Filomat 37:15 (2023), 4943–4957
https://doi.org/10.2298/FIL2315943N

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

New results on Hermite–Hadamard type inequalities via Caputo-Fabrizio fractional integral for s-convex function

Jamshed Nasir[a], Shahid Qaisar[b], Ather Qayyum[c], Hüseyin Budak[d]

[a] *Virtual University of Pakistan, Department of Mathematics, 54000, Pakistan*
[b] *Department of Mathematics, COMSATS University Islamabad, Sahiwal Campus, Sahiwal 57000, Pakistan*
[c] *Department of Mathematics, Institute of Southern Punjab Multan-Pakistan*
[d] *Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, 81620, Türkiye*

Abstract. The purpose of this article is to construct on Hermite–Hadamard type inequalities via Caputo-Fabrizio fractional integral for s-convex function. The results are applied to fractional variations of Hermite–Hadamard type inequalities for differentiable mapping φ with s-convex absolute value derivatives. The findings also provide a new lemma for φ' and new limits via Caputo-Fabrizio fractional operator by using the well-known Hölder's integral inequalities. Moreover some new bounds for applications of matrix and special means of different positive real numbers are also discussed.

1. Introduction and Preliminaries

Convexity is well known to play an important and vital role in many areas, including economics, finance, optimization, game theory and different sciences. This concept has been extended and generalized in several directions due to its diverse applications. For more than a century, this theory has been the focus and motivation of outstanding mathematical research. Convex analysis theory provides powerful principles and techniques for studying a wide range of problems in both pure and applied mathematics. Numerous mathematicians and applied scientists are constantly attempting to apply and make available novel ideas for the enjoyment and beautification of convexity theory. Because of their importance in traditional calculus, fractional calculus, quantum calculus, interval-valued, stochastic, time-scale calculus, fractal sets, and other fields, inequalities have an intriguing mathematical model.

Definition 1.1. *[1] Consider an extended real valued function $\varphi : \mathcal{I} \to \mathfrak{R}$, where $\mathcal{I} \subset \mathfrak{R}^n$ is any convex set, then the function φ is convex on $\mathcal{I}$, if*

$$\varphi(\chi\eta_1 + (1-\chi)\eta_2) \leq \chi\varphi(\eta_1) + (1-\chi)\varphi(\eta_2) \tag{1}$$

holds for all $\eta_1, \eta_2 \in \mathcal{I}$ and $\chi \in (0, 1)$.

2020 *Mathematics Subject Classification*. Primary 26D10; Secondary 26D15.
Keywords. Convex function; s-Convex function; Caputo-Fabrizio operator; Hermite-Hadamard inequality; Hölder's inequality; Power mean inequality; Young's inequality.
Received: 08 September 2022; Accepted: 23 December 2022
Communicated by Dragan S. Djordjević
Email addresses: jamshed@vu.edu.pk (Jamshed Nasir), shahidqaisar90@cuisahiwal.edu.pk (Shahid Qaisar), atherqayyum@isp.edu.pk (Ather Qayyum), hsyn.budak@gmail.com (Hüseyin Budak)

The Hermite–Hadamard(H–H) inequality assert that, if a mapping $\varphi : \mathcal{I} \subseteq \Re \to \Re$ is convex function on an interval $\mathcal{I}$ of a real numbers and $\eta_1, \eta_2 \in \mathcal{I}$ and $\eta_2 > \eta_1$, then

$$\varphi\Big(\frac{\eta_1+\eta_2}{2}\Big) \leq \frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\varphi(\chi)d\chi \leq \frac{\varphi(\eta_1)+\varphi(\eta_2)}{2} \tag{2}$$

interested readers can refer to [2] and [3].

Both inequalities hold in the reversed direction if φ is concave. The inequality (2) is known in the literature as the Hermite–Hadamard's inequality.

We note that the Hermite–Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. The classical Hermite–Hadamard's inequality provides estimates of the mean value of a continuous convex function $\varphi : [\eta_1, \eta_2] \to \Re$.

The following concept was introduced by Orlicz in [4]:

Definition 1.2. *A mapping $\varphi : [0, +\infty) \to \Re$ is said to be s-convex in the first sense.*

$$\varphi(\alpha x + \beta y) \leq \alpha^s \varphi(x) + \beta^s \varphi(y)$$

holds for all $x, y \in [0, +\infty)$ and $\alpha, \beta \geq 0$ with $\alpha^s + \beta^s = 1$, for some fixed $s \in (0, 1]$. The class of s-convex mappings in the sense is usually denoted this class of real functions by $\mathcal{K}_s^1$.

In [5], Hudzik and Maligranda consider the following class of function:

Definition 1.3. *A mapping $\varphi : [0, +\infty) \to \Re$ is said to be s-convex in the first sense.*

$$\varphi(\alpha x + \beta y) \leq \alpha^s \varphi(x) + \beta^s \varphi(y)$$

holds for all $x, y \in [0, +\infty)$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$, for some fixed $s \in (0, 1]$. The class of s-convex mappings in the sense is usually denoted this class of real functions by $\mathcal{K}_s^2$.

Definition 1.4. *A mapping $\varphi : \Re^+ \to \Re$, where $\Re^+ = [0, +\infty)$, is called to be s-convex in the second sense for a real number $s \in (0, 1]$ or φ belongs to the class of $\mathcal{K}_s^2$, if*

$$\varphi(\chi\eta_1 + (1-\chi)\eta_2) \leq \chi^s \varphi(\eta_1) + (1-\chi)^s \varphi(\eta_2) \tag{3}$$

holds, $\forall\ \eta_1, \eta_2 \in [0, +\infty)$ and $\chi \in [0, 1]$.

Dragomir and Fatzpatrick in [2], proved the following Hadamard's inequality which holds for the s-convex in the second sense as:

Theorem 1.5. *Suppose a mapping $\varphi : [0, +\infty) \to [0, +\infty)$ is an s-convex in the second sense, where $s \in (0, 1)$ and let $\eta_1, \eta_2 \in [0, +\infty)$, $\eta_1 < \eta_2$. If $\varphi \in L[\eta_1, \eta_2]$, then the following inequalities hold*

$$2^{s-1}\varphi\left(\frac{\eta_1+\eta_2}{2}\right) \leq \frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\varphi(x)\ dx \leq \frac{\varphi(\eta_1)+\varphi(\eta_2)}{s+1}, \tag{4}$$

the constant $k = \frac{1}{s+1}$ is the best possible value in the second sense in (4). The inequalities (4) are sharp. (see [6],[7], [8], [9]).

Here $\Gamma(.)$, $\beta(.,.)$ are the classical Gamma, Beta functions as described by

$$\Gamma(\eta) = \int_0^1 e^{-\chi}\chi^{\eta-1}d\chi$$

and

$$\beta(\eta_*, \eta^*) = \int_0^1 \chi^{\eta_*-1}(1-\chi)^{\eta^*-1}\,d\chi.$$

In recent years, several scholars have been interested in the definition of fractional derivative. Nonlocal fractional derivatives are classified into two types: those with singular kernels, such as the Riemann-Liouville and Caputo derivatives, and those with nonsingular kernels, such as the Caputo-Fabrizio and Atangana-Baleanu derivatives.

However, fractional derivative operators with non-singular kernels are particularly successful in solving non-locality in real world issues in the desired manner. We'll return to the Caputo-Fabrizio integral operator later. We would like to refer the reader to (see [10]-[15]) and references therein for more information.

New studies on many modeling and real-world problems have been conducted with the assistance of the Caputo-Fabrizio operator. This is because the Caputo-Fabrizio definition is very effective in better describing heterogeneity and systems with different scales with memory effects. The Caputo-Fabrizio definition's main basic feature can be explained (see [16], [17]).

Definition 1.6. *[18] Let $\varphi \in \mathcal{H}^1(\mu,\tau)$(where $\mathcal{H}^1$ is class of first order differentiable function), $\tau > \mu$, $\kappa \in (0,1)$ then, the definition of the new Caputo fractional derivative is:*

$$^{CF}D^{\kappa}\varphi(\chi) = \frac{\mathcal{M}(\kappa)}{1-\kappa}\int_{\mu}^{\chi}\varphi'(s)exp\left[-\frac{\kappa}{(1-\kappa)}(\tau-s)\right]ds, \tag{5}$$

where $\mathcal{M}(\kappa)$ is normalization function.

Moreover, the corresponding Caputo-Fabrizio fractional integral operator is given as:

Definition 1.7. *[19] Let $\varphi \in \mathcal{H}^1(\mu,\tau)$, $\tau > \mu$, $\kappa \in [0,1]$.*

$$\left({}^{CF}_{\mu}I^{\kappa}\varphi\right)(\chi) = \frac{1-\kappa}{\mathcal{M}(\kappa)}\varphi(\chi) + \frac{\kappa}{M(\kappa)}\int_{\mu}^{\chi}\varphi(y)dy$$

and

$$\left({}^{CF}\,\mathcal{I}^{\kappa}_{\tau}\varphi\right)(\chi) = \frac{1-\kappa}{\mathcal{M}(\kappa)}\varphi(\chi) + \frac{\kappa}{\mathcal{M}(\kappa)}\int_{\chi}^{\tau}\varphi(y)dy,$$

where $\mathcal{M}: [0,1] \to (0,\infty)$ is normalization function that satisfying $\mathcal{M}(0) = \mathcal{M}(1) = 1$. (see [17], [20])

İ. İşcan found the following inequality for integrals in [21], which outperforms the classical Hölder inequality.

The integral of Hölder İ. İşcan integral inequality is the following theorem .

Theorem 1.8 (Hölder İ. İşcan integral inequality). *Suppose $p > 1$ and $p^{-1} = 1-q^{-1}$. If Φ and Ψ are real functions defined on $[\eta_1,\eta_2]$ and if $|\Phi|^p$, and $|\Psi|^q$ are integrable on $[\eta_1,\eta_2]$, then*

$$\left(\int_{\eta_1}^{\eta_2}|\Phi(\chi)|^p\,d\chi\right)^{\frac{1}{p}}\left(\int_{\eta_1}^{\eta_2}|\Psi(\chi)|^q\,d\chi\right)^{\frac{1}{q}}$$

$$\geq \frac{1}{\eta_2-\eta_1}\left\{\left(\int_{\eta_1}^{\eta_2}(\eta_2-\chi)|\Phi(\chi)|^p d\chi\right)^{\frac{1}{p}}\left(\int_{\eta_1}^{\eta_2}(\eta_2-\chi)\left|\psi(\chi)\right|^q d\chi\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\eta_1}^{\eta_2}(\chi-\eta_1)|\Phi(\chi)|^p d\chi\right)^{\frac{1}{p}}\left(\int_{\eta_1}^{\eta_2}(\chi-\eta_1)|\Psi(\chi)|^q d\chi\right)^{\frac{1}{q}}\right\}$$
$$\geq \int_{\eta_1}^{\eta_2}|\Phi(\chi)\Psi(\chi)|\,d\chi. \tag{6}$$

The following is Improved power–mean integral inequality:

Theorem 1.9. *Suppose $q \geq 1$. If Φ and Ψ are real functions defined on $[\eta_1,\eta_2]$ and if $|\Phi|$, and $|\Phi||\Psi|^q$ are integrable on $[\eta_1,\eta_2]$, then*

$$\left(\int_{\eta_1}^{\eta_2}|\Phi(\chi)|\,d\chi\right)^{1-\frac{1}{q}}\left(\int_{\eta_1}^{\eta_2}|\Phi(\chi)|\,|\Psi(\chi)|^q d\chi\right)^{\frac{1}{q}}$$
$$\geq \frac{1}{\eta_2-\eta_1}\left\{\left(\int_{\eta_1}^{\eta_2}(\eta_2-\chi)|\Phi(\chi)|\,d\chi\right)^{1-\frac{1}{q}}\left(\int_{\eta_1}^{\eta_2}(\eta_2-\chi)|\Phi(\chi)|\,|\Psi(\chi)|^q d\chi\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\eta_1}^{\eta_2}(\chi-\eta_1)|\Phi(\chi)|\,d\chi\right)^{1-\frac{1}{q}}\left(\int_{\eta_1}^{\eta_2}(\chi-\eta_1)|\Phi(\chi)|\,|\Psi(\chi)|^q d\chi\right)^{\frac{1}{q}}\right\}$$
$$\geq \int_{\eta_1}^{\eta_2}|\Phi(\chi)\Psi(\chi)|\,d\chi. \tag{7}$$

2. Main Results

In this section, we give Hermite–Hadamard's type inequalities for Caputo–Fabrizio fractional integral operator are obtained for a differentiable functions on (η_1,η_2). For this, we give a new Caputo–Fabrizio(CF) fractional integral identity that will serve as an auxiliary to produce subsequent results for improvements.

2.1. Hermite–Hadamard's type inequality via the Caputo–Fabrizio(CF) fractional operator

Theorem 2.1. *Suppose a positive mapping $\varphi : \mathcal{I} = [\eta_1,\eta_2] \to \Re$, $\varphi \in \mathcal{L}[\eta_1,\eta_2]$ If φ is s-convex mapping in the second sense on $[\eta_1,\eta_2]$, $s \in (0,1]$, then the following double inequality holds*

$$2^{s-1}\varphi\left(\frac{\eta_1+\eta_2}{2}\right) \leq \frac{M(\alpha)}{\alpha(\eta_2-\eta_1)}\left\{\left({}^{CF}_{\eta_1}I^{\alpha}\varphi\right)(k)+\left({}^{CF}I^{\alpha}_{\eta_2}\varphi\right)(k)-\frac{2(1-\alpha)}{\mathcal{M}(\alpha)}\varphi(k)\right\}$$
$$\leq \frac{\varphi(\eta_1)+\varphi(\eta_2)}{s+1}, \tag{8}$$

where $k \in [\eta_1,\eta_2]$ and $\mathcal{M}(\alpha) > 0$ is a normalization function.

Proof. Let $[\eta_1,\eta_2]$, with $\eta_1 < \eta_2$. Given that the function φ is s-convex. We get from (4)

$$2^{s}\varphi\left(\frac{\eta_1+\eta_2}{2}\right) \leq \frac{2}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\varphi(z)\,dz \leq 2.\frac{\varphi(\eta_1)+\varphi(\eta_2)}{s+1}, \tag{9}$$

$$2^{s}\varphi\left(\frac{\eta_1+\eta_2}{2}\right) \leq \frac{2}{\eta_2-\eta_1}\left\{\int_{\eta_1}^{k}\varphi(z)\,dz+\int_{k}^{\eta_2}\varphi(z)\,dz\right\}.$$

By multiplying on both sides above with $\frac{\alpha(\eta_2-\eta_1)}{2\mathcal{M}(\alpha)}$ and adding $\frac{2(1-\alpha)}{\mathcal{M}(\alpha)}\varphi(k)$, we have

$$2^{s-1}\varphi\left(\frac{\eta_1+\eta_2}{2}\right).\frac{\alpha(\eta_2-\eta_1)}{\mathcal{M}(\alpha)}+\frac{2(1-\alpha)}{\mathcal{M}(\alpha)}\varphi(k) \leq \left\{\left({}^{CF}_{\eta_1}I^{\alpha}\varphi\right)(k)+\left({}^{CF}I^{\alpha}_{\eta_2}\varphi\right)(k)\right\}. \tag{10}$$

After suitable rearrangement of (10), we arrive at left inequality of (8)

Now we will prove the right side (8). The Hadamard inequality for s-convex function

$$\frac{2}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\varphi(z)\,dz \leq 2.\frac{\varphi(\eta_1)+\varphi(\eta_2)}{s+1}$$
$$\frac{2}{\eta_2-\eta_1}\left\{\int_{\eta_1}^{k}\varphi(z)\,dz+\int_{k}^{\eta_2}\varphi(z)\,dz\right\} \leq 2.\frac{\varphi(\eta_1)+\varphi(\eta_2)}{s+1}. \tag{11}$$

By multiplying on both sides above with $\frac{\alpha(\eta_2-\eta_1)}{2\mathcal{M}(\alpha)}$ and adding $\frac{2(1-\alpha)}{\mathcal{M}(\alpha)}\varphi(k)$, we have

$$\left\{\left({}^{CF}_{\eta_1}I^{\alpha}\varphi\right)(k)+\left({}^{CF}I^{\alpha}_{\eta_2}\varphi\right)(k)\right\} \leq \left(\varphi(\eta_1)+\varphi(\eta_2)\right).\frac{\alpha(\eta_2-\eta_1)}{(s+1)M(\alpha)}+\frac{2(1-\alpha)}{M(\alpha)}\varphi(k). \tag{12}$$

After rearrangement of (12), we get the required right side the inequality of (8), which completes the proof. □

Remark 2.2. *If we choose the $s=1$ in Theorem 2.1, then the inequality* (8) *becomes the inequality (1) of Theorem 2 in [22].*

Lemma 2.3. *Suppose a mapping $\varphi:\mathcal{I}=[\eta_1,\eta_2]\to\mathfrak{R}$ is differentiable on (η_1,η_2) with $\eta_2>\eta_1$. If $\varphi'\in\mathcal{L}[\eta_1,\eta_2]$, then $\alpha\in[0,1]$, the following Caputo–Fabrizio(CF) fractional identity holds*

$$\begin{aligned}
&\frac{\varphi(\eta_1)+\varphi(\eta_2)}{2}-\frac{\mathcal{M}(\alpha)}{\alpha(\eta_2-\eta_1)}\\
&\times\left\{\left({}^{CF}_{\eta_1}I^{\alpha}\varphi\right)(k)+\left({}^{CF}I^{\alpha}_{\left(\frac{\eta_1+\eta_2}{2}\right)}\varphi\right)(k)+\left({}^{CF}_{\left(\frac{\eta_1+\eta_2}{2}\right)}I^{\alpha}\varphi\right)(k)+\left({}^{CF}I^{\alpha}_{\eta_2}\varphi\right)(k)\right\}\\
&=\frac{\eta_2-\eta_1}{4}\left\{\int_0^1(-\chi)\,\varphi'\left(\frac{1+\chi}{2}\eta_1+\frac{1-\chi}{2}\eta_2\right)d\chi+\int_0^1\chi\,\varphi'\left(\frac{1-\chi}{2}\eta_1+\frac{1+\chi}{2}\eta_2\right)d\chi\right\}\\
&-\frac{4(1-\alpha)}{\alpha(\eta_2-\eta_1)}\varphi(k),
\end{aligned} \tag{13}$$

where $k\in[\eta_1,\eta_2]$ and $\mathcal{M}(\alpha)>0$ is a normalization function.

Proof. It can be write that

$$\begin{aligned}
&\frac{\eta_2-\eta_1}{4}\left\{\int_0^1(-\chi)\,\varphi'\left(\frac{1+\chi}{2}\eta_1+\frac{1-\chi}{2}\eta_2\right)d\chi+\int_0^1\chi\,\varphi'\left(\frac{1-\chi}{2}\eta_1+\frac{1+\chi}{2}\eta_2\right)d\chi\right\}\\
&-\frac{4(1-\alpha)}{\alpha(\eta_2-\eta_1)}\varphi(k).
\end{aligned} \tag{14}$$

Integrating by parts, by taking

$$\begin{aligned}
I_1&=\int_0^1(-\chi)\,\varphi'\left(\frac{1+\chi}{2}x+\frac{1-\chi}{2}y\right)d\chi\\
&=-\left\{\frac{\chi\,\varphi\left(\frac{1+\chi}{2}\eta_1+\frac{1-\chi}{2}\eta_2\right)}{\frac{\eta_1-\eta_2}{2}}\Bigg|_0^1-\int_0^1\frac{\varphi\left(\frac{1+\chi}{2}\eta_1+\frac{1-\chi}{2}\eta_2\right)}{\frac{\eta_1-\eta_2}{2}}.1\,d\chi\right\}\\
&=\frac{2}{\eta_2-\eta_1}\varphi(\eta_1)-\frac{4}{(\eta_2-\eta_1)^2}\left\{\int_{\eta_1}^{k}\varphi(z)\,dz+\int_{k}^{\frac{\eta_1+\eta_2}{2}}\varphi(z)\,dz\right\}.
\end{aligned}$$

By multiplying on both sides above with $\frac{\alpha(\eta_2-\eta_1)^2}{4\mathcal{M}(\alpha)}$ and subtracting $\frac{2(1-\alpha)}{\mathcal{M}(\alpha)}\varphi(k)$, we have

$$= \frac{\varphi(\eta_1)}{2} - \frac{\mathcal{M}(\alpha)}{\alpha(\eta_2-\eta_1)}\left\{\left({}^{CF}_{\eta_1}I^{\alpha}\varphi\right)(k) + \left({}^{CF}I^{\alpha}_{\left(\frac{\eta_1+\eta_2}{2}\right)}\varphi\right)(k)\right\}. \tag{15}$$

Similarly, we can write that

$$\begin{aligned} I_2 &= \int_0^1 \chi\,\varphi'\left(\frac{1-\chi}{2}\eta_1 + \frac{1+\chi}{2}\eta_2\right)d\chi \\ &= \frac{\varphi(\eta_2)}{2} - \frac{\mathcal{M}(\alpha)}{\alpha(\eta_2-\eta_1)}\left\{\left({}^{CF}_{\left(\frac{\eta_1+\eta_2}{2}\right)}I^{\alpha}\varphi\right)(k) + \left({}^{CF}I^{\alpha}_{\eta_2}\varphi\right)(k)\right\}. \end{aligned} \tag{16}$$

By using the values of I_1 and I_2 with equation (14), we can get the (13). Thus, the proof is completed. □

Theorem 2.4. *Suppose a mapping $\varphi : \mathcal{I} \subset [0,\infty] \to \Re$ is differentiable on (η_1,η_2) with $\eta_2 > \eta_1$. If $\varphi' \in \mathcal{L}[\eta_1,\eta_2]$, and $\left|\varphi'\right|$ is s-convex on $[\eta_1,\eta_2]$ for some fixed $s \in (0,1]$, then the following inequality holds*

$$\begin{aligned} &\Bigg|\frac{\varphi(\eta_1)+\varphi(\eta_2)}{2} - \frac{\mathcal{M}(\alpha)}{\alpha(\eta_2-\eta_1)} \\ &\times\left\{\left({}^{CF}_{\eta_1}I^{\alpha}\varphi\right)(k) + \left({}^{CF}I^{\alpha}_{\left(\frac{\eta_1+\eta_2}{2}\right)}\varphi\right)(k) + \left({}^{CF}_{\left(\frac{\eta_1+\eta_2}{2}\right)}I^{\alpha}\varphi\right)(k) + \left({}^{CF}I^{\alpha}_{\eta_2}\varphi\right)(k)\right\} \\ &+ \frac{4(1-\alpha)}{\alpha(\eta_2-\eta_1)}\varphi(k)\Bigg| \\ &\le \frac{\eta_2-\eta_1}{4}\left\{\left(\frac{2se^{\ln(2)s}+1}{(s+1)(s+2)2^s} + \frac{\beta(2,s+1)}{2^s}\right)\left(\left|\varphi'(\eta_1)\right| + \left|\varphi'(\eta_2)\right|\right)\right\}, \end{aligned} \tag{17}$$

where $k \in [\eta_1,\eta_2]$ and $\mathcal{M}(\alpha) > 0$ is a normalization function.

Proof. By using Lemma 2.3, the property of the absolute value and the s-convexity of $\left|\varphi'\right|$, we have

$$\begin{aligned} &\Bigg|\frac{\varphi(\eta_1)+\varphi(\eta_2)}{2} - \frac{\mathcal{M}(\alpha)}{\alpha(\eta_2-\eta_1)} \\ &\times\left\{\left({}^{CF}_{\eta_1}I^{\alpha}\varphi\right)(k) + \left({}^{CF}I^{\alpha}_{\left(\frac{\eta_1+\eta_2}{2}\right)}\varphi\right)(k) + \left({}^{CF}_{\left(\frac{\eta_1+\eta_2}{2}\right)}I^{\alpha}\varphi\right)(k) + \left({}^{CF}I^{\alpha}_{\eta_2}\varphi\right)(k)\right\} \\ &+ \frac{4(1-\alpha)}{\alpha(\eta_2-\eta_1)}\varphi(k)\Bigg| \end{aligned}$$

$$\begin{aligned} &\le \frac{\eta_2-\eta_1}{4}\left\{\int_0^1(-\chi)\left|\varphi'\left(\frac{1+\chi}{2}\eta_1 + \frac{1-\chi}{2}\eta_2\right)\right|d\chi + \int_0^1 \chi\left|\varphi'\left(\frac{1-\chi}{2}\eta_1 + \frac{1+\chi}{2}\eta_2\right)\right|d\chi\right\} \\ &\le \frac{\eta_2-\eta_1}{4}\Bigg[\int_0^1 \chi\left\{\left(\frac{1+\chi}{2}\right)^s\left|\varphi'(\eta_1)\right| + \left(\frac{1-\chi}{2}\right)^s\left|\varphi'(\eta_2)\right|\right\}d\chi \\ &+ \int_0^1 \chi\left\{\left(\frac{1-\chi}{2}\right)^s\left|\varphi'(\eta_1)\right| + \left(\frac{1+\chi}{2}\right)^s\left|\varphi'(\eta_2)\right|\right\}d\chi\Bigg] \end{aligned}$$

By using the calculus tools, we get

$$\Bigg|\frac{\varphi(\eta_1)+\varphi(\eta_2)}{2}-\frac{\mathcal{M}(\alpha)}{\alpha(\eta_2-\eta_1)}$$
$$\times\left\{\left({}^{CF}_{\eta_1}I^{\alpha}\varphi\right)(k)+\left({}^{CF}I^{\alpha}_{\left(\frac{\eta_1+\eta_2}{2}\right)}\varphi\right)(k)+\left({}^{CF}_{\left(\frac{\eta_1+\eta_2}{2}\right)}I^{\alpha}\varphi\right)(k)+\left({}^{CF}I^{\alpha}_{\eta_2}\varphi\right)(k)\right\}$$
$$+\ \frac{4(1-\alpha)}{\alpha(\eta_2-\eta_1)}\varphi(k)\Bigg|$$
$$\leq\frac{\eta_2-\eta_1}{4}\left\{\left(\frac{2se^{\ln(2)s}+1}{(s+1)(s+2)2^s}+\frac{\beta(2,s+1)}{2^s}\right)\ \left(\left|\varphi'(\eta_1)\right|+\left|\varphi'(\eta_2)\right|\right)\right\}.$$

Thus, the proof is completed. □

Theorem 2.5. *Suppose a mapping $\varphi:\mathcal{I}\subset[0,\infty]\to\Re$ is differentiable on (η_1,η_2) with $\eta_2>\eta_1$. If $\varphi'\in\mathcal{L}[\eta_1,\eta_2]$, and $\left|\varphi'\right|^q$ is s-convex on $[\eta_1,\eta_2]$ for some fixed $s\in(0,1]$, $q>1$, then the following inequality holds*

$$\Bigg|\frac{\varphi(\eta_1)+\varphi(\eta_2)}{2}-\frac{\mathcal{M}(\alpha)}{\alpha(\eta_2-\eta_1)}$$
$$\times\left\{\left({}^{CF}_{\eta_1}I^{\alpha}\varphi\right)(k)+\left({}^{CF}I^{\alpha}_{\left(\frac{\eta_1+\eta_2}{2}\right)}\varphi\right)(k)+\left({}^{CF}_{\left(\frac{\eta_1+\eta_2}{2}\right)}I^{\alpha}\varphi\right)(k)+\left({}^{CF}I^{\alpha}_{\eta_2}\varphi\right)(k)\right\}$$
$$+\ \frac{4(1-\alpha)}{\alpha(\eta_2-\eta_1)}\varphi(k)\Bigg|$$
$$\leq\ \frac{\eta_2-\eta_1}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left\{\frac{2^{s+1}-1}{(s+1)2^s}\left|\varphi'(\eta_1)\right|^q+\frac{1}{(s+1)2^s}\left|\varphi'(\eta_2)\right|^q\right\}^{\frac{1}{q}}\right.$$
$$\left.+\left\{\frac{1}{(s+1)2^s}\left|\varphi'(\eta_1)\right|^q+\frac{2^{s+1}-1}{(s+1)2^s}\left|\varphi'(\eta_2)\right|^q\right\}^{\frac{1}{q}}\right],\tag{18}$$

where $k\in[\eta_1,\eta_2]$ and $\mathcal{M}(\alpha)>0$ is a normalization function, $p^{-1}=1-q^{-1}$.

Proof. By using Lemma 2.3, the well-known Hölder's integral inequality and the s-convexity of $\left|\varphi'\right|^q$, we have

$$\Bigg|\frac{\varphi(\eta_1)+\varphi(\eta_2)}{2}-\frac{\mathcal{M}(\alpha)}{\alpha(\eta_2-\eta_1)}$$
$$\times\left\{\left({}^{CF}_{\eta_1}I^{\alpha}\varphi\right)(k)+\left({}^{CF}I^{\alpha}_{\left(\frac{\eta_1+\eta_2}{2}\right)}\varphi\right)(k)+\left({}^{CF}_{\left(\frac{\eta_1+\eta_2}{2}\right)}I^{\alpha}\varphi\right)(k)+\left({}^{CF}I^{\alpha}_{\eta_2}\varphi\right)(k)\right\}$$
$$+\ \frac{4(1-\alpha)}{\alpha(\eta_2-\eta_1)}\varphi(k)\Bigg|$$

$$\leq\ \frac{\eta_2-\eta_1}{4}\left\{\int_0^1(-\chi)\ \left|\varphi'\left(\frac{1+\chi}{2}\eta_1+\frac{1-\chi}{2}\eta_2\right)\right|d\chi\right.$$
$$\left.+\int_0^1\chi\left|\varphi'\left(\frac{1-\chi}{2}\eta_1+\frac{1+\chi}{2}\eta_2\right)\right|d\chi\right\}$$
$$\leq\ \frac{\eta_2-\eta_1}{4}\left[\left(\int_0^1\chi^p d\chi\right)^{\frac{1}{p}}\left(\int_0^1\left|\varphi'\left(\frac{1+\chi}{2}\eta_1+\frac{1-\chi}{2}\eta_2\right)\right|^q d\chi\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\chi^p d\chi\right)^{\frac{1}{p}}\left(\int_0^1\left|\varphi'\left(\frac{1-\chi}{2}\eta_1+\frac{1+\chi}{2}\eta_2\right)\right|^q d\chi\right)^{\frac{1}{q}}\ \right]$$

$$\leq \frac{\eta_2-\eta_1}{4}\Big[\Big(\int_0^1 \chi^p d\chi\Big)^{\frac{1}{p}}\Big(\int_0^1\Big\{\Big(\frac{1+\chi}{2}\Big)^s\left|\varphi'(\eta_1)\right|^q+\Big(\frac{1-\chi}{2}\Big)^s\left|\varphi'(\eta_2)\right|^q\Big\}d\chi\Big)^{\frac{1}{q}}$$

$$+\Big(\int_0^1 \chi^p d\chi\Big)^{\frac{1}{p}}\Big(\int_0^1\Big\{\Big(\frac{1-\chi}{2}\Big)^s\left|\varphi'(\eta_1)\right|^q+\Big(\frac{1+\chi}{2}\Big)^s\left|\varphi'(\eta_2)\right|^q\Big\}d\chi\Big)^{\frac{1}{q}}\Big]$$

$$\leq \frac{\eta_2-\eta_1}{4}\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big[\Big\{\frac{2^{s+1}-1}{(s+1)2^s}\left|\varphi'(\eta_1)\right|^q+\frac{1}{(s+1)2^s}\left|\varphi'(\eta_2)\right|^q\Big\}^{\frac{1}{q}}$$

$$+\Big\{\frac{1}{(s+1)2^s}\left|\varphi'(\eta_1)\right|^q+\frac{2^{s+1}-1}{(s+1)2^s}\left|\varphi'(\eta_2)\right|^q\Big\}^{\frac{1}{q}}\Big].$$

Thus, the proof is completed. □

Theorem 2.6. *Suppose a mapping $\varphi : \mathcal{I} \subset [0,\infty] \to \Re$ is differentiable on (η_1, η_2) with $\eta_2 > \eta_1$. If $\varphi' \in \mathcal{L}[\eta_1, \eta_2]$, and $\left|\varphi'\right|^q$ is s-convex on $[\eta_1, \eta_2]$ for some fixed $s \in (0, 1]$, $q \geq 1$, then the following inequality holds*

$$\Big|\frac{\varphi(\eta_1)+\varphi(\eta_2)}{2}-\frac{\mathcal{M}(\alpha)}{\alpha(\eta_2-\eta_1)}$$

$$\times\Big\{\Big({}^{CF}_{\eta_1}I^{\alpha}\varphi\Big)(k)+\Big({}^{CF}I^{\alpha}_{\left(\frac{\eta_1+\eta_2}{2}\right)}\varphi\Big)(k)+\Big({}^{CF}_{\left(\frac{\eta_1+\eta_2}{2}\right)}I^{\alpha}\varphi\Big)(k)+\Big({}^{CF}I^{\alpha}_{\eta_2}\varphi\Big)(k)\Big\}$$

$$+\frac{4(1-\alpha)}{\alpha(\eta_2-\eta_1)}\varphi(k)\Big|$$

$$\leq \frac{\eta_2-\eta_1}{4}\Big(\frac{1}{2}\Big)^{1-\frac{1}{q}}\Big[\Big\{\frac{2se^{\ln(2)s}+1}{(s+1)(s+2)2^s}\left|\varphi'(\eta_1)\right|^q+\frac{1}{(s+1)(s+2)2^s}\left|\varphi'(\eta_2)\right|^q\Big\}^{\frac{1}{q}}$$

$$+\Big\{\frac{1}{(s+1)(s+2)2^s}\left|\varphi'(\eta_1)\right|^q+\frac{2se^{\ln(2)s}+1}{(s+1)(s+2)2^s}\left|\varphi'(\eta_2)\right|^q\Big\}^{\frac{1}{q}}\Big], \quad (19)$$

where $k \in [\eta_1, \eta_2]$ and $M(\alpha) > 0$ is a normalization function, $p^{-1} = 1 - q^{-1}$.

Proof. By using Lemma 2.3, the power-mean integral inequality and the *s*-convexity of $\left|\varphi'\right|^q$, we have

$$\Big|\frac{\varphi(\eta_1)+\varphi(\eta_2)}{2}-\frac{\mathcal{M}(\alpha)}{\alpha(\eta_2-\eta_1)}$$

$$\times\Big\{\Big({}^{CF}_{\eta_1}I^{\alpha}\varphi\Big)(k)+\Big({}^{CF}I^{\alpha}_{\left(\frac{\eta_1+\eta_2}{2}\right)}\varphi\Big)(k)+\Big({}^{CF}_{\left(\frac{\eta_1+\eta_2}{2}\right)}I^{\alpha}\varphi\Big)(k)+\Big({}^{CF}I^{\alpha}_{\eta_2}\varphi\Big)(k)\Big\}$$

$$+\frac{4(1-\alpha)}{\alpha(\eta_2-\eta_1)}\varphi(k)\Big|$$

$$\leq \frac{\eta_2-\eta_1}{4}\Big\{\int_0^1(-\chi)\left|\varphi'\Big(\frac{1+\chi}{2}\eta_1+\frac{1-\chi}{2}\eta_2\Big)\right|d\chi$$

$$+\int_0^1\chi\left|\varphi'\Big(\frac{1-\chi}{2}\eta_1+\frac{1+\chi}{2}\eta_2\Big)\right|d\chi\Big\}$$

$$\leq \frac{\eta_2-\eta_1}{4}\Big[\Big(\int_0^1\chi d\chi\Big)^{1-\frac{1}{q}}\Big(\int_0^1\chi\left|\varphi'\Big(\frac{1+\chi}{2}\eta_1+\frac{1-\chi}{2}\eta_2\Big)\right|^q d\chi\Big)^{\frac{1}{q}}$$

$$+\Big(\int_0^1\chi d\chi\Big)^{1-\frac{1}{q}}\Big(\int_0^1\chi\left|\varphi'\Big(\frac{1-\chi}{2}\eta_1+\frac{1+\chi}{2}\eta_2\Big)\right|^q d\chi\Big)^{\frac{1}{q}}\Big]$$

$$\leq \frac{\eta_2-\eta_1}{4}\Bigg[\left(\int_0^1 \chi d\chi\right)^{1-\frac{1}{q}}\left(\int_0^1 \chi\left\{\left(\frac{1+\chi}{2}\right)^s\left|\varphi'(\eta_1)\right|^q+\left(\frac{1-\chi}{2}\right)^s\left|\varphi'(\eta_2)\right|^q\right\}d\chi\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 \chi d\chi\right)^{1-\frac{1}{q}}\left(\int_0^1 \chi\left\{\left(\frac{1-\chi}{2}\right)^s\left|\varphi'(\eta_1)\right|^q+\left(\frac{1+\chi}{2}\right)^s\left|\varphi'(\eta_2)\right|^q\right\}d\chi\right)^{\frac{1}{q}}\Bigg]$$

$$\leq \frac{\eta_2-\eta_1}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\Bigg[\left\{\frac{2se^{\ln(2)s}+1}{(s+1)(s+2)2^s}\left|\varphi'(\eta_1)\right|^q+\frac{1}{(s+1)(s+2)2^s}\left|\varphi'(\eta_2)\right|^q\right\}^{\frac{1}{q}}$$

$$+\left\{\frac{1}{(s+1)(s+2)2^s}\left|\varphi'(\eta_1)\right|^q+\frac{2se^{\ln(2)s}+1}{(s+1)(s+2)2^s}\left|\varphi'(\eta_2)\right|^q\right\}^{\frac{1}{q}}\Bigg].$$

Thus, the proof is completed. □

Theorem 2.7. *Suppose a mapping $\varphi : \mathcal{I} \subset [0,\infty] \to \Re$ is differentiable on (η_1,η_2) with $\eta_2 > \eta_1$. If $\varphi' \in \mathcal{L}[\eta_1,\eta_2]$, and $\left|\varphi'\right|^q$ is s-convex on $[\eta_1,\eta_2]$ for some fixed $s \in (0,1]$, $q > 1$, then the following inequality holds*

$$\Bigg|\frac{\varphi(\eta_1)+\varphi(\eta_2)}{2}-\frac{\mathcal{M}(\alpha)}{\alpha(\eta_2-\eta_1)}$$

$$\times\left\{\left({}_{\eta_1}^{CF}I^{\alpha}\varphi\right)(k)+\left({}^{CF}I^{\alpha}_{\left(\frac{\eta_1+\eta_2}{2}\right)}\varphi\right)(k)+\left({}^{CF}_{\left(\frac{\eta_1+\eta_2}{2}\right)}I^{\alpha}\varphi\right)(k)+\left({}^{CF}I^{\alpha}_{\eta_2}\varphi\right)(k)\right\}$$

$$+\frac{4(1-\alpha)}{\alpha(\eta_2-\eta_1)}\varphi(k)\Bigg|$$

$$\leq \frac{\eta_2-\eta_1}{4}\Bigg[\Bigg\{\left(\frac{1}{(p+1)(p+2)}\right)^{\frac{1}{p}}\Bigg\{\Bigg\{-\frac{(s+2)2^{s+1}+2se^{\ln(2)s}+s+3}{(s+1)(s+2)2^s}\left|\varphi'(a)\right|^q$$

$$+\frac{1}{(s+2)2^s}\left|\varphi'(b)\right|^q\Bigg\}^{\frac{1}{q}}$$

$$+\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left\{\frac{2se^{\ln(2)s}+1}{(s+1)(s+2)2^s}\left|\varphi'(\eta_1)\right|^q+\frac{1}{(s+1)(s+2)2^s}\left|\varphi'(\eta_2)\right|^q\right\}^{\frac{1}{q}}\Bigg\}$$

$$+\Bigg\{\left(\frac{1}{(p+1)(p+2)}\right)^{\frac{1}{p}}$$

$$\times\left\{\frac{1}{(s+2)2^s}\left|\varphi'(\eta_1)\right|^q-\frac{(s+2)2^{s+1}+2se^{\ln(2)s}+s+3}{(s+1)(s+2)2^s}\left|\varphi'(\eta_2)\right|^q\right\}^{\frac{1}{q}}$$

$$+\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left\{\frac{1}{(s+1)(s+2)2^s}\left|\varphi'(\eta_1)\right|^q+\frac{2se^{\ln(2)s}+1}{(s+1)(s+2)2^s}\left|\varphi'(\eta_2)\right|^q\right\}^{\frac{1}{q}}\Bigg\}\Bigg], \quad (20)$$

where $k \in [\eta_1,\eta_2]$ and $\mathcal{M}(\alpha) > 0$ is a normalization function, $p^{-1} = 1 - q^{-1}$.

Proof. By using Lemma 2.3, the well-known Hölder İşcan integral inequality and the s-convexity of $\left|\varphi'\right|^q$, we have

$$\Bigg|\frac{\varphi(\eta_1)+\varphi(\eta_2)}{2}-\frac{\mathcal{M}(\alpha)}{\alpha(\eta_2-\eta_1)}$$

$$\times\left\{\left({}_{\eta_1}^{CF}I^{\alpha}\varphi\right)(k)+\left({}^{CF}I^{\alpha}_{\left(\frac{\eta_1+\eta_2}{2}\right)}\varphi\right)(k)+\left({}^{CF}_{\left(\frac{\eta_1+\eta_2}{2}\right)}I^{\alpha}\varphi\right)(k)+\left({}^{CF}I^{\alpha}_{\eta_2}\varphi\right)(k)\right\}$$

$$+\frac{4(1-\alpha)}{\alpha(\eta_2-\eta_1)}\varphi(k)\Bigg|$$

$$\leq \frac{\eta_2-\eta_1}{4}\left\{\int_0^1(-\chi)\left|\varphi'\left(\frac{1+\chi}{2}\eta_1+\frac{1-\chi}{2}\eta_2\right)\right|d\chi\right.$$

$$+\int_0^1 \chi\left|\varphi'\left(\frac{1-\chi}{2}\eta_1+\frac{1+\chi}{2}\eta_2\right)\right|d\chi\Big\}$$

$$\begin{aligned}
&\leq \frac{\eta_2-\eta_1}{4}\Bigg[\Bigg\{\left(\int_0^1(1-\chi)\chi^p d\chi\right)^{\frac{1}{p}}\left(\int_0^1(1-\chi)\left|\varphi'\left(\frac{1+\chi}{2}\eta_1+\frac{1-\chi}{2}\eta_2\right)\right|^q d\chi\right)^{\frac{1}{q}}\\
&+\left(\int_0^1 \chi^{1+p}d\chi\right)^{\frac{1}{p}}\left(\int_0^1 \chi\left|\varphi'\left(\frac{1+\chi}{2}\eta_1+\frac{1-\chi}{2}\eta_2\right)\right|^q d\chi\right)^{\frac{1}{q}}\Bigg\}\\
&+\Bigg\{\left(\int_0^1(1-\chi)\chi^p d\chi\right)^{\frac{1}{p}}\left(\int_0^1(1-\chi)\left|\varphi'\left(\frac{1-\chi}{2}\eta_1+\frac{1+\chi}{2}\eta_2\right)\right|^q d\chi\right)^{\frac{1}{q}}\\
&+\left(\int_0^1 \chi^{1+p}d\chi\right)^{\frac{1}{p}}\left(\int_0^1 \chi\left|\varphi'\left(\frac{1-\chi}{2}\eta_1+\frac{1+\chi}{2}\eta_2\right)\right|^q d\chi\right)^{\frac{1}{q}}\Bigg\}\Bigg]\\
&\leq \frac{\eta_2-\eta_1}{4}\Bigg[\Bigg\{\left(\int_0^1(1-\chi)\chi^p d\chi\right)^{\frac{1}{p}}\\
&\times\left(\int_0^1(1-\chi)\left\{\left(\frac{1+\chi}{2}\right)^s|\varphi'(\eta_1)|^q+\left(\frac{1-\chi}{2}\right)^s|\varphi'(\eta_2)|^q\right\}d\chi\right)^{\frac{1}{q}}\\
&+\left(\int_0^1 \chi^{1+p}d\chi\right)^{\frac{1}{p}}\left(\int_0^1 \chi\left\{\left(\frac{1+\chi}{2}\right)^s|\varphi'(\eta_1)|^q+\left(\frac{1-\chi}{2}\right)^s|\varphi'(\eta_2)|^q\right\}d\chi\right)^{\frac{1}{q}}\Bigg\}\\
&+\Bigg\{\left(\int_0^1(1-\chi)\chi^p d\chi\right)^{\frac{1}{p}}\\
&\times\left(\int_0^1(1-\chi)\left\{\left(\frac{1-\chi}{2}\right)^s|\varphi'(\eta_1)|^q+\left(\frac{1+\chi}{2}\right)^s|\varphi'(\eta_2)|^q\right\}d\chi\right)^{\frac{1}{q}}\\
&+\left(\int_0^1 \chi^{1+p}d\chi\right)^{\frac{1}{p}}\left(\int_0^1 \chi\left\{\left(\frac{1-\chi}{2}\right)^s|\varphi'(\eta_1)|^q+\left(\frac{1+\chi}{2}\right)^s|\varphi'(\eta_2)|^q\right\}d\chi\right)^{\frac{1}{q}}\Bigg\}\Bigg].
\end{aligned}$$

By using the calculus tools, we can get (20). Thus, the proof is completed. □

Theorem 2.8. *Suppose a mapping $\varphi : \mathcal{I} \subset [0,\infty] \to \mathfrak{R}$ is differentiable on (η_1,η_2) with $\eta_2 > \eta_1$. If $\varphi' \in \mathcal{L}[\eta_1,\eta_2]$, and $|\varphi'|^q$ is s-convex on $[\eta_1,\eta_2]$ for some fixed $s \in (0,1]$, $q \geq 1$, then the following inequality holds*

$$\begin{aligned}
&\Bigg|\frac{\varphi(\eta_1)+\varphi(\eta_2)}{2}-\frac{\mathcal{M}(\alpha)}{\alpha(\eta_2-\eta_1)}\\
&\times\left\{\left({}^{CF}_{\eta_1}I^{\alpha}\varphi\right)(k)+\left({}^{CF}I^{\alpha}_{\left(\frac{\eta_1+\eta_2}{2}\right)}\varphi\right)(k)+\left({}^{CF}_{\left(\frac{\eta_1+\eta_2}{2}\right)}I^{\alpha}\varphi\right)(k)+\left({}^{CF}I^{\alpha}_{\eta_2}\varphi\right)(k)\right\}\\
&+\frac{4(1-\alpha)}{\alpha(\eta_2-\eta_1)}\varphi(k)\Bigg|\\
&\leq \frac{\eta_2-\eta_1}{4}\\
&\times\Bigg[\Bigg\{\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left\{\frac{4(s-1)e^{\ln(2)s}+s+5}{(s+1)(s+2)(s+3)2^s}|\varphi'(\eta_1)|^q+\frac{1}{(s+2)(s+3)2^s}|\varphi'(\eta_2)|^q\right\}^{\frac{1}{q}} \qquad (21)\\
&+\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left\{\frac{2^{1-s}\left(\left(s^2s+2\right)e^{\ln(2)s}-1\right)}{(s+1)(s+2)(s+3)}|\varphi'(\eta_1)|^q+\frac{2^{1-s}}{(s+1)(s+2)(s+3)}|\varphi'(\eta_2)|^q\right\}^{\frac{1}{q}}\Bigg\}
\end{aligned}$$

$$+\left\{\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left\{\frac{1}{(s+2)(s+3)2^{s}}\left|\varphi'(\eta_1)\right|^{q}+\frac{4(s-1)e^{\ln(2)s}+s+5}{(s+1)(s+2)(s+3)2^{s}}\left|\varphi'(\eta_2)\right|^{q}\right\}^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left\{\frac{2^{1-s}}{(s+1)(s+2)(s+3)}\left|\varphi'(\eta_1)\right|^{q}+\frac{2^{1-s}\left(\left(s^{2}s+2\right)e^{\ln(2)s}-1\right)}{(s+1)(s+2)(s+3)}\left|\varphi'(\eta_2)\right|^{q}\right\}^{\frac{1}{q}}\right\}\Bigg], \tag{22}$$

where $k\in[\eta_1,\eta_2]$ and $\mathcal{M}(\alpha)>0$ is a normalization function, $p^{-1}=1-q^{-1}$.

Proof. By using Lemma 2.3, the Improved power-mean integral inequality and the s-convexity of $\left|\varphi'\right|^{q}$, we have

$$\left|\frac{\varphi(\eta_1)+\varphi(\eta_2)}{2}-\frac{\mathcal{M}(\alpha)}{\alpha(\eta_2-\eta_1)}\right.$$

$$\times\left\{\left({}^{CF}_{\eta_1}I^{\alpha}\varphi\right)(k)+\left({}^{CF}I^{\alpha}_{\left(\frac{\eta_1+\eta_2}{2}\right)}\varphi\right)(k)+\left({}^{CF}_{\left(\frac{\eta_1+\eta_2}{2}\right)}I^{\alpha}\varphi\right)(k)+\left({}^{CF}I^{\alpha}_{\eta_2}\varphi\right)(k)\right\}$$

$$\left.+\frac{4(1-\alpha)}{\alpha(\eta_2-\eta_1)}\varphi(k)\right|$$

$$\leq\frac{\eta_2-\eta_1}{4}$$

$$\times\left\{\int_0^1(-\chi)\left|\varphi'\left(\frac{1+\chi}{2}\eta_1+\frac{1-\chi}{2}\eta_2\right)\right|d\chi+\int_0^1\chi\left|\varphi'\left(\frac{1-\chi}{2}\eta_1+\frac{1+\chi}{2}\eta_2\right)\right|d\chi\right\}$$

$$\leq\frac{\eta_2-\eta_1}{4}\left[\left\{\left(\int_0^1(1-\chi)\chi d\chi\right)^{1-\frac{1}{q}}\left(\int_0^1(1-\chi)\chi\left|\varphi'\left(\frac{1+\chi}{2}\eta_1+\frac{1-\chi}{2}\eta_2\right)\right|^{q}d\chi\right)^{\frac{1}{q}}\right.\right.$$

$$\left.+\left(\int_0^1\chi^{2}d\chi\right)^{1-\frac{1}{q}}\left(\int_0^1\chi^{2}\left|\varphi'\left(\frac{1+\chi}{2}\eta_1+\frac{1-\chi}{2}\eta_2\right)\right|^{q}d\chi\right)^{\frac{1}{q}}\right\}$$

$$+\left\{\left(\int_0^1(1-\chi)\chi d\chi\right)^{1-\frac{1}{q}}\left(\int_0^1(1-\chi)\chi\left|\varphi'\left(\frac{1-\chi}{2}\eta_1+\frac{1+\chi}{2}\eta_2\right)\right|^{q}d\chi\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\int_0^1\chi^{2}d\chi\right)^{1-\frac{1}{q}}\left(\int_0^1\chi^{2}\left|\varphi'\left(\frac{1-\chi}{2}\eta_1+\frac{1+\chi}{2}\eta_2\right)\right|^{q}d\chi\right)^{\frac{1}{q}}\right\}\right]$$

$$\leq\frac{\eta_2-\eta_1}{4}$$

$$\times\left[\left\{\left(\int_0^1(1-\chi)\chi d\chi\right)^{1-\frac{1}{q}}\right.\right.$$

$$\times\left(\int_0^1(1-\chi)\chi\left\{\left(\frac{1+\chi}{2}\right)^{s}\left|\varphi'(\eta_1)\right|^{q}+\left(\frac{1-\chi}{2}\right)^{s}\left|\varphi'(\eta_2)\right|^{q}\right\}d\chi\right)^{\frac{1}{q}}$$

$$\left.+\left(\int_0^1\chi^{2}d\chi\right)^{1-\frac{1}{q}}\left(\int_0^1\chi^{2}\left\{\left(\frac{1+\chi}{2}\right)^{s}\left|\varphi'(\eta_1)\right|^{q}+\left(\frac{1-\chi}{2}\right)^{s}\left|\varphi'(\eta_2)\right|^{q}\right\}d\chi\right)^{\frac{1}{q}}\right\}$$

$$+\left\{\left(\int_0^1(1-\chi)\chi d\chi\right)^{1-\frac{1}{q}}\right.$$

$$\times\left(\int_0^1(1-\chi)\chi\left\{\left(\frac{1-\chi}{2}\right)^{s}\left|\varphi'(\eta_1)\right|^{q}+\left(\frac{1+\chi}{2}\right)^{s}\left|\varphi'(\eta_2)\right|^{q}\right\}d\chi\right)^{\frac{1}{q}}$$

$$\left.\left.+\left(\int_0^1\chi^{2}d\chi\right)^{1-\frac{1}{q}}\left(\int_0^1\chi^{2}\left\{\left(\frac{1-\chi}{2}\right)^{s}\left|\varphi'(\eta_1)\right|^{q}+\left(\frac{1+\chi}{2}\right)^{s}\left|\varphi'(\eta_2)\right|^{q}\right\}d\chi\right)^{\frac{1}{q}}\right\}\right]$$

$$
\begin{aligned}
\leq\ & \frac{\eta_2-\eta_1}{4} \\
\times & \left[\left\{\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\right.\right. \\
\times & \left\{\frac{4(s-1)e^{\ln(2)s}+s+5}{(s+1)(s+2)(s+3)2^s}\left|\varphi'(\eta_1)\right|^q+\frac{1}{(s+2)(s+3)2^s}\left|\varphi'(\eta_2)\right|^q\right\}^{\frac{1}{q}} \\
+ & \left(\frac{1}{3}\right)^{1-\frac{1}{q}} \\
\times & \left.\left\{\frac{2^{1-s}\left(\left(s^2s+2\right)e^{\ln(2)s}-1\right)}{(s+1)(s+2)(s+3)}\left|\varphi'(\eta_1)\right|^q+\frac{2^{1-s}}{(s+1)(s+2)(s+3)}\left|\varphi'(\eta_2)\right|^q\right\}^{\frac{1}{q}}\right\} \\
+ & \left\{\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\right. \\
\times & \left\{\frac{1}{(s+2)(s+3)2^s}\left|\varphi'(\eta_1)\right|^q+\frac{4(s-1)e^{\ln(2)s}+s+5}{(s+1)(s+2)(s+3)2^s}\left|\varphi'(\eta_2)\right|^q\right\}^{\frac{1}{q}} \\
+ & \left(\frac{1}{3}\right)^{1-\frac{1}{q}} \\
\times & \left.\left.\left\{\frac{2^{1-s}}{(s+1)(s+2)(s+3)}\left|\varphi'(\eta_1)\right|^q+\frac{2^{1-s}\left(\left(s^2s+2\right)e^{\ln(2)s}-1\right)}{(s+1)(s+2)(s+3)}\left|\varphi'(\eta_2)\right|^q\right\}^{\frac{1}{q}}\right\}\right].
\end{aligned}
$$

Thus, the proof is completed. □

Theorem 2.9. *Suppose a mapping $\varphi : \mathcal{I} \subset [0,\infty] \to \mathfrak{R}$ is differentiable on (η_1,η_2) with $\eta_2 > \eta_1$. If $\varphi' \in \mathcal{L}[\eta_1,\eta_2]$, and $\left|\varphi'\right|^q$ is s-convex on $[\eta_1,\eta_2]$ for some fixed $s \in (0,1]$, $q > 1$, then the following inequality holds*

$$
\begin{aligned}
& \left|\frac{\varphi(\eta_1)+\varphi(\eta_2)}{2}-\frac{\mathcal{M}(\alpha)}{\alpha(\eta_2-\eta_1)}\right. \\
& \times\left\{\left({}^{CF}_{\eta_1}I^{\alpha}\varphi\right)(k)+\left({}^{CF}I^{\alpha}_{\left(\frac{\eta_1+\eta_2}{2}\right)}\varphi\right)(k)+\left({}^{CF}_{\left(\frac{\eta_1+\eta_2}{2}\right)}I^{\alpha}\varphi\right)(k)+\left({}^{CF}I^{\alpha}_{\eta_2}\varphi\right)(k)\right\} \\
& \left.+\ \frac{4(1-\alpha)}{\alpha(\eta_2-\eta_1)}\varphi(k)\right| \\
& \leq\ \frac{\eta_2-\eta_1}{2}\left[\frac{1}{p(p+1)}+\frac{1}{q(s+1)}\left(\left|\varphi'(\eta_1)\right|^q+\left|\varphi'(\eta_2)\right|^q\right)\right],
\end{aligned}
\tag{23}
$$

where $k \in [\eta_1,\eta_2]$ and $\mathcal{M}(\alpha) > 0$ is a normalization function, $p^{-1} = 1 - q^{-1}$.

Proof. By using Lemma 2.3, the s-convexity of $\left|\varphi'\right|^q$, we have

$$
\begin{aligned}
& \left|\frac{\varphi(\eta_1)+\varphi(\eta_2)}{2}-\frac{M(\alpha)}{\alpha(\eta_2-\eta_1)}\right. \\
& \times\left\{\left({}^{CF}_{\eta_1}I^{\alpha}\varphi\right)(k)+\left({}^{CF}I^{\alpha}_{\left(\frac{\eta_1+\eta_2}{2}\right)}\varphi\right)(k)+\left({}^{CF}_{\left(\frac{\eta_1+\eta_2}{2}\right)}I^{\alpha}\varphi\right)(k)+\left({}^{CF}I^{\alpha}_{\eta_2}\varphi\right)(k)\right\} \\
& \left.+\ \frac{4(1-\alpha)}{\alpha(\eta_2-\eta_1)}\varphi(k)\right| \\
& \leq\ \frac{\eta_2-\eta_1}{4}
\end{aligned}
$$

$$\times\left\{\int_0^1(-\chi)\left|\varphi'\left(\frac{1+\chi}{2}\eta_1+\frac{1-\chi}{2}\eta_2\right)\right|d\chi+\int_0^1\chi\left|\varphi'\left(\frac{1-\chi}{2}\eta_1+\frac{1+\chi}{2}\eta_2\right)\right|d\chi\right\}.$$

By using the Young's inequality as

$$uv\leq\frac{1}{p}u^p+\frac{1}{q}v^q.$$

$$\begin{aligned}
&\Bigg|\frac{\varphi(\eta_1)+\varphi(\eta_2)}{2}-\frac{\mathcal{M}(\alpha)}{\alpha(\eta_2-\eta_1)}\\
&\times\left\{\left({}^{CF}_{\eta_1}I^{\alpha}\varphi\right)(k)+\left({}^{CF}I^{\alpha}_{\left(\frac{\eta_1+\eta_2}{2}\right)}\varphi\right)(k)+\left({}^{CF}_{\left(\frac{\eta_1+\eta_2}{2}\right)}I^{\alpha}\varphi\right)(k)+\left({}^{CF}I^{\alpha}_{\eta_2}\varphi\right)(k)\right\}\\
&+\ \frac{4(1-\alpha)}{\alpha(\eta_2-\eta_1)}\varphi(k)\Bigg|\\
&\leq\ \frac{\eta_2-\eta_1}{4}\Bigg[\left\{\frac{1}{p}\int_0^1\chi^pd\chi+\frac{1}{q}\int_0^1\left|\varphi'\left(\frac{1+\chi}{2}\eta_1+\frac{1-\chi}{2}\eta_2\right)\right|^q d\chi\right\}\\
&+\ \left\{\frac{1}{p}\int_0^1\chi^pd\chi+\frac{1}{q}\int_0^1\left|\varphi'\left(\frac{1-\chi}{2}\eta_1+\frac{1+\chi}{2}\eta_2\right)\right|^q d\chi\right\}\Bigg]
\end{aligned}$$

$$\begin{aligned}
&\leq\ \frac{\eta_2-\eta_1}{4}\Bigg[\left\{\frac{1}{p}\int_0^1\chi^pd\chi+\frac{1}{q}\int_0^1\left\{\left(\frac{1+\chi}{2}\right)^s\left|\varphi'(\eta_1)\right|^q+\left(\frac{1-\chi}{2}\right)^s\left|\varphi'(\eta_2)\right|^q\right\}d\chi\right\}\\
&+\ \left\{\frac{1}{p}\int_0^1\chi^pd\chi+\frac{1}{q}\int_0^1\left\{\left(\frac{1-\chi}{2}\right)^s\left|\varphi'(\eta_1)\right|^q+\left(\frac{1+\chi}{2}\right)^s\left|\varphi'(\eta_2)\right|^q\right\}d\chi\right\}\Bigg]\\
&\leq\ \frac{\eta_2-\eta_1}{2}\left[\frac{1}{p(p+1)}+\frac{1}{q(s+1)}\left(\left|\varphi'(\eta_1)\right|^q+\left|\varphi'(\eta_2)\right|^q\right)\right].
\end{aligned}$$

Thus, the proof is completed.

□

3. Applications to matrix and special means

Consider that $s\in(0,1]$ and $u,v,w\in\mathfrak{R}$. We define a mapping $\varphi:[0,\infty)\ \to\mathfrak{R}$ as

$$\varphi(\chi)=\begin{cases} u, & \chi=0\\ v\chi^s+w, & \chi>0.\end{cases}$$

If $v\geq0$ and $0\leq w\leq u$, then $\varphi\in K_s^2$ in [5]. Thus, for $u=\ w=\ 0$, and $v=1$, we have $\varphi:[a,b]\to\mathfrak{R}\ \varphi(\chi)=\chi^s$ with $\varphi\in K_s^2$.

In [24], the following result is mentioned. Suppose $\varphi:I_1\to\mathfrak{R}_+$ be a non-decreasing and s-convex function on I_1 and $\psi:J\to I_2\subseteq I_1$ is a non-negative convex function on J, then $\varphi o\psi$ is s-convex on I_1.

Corollary 3.1. *Suppose $\psi:I\to I_1\subseteq[0,\infty)$ is a non-negative convex function on I, then $\psi^s(x)$ is s-convex on $[0,\infty)$, $0<s<1$.*

Example 3.2. *We denote by C^n the set of $n\times n$ complex matrices, M_n the algebra of $n\times n$ complex matrices, and by M_n^+ the strictly positive matrices in M_n. That is, $A\in M_n^+$ if $\langle\mathcal{A}x,x\rangle>0$ for all nonzero $x\in C^n$. In [23], Sababheh proved* *that the function* $\psi(\theta) = \left\|\mathcal{A}^{\theta}\mathcal{X}\mathcal{B}^{1-\theta} + \mathcal{A}^{1-\theta}\mathcal{X}\mathcal{B}^{\theta}\right\|$, $\mathcal{A},\mathcal{B} \in M_n^+$, $\mathcal{X} \in M_n$, *is convex for all* $\theta \in [0,1], s \in (0,1)$. *Then by using Theorem 2.1, we have*

$$\begin{aligned}
&2^{s-1}\left\|\mathcal{A}^{\frac{x+y}{2}}\mathcal{X}\mathcal{B}^{1-\frac{x+y}{2}} + \mathcal{A}^{1-\frac{x+y}{2}}\mathcal{X}\mathcal{B}^{\frac{x+y}{2}}\right\| \\
&\leq \frac{M(\alpha)}{\alpha(y-x)}\Big\{ {}^{CF}_{\ x}I^{\alpha}\left\|\mathcal{A}^{k}\mathcal{X}\mathcal{B}^{1-k} + \mathcal{A}^{1-k}\mathcal{X}\mathcal{B}^{k}\right\| + {}^{CF}I^{\alpha}_{y}\left\|\mathcal{A}^{k}\mathcal{X}\mathcal{B}^{1-k} + \mathcal{A}^{1-k}\mathcal{X}\mathcal{B}^{k}\right\| \\
&- \frac{2(1-\alpha)}{M(\alpha)}\left\|\mathcal{A}^{k}\mathcal{X}\mathcal{B}^{1-k} + \mathcal{A}^{1-k}\mathcal{X}\mathcal{B}^{k}\right\|\Big\} \\
&\leq \frac{1}{s+1}\Big\{\left\|\mathcal{A}^{x}\mathcal{X}\mathcal{B}^{1-x} + \mathcal{A}^{1-x}\mathcal{X}\mathcal{B}^{x}\right\| + \left\|\mathcal{A}^{y}\mathcal{X}\mathcal{B}^{1-y} + \mathcal{A}^{1-y}\mathcal{X}\mathcal{B}^{y}\right\|\Big\}.
\end{aligned}$$

Now for arbitrary real numbers $c, d (c \neq d)$, let us consider the following means:

$$\begin{aligned}
A(c,d) &= \frac{c+d}{2}, \\
H(c,d) &= \frac{2cd}{c+d}, \\
L_r(c,d) &= \left[\frac{d^{r+1} - c^{r+1}}{(r+1)(d-c)}\right]^{\frac{1}{r}}.
\end{aligned}$$

Proposition 3.3. *Suppose* $c, d \in \Re^+$, $0 < c < d$, $s \in (0,1)$. *Then*

$$\left|A(c^s, d^s) - L_s^s(c,d)\right| \leq s(d-c)\left\{\frac{2se^{\ln 2(s)} + 1}{(s+1)(s+2)2^s} + \frac{\beta(2, s+1)}{2^s}\right\} A\left(|c|^{s-1}, |d|^{s-1}\right).$$

Proof. *In Theorem 2.4, if we set* $\varphi(z) = z^s$, $z \in \Re$ *and* $s \in (0,1), \alpha = 1$, *and* $\mathcal{M}(0) = \mathcal{M}(1) = 1$, *then we obtain the result immediately.* □

Proposition 3.4. *Suppose* $c, d \in \Re^+$, $0 < c < d$, $s \in (0,1)$. *Then*

$$\begin{aligned}
&\left|H^{-1}(c^s, d^s) - L_{-s}^{-s}(c,d)\right| \\
&\leq \frac{s(d-c)}{2}\left\{\frac{2se^{\ln 2(s)} + 1}{(s+1)(s+2)2^s} + \frac{\beta(2, s+1)}{2^s}\right\} A\left(|c|^{-s-1}, |d|^{-s-1}\right).
\end{aligned}$$

Proof. *In Theorem 2.4, if we set* $\varphi(z) = z^{-s}$, $z \in \Re$ *and* $s \in (0,1), \alpha = 1$, *and* $\mathcal{M}(0) = \mathcal{M}(1) = 1$, *then we obtain the result immediately.* □

4. Conclusion

This article, we have generated and explored some new Hermite–Hadamard's type inequalities via Caputo-Fabrizio(CF) fractional integral for s-convex function. New bounds and novel connections are developed of Hermite–Hadamard's type inequalities for differentiable mappings whose derivatives in absolute value at certain powers are s-convex. In the second last section, we have also developed some new Hölder İşcan and Improved power mean integral inequalities(For detailed see [21]). We hope that the strategies of this paper will motivate the researchers working in functional analysis, information theory and statistical theory.

References

[1] L. Chen, J. Zhang, M. S. Saleem, I. Ahmed, S. Waheed and L. Pan, *Fractional integral inequalities for h-convex functions via Caputo-Fabrizio operator*, AIMS Mathematics, 6(6), 6377-6389, (2021)..

[2] S. S. Dragomir and S. Fitzpatrik, *The Hadamard inequality for s-convex functions in the second sense*, Demonstratio Math., (32)(4), 687-696, (1999).

[3] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, (2006).

[4] W. Orlicz, *A note on modular spaces* I, Bull. Acad. Polon. Sci. Ser. Math. Astronom. Phys., 9, 157-162, (1961).

[5] H. Hudzik and L. Maligranda, *Some remarks on s-convex functions, Aequationes Math*, 48100-111,(1994) .

[6] P. Burai, A. Házy, and T. Juhász, *On approximately Breckner s-convex functions*, Control Cybernet. 40(1) , 91–99, (2011).

[7] M. Alomari, M. Darus and S. S. Dragomir, *New inequalities of Simpson's type for s-convex functions with applications*, Research report collection, 12(4), (2009).

[8] S. I. Butt, J. Nasir, S. Qaisar, and K. M. Abualnaja, *k-Fractional Variants of Hermite-Mercer-Type Inequalities via s-convexity with Applications*. Journal of Function Spaces,(2021).

[9] M. Z. Sarikaya, E. Set and M. E. Özdemir, *On new inequalities fo simpson's type for s−convex functions*, Computer and Mathematics with Applications, 60(8), 2191-2199, (2010).

[10] N. S. Barnett, P. Cerone and S.S. Dragomir, *Some new inequalities for Hermite-Hadamard divergence in information theory, Stochastic analysis and applications*, Vol. 3, 7-19, Nova Sci. Publ., Hauppauge, NY, (2003).

[11] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite–Hadamard Inequalities and applications*, (RGMIA Monographs http://rgmia.vu.edu.au/ monographs/ hermite hadamard.html), Victoria University, (2000).

[12] S. Wu, *On the weighted generalization of the Hermite-Hadamard inequality and its applications*, Rocky Mountain J. Math. 39, no. 5, 1741-1749, (2009).

[13] H. Kalsoom, M. Vivas-Cortez, M. A. Latif, and H. Ahmad, *Weighted midpoint Hermite-Hadamard-Fejer type inequalities in fractional calculus for harmonically convex functions*, Fractal and Fractional, 5(4), 2021, 252.

[14] M. A. Latif, H. Kalsoom, and M. Z. Abidin, *Hermite–Hadamard-Type Inequalities Involving Harmonically Convex Function via the Atangana–Baleanu Fractional Integral Operator*, Symmetry, 14(9), 2022, 1774.

[15] H. Kalsoom, M. A. Latif, Z. A. Khan, and M. Vivas-Cortez, *Some New Hermite-Hadamard-Fejér fractional type inequalities for h-convex and harmonically h-Convex interval-valued Functions*, Mathematics, 10(1), 2021, 74.

[16] M. Tariq, H. Ahmad, A. G. Shaikh, S. K. Sahoo, K. M. Khedher and T. N. Gia, *New fractional integral inequalities for preinvex functions involving caputo fabrizio operator*, AIMS Mathematics, 7(3): 3440-3455, (2021).

[17] M. Vivas-Cortez, M. U. Awan, M. Z. Javed, M. A. Noor and K. I. Noor, *A Study of Uniform Harmonic-Convex Functions with respect to Hermite-Hadamard's Inequality and Its Caputo-Fabrizio Fractional Analogue and Applications* Journal of Function Spaces, 2021, (2021).
%bibitemAwan

[18] M. Caputo and M. Fabrizio, A new definition of fractional derivative without singular kernel, *Progress in Fractional Differentiation and Applications*, 1(2), 73-85, (2015).

[19] T. Abdeljawad and D. Baleanu, *On fractional derivatives with exponential kernel and their discrete versions*, Reports on Mathematical Physics, 80(1), 11-27, (2017).

[20] M. Vivas-Cortez, M. S. Saleem, S. Sajid, M. S. Zahoor and A. Kashuri, *Hermite-Jensen-Mercer-Type Inequalities via Caputo-Fabrizio Fractional Integral for h-Convex Function*, Fractal and Fractional, 5(4), 269, (2021).

[21] İ. İşcan, *New refinements for integral and sum forms of Hölder inequality*, Journal of Inequalities and Applications, **1**(2019), 1–11.

[22] M. Gürbüz, A. O. Akdemir, S. Rashid, E. Set, *Hermite–Hadamard inequality for fractional integrals of Caputo-Fabrizio type and related inequalities*, Journal of Inequalities and Applications, 2020(1), 1-10, (2020).

[23] M. Sababheh, *Convex functions and means of matrices*, arXiv preprint arXiv:1606.08099, (2016).

[24] M. Alomari, M. Darus, S. S. Dragomir, *New inequalities of Simpson's type for s-convex functions with applications*, RGMIA Res. Rep. Coll. 12 (4) (2009) Article 9. Online http://ajmaa.org/RGMIA/v12n4.php.