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Vector-valued nonuniform multiresolution analysis associated with linear canonical transform domain

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Abstract. A generalization of Mallat's classical multiresolution analysis, based on the theory of spectral pairs, was considered in two articles by Gabardo and Nashed. In this setting, the associated translation set is no longer a discrete subgroup of $\mathbb{R}$ but a spectrum associated with a certain one-dimensional spectral pair and the associated dilation is an even positive integer related to the given spectral pair. In this paper, we continue the study based on this nonstandard setting and introduce vector-valued nonuniform multiresolution analysis associated with linear canonical transform (LCT-VNUMRA) where the associated subspace V_0^{μ} of the function space $L^2\left(\mathbb{R}, \mathbb{C}^M\right)$ has an orthonormal basis of the form $\left\{\mathbf{\Phi}(x-\lambda)e^{-\frac{\iota\pi A}{B}(t^2-\lambda^2)}\right\}_{\lambda\in\Lambda}$ where $\Lambda = \{0, r/N\} + 2\mathbb{Z}, N \geq 1$ is an integer and r is an odd integer such that r and N are relatively prime. We establish a necessary and sufficient condition for the existence of associated wavelets and derive an algorithm for the construction of vector-valued nonuniform multiresolution analysis starting from a vector refinement mask with appropriate conditions

1. Introduction

Multiresolution analysis (MRA) is an important mathematical tool since it provides a natural framework for understanding and constructing discrete wavelet systems. An MRA is an increasing family of closed subspaces $\left\{V_j : j \in \mathbb{Z}\right\}$ of $L^2(\mathbb{R})$ such that $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$, $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$ and which satisfies $f \in V_j$ if and only if $f(2\cdot) \in V_{j+1}$. Furthermore, there exists an element $\varphi \in V_0$ such that the collection of integer translates of function φ, $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ represents a complete orthonormal system for V_0. The function φ is called the *scaling function* or the *father wavelet*. The concept of MRA has been extended in various ways in recent years. These concepts are generalized to $L^2\left(\mathbb{R}^d\right)$, to lattices different from $\mathbb{Z}^d$, allowing the subspaces of MRA to be generated by Riesz basis instead of orthonormal basis, admitting a finite number of scaling functions, replacing the dilation factor 2 by an integer $M \geq 2$ or by an expansive matrix $A \in GL_d(\mathbb{R})$ as long as $A \subset A\mathbb{Z}^d$. On the other hand, Xia and Suter [20] introduced the concept of vector-valued MRA and orthogonal vector-valued wavelet basis and showed that vector-valued wavelets are a class of generalized multiwavelets. Chen and Cheng [11] presented the construction of a class of compactly supported orthogonal vector-valued wavelets and investigated the properties of vector-valued wavelet packets. Vector-valued wavelets are a class of generalized multiwavelets and multiwavelets can be generated from the component function in vector-valued wavelets. Vector-valued wavelets and

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multiwavelets are different in the following sense. Vector-valued wavelets can be used to decorrelate a vector-valued signal not only in the time domain but also between components for a fixed time where as multiwavelets focuses only on the decorrelation of signals in time domain. Moreover, prefiltering is usually required for discrete multiwavelet transform but not necessary for discrete vector-valued wavelet transforms. But all these concepts are developed on regular lattices, that is the translation set is always a group. Recently, Gabardo and Nashed [13, 14] considered a generalization of Mallat's [15] celebrated theory of MRA based on spectral pairs, in which the translation set acting on the scaling function associated with the MRA to generate the subspace V_0 is no longer a group, but is the union of $\mathbb{Z}$ and a translate of $\mathbb{Z}$. More results in this direction can be found in [16–18].

The concept of novel MRA in nonuniform settings was established by Shah et.al [19]. They call it Nonuniform Multiresolution analysis associated with linear canonical transform (LCT-NUMRA). They also constructed associated wavelet packets and presented orthogonal decomposition. In this paper, we continue the study based on this nonstandard setting and introduce vector-valued nonuniform multiresolution analysis associated with linear canonical transform (LCT-VNUMRA) where the associated subspace V_0^{μ} of $L^2\Big(\mathbb{R}, \mathbb{C}^M\Big)$ has an orthonormal basis of the form $\Big\{\mathbf{\Phi}(x-\lambda)e^{-\frac{-\iota\pi A}{B}(t^2-\lambda^2)}\Big\}_{\lambda\in\Lambda}$ where $\Lambda = \{0, r/N\} + 2\mathbb{Z}, N \geq 1$ is an integer and r is an odd integer such that r and N are relatively prime. We establish a necessary and sufficient condition for the existence of associated wavelets and derive an algorithm for the construction of vector-valued MRA analysis starting from a vector refinement mask with appropriate conditions. For more about in the direction of linear canonical transform domains, we refer to [1–9, 12] and the references therein.

This paper is organized as follows. In Section 2, we review the uniform and non-uniform MRA associated with LCT and certain properties related to the construction of associated wavelets. In Section 3, we introduce the notion of LCT-VNUMRA and establish a necessary and sufficient condition for the existence of associated wavelet. In Section 4, we construct a LCT-VNUMRA starting from a vector refinement mask satisfying appropriate conditions. In Section 5, we provided the conclusion of the paper.

2. Nonuniform Multiresolution Analysis Associated with Linear Canonical Transform

For the sake of simplicity, we consider the second order matrix $\mu_{2\times 2} = (A, B, C, D)$ with its transpose defined by $\mu_{2\times 2}^T = (A, B, C, D)^T$. Let us first introduce the definition of LCT.

Definition 2.1. *The LCT of any $f \in L^2(\mathbb{R})$ with respect to the unimodular matrix $\mu_{2\times 2} = (A, B, C, D)$ is defined by*

$$\mathcal{L}[f](\xi) = \begin{cases} \int_{\mathbb{R}} f(t)\mathcal{K}_{\mu}(t,\xi)dt & B \neq 0 \\ \\ \sqrt{D}\exp\frac{CD\xi^2}{2}f(D\xi) & B = 0. \end{cases} \tag{1}$$

where $\mathcal{K}_{\mu}(t,\xi)$ is the kernel of linear canonical transform and is given by

$$\mathcal{K}_{\mu}(t,\xi) = \frac{1}{\sqrt{2\pi\iota B}}\exp\left\{\frac{\iota(At^2 - 2t\xi + D\xi^2)}{2B}\right\}, \quad B \neq 0$$

It is here noted that for the case $B = 0$, the LCT defined by equation (1) corresponds to a chirp multiplication operation and is therefore of no particular interest to us. As such, in the rest of the article, we will keep our focus on the case when $B \neq 0$. It is here worth noticing that the phase-space transform (1) is lossless if and only if the matrix μ is unimodular; that is, $AD - BC = 1$[21]. Several special transforms can be obtained from the LCT (1). For example, for $\mu = (1, B, 0, 1)$, gives the Fresnel transform, for $\mu = (\cos\theta, \sin\theta, -\sin\theta, \cos\theta)$ the LCT yields us the fractional Fourier transform whereas for $\mu = (0, 1, -1, 0)$, we reach at the classical Fourier transform. Moreover, Bi-lateral Laplace, Gauss-Weierstrass, and Bargmann transform are also its special cases [10].

The inversion formula corresponding to LCT (1) is defined by

$$f(t) = \int_{\mathbb{R}} \mathcal{L}[f](\xi)\overline{\mathcal{K}_{\mu}(t,\xi)}d\xi.$$

Morever the well known Parsevel's formula of the linear canonical transform (1) may be stated as

$$\langle \mathcal{L}[f], \mathcal{L}[g] \rangle = \langle f, g \rangle, \quad \text{for all} \quad f, g, L^2(\mathbb{R}).$$

Recently, Shah et.al [19] considered a generalization of the notion of LCT associated with linear canonical transform, which is called *nonuniform multiresolution analysis associated with linear canonical transform* (LCT-NUMRA) and is based on the theory of spectral pairs. In this set up, the associated subspace V_0^μ of $L^2(\mathbb{R})$ has an orthonormal basis, a collection of translates of the scaling function φ of the form $\left\{\varphi(t-\lambda)e^{-\frac{\iota\pi A}{B}(t^2-\lambda^2)}\right\}_{\lambda\in\Lambda}$ where $\Lambda = \{0, r/N\} + 2\mathbb{Z}, N \geq 1$ is an integer and r is an odd integer such that r and N are relatively prime.

We first recall the definition of a LCT-VNUMRA (as defined in [19]) and some of its properties.

Definition 2.2. *For an integer $N \geq 1$ and an odd integer r with $1 \leq r \leq 2N-1$ such that r and N are relatively prime, a nonuniform multiresolution analysis associated with linear canonical transform is a sequence of closed subspaces $\left\{V_j^\mu : j \in \mathbb{Z}\right\}$ of $L^2(\mathbb{R})$ such that the following properties hold:*

(a) $V_j^\mu \subset V_{j+1}^\mu$ *for all* $j \in \mathbb{Z}$;

(b) $\bigcup_{j\in\mathbb{Z}} V_j^\mu$ *is dense in* $L^2(\mathbb{R})$;

(c) $\bigcap_{j\in\mathbb{Z}} V_j^\mu = \{0\}$;

(d) $f(t) \in V_j^\mu$ *if and only if* $f(2N\cdot)\, e^{-\iota\pi A\left(1-(2N)^2\right)t^2/B} \in V_{j+1}^\mu$ *for all* $j \in \mathbb{Z}$;

(e) There exists a function φ *in* V_0^μ *such that* $\left\{\varphi(t-\lambda)e^{-\frac{\iota\pi A}{B}(t^2-\lambda^2)} : \lambda \in \Lambda\right\}$, *is a complete orthonormal basis for* V_0^μ.

Given a LCT- NUMRA $\left\{V_j^\mu : j \in \mathbb{Z}\right\}$, we define another sequence $\left\{W_j^\mu : j \in \mathbb{Z}\right\}$ of closed subspaces of $L^2(\mathbb{R})$ by $W_j^\mu := V_{j+1}^\mu \ominus V_j^\mu, j \in \mathbb{Z}$. These subspaces inherit the scaling property of V_j^μ, namely,

$$f(\cdot) \in W_j^\mu \quad \text{if and only if} \quad f(2N\cdot)\, e^{2\iota\pi\lambda\xi/B} \in W_{j+1}^\mu. \tag{2}$$

Moreover, the subspaces $\left\{W_j^\mu : j \in \mathbb{Z}\right\}$ are mutually orthogonal, and we have the following orthogonal decomposition:

$$L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j^\mu = V_0^\mu \oplus \left(\bigoplus_{j\geq 0} W_j^\mu\right). \tag{3}$$

A set of functions $\left\{\psi_1^\mu, \psi_1^\mu, \ldots, \psi_{2N-1}^\mu\right\}$ in $L^2(\mathbb{R})$ is said to be a *set of basic wavelets* associated with the LCT-NUMRA $\left\{V_j^\mu : j \in \mathbb{Z}\right\}$ if the family of functions $\left\{\psi_\ell(t-\lambda)e^{-\frac{\iota\pi A}{B}(t^2-\lambda^2)} : 1 \leq \ell \leq 2N-1, \lambda \in \Lambda\right\}$ forms an orthonormal basis for W_0^μ. In view of (2) and (3), it is clear that if $\{\psi_1, \psi_1, \ldots, \psi_{2N-1}\}$ is a set of basic wavelets, then $\left\{(2N)^{j/2}\psi_\ell\left((2N)^j t - \lambda\right)e^{-\frac{\iota\pi A}{B}(t^2-\lambda^2)} : 1 \leq \ell \leq 2N-1, \lambda \in \Lambda\right\}$ constitutes an orthonormal basis for $L^2(\mathbb{R})$.

3. Vector-valued Nonuniform Multiresolution Associated with Linear Canonical Transform

In this section, we introduce the notion of vector-valued nonuniform multiresolution analysis associated with linear canonical transform (LCT-VNUMRA) and establish a necessary and sufficient condition for the existence of associated wavelets.

Let M be a constant and $2 \leq M \in \mathbb{Z}$. By $L^2\left(\mathbb{R}, \mathbb{C}^M\right)$, we denote the set of all vector-valued functions $\mathbf{f}(x)$ i.e.,

$$L^2\left(\mathbb{R}, \mathbb{C}^M\right) = \left\{\mathbf{f}(x) = \left(f_1(x), f_2(x), \ldots, f_M(x)\right)^T : x \in \mathbb{R}, f_t(x) \in L^2(\mathbb{R}), t = 1, 2, \ldots, M\right\}$$

where T means the transpose of a vector. The space $L^2\Big(\mathbb{R}, \mathbb{C}^M\Big)$ is called *vector-valued function space*. For $\mathbf{f}(x) \in L^2\Big(\mathbb{R}, \mathbb{C}^M\Big)$, $\big\|\mathbf{f}\big\|$ denotes the norm of vector-valued function $\mathbf{f}$ and is defined as:

$$\big\|\mathbf{f}\big\|_2 = \left(\sum_{t=1}^{M} \int_{\mathbb{R}} \big|f_t(x)\big|^2 dx\right)^{1/2}. \tag{4}$$

For a vector-valued function $\mathbf{f}(x) \in L^2\Big(\mathbb{R}, \mathbb{C}^M\Big)$, the integration of $\mathbf{f}(x)$ is defined as:

$$\int_{\mathbb{R}} \mathbf{f}(x)dx = \left(\int_{\mathbb{R}} f_1(x)dx, \int_{\mathbb{R}} f_2(x)dx, \ldots, \int_{\mathbb{R}} f_M(x)dx\right)^T.$$

For any two vector-valued functions $\mathbf{f}, \mathbf{g} \in L^2\Big(\mathbb{R}, \mathbb{C}^M\Big)$, their vector-valued inner product $\langle \mathbf{f}, \mathbf{g}\rangle$ is defined as:

$$\langle \mathbf{f}, \mathbf{g}\rangle = \int_{\mathbb{R}} \mathbf{f}(x)\overline{\mathbf{g}(x)}\, dx. \tag{5}$$

With $\Lambda = \{0, r/N\} + 2\mathbb{Z}$ as defined above, we define the *vector-valued nonuniform multiresolution analysis associated with linear canonical transform* (LCT-VNUMRA) as follows:

Definition 3.1. *Given a real uni-modular matrix $\mu = (A, B, C, D)$ and an integer $N \geq 1$ and an odd integer r with $1 \leq r \leq 2N - 1$ such that r and N are relatively prime, an associated linear canonical vector-valued non-uniform multiresolution analysis (LCT-LCT-VNUMRA) is a sequence of closed subspaces $\Big\{V_j^\mu : j \in \mathbb{Z}\Big\}$ of $L^2\Big(\mathbb{R}, \mathbb{C}^M\Big)$ such that the following properties hold:*

- *(a) $V_j^\mu \subset V_{j+1}^\mu$ for all $j \in \mathbb{Z}$;*
- *(b) $\bigcup_{j\in\mathbb{Z}} V_j^\mu$ is dense in $L^2\Big(\mathbb{R}, \mathbb{C}^M\Big)$;*
- *(c) $\bigcap_{j\in\mathbb{Z}} V_j^\mu = \{\mathbf{0}\}$, where $\mathbf{0}$ is the zero vector of $L^2\Big(\mathbb{R}, \mathbb{C}^M\Big)$;*
- *(d) $\mathbf{\Phi}(t) \in V_j^\mu$ if and only if $\mathbf{\Phi}(2Nt)e^{-\iota\pi A\left(1-(2N)^2\right)t^2/B} \in V_{j+1}^\mu$ for all $j \in \mathbb{Z}$;*
- *(e) There exists a function Φ in V_0^μ such that $\Big\{\Phi_{0,\lambda}^\mu(t) = \mathbf{\Phi}(t-\lambda)e^{-\frac{\iota\pi A}{B}(t^2-\lambda^2)} : \lambda \in \Lambda\Big\}$, is a complete orthonormal basis for V_0^μ. The vector valued function $\mathbf{\Phi}(x)$ is called a vector-valued scaling function of the LCT-VNUMRA.*

For every $j \in \mathbb{Z}$, define W_j^μ to be the orthogonal complement of V_j^μ in V_{j+1}^μ. Then we have

$$V_{j+1}^\mu = V_j^\mu \oplus W_j^\mu \quad \text{and} \quad W_\ell^\mu \perp W_\ell'^\mu \quad \text{if } \ell \neq \ell'. \tag{6}$$

It follows that for $j > J$,

$$V_j^\mu = V_J^\mu \oplus \bigoplus_{\ell=0}^{j-J-1} W_{j-\ell}^\mu \tag{7}$$

where all these subspaces are orthogonal. By virtue of condition (b) in the Definition 3.1, this implies

$$L^2\Big(\mathbb{R}, \mathbb{C}^M\Big) = \bigoplus_{j\in\mathbb{Z}} W_j^\mu, \tag{8}$$

a decomposition of $L^2\Big(\mathbb{R}, \mathbb{C}^M\Big)$ into mutually orthogonal subspaces.

As in the standard case, one expects the existence of $2N - 1$ number of functions so that their translation by elements of Λ and dilations by the integral powers of $2N$ form an orthonormal basis for $L^2\Big(\mathbb{R}, \mathbb{C}^M\Big)$.

Definition 3.2. *A set of functions* $\left\{\mathbf{\Psi}_1^{\mu}, \mathbf{\Psi}_2^{\mu}, \ldots, \mathbf{\Psi}_{2N-1}^{\mu}\right\}$ *in* $L^2\left(\mathbb{R}, \mathbb{C}^M\right)$ *will be called a set of basic wavelets associated with a given LCT-VNUMRA if the family of functions* $\left\{\mathbf{\Psi}_{\ell}(t-\lambda)e^{-\frac{-\iota\pi A}{B}(t^2-\lambda^2)} : 1 \leq \ell \leq 2N-1, \lambda \in \Lambda\right\}$ *forms an orthonormal basis for* W_0^{μ}.

In the following, we want to seek a set of wavelet functions $\left\{\mathbf{\Psi}_1^{\mu}, \mathbf{\Psi}_2^{\mu}, \ldots, \mathbf{\Psi}_{2N-1}^{\mu}\right\}$ in W_0^{μ} such that $\left\{(2N)^{j/2}\mathbf{\Psi}_{\ell}\left((2N)^j t-\lambda\right)e^{-\frac{\iota\pi A}{B}(t^2-\lambda^2)} : 1 \leq \ell \leq 2N-1, \lambda \in \Lambda\right\}$ form an orthonormal basis of W_j^{μ}. By the nested structure of LCT-LCT-VNUMRA, this task can be reduce to find $\mathbf{\Psi}_{\ell}\mu \in W_0^{\mu}$ such that $\left\{\mathbf{\Psi}_{\ell}\left(t-\lambda\right)e^{-\frac{\iota\pi A}{B}(t^2-\lambda^2)} : 1 \leq \ell \leq 2N-1, \lambda \in \Lambda\right\}$ constitutes an orthonormal basis of W_0^{μ}.

Let $\mathbf{\Phi} = \left(\varphi_1^{\mu}, \varphi_2^{\mu}, \ldots, \varphi_M^{\mu}\right)^T$ be a scaling vector of the given LCT-VNUMRA. Since $\mathbf{\Phi} \in V_0^{\mu} \subset V_1^{\mu}$, there exist $M \times M$ constant matrix sequence $\{G_{\lambda}\}_{\lambda\in\Lambda}$ such that

$$\mathbf{\Phi}(t) = \sqrt{2N}\sum_{\lambda\in\Lambda} G_{\lambda}\mathbf{\Phi}\left(2Nt-\lambda\right)e^{-\frac{\iota\pi A}{B}(t^2-\lambda^2)}. \tag{9}$$

where $G_{\lambda} = \displaystyle\int_{\mathbb{R}} \mathbf{\Phi}(t)e^{-\iota\pi A\left(1-(2N)^2\right)t^2/B}\overline{\phi_{1,\lambda}^{\mu}(t)}dt.$

Taking LCT on both sides of equation (9), we obtain

$$\mathcal{L}[\mathbf{\Phi}(t)](\xi) = \hat{\mathbf{\Phi}}\left(\frac{\xi}{B}\right) = G^{\mu}\left(\frac{\xi}{2NB}\right)\hat{\mathbf{\Phi}}\left(\frac{\xi}{2NB}\right), \tag{10}$$

where $G^{\mu}\left(\frac{\xi}{B}\right) = \frac{1}{\sqrt{2N}}\sum_{\lambda\in\Lambda} G_{\lambda}^{\mu}\overline{e^{-2\pi\iota\lambda\xi/B}}$, is called *symbol* or *vector refinement mask* of the scaling function Φ. By replacing ξ by $\xi/2NB$ in relation (10), we obtain

$$\hat{\mathbf{\Phi}}\left(\frac{\xi}{2NB}\right) = G^{\mu}\left(\left(\frac{1}{2NB}\right)^2\xi\right)\hat{\mathbf{\Phi}}\left(\left(\frac{1}{2NB}\right)^2\xi\right),$$

and then

$$\hat{\mathbf{\Phi}}(\xi) = G^{\mu}\left(\frac{\xi}{2NB}\right)G^{\mu}\left(\left(\frac{1}{2NB}\right)^2\xi\right)\hat{\mathbf{\Phi}}\left(\left(\frac{1}{2NB}\right)^2\xi\right).$$

We can continue this and obtain, for any $n \in \mathbb{N}$,

$$\begin{aligned}\hat{\mathbf{\Phi}}(\xi) &= G^{\mu}\left(\frac{\xi}{2NB}\right)G^{\mu}\left(\left(\frac{1}{2NB}\right)^2\xi\right)\cdots G^{\mu}\left(\left(\frac{1}{2NB}\right)^n\xi\right)\hat{\mathbf{\Phi}}\left(\left(\frac{1}{2NB}\right)^n\xi\right)\\ &= \hat{\mathbf{\Phi}}\left(\left(\frac{1}{2NB}\right)^n\xi\right)\prod_{m=1}^{n} G^{\mu}\left(\left(\frac{1}{2NB}\right)^m\xi\right).\end{aligned}$$

By taking $n \to \infty$ and noting that $\left|\left(\frac{1}{2NB}\right)^n\right| = \frac{1}{(2NB)^n} \to 0$ as $n \to \infty$, the above relation reduces to

$$\hat{\mathbf{\Phi}}(\xi) = \hat{\mathbf{\Phi}}(0)\prod_{m=1}^{\infty} G^{\mu}\left(\left(\frac{1}{2NB}\right)^m\xi\right). \tag{11}$$

As usual, we assume $\hat{\mathbf{\Phi}}(\xi)$ is continuous at zero, and $\hat{\mathbf{\Phi}}(0) = I_M$, where I_M denotes the identity matrix of order $M \times M$. Therefore, equation (11) becomes

$$\hat{\mathbf{\Phi}}(\xi) = \prod_{m=1}^{\infty} G^{\mu}\left(\left(\frac{1}{2NB}\right)^m\xi\right) \tag{12}$$

Moreover, it is immediate from (10) that $G(0) = I_M$, which is essential for convergence of the infinite product $\prod_{m=1}^{\infty} G^{\mu}\left(\left(\frac{1}{2NB}\right)^m \xi\right)$.

We now investigate the orthogonal property of the scaling function $\mathbf{\Phi}$ by means of the vector refinement mask $G(\xi)$.

Lemma 3.3. *If $\mathbf{\Phi} \in L^2\left(\mathbb{R}, \mathbb{C}^M\right)$ defined by Equation (9) is an orthogonal vector-valued scaling function, then we have*

$$\sum_{m\in 2\mathbb{Z}} G^{\mu}_m \overline{G^{\mu}_{2NB(\lambda-\lambda')+m}} = 2NB\delta_{\lambda,\lambda'} I_M, \quad \forall\, \lambda, \lambda' \in \Lambda, \tag{13}$$

where $\delta_{\lambda,\lambda'}$ denotes the Kronecker's delta.

Proof. Since the scaling function is orthogonal vector-valued, we have

$$\begin{aligned}
\delta_{\lambda,\lambda'} I_M &= \int_{\mathbb{R}} \mathbf{\Phi}(t-\lambda) e^{-\frac{-\iota\pi A}{B}(t^2-\lambda^2)} \overline{\mathbf{\Phi}(t-\lambda') e^{-\frac{-\iota\pi A}{B}(t^2-\lambda'2)}} dt \\
&= \sum_{\sigma\in\Lambda} \int_{\mathbb{R}} G^{\mu}_{\sigma} \mathbf{\Phi}\Big(2NBt - 2NB\lambda - \sigma\Big) \sum_{\sigma\in\Lambda} \overline{G^{\mu}_{\sigma}}\, \overline{\mathbf{\Phi}\Big(2NBt - 2NB\lambda' - \sigma\Big)} dt \\
&= \sum_{\sigma\in\Lambda}\sum_{\sigma\in\Lambda} G^{\mu}_{\sigma} \left\{ \int_{\mathbb{R}} \mathbf{\Phi}\Big(2NBt - 2NB\lambda - \sigma\Big) \overline{\mathbf{\Phi}\Big(2NBt - 2NB\lambda' - \sigma\Big)} \right\} dt \overline{G^{\mu}_{\sigma}} \\
&= \frac{1}{2NB} \sum_{\sigma\in\Lambda}\sum_{\sigma\in\Lambda} G^{\mu}_{\sigma} \left\{ \int_{\mathbb{R}} \mathbf{\Phi}\Big(t - 2NB\lambda - \sigma\Big) \overline{\mathbf{\Phi}\Big(t - 2NB\lambda' - \sigma\Big)} \right\} dt \overline{G^{\mu}_{\sigma}}.
\end{aligned}$$

Taking $\sigma = 2m$ and $\sigma = 2n$, where $m, n \in \mathbb{Z}$, we have

$$\begin{aligned}
\delta_{\lambda,\lambda'} I_M &= \frac{1}{2NB} \sum_{\sigma\in\Lambda}\sum_{\sigma\in\Lambda} G^{\mu}_{\sigma} \left\langle \mathbf{\Phi}\Big(t - 2NB\lambda - \sigma\Big), \overline{\mathbf{\Phi}\Big(t - 2NB\lambda' - \sigma\Big)} \right\rangle \overline{G^{\mu}_{\sigma}} \\
&= \frac{1}{2NB} \sum_{m\in\mathbb{N}_0}\sum_{n\in\mathbb{N}_0} G^{\mu}_{2m} \left\langle \mathbf{\Phi}\Big(t - 2NB\lambda - 2m\Big), \overline{\mathbf{\Phi}\Big(t - 2NB\lambda' - 2n\Big)} \right\rangle \overline{G^{\mu}_{2n}} \\
&= \frac{1}{2NB} \sum_{m\in\mathbb{N}_0} G^{\mu}_{2m} \overline{G^{\mu}_{2NB(\lambda-\lambda')+2m}}.
\end{aligned}$$

Therefore, identity (13) follows.

Taking $\sigma = \dfrac{r}{NB} + 2m$ and $\sigma = 2n$, where $m, n \in \mathbb{Z}$, we have

$$\begin{aligned}
\delta_{\lambda,\lambda'} I_M &= \frac{1}{2NB} \sum_{\sigma\in\Lambda}\sum_{\sigma\in\Lambda} G^{\mu}_{\sigma} \left\langle \mathbf{\Phi}\Big(t - 2NB\lambda - \sigma\Big), \overline{\mathbf{\Phi}\Big(t - 2NB\lambda' - \sigma\Big)} \right\rangle \overline{G^{\mu}_{\sigma}} \\
&= \frac{1}{2NB} \sum_{m\in2\mathbb{Z}}\sum_{n\in2\mathbb{Z}} G^{\mu}_{\frac{r}{NB}+2m} \Big\langle \mathbf{\Phi}\Big(t - 2NB\lambda - \frac{r}{NB} - 2m\Big), \\
&\qquad\qquad \overline{\mathbf{\Phi}\Big(t - 2NB\lambda' - 2n\Big)} \Big\rangle \overline{G^{\mu}_{2n}} \\
&= \frac{1}{2NB} \sum_{m\in2\mathbb{Z}} G^{\mu}_{2m} \overline{G^{\mu}_{2NB(\lambda-\lambda')+2m}}.
\end{aligned}$$

Thus, in both the cases, we get the desired result. □

We denote $\mathbf{\Psi}_0 = \mathbf{\Phi}$, the scaling function, and consider $2N-1$ functions $\mathbf{\Psi}^{\mu}_{\ell}, 1 \leq \ell \leq 2N-1$, in W^{μ}_0 as possible candidates for wavelets. Since $(1/2NB)\,\mathbf{\Psi}^{\mu}_{\ell}(1/2NBt) \in V^{\mu}_{-1} \subset V^{\mu}_{0}$, it follows from property (d) of Definition 3.1 that for each ℓ, $0 \leq \ell \leq 2N-1$, there exists a uniquely supported sequence $\left\{H^{\ell}_{\lambda}\right\}_{\lambda\in\Lambda,\, 1\leq\ell\leq 2N-1}$ of $M \times M$ constant matrices such that

$$\mathbf{\Psi}^{\mu}_{\ell}(t) = \sqrt{2N}\sum_{\lambda\in\Lambda} H_{\lambda,\ell}\,\mathbf{\Phi}\,(2Nt-\lambda)\,e^{-\frac{\iota\pi A}{B}(t^2-\lambda^2)}. \tag{14}$$

On taking the LCT on both sides of Equation (14), we have

$$\hat{\mathbf{\Psi}}^{\mu}_{\ell}\left(2N\frac{\xi}{B}\right) = H^{\mu}_{\ell}\left(\frac{\xi}{B}\right)\hat{\mathbf{\Phi}}\left(\frac{\xi}{B}\right), \tag{15}$$

where

$$H^{\mu}_{\ell}\left(\frac{\xi}{B}\right) = \frac{1}{\sqrt{2N}}\sum_{\lambda\in\Lambda} H^{\mu}_{\lambda,\ell}\,e^{-2\pi\iota\lambda\xi/B}. \tag{16}$$

In view of the specific form of $\Lambda = \left\{0, \dfrac{r}{N}\right\} + 2\mathbb{Z}$, we observe that

$$H^{\mu}_{\ell}\left(\frac{\xi}{B}\right) = H^{\mu,1}_{\ell}\left(\frac{\xi}{B}\right) + e^{-2\pi\iota\xi/NB}H^{\mu,2}_{\ell}\left(\frac{\xi}{B}\right), \quad 0 \leq \ell \leq 2N-1, \tag{17}$$

where $H^{\mu,1}_{\ell}$ and $H^{\mu,2}_{\ell}$ are $M \times M$ constant symmetric matrix sequences.

Lemma 3.4. *Consider a LCT-VNUMRA on $\mathbb{R}$ as in Definition 3.1. Suppose that there exist $2N-1$ functions $\mathbf{\Psi}_k$, $k = 1, 2, \ldots, 2N-1$ in V_1. Then the family of functions*

$$\left\{\mathbf{\Psi}_k(t-\lambda)e^{-\frac{\iota\pi A}{B}(t^2-\lambda^2)} : \lambda \in \Lambda,\, k = 0, 1, \ldots, 2N-1\right\} \tag{18}$$

forms an orthonormal system in V_1 if and only if

$$\sum_{r=0}^{2N-1} H^{\mu}_{k}\left(\frac{\xi}{2NB} + \frac{r}{4N}\right)\overline{H^{\mu}_{\ell}\left(\frac{\xi}{2NB} + \frac{r}{4N}\right)} = \delta_{k,\ell}\,I_M, \quad 0 \leq k, \ell \leq 2N-1. \tag{19}$$

Proof. Firstly, we will prove the necessary condition. By the orthonormality of the system $\{\mathbf{\Psi}_k(t-\lambda)e^{-\frac{\iota\pi A}{B}(t^2-\lambda^2)}\}_{\lambda\in\Lambda,\, k=0,1,\ldots,2N-1}$, we have

$$\left\langle \mathbf{\Psi}_k(t-\lambda), \mathbf{\Psi}_\ell(t-\sigma)\right\rangle = \int_{\mathbb{R}} \mathbf{\Psi}_k(t-\lambda)e^{-\frac{\iota\pi A}{B}(t^2-\lambda^2)}\,\overline{\mathbf{\Psi}_\ell(t-\sigma)e^{-\frac{\iota\pi A}{B}(t^2-\sigma^2)}}\,dt = e^{\iota\pi\frac{A}{B}(\lambda^2-\sigma^2)}\delta_{k,\ell}\,\delta_{\lambda,\sigma}\,I_M,$$

where $\lambda, \sigma \in \Lambda$ and $k, \ell \in \{0, 1, 2, \ldots, 2N-1\}$. Above relation can be recast in LCT domain as

$$\begin{aligned}\delta_{k,\ell}\,\delta_{\lambda,\sigma}I_M &= \int_{\mathbb{R}} \hat{\mathbf{\Psi}}_k\left(\frac{\xi}{B}\right)e^{\frac{-2\pi\iota\xi\lambda}{B}}\,\overline{\hat{\mathbf{\Psi}}_\ell\left(\frac{\xi}{B}\right)}e^{\frac{2\pi\iota\xi\sigma}{B}}d\xi \\ &= \int_{\mathbb{R}} \hat{\mathbf{\Psi}}_k\left(\frac{\xi}{B}\right)\overline{\hat{\mathbf{\Psi}}_\ell\left(\frac{\xi}{B}\right)}e^{\frac{-2\pi\iota\xi}{B}(\lambda-\sigma)}d\xi.\end{aligned}$$

Taking $\lambda = 2m$ and $\sigma = 2n$, where $m, n \in \mathbb{Z}$, we have

$$\begin{aligned}\delta_{k,\ell}\,\delta_{m,n}I_M &= \frac{1}{B}\int_{\mathbb{R}} e^{\frac{-4\pi\iota\xi}{B}(m-n)}\hat{\mathbf{\Psi}}_k\left(\frac{\xi}{B}\right)\overline{\hat{\mathbf{\Psi}}_\ell\left(\frac{\xi}{B}\right)}d\xi \\ &= \frac{1}{B}\int_{[0,BN]} e^{\frac{-4\pi\iota\xi}{B}(m-n)}\sum_{j\in\mathbb{Z}}\hat{\mathbf{\Psi}}_k\left(\frac{\xi}{B}+Nj\right)\overline{\hat{\mathbf{\Psi}}_\ell\left(\frac{\xi}{B}+Nj\right)}d\xi.\end{aligned}$$

Define

$$F_{k,\ell}\left(\frac{\xi}{B}\right)=\sum_{j\in\mathbb{Z}}\hat{\mathbf{\Psi}}_k\left(\frac{\xi}{B}+Nj\right)\overline{\hat{\mathbf{\Psi}}_\ell\left(\frac{\xi}{B}+Nj\right)},\quad 0\le k,\ell\le 2N-1.$$

Then, we have

$$\begin{aligned}\delta_{k,\ell}\,\delta_{m,n}I_M &= \frac{1}{B}\int_{[0,BN]}e^{\frac{-4\pi\iota\xi}{B}(m-n)}F_{k,\ell}\left(\frac{\xi}{B}\right)d\xi\\ &= \frac{1}{B}\int_{[0,BN]}e^{\frac{-4\pi\iota\xi}{B}(m-n)}\left\{\sum_{s=0}^{2N-1}F_{k,\ell}\left(\frac{\xi}{B}+\frac{s}{2}\right)\right\}d\xi,\end{aligned}$$

and

$$\sum_{s=0}^{2N-1}F_{k,\ell}\left(\frac{\xi}{B}+\frac{s}{2}\right)=2\delta_{k,\ell}\,I_M. \tag{20}$$

On taking $\lambda=\dfrac{r}{N}+2m$ and $\sigma=2n$, where $m,n\in\mathbb{Z}$, we obtain

$$\begin{aligned}0 &= \int_{\mathbb{R}}\hat{\mathbf{\Psi}}_k\left(\frac{\xi}{B}\right)e^{\frac{-2\pi\iota\xi\lambda}{B}}\overline{\hat{\mathbf{\Psi}}_\ell\left(\frac{\xi}{B}\right)}e^{\frac{2\pi\iota\xi\sigma}{B}}d\xi\\ &= \frac{1}{B}\int_{[0,BN]}e^{\frac{-2\pi\iota\xi}{B}(\frac{r}{N}+2m+2n)}\hat{\mathbf{\Psi}}_k\left(\frac{\xi}{B}\right)\overline{\hat{\mathbf{\Psi}}_\ell\left(\frac{\xi}{B}\right)}d\xi\\ &= \frac{1}{B}\int_{[0,BN]}e^{-4\pi\iota\frac{\xi}{B}(m-n)}e^{-2\pi\iota\frac{\xi}{B}\frac{r}{N}}\sum_{j\in\mathbb{Z}}\hat{\mathbf{\Psi}}_k\left(\frac{\xi}{B}+Nj\right)\overline{\hat{\mathbf{\Psi}}_\ell\left(\frac{\xi}{B}+Nj\right)}d\xi\\ &= \frac{1}{B}\int_{[0,BN]}e^{-4\pi\iota\frac{\xi}{B}(m-n)}e^{-2\pi\iota\frac{\xi}{B}\frac{r}{N}}F_{k,\ell}\left(\frac{\xi}{B}\right)d\xi\\ &= \frac{1}{B}\int_{[0,B/2)}e^{-4\pi\iota\frac{\xi}{B}(m-n)}e^{-2\pi\iota\frac{\xi}{B}\frac{r}{N}}\left\{\sum_{s=0}^{2N-1}e^{-2\pi\iota\frac{r}{N}s}F_{k,\ell}\left(\frac{\xi}{B}+\frac{s}{2}\right)\right\}d\xi.\end{aligned}$$

We conclude that

$$\sum_{s=0}^{2N-1}e^{-2\pi\iota\frac{r}{N}s}F_{k,\ell}\left(\frac{\xi}{B}+\frac{s}{2}\right)=\mathbf{0}. \tag{21}$$

Also we have

$$\sum_{j=0}^{2N-1}F_{k,\ell}\left(\frac{\xi}{B}+\frac{j}{2}\right)=\sum_{j\in\mathbb{Z}}\hat{\mathbf{\Psi}}_k\left(\frac{\xi}{B}+\frac{j}{2}\right)\overline{\hat{\mathbf{\Psi}}_\ell\left(\frac{\xi}{B}+\frac{j}{2}\right)}.$$

Therefore, equations (20) reduces to

$$\sum_{s\in\mathbb{Z}}\hat{\mathbf{\Psi}}_k\left(\frac{\xi}{B}+\frac{j}{2}\right)\overline{\hat{\mathbf{\Psi}}_\ell\left(\frac{\xi}{B}+\frac{j}{2}\right)}=2\delta_{k,\ell}\,I_M. \tag{22}$$

Moreover, we have

$$
\begin{aligned}
F_{k,\ell}\left(\tfrac{2N\xi}{B}\right) &= \sum_{j\in\mathbb{Z}} \hat{\mathbf{\Psi}}_k\left(2N\left(\frac{\xi}{B}+\frac{j}{2}\right)\right)\overline{\hat{\mathbf{\Psi}}_\ell\left(2N\left(\frac{\xi}{B}+\frac{j}{2}\right)\right)} \\
&= \sum_{j\in\mathbb{Z}} H_k^\mu\left(\frac{\xi}{B}+\frac{j}{2}\right)\hat{\mathbf{\Phi}}\left(\frac{\xi}{B}+\frac{j}{2}\right)\overline{\hat{\mathbf{\Phi}}\left(\frac{\xi}{B}+\frac{j}{2}\right)}\,\overline{H_\ell^\mu\left(\frac{\xi}{B}+\frac{j}{2}\right)} \\
&= \sum_{j=n.2N} H_k^\mu\left(\frac{\xi}{B}+nN\right)\hat{\mathbf{\Phi}}\left(\frac{\xi}{B}+nN\right)\overline{\hat{\mathbf{\Phi}}\left(\frac{\xi}{B}+nN\right)}\,\overline{H_\ell^\mu\left(\frac{\xi}{B}+nN\right)} \\
&\quad + \sum_{j=n.2N+1} H_k^\mu\left(\frac{\xi}{B}+nN+\frac{1}{2}\right)\hat{\mathbf{\Phi}}\left(\frac{\xi}{B}+nN+\frac{1}{2}\right) \\
&\qquad \overline{\hat{\mathbf{\Phi}}\left(\tfrac{\xi}{B}+nN+\tfrac{1}{2}\right)}\,\overline{H_\ell^\mu\left(\tfrac{\xi}{B}+nN+\tfrac{1}{2}\right)} \\
&\quad + \cdots + \\
&\quad + \sum_{j=n.2N+(2N-1)} H_k^\mu\left(\frac{\xi}{B}+nN+\frac{2N-1}{2}\right)\hat{\mathbf{\Phi}}\left(\frac{\xi}{B}+nN+\frac{2N-1}{2}\right) \\
&\qquad \overline{\hat{\mathbf{\Phi}}\left(\tfrac{\xi}{B}+nN+\tfrac{2N-1}{2}\right)}\,\overline{H_\ell^\mu\left(\tfrac{\xi}{B}+nN+\tfrac{2N-1}{2}\right)} \\
&= H_k^\mu\left(\tfrac{\xi}{B}\right)\left\{\sum_{j=n.2N}\hat{\mathbf{\Phi}}\left(\frac{\xi}{B}+nN\right)\overline{\hat{\mathbf{\Phi}}\left(\frac{\xi}{B}+nN\right)}\right\}\overline{H_\ell^\mu\left(\tfrac{\xi}{B}\right)} \\
&\quad + H_k^\mu\left(\tfrac{\xi}{B}+\tfrac{1}{2}\right)\left\{\sum_{j=n.2N+1}\hat{\mathbf{\Phi}}\left(\frac{\xi}{B}+nN+\frac{1}{2}\right)\overline{\hat{\mathbf{\Phi}}\left(\frac{\xi}{B}+nN+\frac{1}{2}\right)}\right\}\overline{H_\ell^\mu\left(\tfrac{\xi}{B}+\tfrac{1}{2}\right)} \\
&\quad + \cdots \\
&\quad + H_k^\mu\left(\tfrac{\xi}{B}+\tfrac{2N-1}{2}\right)\left\{\sum_{j=n.2N+(2N-1)}\hat{\mathbf{\Phi}}\left(\frac{\xi}{B}+nN+\frac{2N-1}{2}\right)\overline{\hat{\mathbf{\Phi}}\left(\frac{\xi}{B}+nN+\frac{2N-1}{2}\right)}\right\} \\
&\qquad \overline{H_\ell^\mu\left(\tfrac{\xi}{B}+\tfrac{2N-1}{2}\right)} \\
&= 2\Big\{H_k^\mu\left(\tfrac{\xi}{B}\right)\overline{H_\ell^\mu\left(\tfrac{\xi}{B}\right)} + H_k^\mu\left(\tfrac{\xi}{B}+\tfrac{1}{2}\right)\overline{H_\ell^\mu\left(\tfrac{\xi}{B}+\tfrac{1}{2}\right)} + \cdots + \\
&\qquad H_k^\mu\left(\tfrac{\xi}{B}+\tfrac{2N-1}{2}\right)\overline{H_\ell^\mu\left(\tfrac{\xi}{B}+\tfrac{2N-1}{2}\right)}\Big\} \\
&= 2\sum_{j=0}^{2N-1} H_k^\mu\left(\frac{\xi}{B}+\frac{j}{2}\right)\overline{H_\ell^\mu\left(\frac{\xi}{B}+\frac{j}{2}\right)}.
\end{aligned}
$$

Therefore, we have

$$
\sum_{j\in\mathbb{Z}} \hat{\mathbf{\Psi}}_k\left(\frac{\xi}{B}+\frac{j}{2}\right)\overline{\hat{\mathbf{\Psi}}_\ell\left(\frac{\xi}{B}+\frac{j}{2}\right)} = 2\sum_{j=0}^{2N-1} H_k^\mu\left(\frac{\xi}{2NB}+\frac{j}{4N}\right)\overline{H_\ell^\mu\left(\frac{\xi}{2NB}+\frac{j}{4N}\right)}.
$$

By using (22), we conclude that

$$\sum_{j=0}^{2N-1} H_k^{\mu}\left(\frac{\xi}{2NB}+\frac{j}{4N}\right)\overline{H_\ell^{\mu}\left(\frac{\xi}{2NB}+\frac{j}{4N}\right)}=\delta_{k,\ell}\, I_M.$$

Now we will prove the sufficiency.

By equations (15), we have

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}} \hat{\mathbf{\Psi}}_k\left(\frac{\xi}{B}+\frac{j}{2}\right)\overline{\hat{\mathbf{\Psi}}_\ell\left(\frac{\xi}{B}+\frac{j}{2}\right)}\\
&=\sum_{j\in\mathbb{Z}} H_k^{\mu}\left(\frac{\xi}{2NB}+\frac{j}{4N}\right)\hat{\mathbf{\Phi}}\left(\frac{\xi}{2NB}+\frac{j}{4N}\right)\overline{H_\ell^{\mu}\left(\frac{\xi}{2NB}+\frac{j}{4N}\right)}\,\overline{\hat{\mathbf{\Phi}}\left(\frac{\xi}{2NB}+\frac{j}{4N}\right)}\\
&=H_k^{\mu}\left(\tfrac{\xi}{2NB}+\tfrac{n}{2}\right)\left\{\sum_{j=n.2N}\hat{\mathbf{\Phi}}\left(\frac{\xi}{2NB}+\frac{n}{2}\right)\overline{\hat{\mathbf{\Phi}}\left(\frac{\xi}{2NB}+\frac{n}{2}\right)}\right\}\overline{H_\ell^{\mu}\left(\tfrac{\xi}{2NB}+\tfrac{n}{2}\right)}\\
&+H_k^{\mu}\left(\tfrac{\xi}{2NB}+\tfrac{n}{2}+\tfrac{1}{4N}\right)\left\{\sum_{j=n.2N}\hat{\mathbf{\Phi}}\left(\frac{\xi}{2NB}+\frac{n}{2}+\frac{1}{4N}\right)\overline{\hat{\mathbf{\Phi}}\left(\frac{\xi}{2NB}+\frac{n}{2}+\frac{1}{4N}\right)}\right\}\overline{H_\ell^{\mu}\left(\tfrac{\xi}{2NB}+\tfrac{n}{2}+\tfrac{1}{4N}\right)}\\
&+\cdots\\
&+H_k^{\mu}\left(\tfrac{\xi}{2NB}+\tfrac{n}{2}+\tfrac{2N-1}{4N}\right)\left\{\sum_{j=n.2N}\hat{\mathbf{\Phi}}\left(\frac{\xi}{2NB}+\frac{n}{2}+\frac{2N-1}{4N}\right)\overline{\hat{\mathbf{\Phi}}\left(\frac{\xi}{2NB}+\frac{n}{2}+\frac{2N-1}{4N}\right)}\right\}\\
&\overline{H_\ell^{\mu}\left(\tfrac{\xi}{2NB}+\tfrac{n}{2}+\tfrac{2N-1}{4N}\right)}\\
&=2\left\{H_k^{\mu}\left(\tfrac{\xi}{2NB}\right)\overline{H_\ell^{\mu}\left(\tfrac{\xi}{2NB}\right)}+H_k^{\mu}\left(\tfrac{\xi}{2NB}+\tfrac{1}{4N}\right)\overline{H_\ell^{\mu}\left(\tfrac{\xi}{2NB}+\tfrac{1}{4N}\right)}+\cdots+\right.\\
&\qquad\left.H_k^{\mu}\left(\tfrac{\xi}{2NB}+\tfrac{2N-1}{4N}\right)\overline{H_\ell^{\mu}\left(\tfrac{\xi}{2NB}+\tfrac{2N-1}{4N}\right)}\right\}\\
&=2\delta_{k,\ell}\, I_M.
\end{aligned}$$

It proves the orthonormality of the system $\left\{\mathbf{\Psi}_k(x-\lambda)e^{-\frac{\iota\pi A}{B}(t^2-\lambda^2)}:\lambda\in\Lambda, k=0,1,\ldots,2N-1\right\}$. □

Theorem 3.5. *Suppose* $\left\{\mathbf{\Psi}_k(x-\lambda)e^{-\frac{\iota\pi A}{B}(t^2-\lambda^2)}\right\}_{\lambda\in\Lambda,\ k=0,1,\ldots,2N-1}$ *is the system as defined in Lemma 3.4 and orthonormal in* V_1*. Then this system is complete in* $W_0^{\mu}\equiv V_1^{\mu}\ominus V_0^{\mu}$*.*

Proof. Since the system (18) is orthonormal in V_1. By Lemma 3.4 we have,

$$\left\{H_k^{\mu}\left(\frac{\xi}{2NB}\right)\overline{H_\ell^{\mu}\left(\frac{\xi}{2NB}\right)}+H_k^{\mu}\left(\frac{\xi}{2NB}+\frac{1}{4N}\right)\overline{H_\ell^{\mu}\left(\frac{\xi}{2NB}+\frac{1}{4N}\right)}+\cdots+H_k^{\mu}\left(\frac{\xi}{2NB}+\frac{2N-1}{4N}\right)\overline{H_\ell^{\mu}\left(\frac{\xi}{2NB}+\frac{2N-1}{4N}\right)}\right\}$$
$$=\delta_{k,\ell}\, I_M.$$

We will now prove its completeness.

For $\mathbf{f}_k\in W_0^{\mu}$, there exists constant matrices $\left\{P_{\lambda,k}^{\mu}\right\}$ such that

$$\mathbf{f}_k(t) = \sqrt{2N}\sum_{\lambda\in\Lambda} P^{\mu}_{\lambda,k}\,\mathbf{\Phi}\Big(2Nt-\lambda\Big)e^{-\frac{-\iota\pi A}{B}(t^2-\lambda^2)},\quad 0\le k\le 2N-1.$$

Above relation can be written in the LCT domain as

$$\hat{\mathbf{f}}_k\left(\frac{\xi}{B}\right) = P^{\mu}_k\left(\frac{\xi}{2NB}\right)\hat{\mathbf{\Phi}}\left(\frac{\xi}{2NB}\right), \tag{23}$$

where

$$P^{\mu}_k(\xi) = \frac{1}{\sqrt{qN}}\sum_{\lambda\in\Lambda} P^{\mu}_{\lambda,k}e^{-2\pi\iota\lambda\xi/B}.$$

On the other hand, $\mathbf{f}_k \notin V^{\mu}_0$ and $\mathbf{f}_k \in W^{\mu}_0$ implies

$$\int_{\mathbb{R}} \mathbf{f}_k(t)\overline{\mathbf{\Phi}(t-\lambda)}e^{-\frac{-\iota\pi A}{B}(t^2-\lambda^2)}\,dt = \mathbf{0},\quad \lambda\in\Lambda.$$

This condition is equivalent to

$$\sum_{n\in\mathbb{Z}} \hat{\mathbf{f}}_k\left(\frac{\xi}{B}+\frac{n}{2}\right)\overline{\hat{\mathbf{\Phi}}\left(\frac{\xi}{B}+\frac{n}{2}\right)} = \mathbf{0},\quad \xi\in\mathbb{R}.$$

Therefore, the identities (10) and (23) give for all $\xi\in\mathbb{R}$,

$$\sum_{n\in\mathbb{Z}} P^{\mu}_k\left(\frac{\xi}{2NB}+\frac{j}{4N}\right)\hat{\mathbf{\Phi}}\left(\frac{\xi}{2NB}+\frac{j}{4N}\right)\overline{G^{\mu}\left(\frac{\xi}{2NB}+\frac{j}{4N}\right)}\,\overline{\hat{\mathbf{\Phi}}\left(\frac{\xi}{2NB}+\frac{j}{4N}\right)} = \mathbf{0}.$$

As similar to the identity (19) in Lemma 3.4, we have for $0\le k\le 2N-1$,

$$P^{\mu}_k\left(\frac{\xi}{2NB}\right)\overline{G^{\mu}\left(\frac{\xi}{2NB}\right)}+P^{\mu}_k\left(\frac{\xi}{2NB}+\frac{1}{4N}\right)\overline{G^{\mu}\left(\frac{\xi}{2NB}+\frac{1}{4N}\right)}+\cdots+P^{\mu}_k\left(\frac{\xi}{2NB}+\frac{2N-1}{4N}\right)\overline{G^{\mu}\left(\frac{\xi}{2NB}+\frac{2N-1}{4N}\right)} = \mathbf{0}. \tag{24}$$

Let

$$\begin{aligned}
P^{\mu}_{k'}\left(\frac{\xi}{2NB}\right) &= \overline{\left(P^{\mu}_k\left(\frac{\xi}{2NB}\right), P^{\mu}_k\left(\frac{\xi}{2NB}+\frac{1}{4N}\right),\ldots,P^{\mu}_k\left(\frac{\xi}{2NB}+\frac{2N-1}{4N}\right)\right)},\\
\tilde{G}^{\mu}\left(\frac{\xi}{2NB}\right) &= \overline{\left(G^{\mu}\left(\frac{\xi}{2NB}\right), G^{\mu}\left(\frac{\xi}{2NB}+\frac{1}{4N}\right),\ldots,G^{\mu}\left(\frac{\xi}{2NB}+\frac{2N-1}{4N}\right)\right)},\\
H^{\mu}_{k'}\left(\frac{\xi}{2NB}\right) &= \overline{\left(H^{\mu}_k\left(\frac{\xi}{2NB}\right), H^{\mu}_k\left(\frac{\xi}{2NB}+\frac{1}{4N}\right),\ldots,H^{\mu}_k\left(\frac{\xi}{2NB}+\frac{2N-1}{4N}\right)\right)}.
\end{aligned}$$

Then, equation (19) implies that for any $\xi\in\mathbb{R}$, the column vectors in $2NM\times M$ matrix $\tilde{G}^{\mu}$ and the column vectors in $2NM\times M$ matrix $H^{\mu}_{k'}$ are orthogonal for $k=0,1,\ldots,2N-1$ and these vectors form an orthogonal basis of $2NM$ dimensional complex Euclidean space $\mathbb{C}^{2NM}$.

Equation (24) implies that the column vectors in $2NM\times M$ matrix $P^{\mu}_{k'}$ and the column vectors of $2NM\times M$ matrix $\tilde{G}^{\mu}$ are orthogonal. Therefore, there exists an $M\times M$ matrix $Q_k(\xi)$ such that

$$P^{\mu}_k\left(\frac{\xi}{B}\right) = Q^{\mu}_k\left(\frac{\xi}{B}\right)H^{\mu}_k\left(\frac{\xi}{B}\right),\quad \xi\in\mathbb{R},\ 0\le k\le 2N-1.$$

Therefore, from equations (15) and (23), we have

$$\begin{aligned}\hat{\mathbf{f}}_k\left(\tfrac{\xi}{B}\right) &= P_k^{\mu}\left(\tfrac{\xi}{2NB}\right)\hat{\mathbf{\Phi}}\left(\tfrac{\xi}{2NB}\right)\\ &= Q_k^{\mu}\left(\tfrac{\xi}{2NB}\right)H_k^{\mu}\left(\tfrac{\xi}{2NB}\right)\hat{\mathbf{\Phi}}\left(\tfrac{\xi}{2NB}\right)\\ &= Q_k^{\mu}\left(\tfrac{\xi}{2NB}\right)\hat{\mathbf{\Psi}}_k\left(\tfrac{\xi}{B}\right).\end{aligned}$$

By using the orthonormality of the system (18), we have

$$\int_{\mathbb{R}}\hat{\mathbf{f}}_k\left(\frac{2N\xi}{B}\right)\overline{\hat{\mathbf{f}}_k\left(\frac{2N\xi}{B}\right)}d\xi = \int_{\mathbb{R}}Q_k^{\mu}\left(\frac{\xi}{B}\right)\hat{\mathbf{\Psi}}_k\left(\frac{2N\xi}{B}\right)\overline{\hat{\mathbf{\Psi}}_k\left(\frac{2N\xi}{B}\right)}\,\overline{Q_k^{\mu}\left(\frac{\xi}{B}\right)}d\xi.$$

Therefore, we have

$$\int_{\mathbb{R}}\hat{\mathbf{f}}_k\left(\frac{2N\xi}{B}\right)\overline{\hat{\mathbf{f}}_k\left(\frac{2N\xi}{B}\right)}d\xi = 2\int_0^{1/2}Q_k^{\mu}\left(\frac{\xi}{B}\right)\overline{Q_k^{\mu}\left(\frac{\xi}{B}\right)}d\xi.$$

This shows that $P_k(^{\mu}\xi)$ has the series expansion and let the constant $M\times M$ matrices $\left\{R_{\lambda,k}^{\mu}\right\}_{\lambda\in\Lambda,k=0,1,\ldots,2N-1}$ be its coefficients. Therefore, we have

$$\mathbf{f}_k(t) = \sum_{\lambda\in\Lambda}R_{\lambda,k}^{\mu}\,\mathbf{\Psi}_k(t-\lambda)e^{-\frac{-\iota\pi A}{B}(t^2-\lambda^2)}.$$

It proves the completeness of the system $\left\{\mathbf{\Psi}_k(x-\lambda)e^{-\frac{-\iota\pi A}{B}(t^2-\lambda^2)}\right\}_{\lambda\in\Lambda,\ k=0,1,\ldots,2N-1}$ in W_0. $\square$

If $\mathbf{\Psi}_0^{\mu},\mathbf{\Psi}_1^{\mu},\ldots,\mathbf{\Psi}_{2N-1}^{\mu}\in V_1^{\mu}$ are as in Lemma 3.4, one can obtain from them as orthonormal basis for $L^2\left(\mathbb{R},\mathbb{C}^M\right)$ by following the standard procedure for construction of wavelet from a given MRA. It can be easily checked that for every $j\in\mathbb{Z}$, the collection $\left\{\sqrt{2N}\mathbf{\Psi}_k\left((2N)^j\,t-\lambda\right)e^{-\frac{-\iota\pi A}{B}(t^2-\lambda^2)}:\lambda\in\Lambda,k=0,1,\ldots,2N-1\right\}$ is a complete orthogonal system for V_{j+1}^{μ}. Therefore, it follows immediately from (8) that the collection

$$\left\{\sqrt{2N}\mathbf{\Psi}_k\left((2N)^j\,t-\lambda\right)e^{-\frac{-\iota\pi A}{B}(t^2-\lambda^2)}:\lambda\in\Lambda,k=0,1,\ldots,2N-1\right\}$$

forms a complete orthonormal system for $L^2\left(\mathbb{R},\mathbb{C}^M\right)$.

4. Construction of LCT-VNUMRA

The main goal of this section is to construct a LCT-VNUMRA starting from a vector-valued refinement mask of the form

$$G^{\mu}\left(\frac{\xi}{B}\right) = G_{\lambda,1}^{\mu}\left(\frac{\xi}{B}\right) + e^{-2\pi\iota\frac{r}{N}\frac{\xi}{B}}G_{\lambda,2}^{\mu}\left(\frac{\xi}{B}\right), \tag{25}$$

where $N>1$ is an integer and r is an odd integer with $1\le r\le 2N-1$ such that r and N are relatively prime and $G_{\lambda,1}^{\mu}\left(\frac{\xi}{B}\right)$ and $G_{\lambda,2}^{\mu}\left(\frac{\xi}{B}\right)$ are $M\times M$ constant symmetric matrix sequences. In other words, we establish conditions under which the solutions of scaling equation (9) generate a LCT-VNUMRA in $L^2(\mathbb{R})$ or equivalently, we find a sufficient for the orthonormality of the system $\left\{\mathbf{\Phi}(t-\lambda)e^{-\frac{-\iota\pi A}{B}(t^2-\lambda^2)}:\lambda\in\Lambda\right\}$, where $\Lambda=\{0,r/N\}+2\mathbb{Z}$. Therefore, the scaling vector Φ associated with given LCT-VNUMRA should satisfy the scaling identity

$$\hat{\mathbf{\Phi}}\left(\frac{2N\xi}{B}\right) = G^{\mu}\left(\frac{\xi}{B}\right)\hat{\mathbf{\Phi}}\left(\frac{\xi}{B}\right). \tag{26}$$

We further assume that:

$$\sum_{s=0}^{2N-1} G^{\mu}\left(\frac{\xi}{2NB}+\frac{s}{4N}\right)\overline{G^{\mu}\left(\frac{\xi}{2NB}+\frac{s}{4N}\right)} = I_M. \tag{27}$$

Theorem 4.1. *Let $G^{\mu}\left(\frac{\xi}{B}\right)$ be the vector-valued refinement mask associated with the vector-valued scaling function $\mathbf{\Phi}$ of LCT-VNUMRA and satisfies the condition (27) together with $G^{\mu}(0) = I_M$ and $G^{\mu}\left(\frac{\xi}{B}\right) = \overline{G^{\mu}\left(\frac{\xi}{B}\right)}, \forall\ \xi \in \mathbb{R}$. Then, a sufficient condition for the collection $\left\{\mathbf{\Phi}(x-\lambda)e^{-\frac{-\iota\pi A}{B}(t^2-\lambda^2)}\right\}_{:\lambda\in\Lambda}$ to be orthonormal in $L^2\left(\mathbb{R},\mathbb{C}^M\right)$ is the existence of a constant $C > 0$ and of a compact set $E \subset \mathbb{R}$ that contains the neighbourhood of the origin such that*

$$\left|G^{\mu}\left(\frac{\xi}{(2N)^k B}\right)\right| \geq C, \quad \forall\ \xi \in \mathbb{R}, k \in \mathbb{Z}. \tag{28}$$

Proof. Let us assume the existence of a constant C and of the compact set $E \subset K$ with properties satisfied above. For any $k \in \mathbb{N}$, we define

$$g_k\left(\frac{\xi}{B}\right) = \left\{\prod_{j=1}^{k} G^{\mu}\left(\frac{\xi}{(2N)^k B}\right)\right\}\chi_E\left(\frac{\xi}{(2N)^k B}\right).$$

As the interior of the compact set E contains $\mathbf{0}, g_k \to \hat{\mathbf{\Phi}}$ pointwise as $k \to \infty$. Therefore, there exists a constant $W > 0$ such that $\left|G^{\mu}\left(\frac{\xi}{B}\right) - G^{\mu}(0)\right| \leq W|\xi|$, for all $\xi \in \mathbb{R}$, and thus $\left|G^{\mu}\left(\frac{\xi}{B}\right)\right| \geq 1 - W|\xi|$. Since E is bounded, we can find an integer $k_0 \in \mathbb{Z}$ such that $W|\xi| < (2N)^k$, for $k > k_0, \xi \in E$ and hence, there exists a constant $C_1 > 0$ such that

$$\chi_E\left(\frac{\xi}{B}\right) \leq C_1\left|\hat{\mathbf{\Phi}}\left(\frac{\xi}{B}\right)\right|, \quad \text{for all } \xi \in \mathbb{R}.$$

Thus, we have

$$\left|g_k\left(\frac{\xi}{B}\right)\right| \leq C_1\left\{\prod_{j=1}^{k}\left|G^{\mu}\left(\frac{\xi}{(2N)^k B}\right)\right|\right\}\left|\hat{\mathbf{\Phi}}\left(\frac{\xi}{(2N)^k B}\right)\right| = C_1\left|\hat{\mathbf{\Phi}}\left(\frac{\xi}{B}\right)\right|.$$

Therefore, by Lebesgue dominated convergence theorem the sequence $\{g_k\}$ converges to $\hat{\mathbf{\Phi}}$ in L^2-norm. We will now compute by induction the integral

$$\int_{\mathbb{R}} g_k(\xi)\,\overline{g_k(\xi)}\,\overline{\chi_{(\lambda-\sigma)}(\xi)}\,d\xi, \quad \text{where } \lambda,\sigma \in \Lambda.$$

For $k = 1$, we have

$$\begin{aligned}
&\int_{\mathbb{R}} g_1\left(\frac{\xi}{B}\right)\overline{g_1\left(\frac{\xi}{B}\right)}e^{-2\pi\iota\frac{\xi}{B}(\lambda-\sigma)}\,d\xi \\
&= \int_{\mathbb{R}} G^{\mu}\left(\frac{\xi}{2NB}\right)\overline{G^{\mu}\left(\frac{\xi}{2NB}\right)}\chi_E\left(\frac{\xi}{2NB}\right)e^{-2\pi\iota\frac{\xi}{B}(\lambda-\sigma)}\,d\xi \\
&= (2N)\int_E G^{\mu}\left(\frac{\xi}{B}\right)\overline{G^{\mu}\left(\frac{\xi}{B}\right)}e^{-2\pi\iota\frac{(2N)\xi}{B}(\lambda-\sigma)}\,d\xi \\
&= 4N\int_0^{B/2}\left\{\sum_{s=0}^{2N-1} G^{\mu}\left(\frac{\xi}{B}+\frac{1}{4N}\right)\overline{G^{\mu}\left(\frac{\xi}{B}+\frac{1}{4N}\right)}e^{-\pi\iota\frac{\xi}{B}(\lambda-\sigma)s}\right\}e^{-2\pi\iota\frac{(2N)\xi}{B}(\lambda-\sigma)}\,d\xi.
\end{aligned}$$

If $\lambda-\sigma\in 2\mathbb{Z}$, then the expression in the brackets in the above integral is equal to I_M by (27) and thus

$$\begin{aligned}\int_{\mathbb{R}} g_1\left(\frac{\xi}{B}\right)\overline{g_1\left(\frac{\xi}{B}\right)}e^{-2\pi\iota\frac{\xi}{B}(\lambda-\sigma)}\,d\xi &= 4N\int_0^{B/4N} I_M e^{-2\pi\iota\frac{(2N)\xi}{B}(\lambda-\sigma)}d\xi\\ &= 2\int_{[0,B/2)} I_M e^{-2\pi\iota\frac{\xi}{B}(\lambda-\sigma)}d\xi\\ &= \delta_{\lambda,\sigma}\, I_M.\end{aligned}$$

On the other hand, if $\lambda=2m$, $\sigma=2n+r/N$, where $m,n\in\mathbb{Z}$, then the same expression will vanish and the integral becomes

$$\int_{\mathbb{R}} g_1\left(\frac{\xi}{B}\right)\overline{g_1\left(\frac{\xi}{B}\right)}e^{-2\pi\iota\frac{\xi}{B}(\lambda-\sigma)}\,d\xi=\mathbf{0}.$$

When $k\geq 2$, we have

$$\begin{aligned}&\int_{\mathbb{R}} g_k\left(\frac{\xi}{B}\right)\overline{g_k\left(\frac{\xi}{B}\right)}e^{-2\pi\iota\frac{\xi}{B}(\lambda-\sigma)}\,d\xi\\
&=\int_{\mathbb{R}} G^\mu\left(\frac{\xi}{(2N)^1B}\right)G^\mu\left(\frac{\xi}{(2N)^2B}\right)\ldots G^\mu\left(\frac{\xi}{(2N)^kB}\right)\overline{G^\mu\left(\frac{\xi}{(2N)^kB}\right)}\,\overline{G^\mu\left(\frac{\xi}{(2N)^{k-1}B}\right)}\\
&\qquad\ldots\overline{G^\mu\left(\frac{\xi}{(2N)^1B}\right)}e^{-2\pi\iota\frac{\xi}{B}(\lambda-\sigma)}d\xi\\
&=(2N)^k\int_E G^\mu\left(\frac{(2N)^{k-1}\xi}{B}\right)G^\mu\left(\frac{(2N)^{k-2}\xi}{B}\right)\ldots G^\mu\left(\frac{\xi}{B}\right)\overline{G^\mu\left(\frac{\xi}{B}\right)}\,\overline{G^\mu\left(\frac{(2N)\xi}{B}\right)}\\
&\qquad\ldots\overline{G^\mu\left(\frac{(2N)^{k-1}\xi}{B}\right)}e^{-2\pi\iota\frac{(2N)^k\xi}{B}(\lambda-\sigma)}d\xi\\
&=(2N)^k\int_E\left\{\prod_{\ell=0}^{k-1}G^\mu\left(\frac{(2N)^\ell\xi}{B}\right)\right\}\overline{\left\{\prod_{\ell=0}^{k-1}G^\mu\left(\frac{(2N)^\ell\xi}{B}\right)\right\}}e^{-2\pi\iota\frac{(2N)^k\xi}{B}(\lambda-\sigma)}d\xi\\
&=(2N)^k\int_E\left\{\prod_{\ell=1}^{k-1}G^\mu\left(\frac{(2N)^\ell\xi}{B}\right)\right\}G^\mu\left(\frac{\xi}{B}\right)\overline{G^\mu\left(\frac{\xi}{B}\right)}\,\overline{\left\{\prod_{\ell=1}^{k-1}G^\mu\left(\frac{(2N)^\ell\xi}{B}\right)\right\}}e^{-2\pi\iota\frac{(2N)^k\xi}{B}(\lambda-\sigma)}d\xi\\
&=2(2N)^k\int_0^{B/2}\left\{\prod_{\ell=1}^{k-1}G^\mu\left(\frac{(2N)^\ell\xi}{B}\right)\right\}G^\mu\left(\frac{\xi}{B}\right)\overline{G^\mu\left(\frac{\xi}{B}\right)}\,\overline{\left\{\prod_{\ell=1}^{k-1}G^\mu\left(\frac{(2N)^\ell\xi}{B}\right)\right\}}e^{-2\pi\iota\frac{(2N)^k\xi}{B}(\lambda-\sigma)}d\xi\\
&=2(2N)^k\int_0^{B/4N}\left\{\prod_{\ell=2}^{k-1}G^\mu\left(\frac{(2N)^\ell\xi}{B}\right)\right\}\left[\sum_{s=0}^{2N-1}G^\mu\left(\frac{2N\xi}{B}+\frac{s}{2}\right)G^\mu\left(\frac{\xi}{B}+\frac{s}{2}\right)\right.\\
&\qquad\left.\overline{G^\mu\left(\frac{2N\xi}{B}+\frac{s}{2}\right)G^\mu\left(\frac{\xi}{B}+\frac{s}{2}\right)}\right]\overline{\left\{\prod_{\ell=1}^{k-1}G^\mu\left(\frac{(2N)^\ell\xi}{B}\right)\right\}}e^{-2\pi\iota\frac{(2N)^k\xi}{B}(\lambda-\sigma)}d\xi\\
&=2(2N)^k\int_0^{B/4N}\left\{\prod_{\ell=2}^{k-1}G^\mu\left(\frac{(2N)^\ell\xi}{B}\right)\right\}S\left(\frac{\xi}{B}\right)\overline{\left\{\prod_{\ell=2}^{k-1}G^\mu\left(\frac{(2N)^\ell\xi}{B}\right)\right\}}e^{-2\pi\iota\frac{(2N)^k\xi}{B}(\lambda-\sigma)}d\xi\end{aligned}$$

where

$$S\left(\frac{\xi}{B}\right) = \left\{\sum_{s=0}^{2N-1} G^\mu\left(\frac{2N\xi}{B}+\frac{s}{2}\right) G^\mu\left(\frac{\xi}{B}+\frac{s}{2}\right)\overline{G^\mu\left(\frac{2N\xi}{B}+\frac{s}{2}\right) G^\mu\left(\frac{\xi}{B}+\frac{s}{2}\right)}\right\}.$$

Since the refinement mask $G^\mu\left(\frac{\xi}{B}\right)$ can be expressed as (25), therefore, the above relation becomes

$$\begin{aligned} S\left(\tfrac{\xi}{B}\right) &= G^\mu_{\lambda,1}\left(\tfrac{\xi}{B}\right)\overline{G^\mu_{\lambda,1}\left(\tfrac{\xi}{B}\right)}G^\mu_{\lambda,2}\left(\tfrac{\xi}{B}\right)\overline{G^\mu_{\lambda,2}\left(\tfrac{\xi}{B}\right)} \\ &= G^\mu\left(\tfrac{2N\xi}{B}\right)\overline{G^\mu\left(\tfrac{2N\xi}{B}\right)}. \end{aligned}$$

Thus, we have

$$\begin{aligned} &\int_{\mathbb{R}} g_k\left(\frac{\xi}{B}\right)\overline{g_k\left(\frac{\xi}{B}\right)}e^{-2\pi\iota\frac{\xi}{B}(\lambda-\sigma)}\,d\xi \\ &= (2N)^k\int_{E/2N}\left\{\prod_{\ell=1}^{k-1}G^\mu\left(\frac{(2N)^\ell\xi}{B}\right)\right\}\overline{\left\{\prod_{\ell=1}^{k-1}G^\mu\left(\frac{(2N)^\ell\xi}{B}\right)\right\}}e^{-2\pi\iota\frac{(2N)^k\xi}{B}(\lambda-\sigma)}d\xi \\ &= (2N)^{k-1}\int_{E}\left\{\prod_{\ell=0}^{k-2}G^\mu\left(\frac{(2N)^\ell\xi}{B}\right)\right\}\overline{\left\{\prod_{\ell=0}^{k-2}G^\mu\left(\frac{(2N)^\ell\xi}{B}\right)\right\}}e^{-2\pi\iota\frac{(2N)^k\xi}{B}(\lambda-\sigma)}d\xi \\ &= \int_{\mathbb{R}} g_{k-1}\left(\frac{\xi}{B}\right)\overline{g_{k-1}\left(\frac{\xi}{B}\right)}e^{-2\pi\iota\frac{\xi}{B}(\lambda-\sigma)}\,d\xi \end{aligned}$$

Therefore for any $k\in\mathbb{Z}$, we have

$$\int_{\mathbb{R}} g_k\left(\frac{\xi}{B}\right)\overline{g_k\left(\frac{\xi}{B}\right)}e^{-2\pi\iota\frac{\xi}{B}(\lambda-\sigma)}\,d\xi = \delta_{\lambda,\sigma}, \quad \lambda,\sigma\in\Lambda.$$

Passing to the limit as $k\to\infty$ and using Plancherel's formula, we obtain

$$\int_{\mathbb{R}}\mathbf{\Phi}(x-\lambda)e^{\frac{-2\iota\pi A}{B}(t^2-\lambda^2)}\overline{\mathbf{\Phi}(x-\sigma)e^{\frac{-2\iota\pi A}{B}(t^2-\sigma^2)}}\,dx = \int_{\mathbb{R}}\hat{\mathbf{\Phi}}(\xi)\overline{\hat{\mathbf{\Phi}}(\xi)}e^{\frac{-2\iota\pi A}{B}(\lambda^2-\sigma^2)}\,d\xi = \delta_{\lambda,\sigma}, \quad \lambda,\sigma\in\Lambda$$

which proves the desired orthonormality. □

5. Conclusion

In this paper, we continue the study based on this nonstandard setting and introduced vector-valued nonuniform multiresolution analysis associated with linear canonical transform (LCT-VNUMRA). We establish a necessary and sufficient condition for the existence of associated wavelets and derive an algorithm for the construction of vector-valued MRA analysis starting from a vector refinement mask with appropriate conditions.

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