ \Big\|x^1 - y^1\Big\|^q,$$

that is,

$$\Big\|R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}(x) - R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}(y)\Big\| \leq \Delta\Big\|x - y\Big\|, \forall\ x, y \in \Upsilon,$$

where $\Delta = \left[\sum \alpha_i - \sum \beta_j + \rho\left(\sum \bar{\mu}_i - \sum \bar{\gamma}_j\right)\right]^{-1}$. By the same procedure, we can obtain the result when p is odd.

3. Generalized Yosida Approximation Mappings

We define the generalized Yosida approximation mapping in terms of proximal-point mapping given by (7)

Definition 3.1. *The generalized Yosida approximation mapping* $J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}$ *is defined as*

$$J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}(x) = \frac{1}{\rho}\left[I - R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}\right](x),\ \forall x \in Y, \tag{8}$$

where I is identity mapping and ρ is non-negative constant.

Lemma 3.2. *The generalized Yosida approximation mapping* $J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}$ *defined by (8), is*
(i) Φ_1*-Lipschitz continuous, where*

$$\Phi_1 = \frac{1+\Delta}{\rho},\ \Delta = \left[\sum \alpha_i - \sum \beta_j + \rho(\sum \bar{\mu}_i - \sum \bar{\gamma}_j)\right]^{-1}$$

and $\sum \alpha_i > \sum \beta_j, \sum \bar{\mu}_i > \sum \bar{\gamma}_j$;
(i) Φ_2*-strongly accretive, where*

$$\Phi_2 = \frac{1-\Delta}{\rho},\ \Delta = \left[\sum \alpha_i - \sum \beta_j + \rho(\sum \bar{\mu}_i - \sum \bar{\gamma}_j)\right]^{-1}$$

and $\sum \alpha_i > \sum \beta_j, \sum \bar{\mu}_i > \sum \bar{\gamma}_j$.

Proof. (i) Let $x, y \in Y$ and $\rho > 0$. Using Proposition 2.3, we have

$$\begin{aligned}
\left\|J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}(x) - J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}(y)\right\| &= \frac{1}{\rho}\left\|I(x) - R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}(x) - \left[I(y) - R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}(y)\right]\right\|,\\
&\leq \frac{1}{\rho}\left[\left\|x-y\right\| + \|R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}(x) - R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}(y)\right\|\Big],\\
&\leq \frac{1}{\rho}\left[\left\|x-y\right\| + \Delta\left\|x-y\right\|\right],\\
&\leq \frac{\Delta+1}{\rho}\left\|x-y\right\|.
\end{aligned}$$

that is,

$$\left\|J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}(x) - J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}(y)\right\| = \Phi_1\left\|x-y\right\|,$$

where $\Phi_1 = \frac{1+\Delta}{\rho}$, $\Delta = \left(\sum \alpha_i - \sum \beta_j + \rho(\sum \bar{\mu}_i - \sum \bar{\gamma}_j)\right)^{-1}$ and $\sum \alpha_i > \sum \beta_j, \sum \bar{\mu}_i > \sum \bar{\gamma}_j$.
(ii) Let $x, y \in Y$ and $\rho > 0$. Using Proposition 2.3, we have

$$\begin{aligned}
&\left\langle J^{H^p(.,.,...)}_{\rho,\mathcal{N}(.,.,...)}(x) - J^{H^p(.,.,...)}_{\rho,\mathcal{N}(.,.,...)}(y), J_q\big(x-y\big)\right\rangle\\
&\quad= \frac{1}{\rho}\left\langle I(x) - R^{H^p(.,.,...)}_{\rho,\mathcal{N}(.,.,...)}(x) - \left[I(y) - R^{H^p(.,.,...)}_{\rho,\mathcal{N}(.,.,...)}(y)\right], J_q\big(x-y\big)\right\rangle,\\
&\quad= \frac{1}{\rho}\left\langle x-y, J_q\big(x-y\big)\right\rangle - \frac{1}{\rho}\left\langle R^{H^p(.,.,...)}_{\rho,\mathcal{N}(.,.,...)}(x) - R^{H^p(.,.,...)}_{\rho,\mathcal{N}(.,.,...)}(y), J_q\big(x-y\big)\right\rangle,\\
&\quad\geq \frac{1}{\rho}\left\|x-y\right\|^q - \frac{1}{\rho}\left\|R^{H^p(.,.,...)}_{\rho,\mathcal{N}(.,.,...)}(x) - R^{H^p(.,.,...)}_{\rho,\mathcal{N}(.,.,...)}(y)\right\|\left\|x-y\right\|^{q-1},\\
&\quad\geq \frac{1}{\rho}\left\|x-y\right\|^q - \frac{1}{\rho}\Delta\left\|x-y\right\|\left\|x-y\right\|^{q-1},\\
&\quad\geq \frac{1-\Delta}{\rho}\left\|x-y\right\|^q.
\end{aligned}$$

that is,

$$\left\langle J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}(x) - J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}(y), J_q\big(x-y\big)\right\rangle \ \geq\ \Phi_2\Big\|x-y\Big\|^q,$$

where $\Phi_2 = \frac{1-\Delta}{\rho}$, $\Delta = \Big(\sum\alpha_i - \sum\beta_j + \rho(\sum\bar{\mu}_i - \sum\bar{\gamma}_j)\Big)^{-1}$ and $\sum\alpha_i > \sum\beta_j, \sum\bar{\mu}_i > \sum\bar{\gamma}_j$.

Note The proximal-point mapping defined by (7) and generalized Yosida approximation mapping defined by (8) are associated by the following relation:

$$J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}(x) = \Big[\rho\mathcal{N}\Big(f_1,f_2,...,f_p\Big) + H^p\Big(A_1,A_2,...,A_p\Big) - I\Big]\, R^{H^p(.,.,...)}_{\rho,\mathcal{N}(.,.,...)}(x). \tag{9}$$

4. Graph convergence of the generalized Yosida approximation mappings

Now, we discuss the graph convergence of the generalized Yosida approximation mappings.

Definition 4.1. *Let $\mathcal{N}: Y^p \multimap Y$ be a multi-valued mapping, then, graph of $\mathcal{N}$ is given as:*

$$\mathrm{Gr}\Big(\mathcal{N}(\tilde{u}_1,\tilde{u}_2,...,\tilde{u}_p)\Big) = \Big\{\Big((\tilde{u}_1,\tilde{u}_2,...,\tilde{u}_p),x^*\Big) : x^* \in \mathcal{N}\Big(\tilde{u}_1,\tilde{u}_2,...,\tilde{u}_p\Big)\Big\}.$$

Definition 4.2. *For $n = 0,1,2,....$, let $\mathcal{N}_n$, $\mathcal{N}: Y \multimap Y$ be the multi-valued mappings such that $\mathcal{N}$, $\mathcal{N}_n$ be the generalized $\alpha_i\beta_j$-$H^p(.,.,...)$-accretive mappings with mappings $\Big(A_1,A_2,...,A_p\Big)$ and $\Big(f_1,f_2,...,f_p\Big)$. Graph convergence of a sequence $\{\mathcal{N}_n\}$ to $\mathcal{N}$ expressed as $\mathcal{N}_n \xrightarrow{G} \mathcal{N}$, if for each $\Big(\Big(f_1(x),f_2(x),...,f_p(x)\Big),y\Big) \in \mathrm{Gr}\Big(\mathcal{N}\Big(f_1,f_2,...,f_p\Big)\Big)$, there exists a sequence*
$\Big(\Big(f_1(x_n),f_2(x_n),...,f_p(x_n)\Big),y_n\Big) \in \mathrm{Gr}\Big(\mathcal{N}_n\Big(f_1,f_2,...,f_p\Big)\Big)$ such that

$$f_1(x_n) \to f_1(x), f_2(x_n) \to f_2(x),...,f_p(x_n) \to f_p(x),\ y_n \to\ y\ \text{ as } n \to\ \infty.$$

Theorem 4.3. *Let us consider the assumptions M_1 and M_2 hold good. For $n = 0,1,2,....$, $\mathcal{N}_n,\mathcal{N}: Y^p \multimap Y$ be the generalized $\alpha_i\beta_j$-$H^p(.,.,...)$-accretive mappings with mappings $\Big(A_1,A_2,...,A_p\Big)$ and $\Big(f_1,f_2,...,f_p\Big)$ and $\sum\bar{\mu}_i > \sum\bar{\gamma}_j, \sum\alpha_i > \sum\beta_j$. For each $i \in \{1,2,...,p\}$, $p \geq 3$, assume that*
(i) $H^p(.,.,...)$ is s_i-Lipschitz continuous with respect to A_i;
(ii) f_i is κ_i-expansive in the ith-argument;
Then $\mathcal{N}_n \xrightarrow{G} \mathcal{N}$ if and only if

$$R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)}(x) \to R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}(x),\ \forall x \in Y,\ \rho > 0,$$

where $R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)} = \Big(H^p\Big(A_1,A_2,...,A_p\Big) + \rho\mathcal{N}_n\Big(f_1,f_2,...,f_p\Big)\Big)^{-1}$,
$R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)} = \Big(H^p\Big(A_1,A_2,...,A_p\Big) + \rho\mathcal{N}\Big(f_1,f_2,...,f_p\Big)\Big)^{-1}$.

Proof. Since, the proximal-point mappings connected with $\alpha_i\beta_j$-$H^p(.,.,...)$-accretive mappings are Lipschitz continuous. That is, $R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}$ and $R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)}$ are Δ-Lipschitz continuous.

If part: Assume that $\mathcal{N}_n \xrightarrow{G} \mathcal{N}$.
Given for any $x \in Y$, let $z_n = R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)}(x)$, $z = R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}(x)$.
Then, $\frac{1}{\rho}\Big[x - H^p\Big(A_1z,A_2z,...,A_pz\Big)\Big] \in \mathcal{N}\Big(f_1,f_2,...,f_p\Big)(z)$,
or $\Big[z,\ \frac{1}{\rho}\Big(x - H^p(A_1z,A_2z,...,A_pz)\Big)\Big] \in \mathrm{Gr}\Big(\mathcal{N}\Big(\mathrm{f}_1,\mathrm{f}_2,...,\mathrm{f}_p\Big)\Big)$.
By definition of $\mathrm{Gr}\Big(\mathcal{N}\Big(\mathrm{f}_1,\mathrm{f}_2,...,\mathrm{f}_p\Big)\Big)$, there exists a sequence $\Big\{f_1(\tilde{z}_n),f_2(\tilde{z}_n),...,f_p(\tilde{z}_n),\tilde{y}_n\Big\}$ such that

$$f_1(\tilde{z}_n) \to f_1(z), f_2(\tilde{z_n}) \to f_2(z),...,f_p(\tilde{z}_n) \to f_p(z), \tilde{y}_n \to\ \frac{1}{\rho}\Big[x - H^p\Big(A_1z,A_2z,...,A_pz\Big)\Big] \tag{10}$$

as $n \to \infty$. Since, $\tilde{y}_n \in \mathcal{N}_n\Big(f_1(\tilde{z}_n), f_2(\tilde{z}_n), ..., f_p(\tilde{z}_n)\Big)$, we have

$$H^p\Big(A_1\tilde{z}_n, A_2\tilde{z}_n, ..., A_p\tilde{z}_n\Big) + \rho\tilde{y}_n \in \Big[H^p\Big(A_1, A_2, ..., A_p\Big) + \rho\mathcal{N}_n\Big(f_1, f_2, ..., f_p\Big)\Big](\tilde{z}_n).$$

Therefore,

$$\tilde{z_n} = R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)}\Big[H^p\Big(A_1\tilde{z}_n, A_2\tilde{z}_n, ..., A_p\tilde{z}_n\Big) + \rho\tilde{y}_n\Big].$$

Using the Δ-Lipschitz continuity of $R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)}$, we have

$$\begin{aligned}
\|z_n - z\| &\leq \|z_n - \tilde{z}_n\| + \|\tilde{z}_n - z\| \\
&= \Big\|R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)}(x) - R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)}\Big[H^p\Big(A_1\tilde{z}_n, A_2\tilde{z}_n, ..., A_p\tilde{z}_n\Big) + \rho\tilde{y}_n\Big]\Big\| + \Big\|z'_n - z\Big\| \\
&\leq \Delta\, \Big\|x - H^p\Big(A_1\tilde{z}_n, A_2\tilde{z}_n, ..., A_p\tilde{z}_n\Big) - \rho\tilde{y}_n\Big\| + \Big\|z'_n - z\Big\| \\
&\leq \Delta\, \Big[\Big\|x - H^p\Big(A_1 z, A_2 z, ..., A_p z\Big) - \rho\tilde{y}_n\Big\| \\
&\quad + \Big\|H^p\Big(A_1 z, A_2 z, ..., A_p z\Big) - H^p\Big(A_1\tilde{z}_n, A_2\tilde{z}_n, ..., A_p\tilde{z}_n\Big)\Big\|\Big] + \Big\|\tilde{z}_n - z\Big\|.
\end{aligned} \tag{11}$$

Using the s_i-Lipschitz continuity of $H^p\Big(., ., A_i, ...\Big)$, we have

$$\Big\|H^p\Big(A_1 z, A_2 z, ..., A_p z\Big) - H^p\Big(A_1\tilde{z}_n, A_2\tilde{z}_n, ..., A_p\tilde{z}_n\Big)\Big\| \leq \Big(s_1 + s_2 + ... + s_p\Big)\, \Big\|z - \tilde{z}_n\Big\|. \tag{12}$$

Using (11) and (12), we have

$$\Big\|z_n - z\Big\| \leq \Delta\Big\|x - H^p\Big(A_1 z, A_2 z, ..., A_p z\Big) - \rho\tilde{y}_n\Big\| + \Big[1 + \Delta\Big(s_1 + s_2 + ... + s_p\Big)\Big]\, \Big\|z - \tilde{z}_n\Big\|. \tag{13}$$

As f_i is κ_i-expansive, then we have

$$\Big\|f_i(\tilde{z}_n) - f_i(z)\Big\| \geq \kappa_i\Big\|\tilde{z}_n - z\Big\| \geq 0. \tag{14}$$

Let $n \to \infty$, we have $f_i(\tilde{z}_n) \to f_i(z)$. Using (11), (14) and let $n \to \infty$ we get $\tilde{z}_n \to z$ and

$$\Big\|\frac{1}{\rho}\Big[x - H^p(A_1 z, A_2 z, ..., A_p z) - \rho\tilde{y}_n\Big]\Big\| \to 0.$$

Let $n \to \infty$ and Using (13), we have $\|z_n - z\| \to 0$ i.e.

$$R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)}(u) \to R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}(u).$$

Only if part:

Suppose that $R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)} \to R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}$, $\forall\, u \in Y$, $\rho > 0$. For any given $\Big(f_1(x), f_2(x), ..., f_p(x), y\Big) \in \mathrm{Gr}\Big(\mathcal{N}\Big(\mathrm{f}_1, \mathrm{f}_2, ..., \mathrm{f}_\mathrm{p}\Big)\Big)$, we have $y \in \mathcal{N}\Big(f_1, f_2, ..., f_p\Big)$ and

$$H^p\Big(A_1 x, A_2 x, ..., A_p x\Big) + \rho y \in \Big[H^p\Big(A_1, A_2, ..., A_p\Big) + \rho\mathcal{N}\Big(f_1, f_2, ..., f_p\Big)\Big](x).$$

Therefore $x = R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}\Big[H^p\Big(A_1 x, A_2 x, ..., A_p x\Big) + \rho y\Big]$. Let $x_n = R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)}\Big[H^p\Big(A_1 x, A_2 x, ..., A_p x\Big) + \rho y\Big]$. Then, we have

$$\frac{1}{\rho}\Big[H^p\Big(A_1 x, A_2 x, ..., A_p x\Big) - H^p\Big(A_1 x_n, A_2 x_n, ..., A_p x_n\Big) + \rho y\Big] \in \mathcal{N}_n\Big(f_1(x_n), f_2(x_n), ..., f_p(x_n)\Big).$$

Let $y_n = \frac{1}{\rho}\Big[H^p\Big(A_1x, A_2x, ..., A_px\Big) - H^p\Big(A_1x_n, A_2x_n, ..., A_px_n\Big) + \rho y\Big]$, now we evaluate

$$\begin{aligned}\Big\|y_n - y\Big\| &= \Big\|\frac{1}{\rho}\Big[H^p\Big(A_1x, A_2x, ..., A_px\Big) - H^p\Big(A_1x_n, A_2x_n, ..., A_px_n\Big) + \rho y\Big] - y\Big\| \\ &= \frac{1}{\rho}\Big\|H^p\Big(A_1x, A_2x, ..., A_px\Big) - H^p\Big(A_1x_n, A_2x_n, ..., A_px_n\Big)\Big\| \\ &\leq \frac{1}{\rho}\Big(s_1 + s_2 + ... + s_p\Big)\Big\|x_n - x\Big\|. \end{aligned} \tag{15}$$

As $R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)} \to R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}}$ for given any $u \in Y$, we have $\|x_n - x\| \to 0$. Let $n \to \infty$, equation (15) gives $y_n \to y$. Therefore, $\mathcal{N}_n \xrightarrow{G} \mathcal{N}$.

Now, we prove the convergence of the generalized Yosida approximation mapping with the help of graph convergence of generalized $\alpha_i\beta_j$-$H^p(.,.,...,.)$-accretive mapping without using the convergence of the proximal-point mapping defined by (7).

Theorem 4.4. *Let us consider the assumptions M_1 and M_2 hold good. For $n = 0, 1, 2, ...$, $\mathcal{N}_n, \mathcal{N} : Y^p \multimap Y$ be the generalized $\alpha_i\beta_j$-$H^p(.,.,...)$-accretive mappings with mappings $(A_1, A_2, ..., A_p)$ and $(f_1, f_2, ..., f_p)$ with $\sum \alpha_i > \sum \beta_j, \sum \bar{\mu}_i > \sum \bar{\gamma}_j$. For each $i \in \{1, 2, ..., p\}$, $p \geq 3$, assume that*
(i) *$H^p(.,.,...)$ is s_i-Lipschitz continuous with respect to A_i;*
(ii) *f_i is κ_i-expansive in the ith-argument.*

Then, $\mathcal{N}_n \xrightarrow{G} \mathcal{N}$ if and only if

$$J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)}(x) \to J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}(x),\ \forall x \in Y,\ \rho > 0,$$

where $J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)}(x) = \frac{1}{\rho}\Big[I - R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)}\Big](x)$, $J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}(x) = \frac{1}{\rho}\Big[I - R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}\Big](x)$.

Proof. If part: Suppose that $\mathcal{N}_n \xrightarrow{G} \mathcal{N}$. For any given $x \in Y$, let

$$z_n = J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)}(x), \quad z = J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}(x).$$

Then

$$z = J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}(x) = \frac{1}{\rho}\Big[I - R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}\Big](x),$$

implies that

$$\Big(x - \rho z\Big) = R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}(x) = \Big(H^p(A_1, A_2, ..., A_p) + \rho\mathcal{N}(f_1, f_2, ..., f_p)\Big)^{-1}(x)$$

that is,

$$H^p\Big(A_1, A_2, ..., A_p\Big)\Big(x - \rho z\Big) + \rho\mathcal{N}\Big(f_1, f_2, ..., f_p\Big)\Big(x - \rho z\Big) = x.$$

Thus, we have

$$\frac{1}{\rho}\Big[x - H^p\Big(A_1, A_2, ..., A_p\Big)\Big(x - \rho z\Big)\Big] \in \mathcal{N}\Big(f_1, f_2, ..., f_p\Big)\Big(x - \rho z\Big).$$

$$\Big[f_1(x - \rho z), f_2(x - \rho z), ..., f_p(x - \rho z),\ \frac{1}{\rho}\Big[x - H^p(A_1, A_2, ..., A_p)(x - \rho z)\Big]\Big] \in \mathrm{Gr}\Big(\mathcal{N}(\mathrm{f}_1, \mathrm{f}_2, ..., \mathrm{f}_\mathrm{p})\Big).$$

By definition of $\mathrm{Gr}\Big(\mathcal{N}(\mathrm{f}_1, \mathrm{f}_2, ..., \mathrm{f}_\mathrm{p})\Big)$, there exists a sequence $\{f_1(\tilde{u}_n), f_2(\tilde{u}_n), ..., f_p(\tilde{u}_n), \tilde{y}_n\}$ such that

$$f_1(\tilde{u}_n) \to f_1(x-\rho z), f_2(\tilde{u_n}) \to f_2(x-\rho z), ..., f_p(\tilde{u}_n) \to f_p(x-\rho z),$$

$$\tilde{y}_n \to \frac{1}{\rho}\Big[x - H^p\Big(A_1, A_2, ..., A_p\Big)\Big(x-\rho z\Big)\Big] \tag{16}$$

as $n \to \infty$. Since, $\tilde{y}_n \in \mathcal{N}_n(f_1(\tilde{u}_n), f_2(\tilde{u}_n), ..., f_p(\tilde{u}_n))$, we have

$$H^p\Big(A_1, A_2, ..., A_p\Big)(\tilde{u}_n) + \rho\tilde{y}_n \in \Big[H^p\Big(A_1, A_2, ..., A_p\Big) + \rho\mathcal{N}_n\Big(f_1, f_2, ..., f_p\Big)\Big](\tilde{u}_n)$$

and so

$$\begin{aligned}
\tilde{u}_n &= \Big[H^p\Big(A_1, A_2, ..., A_p\Big) + \rho\mathcal{N}_n\Big(f_1, f_2, ..., f_p\Big)\Big]^{-1}\Big[H^p\Big(A_1, A_2, ..., A_p\Big)\tilde{u}_n + \rho\tilde{y}_n\Big]\\
\tilde{u}_n &= R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)}\Big[H^p\Big(A_1, A_2, ..., A_p\Big)\tilde{u}_n + \rho\tilde{y}_n\Big]\\
\tilde{u}_n &= \Big[I - \rho J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)}\Big]\Big[H^p\Big(A_1, A_2, ..., A_p\Big)\tilde{u}_n + \rho\tilde{y}_n\Big].
\end{aligned}$$

which shows that

$$\frac{1}{\rho}\tilde{u}_n = \frac{1}{\rho}H^p(A_1, A_2, ..., A_p)\tilde{u}_n + \tilde{y}_n - J^{H^p(.,.,...)}_{\rho,\mathcal{N}_n(.,.,...)}\Big[H^p\Big(A_1, A_2, ..., A_p\Big)\tilde{u}_n + \rho\tilde{y}_n\Big]. \tag{17}$$

We get

$$\begin{aligned}
\Big\|z_n - z\Big\| &= \Big\|J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)}(x) - z\Big\|\\
&= \Big\|J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)}(x) + \frac{1}{\rho}\tilde{u}_n - \frac{1}{\rho}\tilde{u}_n - z\Big\|\\
&= \Big\|J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)}(x) + \frac{1}{\rho}H^p\Big(A_1, A_2, ..., A_p\Big)\tilde{u}_n + \tilde{y}_n - J^{H^p(.,.,...)}_{\rho,\mathcal{N}_n(.,.,...)}\Big[H^p\Big(A_1, A_2, ..., A_p\Big)\tilde{u}_n + \rho\tilde{y}_n\Big] - \frac{1}{\rho}\tilde{u}_n - z\Big\|\\
&\leq \Big\|J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)}(x) - J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)}\Big[H^p\Big(A_1\tilde{z}_n, A_2\tilde{u}_n, ..., A_p\tilde{u}_n\Big) + \rho\tilde{y}_n\Big]\Big\| + \Big\|\frac{1}{\rho}H^p\Big(A_1, A_2, ..., A_p\Big)\tilde{u}_n + \tilde{y}_n - \frac{1}{\rho}\tilde{u}_n - z\Big\|\\
&\leq \Phi_1\Big\|x - H^p\Big(A_1\tilde{z}_n, A_2\tilde{u}_n, ..., A_p\tilde{u}_n\Big) - \rho\tilde{y}_n\Big\| + \Big\|\frac{1}{\rho}H^p\Big(A_1, A_2, ..., A_p\Big)\tilde{u}_n + \tilde{y}_n - \frac{1}{\rho}x\Big\| + \Big\|\frac{1}{\rho}\tilde{u}_n - \frac{1}{\rho}(x-\rho z)\Big\|\\
&\leq \Big(\Phi_1 - \frac{1}{\rho}\Big)\Big\|x - H^p\Big(A_1, A_2, ..., A_p\Big)(\tilde{u}_n) - \rho\tilde{y}_n\Big\| + \frac{1}{\rho}\Big\|\tilde{u}_n - (x-\rho z)\Big\|\\
&\leq \Big(\Phi_1 - \frac{1}{\rho}\Big)\Big\|x - H^p\Big(A_1, A_2, ..., A_p\Big)(\tilde{u}_n) + H^p\Big(A_1, A_2, ..., A_p\Big)\Big(x-\rho z\Big)\\
&\quad - H^p\Big(A_1, A_2, ..., A_p\Big)(x-\rho z) - \rho\tilde{y}_n\Big\| + \frac{1}{\rho}\Big\|\tilde{u}_n - (x-\rho z)\Big\|\\
&\leq \Big(\Phi_1 - \frac{1}{\rho}\Big)\Big\|x - H^p\Big(A_1, A_2, ..., A_p\Big)(x-\rho z) - \rho\tilde{y}_n\Big\| + \frac{1}{\rho}\Big\|\tilde{u}_n - (x-\rho z)\Big\|\\
&\quad + \Big(\Phi_1 - \frac{1}{\rho}\Big)\Big\|H^p\Big(A_1, A_2, ..., A_p\Big)(x-\rho z) - H^p\Big(A_1, A_2, ..., A_p\Big)(\tilde{u}_n)\Big\|.
\end{aligned} \tag{18}$$

Using the Lipschitz continuity of $H^p\Big(A_1, A_2, ..., A_p\Big)$, we have

$$\Big\|H^p\Big(A_1, A_2, ..., A_p\Big)(x-\rho z) - H^p\Big(A_1, A_2, ..., A_p\Big)(\tilde{u}_n))\Big\| \leq s\Big\|x - \rho z - \tilde{z}_n\Big\|, \tag{19}$$

where $s_1 + s_2 + ... + s_p = s$. It follows from (17) and (18) that

$$\left\|z_n - z\right\| \leq \left(\Phi_1 - \frac{1}{\rho}\right)\left\|x - H^p\left(A_1, A_2, ..., A_p\right)\left(x - \rho z\right) - \rho\tilde{y}_n\right\| + \left[\frac{1}{\rho} + s\left(\Phi_1 - \frac{1}{\rho}\right)\right]\left\|\tilde{u}_n - x + \rho z\right\|. \tag{20}$$

Since, f_i is κ_i-expansive mapping, we have

$$\left\|f_i(\tilde{z}_n) - f_i(z)\right\| \geq \kappa_i\left\|\tilde{z}_n - z\right\| \geq 0. \tag{21}$$

Since, $f_i(\tilde{z}_n) \to f_i(z)$ as $n \to \infty$. By (17), (21), we have $\tilde{u}_n \to (x - \rho z)$ and $\tilde{y}_n \to \frac{1}{\rho}\left[x - H^p\left(A_1, A_2, ..., A_p\right)\left(x - \rho z\right)\right]$ i.e.

$$\left\|\tilde{u}_n - x + \rho z\right\| \to 0, \ \frac{1}{\rho}\left\|x - H^p\left(A_1, A_2, ..., A_p\right)\left(x - \rho z\right) - \rho\tilde{y}_n\right\| \to 0$$

as $n \to \infty$. From (21), we have $\|z_n - z\| \to 0$ as $n \to \infty$. Hence

$$J^{H^p(.,.,...)}_{\rho, \mathcal{N}_n(.,.,...)}(x) \to J^{H^p(.,.,...)}_{\rho, \mathcal{N}(.,.,...)}(x).$$

Only if part: Suppose that

$$J^{H^p(.,.,...)}_{\rho, \mathcal{N}_n(.,.,...)}(x) \to J^{H^p(.,.,...)}_{\rho, \mathcal{N}(.,.,...)}(x), \ \forall\, x \in Y, \ \rho > 0.$$

For any given $\left(f_1(x), f_2(x), ..., f_p(x), y\right) \in \mathrm{Gr}\left(\mathcal{N}\left(\mathrm{f}_1, \mathrm{f}_2, ..., \mathrm{f}_\mathrm{p}\right)\right)$, we have $y \in \mathcal{N}\left(f_1, f_2, ..., f_p\right)(x)$

$$H^p\left(A_1x, A_2x, ..., A_px\right) + \rho y \ \in \left[H^p\left(A_1, A_2, ..., A_p\right) + \rho\mathcal{N}\left(f_1, f_2, ..., f_p\right)\right](x)$$

and so $x \ = \ \left[I - J^{H^p(.,.,...)}_{\rho, \mathcal{N}(.,.,...)}\right]\left[H^p\left(A_1x, A_2x, ..., A_px\right) + \rho y\right]$.
Let

$$x_n \ = \ \left[I - J^{H^p(.,.,...)}_{\rho, \mathcal{N}_n(.,.,...)}\right]\left[H^p\left(A_1x_n, A_2x_n, ..., A_px_n\right) + \rho y\right].$$

Then,

$$\frac{1}{\rho}\left[H^p\left(A_1x, A_2x, ..., A_px\right) - H^p\left(A_1x_n, A_2x_n, ..., A_px_n\right) + \rho y\right] \in \mathcal{N}_n\left(f_1(x_n), f_2(x_n), ..., f_p(x_n)\right).$$

Let

$$y_n \ = \ \frac{1}{\rho}\left[H^p\left(A_1x, A_2x, ..., A_px\right) - H^p\left(A_1x_n, A_2x_n, ..., A_px_n\right) + \rho y\right].$$

It follows from (18) that

$$\begin{aligned}\left\|y_n - y\right\| \ &\leq \ \left\|\frac{1}{\rho}\left[H^p\left(A_1x, A_2x, ..., A_px\right) - H^p\left(A_1x_n, A_2x_n, ..., A_px_n\right) + \rho y\right] - y\right\| \\ &= \frac{1}{\rho}\left\|H^p\left(A_1x, A_2x, ..., A_px\right) - H^p\left(A_1x_n, A_2x_n, ..., A_px_n\right)\right\| \\ &\leq \ \frac{s}{\rho}\left\|x_n - x\right\|, \ \text{ where } s_1 + s_2 + ... + s_p = s.\end{aligned} \tag{22}$$

$$\left\|x_n - x\right\| \leq \left\|\left(I - J^{H^p(.,.,...)}_{\rho, \mathcal{N}_n(.,.,...)}\right)\left(H^p\left(A_1x, A_2x, ..., A_px\right) + \rho y\right) - \left(I - J^{H^p(.,.,...)}_{\rho, \mathcal{N}(.,.,...)}\right)\left(H^p\left(A_1x, A_2x, ..., A_px\right) + \rho y\right)\right\| \tag{23}$$

$$\left\|x_n - x\right\| \leq \left\|\left[\left(I - J^{H^p(.,.,...)}_{\rho, \mathcal{N}_n(.,.,...)}\right) - \left(I - J^{H^p(.,.,...)}_{\rho, \mathcal{N}(.,.,...)}\right)\right]\left[H^p\left(A_1x, A_2x, ..., A_px\right) + \rho y\right]\right\| \tag{24}$$

Since, $J^{H^p(.,.,...)}_{\rho, \mathcal{N}_n(.,.,...)} \ \to \ J^{H^p(.,.,...)}_{\rho, \mathcal{N}(.,.,...)}$ for any $u \in Y$, we know that $\|x_n - x\| \to 0$. Now (22) implies that $y_n \to y \ \ as \ n \to \infty$, and so $\mathcal{N}_n \xrightarrow{G} \mathcal{N}$.

Theorem 4.5. *The convergence of the proximal-point mapping*

$$R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)}(x) \to R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}(x),\ \forall x \in Y,\ \rho > 0,$$

and the convergence of the generalized Yosida approximation operator

$$J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)}(x) \to J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}(x),\ \forall x \in Y,\ \rho > 0,$$

are equivalent if and only if the operator $\mathcal{N}_n \xrightarrow{G} \mathcal{N}$.

Proof: Assume that

$$R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)}(x) \to R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}(x),\ \forall x \in Y,\ \rho > 0,$$

We have

$$\begin{aligned}
&R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)}(x) \to R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}(x),\ \forall x \in Y,\ \rho > 0,\\
&\quad \Rightarrow \left(I - J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)}\right)(x) \to \left(I - J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}\right)(x),\\
&\quad \Rightarrow \frac{1}{\rho}\left(I - R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)}\right)(x) \to \frac{1}{\rho}\left(I - R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}\right)(x),\\
&\quad \Rightarrow J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)}(x) \to J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}(x),\ \forall x \in Y,\ \rho > 0.
\end{aligned}$$

On similar way, we can show that

$$J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)}(x) \to J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}(x),\ \forall x \in Y,\ \rho > 0,$$

implies that

$$R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)}(x) \to J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}(x),\ \forall x \in Y,\ \rho > 0.$$

Now we construct the following consolidated illustration which shows that the mapping $\mathcal{N}$ is $\alpha_i\beta_j$-$H^p(.,.,.,..)$-accretive with mappings $(A_1, A_2, ..., A_p)$ and $(f_1, f_2, ..., f_p)$, $\mathcal{N}_n \xrightarrow{G} \mathcal{N}$ and $J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)} \to J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}$. By using MATLAB programming, we presents some graphics for the convergence of generalized Yosida approximation mapping.

Example 4.6. Let Y be 2-uniformly smooth Banach space $Y = \mathbb{R}$ with the usual inner product. Let p is even number and $A_i : \mathbb{R} \to \mathbb{R}$ for each $i \in \{1, 2, ..., p\}$, is given by

$$\begin{aligned}
A_1(x) = A_3(x) = ... = A_{p-1}(x) &= \frac{x^3}{8} + \frac{2x}{3},\\
A_2(x) = A_4(x) = ... = A_p(x) &= \frac{x}{2},
\end{aligned}$$

such that the inequality $xy + x^2 + y^2 > 0$ is satisfied for all $x \in \mathbb{R}^2$.

Let $f_i : \mathbb{R} \to \mathbb{R}$ for each $i \in \{1, 2, ..., p\}$, is given by

$$\begin{aligned}
f_1(x) = f_3(x) = ... = f_{p-1}(x) &= \frac{x}{4},\\
f_2(x) = f_4(x) = ... = f_p(x) &= \frac{5x}{12}.
\end{aligned}$$

Assume that $H^p : \underbrace{\mathbb{R} \times \mathbb{R} \times ... \times \mathbb{R}}_{p\ times} \to \mathbb{R}$ are defined by

$$H\Big(A_1(x), A_2(x), ..., A_{p-1}(x), A_p(x)\Big) = A_1(x) - A_2(x) + ... + A_{p-1}(x) - A_p(x),$$

It can be easily shown that H^p is $\frac{2}{3}$-strongly accretive with A_i for each $i \in \{1, 3, ..., p-1\}$ and H^p is $\frac{3}{2}$-relaxed accretive with A_i for all $i \in \{2, 4, ..., p\}$.

Assume that $\mathcal{N}_n, \mathcal{N} : \underbrace{\mathbb{R} \times \mathbb{R} \times ... \times \mathbb{R}}_{p\ times} \multimap \mathbb{R}$ be the multi-valued mappings and defined by

$$\begin{aligned} \mathcal{N}_n\Big(f_1(x), f_2(x), ..., f_{p-1}(x), f_p(x)\Big) &= \Big(f_1(x) - f_2(x) + ... + f_{p-1}(x) - f_p(x)\Big) + \frac{1}{n^2}, \\ \mathcal{N}\Big(f_1(x), f_2(x), ..., f_{p-1}(x), f_p(x)\Big) &= f_1(x) - f_2(x) + ... + f_{p-1}(x) - f_p(x). \end{aligned}$$

It can be easily shown that $\mathcal{N}$ is $\frac{1}{4}$-strongly accretive with f_i for each $i \in \{1, 3, ..., p-1\}$ and H^p is $\frac{17}{12}$-relaxed accretive with f_i for all $i \in \{2, 4, ..., p\}$.

One can easily verify the following for $\rho = 1$:

$$\Big[H^p(A_1, A_2, ..., A_p) + \rho\mathcal{N}(f_1, f_2, ..., f_p)\Big]\mathbb{R} = \mathbb{R}.$$

Now we will show that $\mathcal{N}_n \xrightarrow{G} \mathcal{N}$, if for each $\Big(f_1(x), f_2(x), ..., f_p(x)), y\Big) \in \text{Gr}\Big(\mathcal{N}(f_1, f_2, ..., f_p)\Big)$, there exists a sequence
$\Big((f_1(x_n), f_2(x_n), ..., f_p(x_n)), y_n\Big) \in \text{Gr}\Big(\mathcal{N}_n(f_1, f_2, ..., f_p)\Big)$ such that $f_1(x_n) \to f_1(x), f_2(x_n) \to f_2(x), ..., f_p(x_n) \to f_p(x),\ y_n \to y$ as $n \to \infty$. For this, we consider

$$\begin{aligned} x_n &= \Big(1 + \frac{1}{n}\Big)x, \\ f_1(x_n) &= f_3(x_n) = ... = f_{p-1}(x_n) = \frac{x_n}{4}, \\ f_2(x_n) &= f_4(x_n) = ... = f_p(x_n) = \frac{5}{12}x_n,\ n \in \mathbb{N}. \end{aligned}$$

Since, $\lim_n x_n = \lim_n \Big(1 + \frac{1}{n}\Big)x = x$ Thus, we have $x_n \to x$ as $n \to \infty$. Now

$$\begin{aligned} &\lim_n f_1(x_n) \to f_1(x), \lim_n f_3(x_n) \to f_3(x), ..., \lim_n f_{p-1}(x_n) \to f_{p-1}(x), \\ &\lim_n f_2(x_n) \to f_2(x), \lim_n f_4(x_n) \to f_4(x), ..., \lim_n f_p(x_n) \to f_p(x). \end{aligned}$$

Since,

$$\begin{aligned} y_n &= \mathcal{N}\Big(f_1(x_n), f_2(x_n), ..., f_p(x_n)\Big) \\ &= \Big(f_1(x_n) - f_2(x_n) + ... + f_{p-1}(x_n) - f_p(x_n)\Big) + \frac{1}{n^2} = \frac{p}{2}\left[\frac{x_n}{4} - \frac{5x_n}{4}\right] + \frac{1}{n^2}. \end{aligned}$$

Now we compute

$$\begin{aligned} \lim_n y_n = \lim_n \frac{p}{2}\left[\frac{x_n}{4} - \frac{5x_n}{12}\right] + \frac{1}{n_2^2} &= \frac{p}{2}\left[\frac{x}{4} - \frac{5x}{12}\right]. \\ &= \underbrace{\Big(\frac{x}{4} - \frac{5x}{12}\Big) + \Big(\frac{x}{4} - \frac{5x}{12}\Big) + ... + \Big(\frac{x}{4} - \frac{5x}{12}\Big)}_{p\ terms}. \\ &= f_1(x) - f_2(x) + ... + f_{p-1}(x) - f_p(x). \\ &= \mathcal{N}(x) = y. \end{aligned}$$

Therefore, $y_n \to y$ as $n \to \infty$ and hence, $\mathcal{N}_n \xrightarrow{G} \mathcal{N}$. Next, we will show that $J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)} \to J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}$ as $\mathcal{N}_n \xrightarrow{G} \mathcal{N}$. For $\rho = 1$, the proximal-point mappings are given by

$$R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)}(x) = \Big[H^p(A_1, A_2, ..., A_p) + \rho\mathcal{N}_n(f_1, f_2, ..., f_p)\Big](x_n) = 2\left(x - \frac{1}{n^2}\right)^{\frac{1}{3}},$$
$$R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}(x) = \Big[H^p(A_1, A_2, ..., A_p) + \rho\mathcal{N}(f_1, f_2, ..., f_p)\Big](x) = 2x^{\frac{1}{3}}$$

and the generalized Yosida approximation are given by

$$J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)}(x) = \frac{1}{\rho}\left[I - R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)}\right](x) = \left[x - 2\left(x - \frac{1}{n^2}\right)^{\frac{1}{3}}\right],$$
$$J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}(x) = \frac{1}{\rho}\left[I - R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}\right](x) = \left[x - 2x^{\frac{1}{3}}\right].$$

We evaluate $\left\|J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)} - J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}\right\| = \left\|\left[x - 2\left(x - \frac{1}{n^2}\right)^{\frac{1}{3}}\right] - \left[x - 2x^{\frac{1}{3}}\right]\right\|$, which shows that $\left\|J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)} - J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}\right\| \to 0$ as $n \to \infty$. i.e. $J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)} \to J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}$ as $\mathcal{N}_n \xrightarrow{G} \mathcal{N}$.

5. Uniqueness and existence of solutions of Yosida inclusion problem:

Let Y be a q-uniformly smooth Banach space. For each $i \in \{1, 2, ..., p\}$, $p \geq 3$, $H^p : Y^p \to Y$, $A_i, f_i : Y \to Y$ be single-valued mappings. Let $\mathcal{N} : Y^p \multimap Y$ be a multi-valued mapping such that $\mathcal{N}$ be a generalized $\alpha_i\beta_j$-$H^p(.,.,...,.)$-accretive mapping. Then, Yosida inclusion problem is to find $x \in Y$ such that

$$\Theta \in J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}(x) + \mathcal{N}\Big(f_1(x), f_2(x), ..., f_p(x)\Big), \forall x \in Y,\ \rho > 0, \tag{25}$$

where $J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}$ is a generalized Yosida approximation mapping given by (8).

Theorem 5.1. *For any given $x \in Y$, x is a solution of (SGVI) (25) if and only if x satisfies*

$$x = R^{H^p(.,.,...)}_{\rho,\mathcal{N}(.,.,...)}\Big[H^p(A_1, A_2, ..., A_p)(x) - \rho J^{H^p(.,.,...)}_{\rho,\mathcal{N}(.,.,...)}(x)\Big], \text{ where constant } \rho > 0. \tag{26}$$

Algorithm 5.2. For any given $x_0^1 \in Y$ and obtain $\{x_n^1\}$, by iterative scheme

$$x^1_{n+1} = R^{H^p(.,.,...)}_{\rho,\mathcal{N}_n(.,.,...)}\Big[H^p(A_1, A_2, ..., A_p)(x^1_n) - \rho J^{H^p(.,.,...)}_{\rho,\mathcal{N}_n(.,.,...)}(x^1_n)\Big], \tag{27}$$

where n=0,1,2,... and constant $\rho > 0$.

Theorem 5.3. *Let us consider the problem (25) with assumption M_1-M_4 hold good and $\mathcal{N}_n, \mathcal{N} : Y^p \multimap Y$ be the generalized $\alpha_i\beta_j$-$H^p(.,.,...)$-accretive mappings with mappings $(A_1, A_2, ..., A_p)$ and $(f_1, f_2, ..., f_p)$ and $\sum\alpha_i > \sum\beta_j, \sum\bar{\mu}_i > \sum\bar{\gamma}_j$. For each $i \in \{1, 2, ..., p\}$, let $H^p(.,.,...)$ is s_i-Lipschitz continuous with A_i. Assume that there exists non-negative constant $\rho > 0$ such that*

$$\Big[\sum\alpha_i - \sum\beta_j + \rho\Big(\sum\bar{\mu}_i - \sum\bar{\gamma}_j\Big)\Big] \geq \Big[1 - q(\sum\alpha_i - \sum\beta_j) + c_q s^q\Big]^{1/q} + \Big[1 - q\rho\Phi_2 + c_q\rho^q\Phi_1\Big]^{1/q}; \tag{28}$$

$$\Big[\sum\alpha_i - \sum\beta_j + \rho\Big(\sum\bar{\mu}_i - \sum\bar{\gamma}_j\Big)\Big] \geq \Big[s_1 + s_2 + ... + s_p + \rho\Phi_1\Big]. \tag{29}$$

Then,
(1) *the general nonlinear operator equation (25) based on generalized $\alpha_i\beta_j$-$H^p(.,.,...)$-accretivity framework has a unique solution x^1 in Y;*

(2) *iterative sequences $\{x_n^1\}$ developed through Algorithm 5.2 is converges strongly to the solution x^1 of problem (25).*

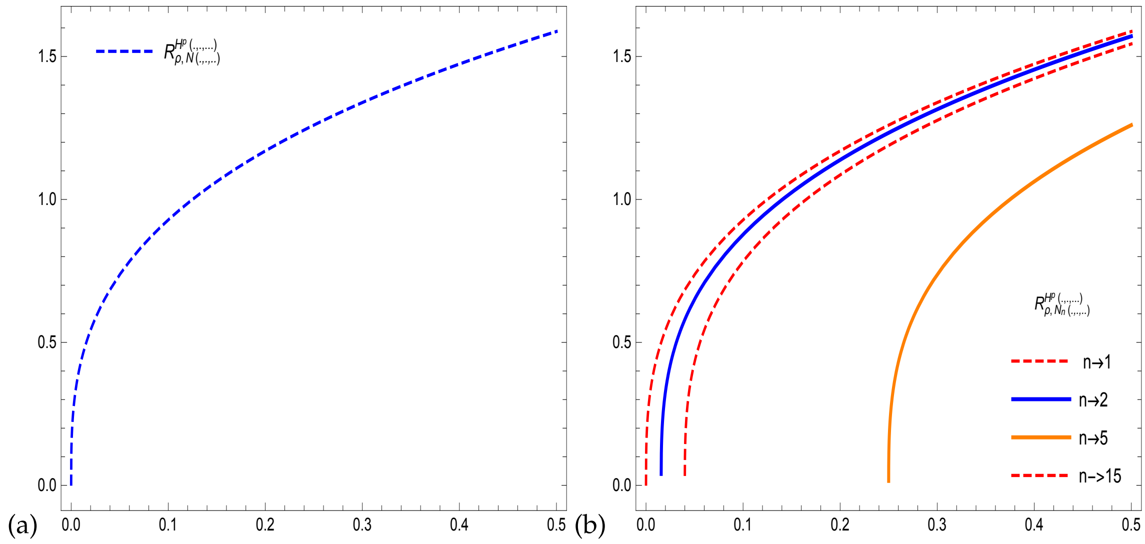

Figure 1: (a) Graph of $R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}$, where $\mathcal{N}(x) = \frac{p}{2}\left[\frac{x}{4} - \frac{5x}{12}\right]$
(b) The convergence of $R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)} \to R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}$ as $\mathcal{N}_n \xrightarrow{G} \mathcal{N}$,where $\mathcal{N}_n(x) = \frac{p}{2}\left[\frac{x}{4} - \frac{5x}{12}\right] + \frac{1}{n^2}$ and $\mathcal{N}(x) = \frac{p}{2}\left[\frac{x}{4} - \frac{5x}{12}\right]$

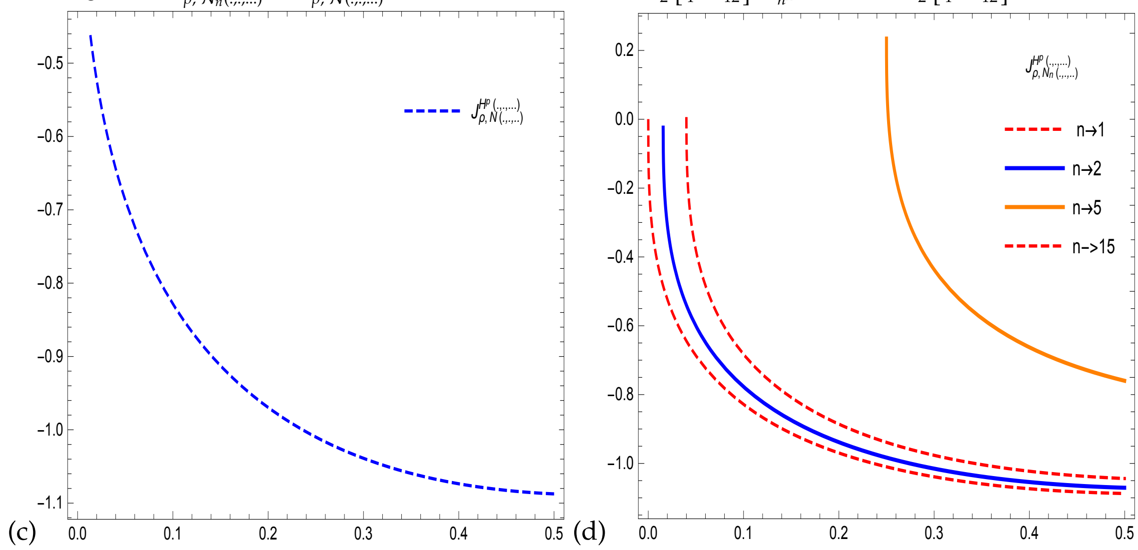

Figure 2: (c) Graph of $J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}$, where $\mathcal{N}(x) = \frac{p}{2}\left[\frac{x}{4} - \frac{5x}{12}\right]$
(d) The convergence of $J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)} \to J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}$ as $\mathcal{N}_n \xrightarrow{G} \mathcal{N}$,where $\mathcal{N}_n(x) = \frac{p}{2}\left[\frac{x}{4} - \frac{5x}{12}\right] + \frac{1}{n^2}$ and $\mathcal{N}(x) = \frac{p}{2}\left[\frac{x}{4} - \frac{5x}{12}\right]$

Proof. Let $P : Y \to Y$ be given as

$$P(x^1) = R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}\left(H^p(A_1x^1, A_2x^1, ..., A_px^1) - \rho J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}(x^1)\right), \forall\ x^1 \in Y.$$

$$P(y^1) = R^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}\left(H^p(A_1y^1, A_2y^1, ..., A_py^1) - \rho J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}(y^1)\right), \forall\ y^1 \in Y.$$

From Proposition 2.3 and using the Lipschitz continuity of proximal-point mapping, we have

$$\begin{aligned}
\left\|P(x^1)-P(y^1)\right\| &= \Big\|R^{H^p(.,.,...)}_{\rho,\,\mathcal{N}(.,.,...)}\Big(H^p(A_1x^1,A_2x^1,...,A_px^1)-\rho J^{H^p(.,.,...)}_{\rho,\,\mathcal{N}(.,.,...)}(x^1)\Big)\\
&\quad -R^{H^p(.,.,...)}_{\rho,\,\mathcal{N}(.,.,...)}\Big(H^p(A_1y^1,A_2y^1,...,A_py^1)-\rho J^{H^p(.,.,...)}_{\rho,\,\mathcal{N}(.,.,...)}(y^1)\Big)\Big\|\\
&\le \Delta\Big\|\Big(H^p(A_1x^1,A_2x^1,...,A_px^1)-\rho J^{H^p(.,.,...)}_{\rho,\,\mathcal{N}(.,.,...)}(x^1)\Big)\\
&\quad -\Big(H^p(A_1y^1,A_2y^1,...,A_py^1)-\rho J^{H^p(.,.,...)}_{\rho,\,\mathcal{N}(.,.,...)}(y^1)\Big)\Big\|\\
&= \Delta\Big\|H^p(A_1,A_2,...,A_p)(x^1)-H^p(A_1,A_2,...,A_p)(y^1)-\rho\Big(J^{H^p(.,.,...)}_{\rho,\,\mathcal{N}(.,.,...)}(x^1)-J^{H^p(.,.,...)}_{\rho,\,\mathcal{N}(.,.,...)}(y^1)\Big)\Big\|\\
&\le \Delta\Big\|H^p(A_1,A_2,...,A_p)(x^1)-H^p(A_1,A_2,...,A_p)(y^1)-\big(x^1-y^1\big)\Big\|\\
&\quad +\Delta\Big\|x^1-y^1-\rho\Big(J^{H^p(.,.,...)}_{\rho,\,\mathcal{N}(.,.,...)}(x^1)-J^{H^p(.,.,...)}_{\rho,\,\mathcal{N}(.,.,...)}(y^1)\Big)\Big\|. \qquad (30)
\end{aligned}$$

By using the s_i-Lipschitz continuity of $H^p(.,...,A_i,..,.)$, we have

$$\begin{aligned}
&\left\|H^p(A_1,A_2,...,A_p)(x^1)-H^p(A_1,A_2,...,A_p)(y^1)\right\|\\
&\le \left\|H^p(A_1x^1,A_2x^1,...,A_px^1))-H^p(A_1y^1,A_2x^1,...,A_px^1)\right\|\\
&\quad +\left\|H^p(A_1y^1,A_2x^1,...,A_px^1)-H^p(A_1y^1,A_2y^1,...,A_px^1)\right\|\\
&\qquad \vdots\\
&\quad +\left\|H^p(A_1y^1,A_2y^1,...,A_px^1)-H^p(A_1y^1,A_2y^1,...,A_py^1)\right\|\\
&\le \big(s_1+s_2+...+s_p\big)\left\|x^1-y^1\right\|.
\end{aligned}$$

$$\left\|H^p(A_1,A_2,...,A_p)(x^1_n)-H^p(A_1,A_2,...,A_p)(x^1_{n-1})\right\| \le s\left\|x^1_{n-1}-x^1_n\right\|,\ \text{where } s=s_1+s_2+...+s_p. \qquad (31)$$

Since, H^p is symmetric accretive with mappings $A_1,A_2,...,A_p$ and s_i-Lipschitz continuous with A_i, using the Lemma 1.5, we have

$$\begin{aligned}
&\left\|x^1-y^1-\big(H^p(A_1,A_2,...,A_p)(x^1)-H^p(A_1,A_2,...,A_p)(y^1)\big)\right\|^q\\
&\le \left\|x^1-y^1\right\|^q-q\big\langle H^p(A_1,A_2,...,A_p)(x^1)-H^p(A_1,A_2,...,A_p)(y^1),x^1-y^1\big\rangle\\
&\quad +c^q\left\|H^p(A_1,A_2,...,A_p)(x^1)-H^p(A_1,A_2,...,A_p)(y^1)\right\|^q\\
&\le \left\|x^1-y^1\right\|^q-q\left[\sum\alpha_i-\sum\beta_j\right]\left\|x^1-y^1\right\|^q\\
&\quad +\left\|H^p(A_1y^1,A_2x^1,...,A_px^1)-H^p(A_1y^1,A_2y^1,...,A_px^1)\right\|^q\\
&\le \left\|x^1-y^1\right\|^q-q\left[\sum\alpha_i-\sum\beta_j\right]\left\|x^1-y^1\right\|^q\\
&\quad +\big(s_1+s_2+...+s_p\big)^q\left\|x^1-y^1\right\|^q\\
&\le \left[1-q\left(\sum\alpha_i-\sum\beta_j\right)+c_qs^q\right]\left\|x^1-y^1\right\|^q.
\end{aligned}$$

Thus, we can write

$$\left\|x^1-y^1-\big(H^p(A_1,A_2,...,A_p)(x^1)-H^p(A_1,A_2,...,A_p)\big)(y^1)\right\|^q \le \left[1-q\left(\sum\alpha_i-\sum\beta_j\right)+c_qs^q\right]\left\|x^1-y^1\right\|^q,$$

which implies that

$$\left\|x^1 - y^1 - \left(H^p(A_1, A_2, ..., A_p)(x^1) - H^p(A_1, A_2, ..., A_p)\right)(y^1)\right\| \leq \left[1 - q\left(\sum \alpha_i - \sum \beta_j\right) + c_q s^q\right]^{1/q} \left\|x^1 - y^1\right\|. \quad (32)$$

Since, $J^{H^p(.,.,...)}_{\rho,\ N(.,.,...)}$ is ϕ_1-Lipschitz continuous and Φ_2-strong accretive, and using the Lemma 1.5, we have

$$\begin{aligned}
&\left\|x^1 - y^1 - \rho\left(J^{H^p(.,.,...)}_{\rho,\ N(.,.,...)}(x^1) - J^{H^p(.,.,...)}_{\rho,\ N(.,.,...)}(y^1)\right)\right\|^q \\
&\leq \left\|x^1 - y^1\right\|^q - q\rho\left\langle J^{H^p(.,.,...)}_{\rho,\ N(.,.,...)}(x^1) - J^{H^p(.,.,...)}_{\rho,\ N(.,.,...)}(y^1), x^1 - y^1\right\rangle \\
&\quad + \rho^q c^q \left\|J^{H^p(.,.,...)}_{\rho,\ N(.,.,...)}(x^1) - J^{H^p(.,.,...)}_{\rho,\ N(.,.,...)}(y^1)\right\|^q \\
&\leq \left\|x^1 - y^1\right\|^q - q\rho\Phi_2\left\|x^1 - y^1\right\|^q + \rho^q c^q \left\|x^1 - y^1\right\|^q \\
&\leq \left[1 - q\rho\Phi_2 + \rho^q c_q \Phi_1\right]\left\|x^1 - y^1\right\|^q.
\end{aligned}$$

Thus, we can write

$$\left\|x^1 - y^1 - \rho\left(J^{H^p(.,.,...)}_{\rho,\ N(.,.,...)}(x^1) - J^{H^p(.,.,...)}_{\rho,\ N(.,.,...)}(y^1)\right)\right\|^q \leq \left[1 - q\rho\Phi_2 + c_q\rho^q\Phi_1\right]\left\|x^1 - y^1\right\|^2,$$

which implies that

$$\left\|x^1 - y^1 - \rho\left(J^{H^p(.,.,...)}_{\rho,\ N(.,.,...)}(x^1) - J^{H^p(.,.,...)}_{\rho,\ N(.,.,...)}(y^1)\right)\right\| \leq \left[1 - q\rho\Phi_2 + c_q\rho^q\Phi_1\right]^{1/q}\left\|x^1 - y^1\right\|. \quad (33)$$

Using (30)-(33) in (31), we have

$$\|P(x^1) - P(y^1)\| = \Upsilon\|x^1 - y^1\|, \quad (34)$$

$$\Upsilon = \Delta\left[\left[1 - q\left(\sum \alpha_i - \sum \beta_j\right) + c_q s^q\right]^{1/q} + \left[1 - q\rho\Phi_2 + c_q\rho^q\Phi_1\right]^{1/q}\right].$$

From condition (28), we have $0 < \Upsilon < 1$, so (34) implies that

$$P = R^{H^p(.,.,...)}_{\rho,\ N(.,.,...)}\left(H^p(A_1, A_2, ..., A_p) - \rho J^{H^p(.,.,...)}_{\rho,\ N(.,.,...)}\right)$$

is a contraction mapping and has a unique fixed point x^1 in Y. Hence x^1 is a unique solution of (25).

Now we prove that x^1_n convergence strongly to x^1. In fact, it follows from Theorem 5.1 and Algorithm 5.2 that

$$\begin{aligned}
\left\|x^1_{n+1} - x^1\right\| &= \Big\|R^{H^p(.,.,...)}_{\rho, N_n(.,.,...)}\left[H^p(A_1, A_2, ..., A_p)(x^1_n) - \rho J^{H^p(.,.,...)}_{\rho,\ N_n(.,.,...)}(x^1_n)\right] \\
&\quad - R^{H^p(.,.,...)}_{\rho, N(.,.,...)}\left[H^p(A_1, A_2, ..., A_p)(x^1) - \rho J^{H^p(.,.,...)}_{\rho,\ N(.,.,...)}(x^1)\right]\Big\| \\
&\leq \Big\|R^{H^p(.,.,...)}_{\rho, N_n(.,.,...)}\left[H^p(A_1, A_2, ..., A_p)(x^1_n) - \rho J^{H^p(.,.,...)}_{\rho,\ N_n(.,.,...)}(x^1_n)\right] \\
&\quad - R^{H^p(.,.,...)}_{\rho, N(.,.,...)}\left[H^p(A_1, A_2, ..., A_p)(x^1_n) - \rho J^{H^p(.,.,...)}_{\rho,\ N_n(.,.,...)}(x^1_n)\right]\Big\| \\
&\quad + \Big\|R^{H^p(.,.,...)}_{\rho, N(.,.,...)}\left[H^p(A_1, A_2, ..., A_p)(x^1_n) - \rho J^{H^p(.,.,...)}_{\rho,\ N_n(.,.,...)}(x^1_n)\right] \\
&\quad - R^{H^p(.,.,...)}_{\rho, N(.,.,...)}\left[H^p(A_1, A_2, ..., A_p)(x^1) - \rho J^{H^p(.,.,...)}_{\rho,\ N(.,.,...)}(x^1)\right]\Big\| \\
&\leq \Big[\Delta \Big\|H^p(A_1, A_2, ..., A_p)(x^1_n) - \rho J^{H^p(.,.,...)}_{\rho,\ N_n(.,.,...)}(x^1_n)\Big] \\
&\quad - \Big[H^p(A_1, A_2, ..., A_p)(x^1) - \rho J^{H^p(.,.,...)}_{\rho,\ N(.,.,...)}(x^1)\Big]\Big\| + k_n, \quad (35)
\end{aligned}$$

where

$$k_n \leq \Big\| R^{H^p(.,.,...)}_{\rho,\mathcal{N}_n(.,.,...)} \left[H^p(A_1,A_2,...,A_p)(x_n^1) - \rho J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)}(x_n^1)\right] - R^{H^p(.,.,...)}_{\rho,\mathcal{N}(.,.,...)} \left[H^p(A_1,A_2,...,A_p)(x_n^1) - \rho J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)}(x_n^1)\right]\Big\|$$

Using the s_i-Lipschitz continuity of $H^p(.,.,.,...)$ with A_i and the generalized Yosida approximation mapping, we get

$$\begin{aligned}
&\Big\|H^p(A_1,A_2,...,A_p)(x_n^1) - H^p(A_1,A_2,...,A_p)(x^1) - \rho\Big[J^{H^p(.,.,...)}_{\rho,\mathcal{N}_n(.,.,...)}(x_n^1) - J^{H^p(.,.,...)}_{\rho,\mathcal{N}(.,.,...)}(x^1)\Big]\Big\| \\
&\leq \Big\|H^p(A_1,A_2,...,A_p)(x_n^1) - H^p(A_1,A_2,...,A_p)(x^1)\Big\| + \rho\Big\|J^{H^p(.,.,...)}_{\rho,\mathcal{N}_n(.,.,...)}(x_n^1) - J^{H^p(.,.,...)}_{\rho,\mathcal{N}(.,.,...)}(x^1)\Big\| \\
&\leq \Big\|H^p(A_1x_n^1,A_2x_n^1,...,A_px_n^1) - H^p(A_1x^1,A_2x_n^1,...,A_px_n^1)\Big\| + \rho\Big\|J^{H^p(.,.,...)}_{\rho,\mathcal{N}_n(.,.,...)}(x_n^1) - J^{H^p(.,.,...)}_{\rho,\mathcal{N}(.,.,...)}(x^1)\Big\| \\
&\quad + \Big\|H^p(A_1x^1,A_2x_n^1,...,A_px_n^1) - H^p(A_1x^1,A_2x^1,...,A_px_n^1)\Big\| \\
&\quad \vdots \\
&\quad + \Big\|H^p(A_1x^1,A_2x^1,...,A_px_n^1) - H^p(A_1x^1,A_2x^1,...,A_px^1)\Big\| \\
&\leq s_1\Big\|x_n^1 - x^1\Big\| + s_2\Big\|x_n^1 - x^1\Big\| + ... + s_p\Big\|x_n^1 - x^1\Big\| \\
&\quad + \rho\Big\|J^{H^p(.,.,...)}_{\rho,\mathcal{N}_n(.,.,...)}(x_n^1) - J^{H^p(.,.,...)}_{\rho,\mathcal{N}(.,.,...)}(x_n^1)\Big\| + \rho\Big\|J^{H^p(.,.,...)}_{\rho,\mathcal{N}(.,.,...)}(x_n^1) - J^{H^p(.,.,...)}_{\rho,\mathcal{N}(.,.,...)}(x^1)\Big\| \\
&\leq (s_1+s_2+...+s_p)\Big\|x_n^1 - x^1\Big\| + +\rho\Big\|J^{H^p(.,.,...)}_{\rho,\mathcal{N}_n(.,.,...)}(x_n^1) - J^{H^p(.,.,...)}_{\rho,\mathcal{N}(.,.,...)}(x_n^1)\Big\| + \rho\Big\|J^{H^p(.,.,...)}_{\rho,\mathcal{N}(.,.,...)}(x_n^1) - J^{H^p(.,.,...)}_{\rho,\mathcal{N}(.,.,...)}(x^1)\Big\| \\
&\leq s\Big\|x_n^1 - x^1\Big\| + \rho\Big\|J^{H^p(.,.,...)}_{\rho,\mathcal{N}_n(.,.,...)}(x_n^1) - J^{H^p(.,.,...)}_{\rho,\mathcal{N}(.,.,...)}(x_n^1)\Big\| + \rho\Big\|J^{H^p(.,.,...)}_{\rho,\mathcal{N}(.,.,...)}(x_n^1) - J^{H^p(.,.,...)}_{\rho,\mathcal{N}(.,.,...)}(x^1)\Big\| \\
&\leq s\Big\|x_n^1 - x^1\Big\| + \rho l_n + \rho\Phi_1\Big\|x_n^1 - x^1\Big\|
\end{aligned} \tag{36}$$

(37)

where $l_n = \Big\|J^{H^p(.,.,...)}_{\rho,\mathcal{N}_n(.,.,...)}(x_n^1) - J^{H^p(.,.,...)}_{\rho,\mathcal{N}(.,.,...)}(x_n^1)\Big\|$.
Using (36) in (35), we get

$$\Big\|x_{n+1}^1 - x^1\Big\| = k_n + \Delta(s+\rho\Phi_1)\Big\|x_n^1 - x^1\Big\| + \Delta\rho l_n,$$

where $\Phi_1 = \frac{1+\Delta}{\rho}$. By Theorem 4.3, we have

$$R^{H^p(.,.,...)}_{\rho,\mathcal{N}_n(.,.,...)}\Big[H^p(A_1,A_2,...,A_p)(x_n^1) - \rho J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)}(x_n^1)\Big] \to R^{H^p(.,.,...)}_{\rho,\mathcal{N}(.,.,...)}\Big[H^p(A_1,A_2,...,A_p)(x_n^1) - \rho J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)}(x_n^1)\Big].$$

Thus we have

$$J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}_n(.,.,...)}(x_n^1) \to J^{H^p(.,.,...)}_{\rho,\ \mathcal{N}(.,.,...)}(x^1).$$

As $n \to \infty$, $k_n, l_n \to \infty$. Thus we have

$$\Big\|x_{n+1}^1 - x^1\Big\| = \Upsilon_n(\Phi_1)\Big\|x_n^1 - x^1\Big\| + \Theta_n,$$

where $\Theta_n = k_n + \Delta\rho l_n$ and $\Upsilon(\Phi_1) = \Delta(s+\rho\Phi_1)$. From (29), we have $0 < \Upsilon(\Phi_1) < 1$ and $\Theta_n \to 0$ as $k_n, l_n \to 0$ $(n \to \infty)$. From Lemma 1.6, we have $\|x_{n+1}^1 - x^1\| \to 0$.

Remark 5.4. Let p be an even number. Let us consider the functional $f : Y \to \mathbb{R}$ on Y, then, vector $\tilde{x}$ such that

$$f(y) - f(z) \geq \left\langle \tilde{x},\ y - z \right\rangle,\ \forall y \in Y, \tag{38}$$

where $f(y)$ is finite for each $y \in Y$, is called the subgradient of f at y. The collections of all such subgradients of f at y holding (38) is called the subdifferential $\partial f(y)$ of f at y. Let $\partial\Big(.,.,f_i,..\Big) : Y^p \multimap Y$ be a multi-valued mapping. We consider the Yosida inclusion problem to find $x \in Y$ such that

$$0 \in J^{H^p(.,.,...)}_{\rho,\delta(.,.,...)}(x) + \partial\Big(f_1, f_2, ..., f_p\Big)(x). \tag{39}$$

Then, it turns out that $\Big[H^p(.,.,...) + \partial\Big(f_1, f_2, ..., f_p\Big)\Big]$ is $\mathcal{L}$-strongly accretive, where $\mathcal{L} = \Big[\sum \alpha_i + \sum \bar{\mu}_i - \Big(\sum \beta_j + \sum \bar{\gamma}_j\Big)\Big] > 0$, if H^p is $\alpha_1\beta_2\alpha_3\beta_4...\alpha_{p-1}\beta_p$-symmetric accretive with $A_1, A_2, ..., A_p$, $f_1, f_2, ..., f_p : Y \to Y$ be the locally Lipschitz functional on Y, and $\partial\Big(f_1, f_2, ..., f_p\Big)$ be $\bar{\mu}_1\bar{\gamma}_2\bar{\mu}_3\bar{\gamma}_4...\bar{\mu}_{p-1}\bar{\gamma}_p$-symmetric accretive with $f_1, f_2, ..., f_p$, equivalently we can state that $\partial\Big(..., f_i, ..\Big)$ is $\alpha_i\beta_j$-$H^p(.,.,...)$-accretive. Now all the assumption of Theorem 5.3 hold, and one can easily find the solution of problem (39) by using Theorem 5.3.

6. Conclusion:

In this manuscript, we focussed on generalized Yosida approximation mappings and Yosida inclusions in Banach spaces. To approximate the solutions to such kinds of inclusion problems, we considered a generalized $\alpha_i\beta_j$-$H^p(.,.,...)$-accretive mapping, which is the generalized form of generalized $\alpha\beta$-$H(...)$-accretive mapping [18]. We proved the graph convergence of generalized $\alpha_i\beta_j$-$H^p(.,.,...)$-accretive mappings and generalized Yosida approximation mappings. As an application part of generalized $\alpha_i\beta_j$-$H^p(.,.,...)$-accretive mappings and generalized Yosida approximation mappings, we designed an iterative algorithm and proved that the iterative sequences generated from the underline algorithm strongly converge to the solution of Yosida inclusion problem in q-uniformly smooth Banach spaces. In future, the results of this manuscript can be continued to the system of Yosida inclusions in the setting of Banach spaces.

Acknowledgment: The authors would like to thank the referees for their valuable comments and suggestions, which improved the original version of the manuscript.

References

[1] H. Attouch, Variational convergence for functions and operators. Applied Mathematics Series. Pitman, London, 1984.
[2] R. Ahmad, M. Ishtyak, M. Rahaman, and I. Ahmad, Graph convergence and generalized Yosida approximation operator with an application, Math. Sci. 11 (2017) 155–163.
[3] I. Ahmad, V.N. Mishra, R. Ahmad, M. Rahaman, An iterative algorithm for a system of generalized implicit variational inclusions, Springer Plus, (2016) 5:1283. DOI:10.1186/s40064-016-2916-8.
[4] J. Balooee, S. Chang and C. Wen, Generalized nearly asymptotically nonexpansive mappings and a system of generalized variational-like inclusions: iterative method and approximation of common solutions, Ann. Funct. Anal. 13 (54) (2022).
[5] M.I. Bhat, B. Zahoor, Existence of solution and iterative approximation of a system of generalized variational-like inclusion problems in semi-inner product spaces, Filomat, 40 (2) (2017) 240–243.
[6] H.W. Cao, Yosida approximation equations technique for system of generalized set-valued variational inclusions, J. Inequal. Appl. 2013 (2013) 455.
[7] M. Dilshad, A.F. Aljohani, M. Akram, A.A. Khidir Yosida, Approximation iterative methods for split monotone variational inclusion problems, J. Funct. Spaces, (2022), Article ID 3665713, 10 pages.
[8] Y.-P. Fang and N.-J. Huang, H-monotone operator and resolvent operator technique for variational inclusions, Appl. Math. Comput. 145(2-3) (2003) 795–803.
[9] S. Gupta, F.A. Khan, Set-valued variational inclusion governed by generalized $\alpha_i\beta_j$-$H^p(.,.,...)$-accretive mapping, Axioms, 11 (10) (2022) Article ID 539. https://doi.org/10.3390/axioms11100539
[10] S. Gupta, S. Husain and V.N. Mishra, Variational inclusion governed by $\alpha\beta$-$H((.,.),(.,.))$-mixed accretive mapping, Filomat. 31 (20) (2017) 6529–6542.
[11] S. Husain, S. Gupta, $H((.,.),(.,.))$-Mixed cocoercive operators with an application for solving variational inclusions in Hilbert spaces, J. Funct. Spaces, 2013 (2013), Article ID 378364, 13 pages.

[12] S. Husain, S. Gupta, V.N. Mishra, Generalized $H(.,.,.)$-η-cocoercive operators and generalized set-valued variational-like inclusions, J. Math. 2013 (2013), Article ID 738491, 10 pages.

[13] S. Gupta, M. Singh, Variational-like inclusion involving infinite family of set-valued mapping governed by resolvent equations, J. Math. Comput. Sci. 1 (10) (2021) 874–892.

[14] S. Gupta, M. Singh, Generalized monotone mapping and resolvent equation techniques with an application, J. Math. Comput. Sci. 2 (11) (2021) 1767–1783.

[15] N.-J. Huang and Y.-P. Fang, Generalized m-accretive mappings in Banach spaces, Journal of Sichuan University 38(4) (2001) 591–592.

[16] S. Husain, S. Gupta and V.N. Mishra, Graph convergence for the $H(.,.)$-mixed mapping with an application for solving the system of generalized variational inclusions, Fixed Point Theory Appl. 2013 ID304 (2013).

[17] J. Iqbal, V.N. Mishra, W.A. Mir, A.H. Dar, M. Ishtyak, L. Rathour, Generalized resolvent operator involving $\mathcal{G}(.,.)$-co-monotone mapping for solving generalized variational inclusion problem, Georgian Math. J. 29(4), (2022), 533-542.

[18] K.R. Kazmi, F.A. Khan and M. Shahzad, A system of generalized variational inclusions involving generalized $H(.,.)$-accretive mapping in real q-uniformly smooth Banach spaces, Appl. Math. Comput. 217(2) (2011) 9679–9688.

[19] R. Ikehata, N. Okozawa, Yosida approximation and Nonlinear Hybrid Equation. Nonlinear Anal. 15(5) (1990) 479–495.

[20] T. Iwamiya, H. Okochi, Monotonicity, resolvents and Yosida approximations of operators on Hilbert manifolds. Nonlinear Anal. 54(2) (2003) 205–214.

[21] H.Y. Lan, Generalized Yosida approximations based on relatively A-maximal m-relaxed monotonicity frameworks. Abstr. Appl. Anal. 2013 ID 157190 (2013).

[22] H.-Y. Lan, Y. J. Cho, and R. U. Verma, Nonlinear relaxed cocoercive variational inclusions involving $(A,\ \eta)$-accretive mappings in Banach spaces, Comput. Math. Appl. 51(9-10) (2006) 1529–1538.

[23] X. Li, N.-J. Huang, Graph convergence for the $H(.,.)$-accretive operator in Banach spaces with an application, Appl. Math. Comput. 217 (2011) 9053–9061.

[24] L.S. Liu, Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive mappings in Banach spaces, J. Math. Anal. Appl. 194 (1995) 114—125.

[25] L.-I. Pişcoran, V.N. Mishra, The variational problem in Lagrange spaces endowed with a special type of (α, β)-metrics, Filomat, 32(2), (2018), 643–652.

[26] S. Z. Nazemi, A new class of monotone mappings and a new class of variational inclusions in Banach spaces, Journal of Optimi. Theory and Appl. 155(3) (2012) 785–795.

[27] J.-P. Penot, R. Ratsimahalo, On the Yosida approximation of operators. Proc. R. Soc. Edinb. Math. 131A (2001) 945–966.

[28] T. Ram and M. Iqbal, $H(.,.,.,.)$-ϕ-η-cocoercive operator with an application to aariational inclusions, Int. J. Nonlinear Anal. Appl. 13 (1) (2022) 1311–1327.

[29] H.A. Rizvi, R. Ahmad, V.N. Mishra, Common solution for a system of generalized unrelated variational Inequalities, Maejo Int. J. Sci. Technol. (2016), 10(03), 354-362.

[30] S. Shafi, V.N. Mishra, $(A(...), \eta)$-Monotone mappings and a system of generalized set-valued variational-like inclusions problem, Jnanabha, 51(1) (2021),245–253.

[31] J. Sun, L. Zhang, X. Xiao, An algorithm based on resolvent operators for solving variational inequalities in Hilbert spaces, Nonlinear Anal. (TMA) 69(10) (2008) 3344-3357.

[32] R.U. Verma, General class of implicit variational inclusions and graph convergence on A-maximal relaxed monotonicity, J. Optim. Theory Appl. vol. 129 (3) (2012) 457–467.

[33] H.K. Xu, Inequalities in Banach spaces with applications, Nonlinear Anal. 16(12) (1991) 1127-1138.

[34] Y.-Z. Zou and N.-J. Huang, $H(.,.)$-accretive operator with an application for solving variational inclusions in Banach spaces, Appl. Math. Comput. 204(2) (2008) 809-816.

[35] Y.-Z. Zou and N.-J. Huang, A new system of variational inclusions involving $H(.,.)$-accretive operator in Banach spaces, Appl. Math. Comput. 212(2) (2009) 135-144.