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On some contractive mappings and a new version of implicit function theorem in topological spaces

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Abstract. Study on existence of fixed points of contraction and contractive (type) mappings in topological spaces is a challenging task. The main goal of this article is to deal with this challenging task. To achieve our goal, we define two new contractive type mappings, namely, h-$\mathcal{A}$-contractive and h-$\mathcal{A}_1$-contractive mappings on a topological space X, where $h : X \times X \to \mathbb{R}_+$ is a function and $\mathcal{A}$, $\mathcal{A}_1$ are two collections of implicit functions. Then, we obtain some fixed point results concerning such contractive type mappings. Finally, as an application of one of the above mentioned fixed point results, we obtain a newer version of the implicit function theorem in topological spaces.

1. Introduction

Investigations of fixed points of various kinds of contraction and contractive type mappings is an essential tool in non-linear analysis. Throughout the last century, a significant number of research have been initiated concerning the finding of fixed points of various classes of contraction or contractive type mappings. Most of such contraction and contractive mappings have been studied at first in metric spaces, and then in several generalized abstract spaces (mainly metric type spaces). However, all such abstract spaces are just simple generalizations of metric spaces and many of them are found to be metrizable. Also we know that among different existing abstract spaces, the topological space is the most general one and it is neither a generalization of metric spaces nor metrizable. So it is now logical to examine whether the notions of contraction and contractive mappings can be extended in topological spaces or not. For early contributions, see [3, 5, 10, 11]. In 1992, Hicks (see [8]) proposed the concept of d-complete topological spaces and obtained some fixed point results concerning some contraction mappings in such spaces. We first recall the definition of d-complete topological spaces.

Definition 1.1. *[8] Let (X,τ) be a topological space and let $d : X \times X \to [0,\infty)$ be a mapping such that $d(x,y) = 0$ if and only if $x = y$. Then X is said to be d-complete if for any sequence $\{x_n\}$ in X, the relation $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$*

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implies that the sequence $\{x_n\}$ is convergent in (X, τ).

After introducing the notion of d-complete topological spaces, Hicks obtained the following interesting fixed point result.

Theorem 1.2. *[8] Let (X, τ) be a d-complete topological space, $T : X \to X$ be an operator and $\phi : X \to [0, \infty)$ be a given mapping. Suppose that there exists an $x \in X$ such that*

$$d(y, Ty) \leq \phi(y) - \phi(Ty) \text{ for all } \quad y \in \{x, Tx, T^2x, \dots\}.$$

Then the following hold:

(a) $\lim_{n\to\infty} T^n x = x^*$ *exists;*

(b) $Tx^* = x^*$ *if and only if the map $G(x) = d(x, Tx)$ is T-orbitally lower semicontinuous.*

After this, a number of authors have shown interests in obtaining fixed point, common fixed point, best proximity point results in topological spaces and therefore many articles related to such direction have been published. For some remarkable results of them, one may see [1, 4, 7, 12–15] and the references therein.

If we go through all such results, then we can note down that all the results are related to different kinds of contraction mappings. So it now becomes logical to think about contractive (type) mappings in topological spaces and obtain some fixed point results. On the other hand, one knows that there are many type of contractive mappings in the literature such as Edelstein type, Kannan type, Chatterjee type, Ćirić type etc. But recently Garai et al. [6] introduced two new contractive type mappings, viz., $\mathcal{A}$-contractive and $\mathcal{A}'$-conractive mappings, which contains all the aforementioned contractive mappings as particular cases. So in order to study different contractive type mappings in topological spaces, it is enough to study some contractive mappings similar to $\mathcal{A}$-contractive and $\mathcal{A}'$-conractive mappings. Influenced by these facts, in this article, our main goal is to introduce two new contractive mappings similar to the above two mappings, which we designate as h-$\mathcal{A}$-contractive and h-$\mathcal{A}_1$-contractive mappings. With the help of these two contractive mappings, we obtain some fixed point results in topological spaces. Also, we authenticate our obtained results by suitable examples. As an application of our derived results, we obtain a new version of *implicit function theorem* in topological spaces.

Before going to our main findings, we recall the notions of orbit and orbital continuity of a mapping in a topological space.

Definition 1.3. *Let X be a topological space and $T : X \to X$ a mapping. Then the set $O(x, \infty) = \{x, Tx, T^2x, \dots\}$ is called the orbit of T at x. T is called orbitally continuous if for any sequence $\{x_n\} \in O(x, \infty)$ that converges to $x^* \in X$ the sequence $\{Tx_n\}$ converges to Tx^* for all $x \in X$.*

2. h-$\mathcal{A}$-contractive mappings

Following Garai et al., we denote by $\mathcal{A}$ the set of all functions $f : \mathbb{R}_+^3 \to \mathbb{R}_+$ satisfying the following conditions:

(a) f is continuous;

(b) if $v > 0$ and $u < f(u, v, v)$ *or* $u < f(v, u, v)$, *or* $u < f(v, v, u)$, *then* $u < v$;

(c) $f(u, v, w) \leq u + v + w$ for all $u, v, w \in \mathbb{R}_+$.

Next, we introduce a new type of contractive mappings on a topological space (X, τ).

Definition 2.1. *Let X be a topological space and let $T : X \to X$ be a mapping. Then T is said to be an h-$\mathcal{A}$-contractive mapping if there exist $f \in \mathcal{A}$ and a continuous mapping $h : X \times X \to \mathbb{R}_+$ such that*

$$h(Tx, Ty) < f(h(x, y), h(x, Tx), h(y, Ty)) \quad \text{for all } x, y \in X \text{ with } x \neq y.$$

Next, we obtain a result concerning fixed points of the newly defined contractive mappings.

Theorem 2.2. *Let X be a topological space and let $T : X \to X$ be an h-$\mathcal{A}$-contractive mapping which is orbitally continuous. Also assume that $h(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$. Let $S = \{s \in X : \{T^n(s)\}$ has a convergent subsequence in X $\}$. If $S \neq \emptyset$ (i.e. there exists $s \in S$ and a sequence of iterates $\{T^n(s)\}$ which has a convergent subsequence $\{T^{n_j}(s)\}$ converging to some $a^* \in X$), then a^* is the unique fixed point of T.*

Proof. Let $s_0 \in S$. Then the sequence of iterates $\{T^n(s_0)\}$ has a convergent subsequence. Let $\{T^{n_j}(s_0)\}$ be a convergent subsequence of $\{T^n(s_0)\}$ such that $\lim_{j\to\infty} T^{n_j}(s_0) = a^*$, $a^* \in X$. Assume that $s_n = T^n(s_0)$ and $p_n = h(s_n, s_{n+1})$ for all $n \in \mathbb{N}$. Then $\lim_{j\to\infty} s_{n_j} = a^*$. The remaining part of the proof of existence and uniqueness of fixed point of T follows from the proof of case (1) of [Theorem 3.4 [6]]. Proceeding in this manner we get that a^* is the unique fixed point of T. □

As a consequence of the above result, we have the following corollary:

Corollary 2.3. *Let X be a topological space and let Y be a sequentially compact topological space. Let $T : X \times Y \to Y$ be an h-$\mathcal{A}$-contractive mapping on Y uniformly in X, i.e.,*

$$h(T(x, y_1), T(x, y_2)) < f(h(y_1, y_2), h(y_1, T(x, y_1)), h(y_2, T(x, y_2))).$$

for all $x \in X$, and for all $y_1, y_2 \in Y$. Also, assume that T is orbitally continuous and $h(x, y) = 0$ if and only if $x = y$. Then, for every $x \in X$, the map $y \to T(x, y)$ has a unique fixed point $g(x)$.

Next, we provide a supporting example.

Example 2.4. *Let us consider the space $Y = \mathbb{Z}_+ \times \mathbb{R}_+$ with dictionary order topology. Let $X = S_\Omega$ be the section of Y by $\Omega = 2 \times \pi$. Then X is a topological space with the subspace topology. Also define $h : X \times X \to \mathbb{R}_+$, $f : \mathbb{R}^3_+ \to \mathbb{R}_+$ and $T : X \to X$ by*

$$h((x_1, y_1), (x_2, y_2)) = (x_1 - x_2)^2 + |y_1 - y_2| \quad \text{for all } (x_1, y_1), (x_2, y_2) \in X;$$
$$f(x_1, x_2, x_3) = x_1 \quad \text{for all } x_1, x_2, x_3 \in \mathbb{R}_+;$$
$$T(x, y) = \left(1, \frac{y}{2}\right) \quad \text{for all } (x, y) \in X.$$

Then h is continuous; $h(x, y) = 0 \iff x = y$; $f \in \mathcal{A}$; and T is orbitally continuous. Let $x = (x_1, y_1)$, $y = (x_2, y_2) \in X$ be arbitrary with $x \neq y$. Then $h(x, y) = (x_1 - x_2)^2 + |y_1 - y_2|$ and $h(Tx, Ty) = \frac{|y_1 - y_2|}{2}$. So

$$h(Tx, Ty) < f(h(x, y), h(x, Tx), h(y, Ty)).$$

Therefore, T is an h-$\mathcal{A}$-contractive mapping. If we take the point $(1, 1) \in X$, then $T(1, 1) = (1, \frac{1}{2}), \ldots, T^{n_j}(1, 1) = (1, \frac{1}{2^{n_j}})$. This implies $\lim_{j\to\infty} T^{n_j}(1, 1) = \lim_{n_j\to\infty}(1, \frac{1}{2^{n_j}}) = (1, 0)$. This shows that $(1, 1) \in S$, i.e., $S \neq \emptyset$. Hence by Theorem 2.2, T has a unique fixed point in X. Note that $(1, 0)$ is the unique fixed point of T.

Here we provide another important example which illuminates that our Theorem 2.2 is more general than some existing other fixed point theorems in topological spaces.

Example 2.5. *Let us take $X = \mathbb{N} \setminus \{3, 5\}$ and consider the discrete topology on X. We define a mapping $T : X \to X$ by*

$$T(x) = \begin{cases} 2 & \text{if } x \in \{2n : n \in \mathbb{N}\} \\ 4 & \text{if } x \in \{2n - 1 : n \in \mathbb{N} \setminus \{2, 3\}\}. \end{cases}$$

Let $h : X \times X \to \mathbb{R}_+$ be defined as

$$h(x, y) = |x - y|.$$

Then clearly, h is continuous and $h(x,y) = 0 \Leftrightarrow x = y$. *Also we take* $f \in \mathcal{A}$ *defined by*

$$f(x_1, x_2, x_3) = \max\{x_1, x_2, x_3\}.$$

Now whenever $x, y \in X$ *with* $x \neq y$, *we get the following cases:*

Case I: *Both* $x, y \in \{2n : n \in \mathbb{N}\}$. *Then* $h(Tx, Ty) = h(2,2) = 0 < h(x,y)$ *and so*

$$h(Tx, Ty) < f(h(x,y), h(x,Tx), h(y,Ty)).$$

Case II: *Both* $x, y \in \{2n-1 : n \in \mathbb{N} \setminus \{2,3\}\}$. *Then* $h(Tx,Ty) = h(4,4) = 0 < h(x,y)$ *and so*

$$h(Tx, Ty) < f(h(x,y), h(x,Tx), h(y,Ty)).$$

Case III: $x \in \{2n : n \in \mathbb{N}\}$ *and* $y \in \{2n-1 : n \in \mathbb{N} \setminus \{2,3\}\}$. *So,* $h(Tx,Ty) = h(2,4) = 2$. *For* $x = 2, y = 1$, *we have* $h(x,y) = 1$, $h(x,Tx) = 0$, $h(y,Ty) = 3$. *So* $h(Tx,Ty) < h(y,Ty)$.

Again for $y \geq 7$, *we have* $h(y,Ty) \geq 3$. *So again* $h(Tx,Ty) < h(y,Ty)$. *Therefore,*

$$h(Tx, Ty) < f(h(x,y), h(x,Tx), h(y,Ty)).$$

Case IV: $x \in \{2n-1 : n \in \mathbb{N} \setminus \{2,3\}\}$ *and* $y \in \{2n : n \in \mathbb{N}\}$. *Then we can similarly show that*

$$h(Tx, Ty) < f(h(x,y), h(x,Tx), h(y,Ty)).$$

Therefore, T is an h-$\mathcal{A}$-contractive mapping. Note that T is orbitally continuous and also the sequence $\{T^n(2)\}$ *converges to* 2. *Thus all the hypotheses of Theorem 2.2 hold. Also T has a unique fixed point* 2.

Next, we note down some interesting facts about the above example.

Remark 2.6. *Note that T is nether (Edelstein's) contractive mapping nor Kannan type contractive mapping. Also, Edelstein's fixed point theorem in topological spaces [10, Theorem 1] is not applicable in this example. So we see that some existing well-known fixed point can't guarantee the existence of the fixed point of T but our obtained result (Theorem 2.2) does.*

Now we introduce the notion of h-completeness of a topological space.

Definition 2.7. *Let* (X, τ) *be a topological space and let* $h : X \times X \to [0, \infty)$ *be a mapping. Then X is said to be h-complete if for any sequence* $\{x_n\}$ *in X with* $\sum_{n=1}^{\infty} h(x_n, x_{n+1}) < \infty$, *the sequence* $\{x_n\}$ *is convergent in* (X, τ).

Next, we show by an example that in Theorem 2.2 if $S = \emptyset$, then an h-$\mathcal{A}$-contractive mapping need not posses a fixed point even if the underlying space is h-complete.

Example 2.8. *Let* $X = \{\frac{1}{n} : n \in \mathbb{N}\}$ *be a topological space with discrete topology. Let* $h : X \times X \to [0, \infty)$ *be defined by*

$$h(x,y) = \begin{cases} 0 & \text{if } x = y; \\ 1 + |x - y| & \text{if } x \neq y. \end{cases}$$

Then h is continuous, $h(x,y) = 0 \Leftrightarrow x = y$, *and, X is h-complete.*

Let $T : X \to X$ *be defined by* $T(x) = \frac{x}{2}$ *for all* $x \in X$. *Then T is continuous and T is an h-$\mathcal{A}$-contractive mapping for* $f(x_1, x_2, x_3) = x_1$ *for all* $x_1, x_2, x_3 \in \mathbb{R}_+$. *Also* $S = \{s \in X : \{T^n(s)\}$ *has a convergent subsequence in* $X\} = \emptyset$. *Here, T has no fixed point.*

Remark 2.9. *We see from the above example that an h-$\mathcal{A}$-contractive mapping on an h-complete topological space need not have a fixed point. Thus, if the underlying topological space is h-complete, then in order to get the guaranty of existence of fixed points of an h-$\mathcal{A}$-contractive mapping, we need some additional assumptions. We present such an additional assumption in the following theorem.*

Theorem 2.10. *Let X be a topological space and let $T : X \to X$ be an h-$\mathcal{A}$-contractive mapping such that either T is orbitally continuous, or T^k is continuous for some $k \in \mathbb{N}$ and h is symmetric in all orbits of T. Also assume that $h(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$ and X is h-complete. Also assume that for any $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$f(h(x,y), h(x,Tx), h(y,Ty)) < \epsilon + \delta \Rightarrow h(T^2x, T^2y) \leq \frac{\epsilon}{4} \text{ for all } x, y \in X. \tag{1}$$

Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary and let $x_n = T^n(x_0)$; $p_n = h(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Our assertion is that, $\{p_n\} \to 0$ as $n \to \infty$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then it is trivial. So, we assume that $x_n \neq x_{n+1}$ for all $n \geq 1$. Then following Theorem 2.2, $\{p_n\}$ is a strictly decreasing sequence and hence it converges to some $p \geq 0$. We claim that $p = 0$. If not, then $p > 0$, and so by (1) there exists $\delta > 0$ such that

$$f(h(x,y), h(x,Tx), h(y,Ty)) < 4p + \delta \Rightarrow h(T^2x, T^2y) \leq p \text{ for all } x, y \in X.$$

Again since $\lim_{n\to\infty} p_n = p$, for the above $\delta > 0$, there exists an $n \in \mathbb{N}$ such that

$$p_n < p + \frac{\delta}{4}.$$

Therefore,

$$\begin{aligned} &f(h(x_n, x_{n+1}), h(x_n, x_{n+1}), h(x_{n+1}, x_{n+2})) \leq h(x_n, x_{n+1}) + h(x_n, x_{n+1}) + h(x_{n+1}, x_{n+2}) \\ &= p_n + p_n + p_{n+1} < 4p + \delta. \end{aligned}$$

Therefore, $h(x_{n+2}, x_{n+3}) \leq p \Rightarrow p_{n+2} \leq p$, which leads to a contradiction to the fact that $\{p_n\}$ is strictly decreasing and $p_n \to p$ as $n \to \infty$. Hence $p = 0$, i.e., $\lim_{n\to\infty} h(x_n, x_{n+1}) = 0$.

Next, we show that $\sum_{n=1}^{\infty} h(x_n, x_{n+1}) < \infty$. For this, let $\epsilon > 0$ be arbitrary. Then there exists $\delta > 0$ such that (1) holds. Without loss of generality, we assume that $0 < \delta < \epsilon$. Since $h(x_n, x_{n+1}) \to 0$ as $n \to \infty$, there exists $N \in \mathbb{N}$ such that

$$h(x_n, x_{n+1}) < \frac{\delta}{3} < \frac{\epsilon}{3} < \epsilon \text{ for all } n \geq N.$$

Now

$$\begin{aligned} &f(h(x_N, x_{N+1}), h(x_N, x_{N+1}), h(x_{N+1}, x_{N+2})) \leq h(x_N, x_{N+1}) + h(x_N, x_{N+1}) + h(x_{N+1}, x_{N+2}) \\ &< \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} < \frac{\epsilon}{3} + \frac{2\delta}{3} < \frac{\epsilon}{3} + \delta. \end{aligned}$$

Then by (1), we get $h(x_{N+2}, x_{N+3}) \leq \frac{\epsilon}{3.4}$. Also,

$$\begin{aligned} &f(h(x_{N+1}, x_{N+2}), h(x_{N+1}, x_{N+2}), h(x_{N+2}, x_{N+3})) \leq h(x_{N+1}, x_{N+2}) + h(x_{N+1}, x_{N+2}) + h(x_{N+2}, x_{N+3}) \\ &< \frac{\delta}{3} + \frac{\delta}{3} + \frac{\epsilon}{3.4} < \frac{\epsilon}{3.4} + \delta. \end{aligned}$$

This implies $h(x_{N+3}, x_{N+4}) \leq \frac{\epsilon}{3.4^2}$. Continuing in this way, we can show that

$$h(x_{N+m}, x_{N+m+1}) < \frac{\epsilon}{3.4^{m-1}} \text{ for all } m \geq 2.$$

Therefore,

$$\begin{aligned}
&\sum_{n=1}^{\infty} h(x_n, x_{n+1})\\
&= M + \sum_{n=N}^{\infty} h(x_n, x_{n+1}), \text{ where } M = h(x_1, x_2) + h(x_2, x_3) + ... + h(x_{N-1}, x_N)\\
&< M + \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3.4} + \frac{\epsilon}{3.4^2} + \cdots\\
&= M + \frac{\epsilon}{3} + \frac{\epsilon}{3}[1 + \frac{1}{4} + \frac{1}{4^2} + \cdots]\\
&= M + \frac{7\epsilon}{9} < \infty.
\end{aligned}$$

Then by h-completeness of X, there exists $z \in X$ such that $\{x_n\}$ converges to z, i.e., $\lim_{n\to\infty} x_n = z$. Next, we prove that $Tz = z$.

First, we assume that T is orbitally continuous. Then $\lim_{n\to\infty} Tx_n = Tz$. Therefore,

$$\lim_{n\to\infty} h(x_n, z) = h(\lim_{n\to\infty} x_n, z) = h(z, z) = 0$$

and so

$$0 = \lim_{n\to\infty} h(x_{n+1}, z) = \lim_{n\to\infty} h(Tx_n, z) = h(\lim_{n\to\infty} Tx_n, z) = h(Tz, z).$$

Thus $Tz = z$.

Next, we assume that T^k is continuous for some $k \in \mathbb{N}$. Then, as above, we can show that $T^k z = z$. If $Tz \neq z$, then $T^{k-1}z \neq z$ also. Therefore,

$$\begin{aligned}
h(Tz, z) &= h(Tz, T^k z)\\
&< f(h(z, T^{k-1}z), h(z, Tz), h(T^{k-1}z, T^k z))\\
&= f(h(T^k z, T^{k-1}z), h(z, Tz), h(T^{k-1}z, T^k z)).
\end{aligned}$$

Using (b), we have $h(Tz, z) < h(T^k z, T^{k-1}z)$. But, $h(T^k z, T^{k-1}z) < h(T^{k-1}z, T^{k-2}z) < h(T^{k-2}z, T^{k-3}z) < \cdots < h(Tz, z)$, which is a contradiction. So $Tz = z$.

Thus z is a fixed point of T. The uniqueness of the fixed point follows from Theorem 2.2. □

Afterwards, we present an example in support of the above theorem.

Example 2.11. *Let us consider the topological space (X, τ) where $X = \mathbb{N}$ and τ is the discrete topology. We define $h : X \times X \to \mathbb{R}_+$, $f : \mathbb{R}_+^3 \to \mathbb{R}_+$ and $T : X \to X$ by*

$$h(x, y) = |x^2 - xy| \quad \text{for all } x, y \in X;$$

$$f(x_1, x_2, x_3) = \max\{x_1, x_2, x_3\} \quad \text{for all } x_1, x_2, x_3 \in \mathbb{R}_+;$$

$$Tx = \begin{cases} \frac{x}{2}, & \text{if } x \text{ is even;}\\ 1, & \text{if } x \text{ is odd.} \end{cases}$$

Clearly h is continuous and $h(x, y) = 0 \Leftrightarrow x = y$. Also note that $\sum_{n=1}^{\infty} h(x_n, x_{n+1}) < \infty$ if and only if $\{x_n\}$ is eventually constant. So X is h-complete.

Now we show that T is an h-$\mathcal{A}$-contractive mapping. Let $x, y \in X$ with $x \neq y$. Then three cases arise.

Case I: *Let x and y both be even. Therefore,*

$$h(Tx, Ty) = h\left(\frac{x}{2}, \frac{y}{2}\right) = \frac{\left|x^2 - xy\right|}{4} < |x^2 - xy| = h(x, y).$$

Case II: *Let x and y both be odd. Therefore,*

$$h(Tx, Ty) = h(1, 1) = |1 - 1| = 0 < |x^2 - xy| = h(x, y).$$

Case III: *Let x be odd and y be even. Therefore,*

$$h(Tx, Ty) = h\left(1, \frac{y}{2}\right) = \left|1 - \frac{y}{2}\right| < \frac{y^2}{2} = h(y, Ty).$$

Thus $h(Tx, Ty) < f(h(x, y), h(x, Tx), h(y, Ty))$ for all $x, y \in X$ with $x \neq y$. Therefore, T is an h-$\mathcal{A}$-contractive mapping. Again for any $\epsilon > 0$, if we choose $\delta = \epsilon$, then one can easily verify that (1) holds good. Further, T is orbitally continuous. Therefore, all the conditions of Theorem 2.10 hold here. So by the consequence of this theorem T has a unique fixed point in X. Indeed, 1 is the unique fixed point of T.

3. h-$\mathcal{A}_1$-contractive mappings

In this section, we introduce another new type of contractive mappings. Before this, we consider another class of functions $\mathcal{A}_1$, which is the collection of all mappings $f : \mathbb{R}^2_+ \to \mathbb{R}_+$ satisfying the following conditions:

(d) f is continuous;

(e) if $v > 0$ and $u < f(u, v)$ *or* $u < f(v, u)$, then $u < v$;

(f) $f(u, u) \leq u$; for all $u \in \mathbb{R}_+$;

(g) $f(u, v) \leq u + v$; for all $u, v \in \mathbb{R}_+$.

Definition 3.1. *Let X be a topological space and let $T : X \to X$ be a mapping. Then T is said to be an h-$\mathcal{A}_1$-contractive mapping if there exist $f \in \mathcal{A}_1$ and a continuous mapping $h : X \times X \to \mathbb{R}_+$ such that*

$$h(Tx, Ty) < f(h(x, y), \min\{h(x, Ty), h(Tx, y)\}) \quad \textit{for all } x, y \in X \textit{ with } x \neq y.$$

The upcoming theorem deals with the existence of fixed points of the above kind of contractive mappings.

Theorem 3.2. *Let X be a topological space and let $T : X \to X$ be an h-$\mathcal{A}_1$-contractive mapping. Also, assume that T is orbitally continuous and $h(x, y) = 0 \Leftrightarrow x = y$. Let $S = \{s \in X : \{T^n(s)\}$ has a convergent subsequence in $X\}$. If $S \neq \emptyset$ (i.e., if there exists $s \in S$ and a sequence of iterates $\{T^n(s)\}$, which has a convergent subsequence $\{T^{n_j}(s)\}$ converging to some $b^* \in X$). Then, b^* is the unique fixed point of T.*

Proof. Let $s \in S$ and let $s_n = T^n(s)$. So the sequence $\{s_n\}$ has a convergent subsequence. Let $\{s_{n_i}\}$ be one such subsequence and let $\lim_{i\to\infty} s_{n_i} = b^*$. Again, we consider a sequence of non-negative real numbers $\{q_n\}$ defined by $q_n = h(s_n, s_{n+1})$ for all $n \in \mathbb{N}$. Some modifications of the proof of first part of [Theorem 3.10, [6]] implies that $q_n \to 0$ as $n \to \infty$, and hence T has a fixed point b^*. Uniqueness of the fixed point of T also follows from aforementioned theorem. □

Next, we present an example in support of the above theorem.

Example 3.3. *Let $X = \mathbb{R}$ and consider the lower limit topology on X. We define $h : X \times X \to \mathbb{R}_+$ by $h(x, y) = (x - y)^2$ for all $x, y \in X$; $f : \mathbb{R}^2_+ \to \mathbb{R}_+$ by $f(x_1, x_2) = x_1$ for all $x_1, x_2, \in \mathbb{R}_+$; and $T : X \to X$ by*

$$T(x) = \begin{cases} 1 & \text{if } x < 0; \\ 1 + \frac{x}{2} & \text{if } x \geq 0. \end{cases}$$

Then h is continuous and $h(x,y)=0 \Leftrightarrow x=y$ and $f \in \mathcal{A}_1$. Next, we show that T is an h-$\mathcal{A}_1$-contractive mapping. Let $x,y \in X$ with $x \neq y$. Then four cases may arise.
Case I: *Let $x,y<0$. Then $Tx=Ty$ and so*

$$h(Tx,Ty) < f(h(x,y), \min\{h(x,Ty), h(Tx,y)\}).$$

Case II: *Let $x,y \geq 0$. Then*

$$h(Tx,Ty) = \left(1+\frac{x}{2}-1-\frac{y}{2}\right)^2 = \frac{(x-y)^2}{4} < (x-y)^2.$$

Therefore,

$$h(Tx,Ty) < f(h(x,y), \min\{h(x,Ty), h(Tx,y)\}).$$

Case III: *Let $x<0, y \geq 0$. Then*

$$h(Tx,Ty) = \left(1-1-\frac{y}{2}\right)^2 = \frac{y^2}{4} < (x-y)^2.$$

Thus

$$h(Tx,Ty) < f(h(x,y), \min\{h(x,Ty), h(Tx,y)\}).$$

Case IV: *Let $x \geq 0, y<0$. Then*

$$h(Tx,Ty) = \left(1+\frac{x}{2}-1\right)^2 = \frac{x^2}{4} < (x-y)^2.$$

Therefore,

$$h(Tx,Ty) < f(h(x,y), \min\{h(x,Ty), h(Tx,y)\}).$$

Hence T is an h-$\mathcal{A}_1$-contractive mapping.

Now for $1 \in X$, $T(1) = 1+\frac{1}{2}$, $T^2(1) = 1+\frac{1}{2}+\frac{1}{2^2}, \ldots, T^{n_j}(1) = \sum_{k=0}^{n_j} \frac{1}{2^k}$ and hence $\lim_{n_j\to\infty} T^{n_j}(1) = \lim_{n_j\to\infty} \sum_{k=0}^{n_j} \frac{1}{2^k} = 2$. Therefore, $1 \in S$ and by Theorem 3.2, 2 is the unique fixed point of T. Indeed, it is so.

In the following example, we show that similar to h-$\mathcal{A}$-contractive mappings, an h-$\mathcal{A}_1$-contractive mapping need not have a fixed point in h-complete topological spaces.

Example 3.4. *Let us choose $X=\mathbb{N}$ and take the discrete topology on X. We define $h : X \times X \to \mathbb{R}_+$ by*

$$h(x,y) = \begin{cases} 0 & \text{if } x=y; \\ 1+\left|\frac{1}{x}-\frac{1}{y}\right| & \text{if } x \neq y; \end{cases}$$

$f : \mathbb{R}_+^2 \to \mathbb{R}_+$ by $f(x_1,x_2) = \max\{x_1,x_2\}$ for all $x_1, x_2, \in \mathbb{R}_+$; and $T : X \to X$ by $Tx = 2x$ for all $x \in X$. Then h is continuous and $h(x,y)=0 \Leftrightarrow x=y$ and $f \in \mathcal{A}_1$. Also, for $x,y \in X$ with $x \neq y$, we have

$$h(Tx,Ty) = 1+\left|\frac{1}{2x}-\frac{1}{2y}\right| = 1+\frac{1}{2}\left|\frac{1}{x}-\frac{1}{y}\right|$$

and

$$h(x,y) = 1+\left|\frac{1}{x}-\frac{1}{y}\right|.$$

Then,

$$h(Tx,Ty) < h(x,y) \leq \max\{h(x,y), \min\{h(x,Ty), h(Tx,y)\}\}.$$

Therefore, T is an h-$\mathcal{A}_1$-contractive mapping. Also, X is h-complete and T has no fixed point.

Remark 3.5. *By the above example, we notice that, in order to reach the existence of fixed points of an h-$\mathcal{A}_1$-contractive mapping in h-complete topological spaces, we need some extra conditions. In the upcoming theorem, we affirm that the extra condition which is considered in Theorem 2.10 also works for h-$\mathcal{A}_1$-contractive mappings.*

Theorem 3.6. *Let X be a topological space and let $T : X \to X$ be an h-$\mathcal{A}_1$-contractive mapping such that either T is orbitally continuous, or T^k is continuous for some $k \in \mathbb{N}$ and h is symmetric in T-orbits. Also assume that $h(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$ and X is h-complete. Also assume that for any $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$f(h(x,y), \min\{h(x,Ty), h(Tx,y)\}) < \epsilon + \delta \Rightarrow h(T^2x, T^2y) \le \frac{\epsilon}{4}. \tag{2}$$

Then T has a unique fixed point.

Proof. Let us choose $x_0 \in X$ and consider two sequences $\{x_n\}$ and $\{q_n\}$ where $x_n = T^n x_0$ and $q_n = h(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. We now prove that $q_n \to 0$ as $n \to \infty$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then we are done. So we assume that $x_n \neq x_{n+1}$ for all $n \ge 1$. Then $\{q_n\}$ is strictly decreasing. Hence $\{q_n\}$ is convergent to some $q \ge 0$. We assume that $q > 0$. Then by (2) there exists $\delta > 0$ such that

$$f(h(x,y), \min\{h(x,Ty), h(Tx,y)\}) < 4q + \delta \Rightarrow h(T^2x, T^2y) \le q.$$

Since $q_n \to q$, we get a natural number n such that

$$q_n < q + \frac{\delta}{2}, \text{ i.e., } h(x_n, x_{n+1}) < q + \frac{\delta}{2}.$$

Then

$$\begin{aligned}
&f(h(x_n, x_{n+1}), \min\{h(x_n, x_{n+2}), h(x_{n+1}, x_{n+1})\}) = f(h(x_n, x_{n+1}), 0) \\
&\le h(x_n, x_{n+1}) + 0 < q + \frac{\delta}{2} < q + \delta.
\end{aligned}$$

This implies that $h(x_{n+2}, x_{n+3}) \le q$, i.e., $q_{n+2} \le q$, which is a contradiction. Therefore, $q = 0$, i.e., $\lim_{n\to\infty} h(x_n, x_{n+1}) = 0$. Next, we show that $\sum_{n=1}^{\infty} h(x_n, x_{n+1}) < \infty$. For this let $\epsilon > 0$ be arbitrary. Then there exists $\delta > 0$ with $\delta < \epsilon$ such that

$$f(h(x,y), \min\{h(x,Ty), h(Tx,y)\}) < 4\epsilon + \delta \Rightarrow h(T^2x, T^2y) \le \epsilon. \tag{3}$$

Since, $\lim_{n\to\infty} h(x_n, x_{n+1}) = 0$, for the above $\delta > 0$, we get $N \in \mathbb{N}$ such that

$$h(x_n, x_{n+1}) < \frac{\delta}{3} < \frac{\epsilon}{3} < \epsilon \text{ for all } n \ge N.$$

Now

$$\begin{aligned}
&f(h(x_N, x_{N+1}), \min\{h(x_N, x_{N+2}), h(x_{N+1}, x_{N+1})\}) = f(h(x_N, x_{N+1}), 0) \\
&< h(x_N, x_{N+1}) + 0 < \frac{\epsilon}{3} < \frac{\epsilon}{3} + \delta.
\end{aligned}$$

Then by (3), we have $h(x_{N+2}, x_{N+3}) \le \frac{\epsilon}{3.4}$. Again

$$\begin{aligned}
&f(h(x_{N+1}, x_{N+2}), \min\{h(x_{N+1}, x_{N+3}), h(x_{N+2}, x_{N+2})\}) = f(h(x_{N+1}, x_{N+2}), 0) \\
&< h(x_{N+1}, x_{N+2}) + 0 < \frac{\epsilon}{3} < \frac{\epsilon}{3} + \delta.
\end{aligned}$$

This implies that $h(x_{N+3}, x_{N+4}) \le \frac{\epsilon}{3.4}$. Also,

$$\begin{aligned}
&f(h(x_{N+2}, x_{N+3}), \min\{h(x_{N+2}, x_{N+4}), h(x_{N+3}, x_{N+3})\}) = f(h(x_{N+2}, x_{N+3}), 0) \\
&< h(x_{N+2}, x_{N+3}) + 0 < \frac{\epsilon}{3.4} < \frac{\epsilon}{3.4} + \delta.
\end{aligned}$$

This implies that $h(x_{N+4}, x_{N+5}) \leq \frac{\epsilon}{3.4^2}$. Further,

$$f(h(x_{N+3}, x_{N+4}), \min\{h(x_{N+3}, x_{N+5}), h(x_{N+4}, x_{N+4})\}) = f(h(x_{N+3}, x_{N+4}), 0)$$
$$< h(x_{N+3}, x_{N+4}) + 0 < \frac{\epsilon}{3.4} < \frac{\epsilon}{3.4} + \delta.$$

Therefore, $h(x_{N+5}, x_{N+6}) \leq \frac{\epsilon}{3.4^2}$. Proceeding in this way, we can show that for any even integers p,

$$h(x_{N+p}, x_{N+p+1}) \leq \frac{\epsilon}{3.4^{\frac{p}{2}}} \text{ and } h(x_{N+p+1}, x_{N+p+2}) \leq \frac{\epsilon}{3.4^{\frac{p}{2}}}.$$

Therefore, if $M := h(x_1, x_2) + h(x_2, x_3) + ... + h(x_{N-1}, x_N)$, then

$$\sum_{n=1}^{\infty} h(x_n, x_{n+1}) < M + \sum_{n=N}^{\infty} h(x_n, x_{n+1}) = M + \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3.4} + \frac{\epsilon}{3.4} + \frac{\epsilon}{3.4^2} + ...$$
$$= M + 2\,[\frac{\epsilon}{3} + \frac{\epsilon}{3.4} + \frac{\epsilon}{3.4^2} + ...] = M + \frac{8\epsilon}{9} < \infty.$$

So by h-completeness of X, there exists $z \in X$, such that $\{x_n\}$ converges to z. The fact that z is the unique fixed point of T can be similarly obtained as that of Theorem 2.10 and Theorem 3.2. □

We complete this section by presenting the following supporting example:

Example 3.7. *Let $X = [0, 1]$ be equipped with the usual topology of $\mathbb{R}$. We define $h : X \times X \to \mathbb{R}_+$ by*

$$h(x, y) = |x^2 - y^2| \quad \text{for all } x, y \in X.$$

Then h is continuous, $h(x, y) = 0 \Leftrightarrow x = y$ and clearly X is h-complete. Further, we define $f : \mathbb{R}_+^2 \to \mathbb{R}_+$ is defined by $f(x_1, x_2) = \max\{x_1, x_2\}$ for all $x_1, x_2 \in \mathbb{R}_+$ and $T : X \to X$ by

$$T(x) = \cos\left(\frac{x^2}{4}\right) \quad \text{for all } x \in X.$$

Then T is orbitally continuous. Let $x, y \in X$ with $x \neq y$. Then

$$h(Tx, Ty) = \left|\cos^2\left(\frac{x^2}{4}\right) - \cos^2\left(\frac{y^2}{4}\right)\right| = \left|\sin^2\left(\frac{x^2}{4}\right) - \sin^2\left(\frac{y^2}{4}\right)\right| = \left|\sin\frac{x^2 + y^2}{4} \sin\frac{x^2 - y^2}{4}\right|$$
$$\leq \frac{|x^2 - y^2|}{4} < f(h(x, y), \min\{h(x, Ty), h(Tx, y)\}).$$

Therefore, T is an h-$\mathcal{A}_1$-contractive mapping. Let $\epsilon > 0$ be arbitrary and take $\delta = \epsilon$. We have

$$h(T^2x, T^2y) = h\left(\cos\left(\frac{\cos^2(\frac{x^2}{4})}{4}\right), \cos^2\left(\frac{\cos^2(\frac{y^2}{4})}{4}\right)\right) = \left|\cos^2\left(\frac{\cos^2(\frac{x^2}{4})}{4}\right) - \cos^2\left(\frac{\cos^2(\frac{y^2}{4})}{4}\right)\right|$$
$$= \left|\sin^2\left(\frac{\cos^2(\frac{x^2}{4})}{4}\right) - \sin^2\left(\frac{\cos^2(\frac{y^2}{4})}{4}\right)\right|$$
$$= \left|\sin\left(\frac{\cos^2(\frac{x^2}{4})}{4} + \frac{\cos^2(\frac{y^2}{4})}{4}\right)\sin\left(\frac{\cos^2(\frac{x^2}{4})}{4} - \frac{\cos^2(\frac{y^2}{4})}{4}\right)\right|$$
$$\leq \left|\sin\left(\frac{\cos^2(\frac{x^2}{4})}{4} - \frac{\cos^2(\frac{y^2}{4})}{4}\right)\right| \leq \left|\frac{\cos^2(\frac{x^2}{4}) - \cos^2(\frac{y^2}{4})}{4}\right| \leq \frac{1}{16}|x^2 - y^2|.$$

Thus

$$h(T^2x, T^2y) \leq \frac{1}{16}|x^2 - y^2| = \frac{h(x, y)}{16}.$$

Therefore, $f(h(x, y), \min\{h(x, Ty), h(Tx, y)\}) < \epsilon + \delta \Rightarrow h(T^2x, T^2y) \leq \frac{\epsilon}{4}$. Thus all the assumptions of Theorem 3.6 hold and so T has a unique fixed point.

4. Application to implicit function theorem

Implicit function theorem has an important application in change of co-ordinate systems (parametrizations) and in dealing with non-linear programming problems. There are a number of different forms of implicit function theorem in various branches of mathematics. Some of these forms depend on Jacobian matrices. In some practical problems, these forms fail to hold because Jacobian matrix becomes singular. In 1978, Jittorntrum [9] introduced the non-differential form of implicit function theorem, which is as follows:

Theorem 4.1. *[9] Suppose that $F : D \subset \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuous mapping with*

$$F(x^0, y^0) = 0.$$

Assume that there exists open neighbourhoods $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^n$ of x^0 and y^0 respectively such that for all $y \in B$, $F(., y) : A \subset \mathbb{R}^n \to \mathbb{R}^n$ is locally one-one. Then, there exist open neighbourhoods $A_0 \subset A$ and $B_0 \subset B$ of x^0 and y^0 respectively such that for all $y \in B_0$, the equation

$$F(x, y) = 0$$

has a unique solution

$$x = Hy \in A_0$$

and the mapping $H : A_0 \to \mathbb{R}^n$ is continuous.

We extend the non-differential form of implicit function theorem from $\mathbb{R}^n$ to linear topological spaces over real field as an application of the Theorem 2.2. In the upcoming theorem, we present such non-differential form.

Theorem 4.2. *Let X_1 be a locally path connected topological space and let X_2 be a connected locally compact linear topological space over the real field. Let $\phi : X_1 \times X_2 \to X_2$ be a continuous function, where $X_1 \times X_2$ is equipped with the product topology. Assume that for each $x \in X_1$, $\phi_x : X_2 \to X_2$ be defined by $\phi_x(y) = \phi(x, y)$ is locally one-one function. Then there exists a neighbourhood $U \times V$ of $(a, b) \in X_1 \times X_2$ and a continuous function $\psi : U \to V$ such that $\phi(x, \psi(x)) = \phi(a, b)$ for all $x \in U$. Furthermore, this ψ is unique.*

Proof. Since X_2 is locally compact and ϕ_x is locally one-one on X_2, there exists a compact neighbourhood V of b in X_2 such that $(\phi_a)^{-1}(\phi(a, b)) \cap V = b$. Let V_k be a compact neighbourhood of b contained in the interior of V. We claim that for any such V_k, there exist a neighbourhood U of a in X_1 such that

$$(\phi_{x_{|_V}})^{-1}(\phi(a, b)) \subset V_k \text{ for all } x \in U. \tag{4}$$

If not then there exists $b' \in V \setminus V_k$ such that $(\phi_{a'_{|_V}})^{-1}(\phi(a, b)) = b'$ for some $a' \in U$. Then we can construct a sequence $\{(a_n, b_n)\}_{n\in\mathbb{N}}$ in $U \times (V \setminus V_k)$ as mentioned in theorem 4.1 of [2] such that $\phi(a_n, b_n) = \phi(a, b)$ and $\lim_{n\to\infty} a_n = a$. Since $\{b_n\}$ is contained in the compact subset $V \setminus V_k^\circ$, so $\{b_n\}$ has a convergent subsequence, which has a limit point $b^* \in V \setminus V_k^\circ$. Therefore (a, b^*) is a limit point of some subsequence $\{(a_{n_k}, b_{n_k})\}_{k\in\mathbb{N}}$ of $\{(a_n, b_n)\}_{n\in\mathbb{N}}$. Now the continuity of f gives $\phi(a, b^*) = \lim_{k\to\infty} \phi(a_{n_k}, b_{n_k}) = \phi(a, b)$. Since ϕ_x is locally one-one we have $b^* = b$, which gives a contradiction. Therefore our assumption (4) holds good. Now we define a mapping $T : U \times V_k \to V_k$ such that,

$$T(x, y) = \frac{1}{2}(y + (\phi_{x_{|_V}})^{-1}(\phi(a, b)).$$

Then T is continuous on $U \times V_k$. We also define, $h : V_k \times V_k \to [0, \infty)$ such that,

$$h(y_1, y_2) = |y_1 - y_2|,$$

for all $y_1, y_2 \in V_k$. Then h is continuous and $h(y_1, y_2) = 0 \Leftrightarrow y_1 = y_2$. Also,

$$h(T(x, y_1), T(x, y_2)) = \frac{1}{2}|y_1 - y_2|.$$

Let $f(u, v, w) = u,$ for all $u, v, w \in \mathbb{R}_+$. Then

$$f(h(y_1, y_2), h(y_1, T(x, y_1)), h(y_2, T(x, y_2)) = |y_1 - y_2|.$$

Therefore,

$$h(T(x, y_1), T(x, y_2)) < f(h(y_1, y_2), h(y_1, T(x, y_1)), h(y_2, T(x, y_2))$$

for all $y_1, y_2 \in V_k$ with $y_1 \neq y_2$ and for all $x \in U$. Therefore, T is an h-$\mathcal{A}$-contractive mapping on X_2 uniformly in X_1. Thus by Corollary 2.3, T has a unique fixed point y^* such that

$$T(x, y^*) = \frac{1}{2}(y^* + (\phi_{x_{|V}})^{-1}(\phi(a, b)) = y^* \Rightarrow y^* = (\phi_{x_{|V}})^{-1}(\phi(a, b)).$$

Therefore, for each $x \in U$, there exists unique $y \in V_k$ such that $y = (\phi_{x_{|V}})^{-1}(\phi(a, b))$, i.e., for each $x \in U$ there exists unique $\psi(x) \in V_k$ such that $\phi(x, \psi(x)) = \phi(a, b)$ for all $x \in U$.

The continuity of ψ follows from the last part of Theorem 4.1 of [2]. $\square$

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