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Total controllability for noninstantaneous impulsive conformable fractional evolution system with nonlinear noise and nonlocal conditions

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Abstract. Noninstantaneous impulsive conformable fractional stochastic differential equation with nonlinear noise and nonlocal condition via Rosenblatt process and Poisson jump is studied in this paper. Sufficient conditions for controllability for the considered problem are established. The required results are obtained based on fractional calculus, stochastic analysis, semigroups and Sadovskii's fixed point theorem. In the end paper, an example is provided to illustrate the applicability of the results.

1. Introduction

Stochastic differential equations are the right tool to model the systems with the random effects and external noises (see [1-10]). Stochastic differential equations with impulse arise from many mathematical model of physical phenomenon in the field of technology, biology, physics and economics [11-13]. Recently, a novel definition named conformable fractional derivative is introduced in [14]. This new fractional derivative quickly becomes the subject of many contributions in several areas of science. In short time, many studies and discussion related to conformable fractional derivative have appeared in several areas of applications, for example, Won Sang [15] discussed the fractional Newton mechanics with conformable fractional derivative. Rosales-García et al [16] applied conformable derivative to experimental Newton's law of cooling. Abdellatif et al [17] studied the stability of conformable stochastic systems depending on a parameter. Abdellatif and Mchiri [18] discussed the partial stability of conformable stochastic systems. Ahmed [19] studied the conformable fractional stochastic differential equations with control function.
On the other hand, controllability problem is searching for a suitable control function that steers the proposed dynamical model to a desired final state [20-25]. Many authors studied controllability of noninstantaneous impulsive stochastic differential equations, for example, Yan and Yang [26] studied the optimal controllability of non-instantaneous impulsive partial stochastic differential systems with fractional sectorial operators. Balasubramaniam [27] discussed the controllability of semilinear noninstantaneous impulsive ABC neutral fractional differential equations. Malik et al [28] obtained the controllability of Sobolev type fuzzy differential equation with non-instantaneous impulsive condition. Xin et al [29] investigated the controllability of nonlinear ordinary differential equations with non-instantaneous impulses. But, the

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controllability of noninstantaneous impulsive conformable fractional stochastic differential equation with Rosenblatt process and Poisson jump via nonlocal condition have not yet been considered in the literature, and this fact motivates this work.
The purpose of this paper is to study the controllability of noninstantaneous impulsive conformable fractional stochastic differential equation with Rosenblatt process, Poisson jump and nonlocal condition in the following form

$$\begin{cases} D^{\hbar}_{0+}y(\varrho) = -Ty(\varrho) + \Re(\varrho, y(\varrho)) + Qu(\varrho) + \Im(\varrho, y(\varrho))\frac{d\omega}{d\varrho} + \sigma(\varrho, y(\varrho))\frac{dZ_H}{d\varrho} \\ + \int_V h(\varrho, y(\varrho), v)\tilde{N}(d\varrho, dv), \quad \varrho \in (s_i, \varrho_{i+1}], \ \ i \in [0, m] \\ y(\varrho) = g_i(\varrho, y(\varrho)), \quad \varrho \in (\varrho_i, s_i], \ \ i \in [1, m] \\ y(0) + \psi(y) = y_0, \end{cases} \tag{1}$$

where $D^{\hbar}_{0+}$ is the conformable fractional derivative with order $\frac{1}{2} < \hbar < 1$, $-T$ is the infinitesimal generator of an analytic semigroup of bounded linear operators $\Delta(\varrho)$, $\varrho \geq 0$, on a separable Hilbert space Φ with inner product $\langle ., . \rangle$ and norm $\| . \|$ and the control function $u(\cdot)$ is given in $L_2(J, U)$, the Hilbert space of admissible control functions with U a Hilbert space. The symbol Q stands for a bounded linear from U into Φ and g_i is noninstataneous impulsive function for all $i = 1, 2, \ldots, m$. Let $J = (0, G]$ is the time interval where, ϱ_i, s_i are fixed number satisfying $0 = s_0 < \varrho_1 \leq s_1 \leq \varrho_2 < \ldots < s_{m-1} < \varrho_m \leq s_m \leq \varrho_{m+1} = G$. Let Ξ be another separable Hilbert space with inner product $\langle ., . \rangle_{\Xi}$ and norm $\| . \|_{\Xi}$. Suppose $\{\omega(\varrho)\}_{\varrho \geq 0}$ is S-Wiener process defined on $(\Omega, \Upsilon, \{\Upsilon_{\varrho}\}_{\varrho \geq 0}, P)$ with values in Hilbert space Ξ and $\{Z_H(\varrho)\}_{\varrho \geq 0}$ is S-Rosenblatt process with parameter $H \in (\frac{1}{2}, 1)$ defined on $(\Omega, \Upsilon, \{\Upsilon_{\varrho}\}_{\varrho \geq 0}, P)$ with values in Hilbert space Y. We are also employing the same notation $\| . \|$ for the norm in Φ, Ξ, Λ, $L(\Xi, \Phi)$ and $L(\Phi, \Lambda)$ where $L(\Xi, \Lambda)$ and $L(\Phi, \Lambda)$ denote respectively the space of all bounded linear operators from Ξ into Λ and Φ into Λ. The functions $\Re$, $\Im$, σ, h, ψ and g_i are given functions to be defined later.

2. Preliminaries

In this section, some definitions and results are given which will be used throughout this paper.
Definition 2.1 (see [14]). Let $0 < \hbar \leq 1$. The conformable fractional derivative of order $\hbar$ of a function $f(\cdot)$ for $\varrho > 0$ is defined as follows

$$\frac{d^{\hbar} f(\varrho)}{d\varrho^{\hbar}} = \lim_{\nu \to 0} \frac{f(\varrho + \nu \varrho^{1-\hbar}) - f(\varrho)}{\nu}.$$

For $\varrho = 0$, we adopt the following definition:

$$\frac{d^{\hbar} f(0)}{d\varrho^{\hbar}} = \lim_{\varrho \to 0^+} \frac{d^{\hbar} f(\varrho)}{d\varrho^{\hbar}}.$$

The fractional integral $I^{\hbar}(\cdot)$ associated with the conformable fractional derivative is defined by

$$I^{\hbar}(f)(\varrho) = \int_0^{\varrho} s^{\hbar-1} f(s) ds.$$

Let (Ω, Υ, P) be a complete probability space equipped with a normal filtration Υ_{ϱ}, $\varrho \in [0, G]$ where Υ_{ϱ} is the σ-algebra generated by random variables $\{\omega(s), Z_H(s), s \in [0, G]\}$ and all P-null sets. Let $(V, \Phi, \rho(dv))$ be a σ-finite measurable space. Given stationary Poisson point process $(p_{\varrho})_{\varrho \geq 0}$, which is defined on (Ω, η, P) with values in V and with characteristic measure ρ. We will denote by $N(\varrho, dv)$ be the counting measure of p_{ϱ} such that $\tilde{N}(\varrho, \Theta) := E(N(\varrho, \Theta)) = \varrho\rho(\Theta)$ for $\Theta \in \Psi$. Define $\tilde{N}(\varrho, dv) := N(\varrho, dv) - \varrho\lambda(dv)$, the Poisson martingale measure generated by p_{ϱ}.
Fix a time interval $[0, G]$ and let $\{Z_H(\varrho), \varrho \in [0, G]\}$ represents one-dimensional Rosenblatt process with parameter $H \in (1/2, 1)$. The Rosenblatt process with parameter $H > \frac{1}{2}$ can be written as

$$Z_H(\varrho) = \zeta(H) \int_0^{\varrho} \int_0^{\varrho} \left[\int_{y_1 \vee y_2}^{\varrho} \frac{\partial K^{H'}}{\partial v}(v, y_1) \frac{\partial K^{H'}}{\partial v}(v, y_2) \right] dB(y_1) dB(y_2)$$

where $K^H(\varrho, s)$ is given by

$$K_H(\varrho, s) = c_H s^{\frac{1}{2}-H} \int_s^{\varrho} (v-s)^{H-\frac{3}{2}} v^{H-\frac{1}{2}} dv, \quad for\ s < \varrho,$$

with $c_H = \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}}$ and $\beta(\cdot,\cdot)$ denotes the Beta function, $K_H(\varrho, s) = 0$ when $\varrho \le s$, $\{B(\varrho), \varrho \in [0, G]\}$ is a Brownian motion, $H' = \frac{H+1}{2}$ and $\zeta(H) = \frac{1}{H+1}\sqrt{\frac{H}{2H-1}}$ is a normalizing constant. The covariance of the Rosenblatt process $Z_H(\varrho), \varrho \in [0, G]$ satisfy
$E(Z_H(\varrho)Z_H(s)) = \frac{1}{2}(s^{2H} + \varrho^{2H} - |s-\varrho|^{2H})$.
Let $S \in L(\Phi, \Phi)$ be an operator defined by $Se_n = \lambda_n e_n$ with finite trace $Tr(S) = \sum_{n=1}^{\infty} \lambda_n < \infty$ where $\lambda_n \ge 0$ $(n = 1, 2, ...)$ are non-negative real numbers and $\{e_n\}$ $(n = 1, 2, ...)$ is a complete orthonormal basis in Φ.
We define the infinite dimensional S-Rosenblatt process on Φ as

$$Z_H(\varrho) = Z_S(\varrho) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n z_n(\varrho)$$

where $(z_n)_{n\ge0}$ is a family of real, independent Rosenblatt process.
In order to define Wiener integrals with respect to the S-Rosenblatt process, we introduce the space $L_2^0 := L_2^0(\Phi, \Lambda)$ of all S-Hilbert Schmidt operators $\chi : \Phi \to \Lambda$. We recall that $\chi \in L(\Phi, \Lambda)$ is called a S-Hilbert-Schmidt operator, if

$$\|\chi\|_{L_2^0}^2 := \sum_{n=1}^{\infty} \| \sqrt{\lambda_n} \chi e_n\|^2 < \infty$$

and that the space L_2^0 equipped with the inner product $\langle \vartheta, \chi \rangle_{L_2^0} = \sum_{n=1}^{\infty} \langle \vartheta e_n, \chi e_n \rangle$ is a separable Hilbert space.
Let $\phi(s)$; $s \in [0, G]$ be a function with values in $L_2^0(\Phi, \Lambda)$, the Wiener integral of ϕ with respect to Z_S is defined by

$$\int_0^{\varrho} \phi(s) dZ_S(s) = \sum_{n=1}^{\infty} \int_0^{\varrho} \sqrt{\lambda} K_H^*(\phi e_n)(y_1, y_2) dB(y_1) dB(y_2)$$

where

$$K_H^*(\vartheta)(y_1, y_2) = \int_{y_1 \vee y_2}^{G} \vartheta(r) \frac{\partial K}{\partial r}(r, y_1, y_2) dr.$$

For more details (see [30).
Lemma 2.1 (see [31]). If $\chi : [0, G] \to L_2^0(\Phi, \Lambda)$ satisfies $\int_0^G \|\chi(s)\|_{L_2^0}^2 < \infty$ then the above sum in (2.2) is well defined as Λ-valued random variable and we have

$$E\left\|\int_0^{\varrho} \chi(s) dZ_H(s)\right\|^2 \le 2H\varrho^{2H-1} \int_0^{\varrho} \|\chi(s)\|_{L_2^0}^2 ds.$$

We suppose that $0 \in \rho(T)$, the resolvent set of T, and $\| \Delta(\varrho) \| \le M$ for some constant $M \ge 1$ and every $\varrho \ge 0$. The collection of all strongly-measurable, square-integrable, X-valued random variables, denoted by $L_2(\Omega, \Lambda)$, is a Banach space equipped with norm

$$\| y(\cdot) \|_{L_2(\Omega,\Lambda)} = (E \| y(., \omega) \|^2)^{\frac{1}{2}},$$

where the expectation, E is defined by $E(y) = \int_{\Omega} y(\omega) dP$.
Let $C(J, L_2(\Omega, \Lambda))$ be the Banach space of all continuous maps from J into $L_2(\Omega, \Lambda)$ satisfying the condition

$\sup_{\varrho \in J} E \parallel y(\varrho) \parallel^2 < \infty$.
Define $\bar{C} = \{y : y(\varrho) \in C(J, L_2(\Omega, \Lambda))\}$, with norm $\| \cdot \|_{\bar{C}}$ defined by

$$\| \cdot \|_{\bar{C}} = (\sup_{\varrho \in J} E|y(\varrho)|^2)^{\frac{1}{2}}.$$

Obviously, $\bar{C}$ is a Banach space.
We impose the following conditions on data of the problem:
$(H1)$ $\mathfrak{R} : J \times \Lambda \to \Lambda$ is a continuous function, and there exists $M_1,\ M_2\ > 0$ such that the function $\mathfrak{R}$ satisfies the Lipschitz condition:

$$E \parallel \mathfrak{R}(\varrho_1, y) - \mathfrak{R}(\varrho_2, x) \parallel^2 \leq M_1(|\varrho_1 - \varrho_2| + E \parallel y - x \parallel^2),$$

for $0 \leq \varrho_1, \varrho_2 \leq G,\ y,\ x \in \Lambda$, and the inequality

$$E \parallel \mathfrak{R}(\varrho, y) \parallel^2 \leq M_2(E \parallel y \parallel^2 + 1) \tag{2}$$

holds for $(\varrho, y) \in J \times \Lambda$.
$(H2)$ The function $\mathfrak{I} : J \times \Lambda \to L(\Xi, \Lambda)$ satisfies the following conditions:
(i) for each $\varrho \in J$, the function $\mathfrak{I}(\varrho, .) : \Lambda \to L(\Xi, \Lambda)$ is continuous and for each $y \in \Lambda$; the function $\mathfrak{I}(., y) : J \to L(\Xi, \Lambda)$ is Υ_ϱ-measurable;
(ii) for each positive number $d \in N$, there is a positive function $h_d(\cdot) : (0, G] \to R^+$ such that

$$\sup_{\|y\|^2 \leq d} E \parallel \mathfrak{I}(\varrho, y) \parallel_S^2 \leq h_d(\varrho),$$

the function $s \to h_d(s) \in L^1((0, G], R^+)$ and there exists a $\Lambda_2 > 0$ such that

$$\liminf_{d \to \infty} \frac{\int_0^\varrho h_d(s) ds}{d} = \Lambda_2 < \infty, \quad \varrho \in (0, G].$$

$(H3)$ The function $\sigma : J \times \Lambda \to L_2^0(\Phi, \Lambda)$ satisfies the following conditions:
(i) for each $\varrho \in J$, the function $\sigma(\varrho, .) : \Lambda \to L_2^0(\Phi, \Lambda)$ is continuous and for each
$y \in \Lambda$; the function $\sigma(., y) : J \to L_2^0(\Phi, \Lambda)$ is Υ_ϱ-measurable;
(ii) for each positive number $d \in N$, there is a positive function $\bar{h}_d(\cdot) : (0, G] \to R^+$ such that

$$\sup_{\|x\|^2 \leq d} E \parallel \sigma(\varrho, y) \parallel_{L_2^0}^2 \leq \bar{h}_d(\varrho),$$

the function $s \to \bar{h}_d(s) \in L^1((0, G], R^+)$ and there exists a $\Lambda_3 > 0$ such that

$$\liminf_{d \to \infty} \frac{\int_0^\varrho \bar{h}_d(s) ds}{d} = \Lambda_3 < \infty,\ \varrho \in (0, G],$$

$(H4)$ The function $h : J \times \Lambda \times V \to \Lambda$ satisfies the following two conditions:
(i) The function $h(\varrho, ., .) : \Lambda \times V \to \Lambda$ is continuous.
(ii) for each positive number $d \in N$, there is a positive function $\chi_d(\cdot) : J \to R^+$ such that

$$\sup_{\|x\|^2 \leq d} \int_V E\|h(\varrho, y, v)\|^2 \lambda(dv) \leq \chi_d(\varrho),$$

the function $s \to \chi_d(s) \in L^1((0, G], R^+)$, and there exists a $\Lambda_4 > 0$ such that

$$\liminf_{d \to \infty} \frac{\int_0^\varrho \chi_d(s) ds}{d} = \Lambda_4 < \infty,\ \varrho \in (0, G].$$

$(H5)$ The function $g_i : (\varrho_i, s_i] \times \Lambda \to \Lambda$ is continuous and satisfies the following two conditions:
(i) There exist a constant $M_3 > 0$, such that

$$E\,\|g_i(\varrho, y)\|^2 \leq M_3(E\|y\|^2 + 1),\ \ \forall\ y \in \Lambda;\ \varrho \in (\varrho_i, s_i],\ i = 1, 2, \ldots, m.$$

(ii) There exist a constant $M_4 > 0$, such that

$$E\,\|g_i(\varrho, y) - g_i(\varrho, x)\|^2 \leq M_4\ E\,\|y - x\|^2,\ \ \forall\ y,\ x \in \Lambda;\ \varrho \in (\varrho_i, s_i],\ i = 1, 2, \ldots, m.$$

$(H6)$ The function $\psi : C(J, \Lambda) \to \Lambda$ satisfies the following two conditions:
(i) There exist a constant $M_5 > 0$, such that

$$E\,\|\psi(y)\|^2 \leq M_5(E\|y\|^2 + 1),\ \ \forall\ y \in \Lambda.$$

(ii) There exist a constant $M_6 > 0$, such that

$$E\,\|\psi(y) - \psi(x)\|^2 \leq M_6\ E\,\|y - x\|^2,\ \ \forall\ y,\ x \in \Lambda.$$

Definition 2.2. An Υ_ϱ-adapted stochastic process $y(\varrho) : J \to \Lambda$ is said to be a mild solution of problem (1) if the following stochastic integral equation is verified:

$$y(\varrho) = \begin{cases} \Delta\left(\frac{\varrho^\hbar}{\hbar}\right)[y_0 - \psi(y)] + \int_0^\varrho s^{\hbar-1}\Delta\left(\frac{\varrho^\hbar - s^\hbar}{\hbar}\right)[\Re(s, y(s)) + Qu(s)]\,ds \\ + \int_0^\varrho s^{\hbar-1}\Delta\left(\frac{\varrho^\hbar - s^\hbar}{\hbar}\right)\Im(s, y(s))d\omega(s) + \int_0^\varrho s^{\hbar-1}\Delta\left(\frac{\varrho^\hbar - s^\hbar}{\hbar}\right)\sigma(s, y(s))dZ_H(s) \\ + \int_0^\varrho s^{\hbar-1}\Delta\left(\frac{\varrho^\hbar - s^\hbar}{\hbar}\right)\int_V h(s, y(s), v)\tilde{N}(ds, dv),\ \ \varrho \in (0, \varrho_1] \\ g_i(\varrho, y(\varrho)),\ \ \varrho \in (\varrho_i, s_i],\ \ i = 1, 2, ..., m, \\ \Delta\left(\frac{\varrho^\hbar - s_i^\hbar}{\hbar}\right)g_i(s_i, y(s_i)) + \int_{s_i}^\varrho s^{\hbar-1}\Delta\left(\frac{\varrho^\hbar - s^\hbar}{\hbar}\right)[\Re(s, y(s)) + Qu(s)]\,ds \\ + \int_{s_i}^\varrho s^{\hbar-1}\Delta\left(\frac{\varrho^\hbar - s^\hbar}{\hbar}\right)\Im(s, y(s))d\omega(s) + \int_{s_i}^\varrho s^{\hbar-1}\Delta\left(\frac{\varrho^\hbar - s^\hbar}{\hbar}\right)\sigma(s, y(s))dZ_H(s) \\ + \int_{s_i}^\varrho s^{\hbar-1}\Delta\left(\frac{\varrho^\hbar - s^\hbar}{\hbar}\right)\int_V h(s, y(s), v)\tilde{N}(ds, dv),\ \ \varrho \in (s_i, \varrho_{i+1}],\ i = 1, 2, \ldots, m. \end{cases}$$

3. Main result

In this section, we study the controllability for the system (1).
Definition 3.1. The system (1) is said to be controllable on J, if for every $y_0, y_1 \in \Lambda$, there exists a control $u \in L^2(J, U)$ such that the mild solution $y(\varrho)$ of the system (1) satisfies $y(G) = y_1$, where y_1 and G are the preassigned terminal state and time respectively.
To establish the result, we need the following additional hypothesis
$(H7)$ The linear operator Π from U into Λ defined by

$$\Pi u = \int_0^G s^{\hbar-1}\Delta\left(\frac{G^\hbar - s^\hbar}{\hbar}\right)Qu(s)ds$$

has an inverse operator Π^{-1} which takes values in $L^2(J, U)\backslash \ker \Pi$, where the kernel space of Π is defined by $\ker \Pi = \{y \in L^2(J, U) : \Pi y = 0\}$ and there exist positive constant M_Q, M_Π such that $\|Q\|^2 = M_Q$, $\|\Pi^{-1}\|^2 = M_\Pi$.
Theorem 3.1. If the assumptions $(H1)$-$(H7)$ are satisfied. Then, the system (1) is controllable on J provided that

$$36\left\{1 + \frac{M_\Pi M_Q M^2 G^{2\hbar-1}}{2\hbar - 1}\right\}\left\{M_5 + M_3(M^2 + 1) + \frac{M^2 G^{2\hbar-1} M_2}{2\hbar - 1} + \frac{Tr(S) M^2 G^{2\hbar-1}}{2\hbar - 1}\Lambda_2 + \frac{2HM^2 G^{2H+2\hbar-2}}{2\hbar - 1}\Lambda_3 + \frac{M^2 G^{2\hbar-1}}{2\hbar - 1}\Lambda_4\right\} < 1. \tag{3}$$

and

$$\ell_1 = 4M_6 + \frac{4M^2 M_1 G^{2\hbar-1}}{2\hbar - 1} + M_4(1 + 4M^2) < 1. \tag{4}$$

Proof. Consider the map $\mho$ on $\bar{C}$ defined by

$$(\mho y)(\varrho) = \begin{cases} \Delta\left(\frac{\varrho^{\hbar}}{\hbar}\right)[y_0 - \psi(y)] + \int_0^{\varrho} s^{\hbar-1}\Delta\left(\frac{\varrho^{\hbar}-s^{\hbar}}{\hbar}\right)[\Re(s, y(s)) + Qu(s)]\,ds \\ + \int_0^{\varrho} s^{\hbar-1}\Delta\left(\frac{\varrho^{\hbar}-s^{\hbar}}{\hbar}\right)\Im(s, y(s))d\omega(s) + \int_0^{\varrho} s^{\hbar-1}\Delta\left(\frac{\varrho^{\hbar}-s^{\hbar}}{\hbar}\right)\sigma(s, y(s))dZ_H(s) \\ + \int_0^{\varrho} s^{\hbar-1}\Delta\left(\frac{\varrho^{\hbar}-s^{\hbar}}{\hbar}\right)\int_V h(s, y(s), v)\tilde{N}(ds, dv),\ \varrho \in (0, \varrho_1] \\ g_i(\varrho, y(\varrho)),\ \ \varrho \in (\varrho_i, s_i],\ \ i = 1, 2, ..., m, \\ \Delta\left(\frac{\varrho^{\hbar}-s_i^{\hbar}}{\hbar}\right)g_i(s_i, y(s_i)) + \int_{s_i}^{\varrho} s^{\hbar-1}\Delta\left(\frac{\varrho^{\hbar}-s^{\hbar}}{\hbar}\right)[\Re(s, y(s)) + Qu(s)]\,ds \\ + \int_{s_i}^{\varrho} s^{\hbar-1}\Delta\left(\frac{\varrho^{\hbar}-s^{\hbar}}{\hbar}\right)\Im(s, y(s))d\omega(s) + \int_{s_i}^{\varrho} s^{\hbar-1}\Delta\left(\frac{\varrho^{\hbar}-s^{\hbar}}{\hbar}\right)\sigma(s, y(s))dZ_H(s) \\ + \int_{s_i}^{\varrho} s^{\hbar-1}\Delta\left(\frac{\varrho^{\hbar}-s^{\hbar}}{\hbar}\right)\int_V h(s, y(s), v)\tilde{N}(ds, dv),\ \varrho \in (s_i, \varrho_{i+1}],\ i = 1, 2, \ldots, m. \end{cases}$$

where

$$u(\varrho) = \begin{cases} \Pi^{-1}\{y_1 - \Delta\left(\frac{G^{\hbar}}{\hbar}\right)[y_0 - \psi(y)] - \int_0^{G} s^{\hbar-1}\Delta\left(\frac{G^{\hbar}-s^{\hbar}}{\hbar}\right)\Re(s, y(s))ds \\ - \int_0^{G} s^{\hbar-1}\Delta\left(\frac{G^{\hbar}-s^{\hbar}}{\hbar}\right)\Im(s, y(s))d\omega(s) - \int_0^{G} s^{\hbar-1}\Delta\left(\frac{G^{\hbar}-s^{\hbar}}{\hbar}\right)\sigma(s, y(s))dZ_H(s) \\ - \int_0^{G} s^{\hbar-1}\Delta\left(\frac{G^{\hbar}-s^{\hbar}}{\hbar}\right)\int_V h(s, y(s), v)\tilde{N}(ds, dv)\}(\varrho),\ \varrho \in (0, \varrho_1] \\ \Pi^{-1}\{y_1 - \Delta\left(\frac{G^{\hbar}-s_i^{\hbar}}{\hbar}\right)g_i(s_i, y(s_i)) - \int_{s_i}^{G} s^{\hbar-1}\Delta\left(\frac{G^{\hbar}-s^{\hbar}}{\hbar}\right)\Re(s, y(s))ds \\ - \int_{s_i}^{G} s^{\hbar-1}\Delta\left(\frac{G^{\hbar}-s^{\hbar}}{\hbar}\right)\Im(s, y(s))d\omega(s) - \int_{s_i}^{G} s^{\hbar-1}\Delta\left(\frac{G^{\hbar}-s^{\hbar}}{\hbar}\right)\sigma(s, y(s))dZ_H(s) \\ - \int_{s_i}^{G} s^{\hbar-1}\Delta\left(\frac{G^{\hbar}-s^{\hbar}}{\hbar}\right)\int_V h(s, y(s), v)\tilde{N}(ds, dv)\}(\varrho),\ \varrho \in (s_i, \varrho_{i+1}],\ i = 1, 2, \ldots, m. \end{cases}$$

It will be shown that the operator $\mho$ from $\bar{C}$ into itself has a fixed point.
For each positive integer d, set $\wp_d = \{y \in \bar{C},\ \ \| y \|_{\bar{C}}^2 \leq d\}$.
We claim that there exists a positive number d such that $\mho(\wp_d) \subset \wp_d$. If it is not true, then for each positive number d, there is a function $y_d(\cdot) \in \wp_d$, but $\mho(y_d) \notin \wp_d$, that is $\| (\mho y_d)(\varrho) \|_{\bar{C}}^2 > d$ for some $\varrho = \varrho(d) \in J$, where $\varrho(d)$ denotes that ϱ is dependent of d.
From $(H1)$-$(H7)$, we have,
for $\varrho \in (0, \varrho_1]$

$$\begin{aligned} \| \mho y_d \|_{\bar{C}}^2 \ \ \leq\ \ & 36 \sup_{\varrho \in J}\Bigg\{E\|\Delta\left(\frac{\varrho^{\hbar}}{\hbar}\right)[y_0 - \psi(y)]\|^2 + E\ \| \int_0^{\varrho} s^{\hbar-1}\Delta\left(\frac{\varrho^{\hbar} - s^{\hbar}}{\hbar}\right)\Re(s, y(s))ds\ \|^2 \\ & + E\ \| \int_0^{\varrho} s^{\hbar-1}\Delta\left(\frac{\varrho^{\hbar} - s^{\hbar}}{\hbar}\right)Qu(s)ds\ \|^2 \\ & + E\ \| \int_0^{\varrho} s^{\hbar-1}\Delta\left(\frac{\varrho^{\hbar} - s^{\hbar}}{\hbar}\right)\Im(s, y(s))d\omega(s)\ \|^2 \\ & + E\ \| \int_0^{\varrho} s^{\hbar-1}\Delta\left(\frac{\varrho^{\hbar} - s^{\hbar}}{\hbar}\right)\sigma(s, y(s))dZ_H(s)\ \|^2 \\ & + \int_0^{\varrho} s^{\hbar-1}\Delta\left(\frac{\varrho^{\hbar} - s^{\hbar}}{\hbar}\right)\int_V h(s, y(s), v)\tilde{N}(ds, dv)\ \|^2\Bigg\} \end{aligned}$$

$$\begin{aligned}
&\leq 36\left\{1+\frac{M_\Pi M_Q M^2 G^{2\hbar-1}}{2\hbar-1}\right\}\left\{M^2[E\|y_0\|^2+M_5(d+1)]\right.\\
&+\frac{M^2G^{2\hbar-1}M_1(d+1)}{2\hbar-1}+\frac{Tr(S)M^2G^{2\hbar-1}}{2\hbar-1}\int_0^G h_d(s)ds\\
&\left.+\frac{2HM^2G^{2H+2\hbar-2}}{2\hbar-1}\int_0^G \bar{h}_d(s)ds+\frac{M^2G^{2\hbar-1}}{2\hbar-1}\int_0^G \chi_d(s)ds\right\}\\
&+\frac{36M_\Pi M_Q M^2G^{2\hbar-1}E\|y_1\|^2}{2\hbar-1}. \qquad (5)
\end{aligned}$$

From $(H5)$, we have for $\varrho \in (\varrho_i, s_i]$

$$\| \mho y_d \|_{\mathbb{C}}^2 \leq \sup_{\varrho\in J} E\|g_i(\varrho, y(\varrho))\|^2 \leq M_3(d+1). \qquad (6)$$

From $(H1)$-$(H7)$, we have, for $\varrho \in (s_i, \varrho_{i+1}]$

$$\begin{aligned}
\| \mho y_d \|_{\mathbb{C}}^2 &\leq 36\sup_{\varrho\in J}\left\{E\|\Delta\left(\frac{\varrho^\hbar - s_i^\hbar}{\hbar}\right)g_i(s_i, y(s_i))\|^2\right.\\
&+E\|\int_{s_i}^{\varrho} s^{\hbar-1}\Delta\left(\frac{\varrho^\hbar - s^\hbar}{\hbar}\right)\Re(s,y(s))ds\|^2+E\|\int_{s_i}^{\varrho} s^{\hbar-1}\Delta\left(\frac{\varrho^\hbar - s^\hbar}{\hbar}\right)Qu(s)ds\|^2\\
&+E\|\int_0^{\varrho} s^{\hbar-1}\Delta\left(\frac{\varrho^\hbar - s^\hbar}{\hbar}\right)\int_{s_i}^{s}\Im(\tau,y(\tau))d\omega(\tau)ds\|^2\\
&+E\|\int_{s_i}^{\varrho} s^{\hbar-1})\Delta\left(\frac{\varrho^\hbar - s^\hbar}{\hbar}\right)\sigma(s,y(s))dZ_H(s)\|^2\\
&\left.+\int_{s_i}^{\varrho} s^{\hbar-1}\Delta\left(\frac{\varrho^\hbar - s^\hbar}{\hbar}\right)\int_V h(s,y(s),v)\tilde{N}(ds,dv)\|^2\right\}\\
&\leq 36\left\{1+\frac{M_\Pi M_Q M^2 G^{2\hbar-1}}{2\hbar-1}\right\}\left\{M^2M_3(d+1)\right.\\
&+\frac{M^2G^{2\hbar-1}M_2(d+1)}{2\hbar-1}+\frac{Tr(S)M^2G^{2\hbar-1}}{2\hbar-1}\int_{s_i}^G h_d(s)ds\\
&\left.+\frac{2HM^2G^{2H+2\hbar-2}}{2\hbar-1}\int_{s_i}^G \bar{h}_d(s)ds+\frac{M^2G^{2\hbar-1}}{2\hbar-1}\int_{s_i}^G \chi_d(s)ds\right\}\\
&+\frac{36M_\Pi M_Q M^2G^{2\hbar-1}E\|y_1\|^2}{2\hbar-1}. \qquad (7)
\end{aligned}$$

Combining (5), (6), (7) in the inequality $d \leq \| (\mho y_d)(t) \|_{\mathbb{C}}^2$ then dividing both sides of the inequality by d and taking the lower limit $d \to +\infty$, we get

$$\begin{aligned}
&36\left\{1+\frac{M_\Pi M_Q M^2G^{2\hbar-1}}{2\hbar-1}\right\}\left\{M_5+M_3(M^2+1)+\frac{M^2G^{2\hbar-1}M_2}{2\hbar-1}+\frac{Tr(S)M^2G^{2\hbar-1}}{2\hbar-1}\Lambda_2\right.\\
&\left.+\frac{2HM^2G^{2H+2\hbar-2}}{2\hbar-1}\Lambda_3+\frac{M^2G^{2\hbar-1}}{2\hbar-1}\Lambda_4\right\} \geq 1.
\end{aligned}$$

This contradicts (3). Hence for positive d, $\mho(\wp_d) \subset \wp_d$.
Next we will show that the operator $\mho$ has a fixed point on $\wp_d$, which implies that equation (1) has a mild solution. We decompose $\mho$ as $\mho = \mho_1 + \mho_2$, where the operators $\mho_1$ and $\mho_2$ are defined on $\wp_d$, respectively, by

$$(\mho_1 y)(\varrho) = \begin{cases} \Delta\left(\frac{\varrho^{\hbar}}{\hbar}\right)[y_0 - \psi(y)] + \int_0^{\varrho} s^{\hbar-1}\Delta\left(\frac{\varrho^{\hbar}-s^{\hbar}}{\hbar}\right)\Re(s, y(s))ds, \ \varrho \in (0, \varrho_1] \\ g_i(\varrho, y(\varrho)), \ \ \varrho \in (\varrho_i, s_i], \ \ i = 1, 2, ..., m, \\ \Delta\left(\frac{\varrho^{\hbar}-s_i^{\hbar}}{\hbar}\right) g_i(s_i, y(s_i)) + \int_{s_i}^{\varrho} s^{\hbar-1}\Delta\left(\frac{\varrho^{\hbar}-s^{\hbar}}{\hbar}\right)\Re(s, y(s))ds, \\ \qquad\qquad \varrho \in (s_i, \varrho_{i+1}], \ i = 1, 2, \ldots, m. \end{cases}$$

$$(\mho_2 y)(\varrho) = \begin{cases} \int_{s_i}^{\varrho} s^{\hbar-1}\Delta\left(\frac{\varrho^{\hbar}-s^{\hbar}}{\hbar}\right) Qu(s)ds \\ + \int_{s_i}^{\varrho} s^{\hbar-1}\Delta\left(\frac{\varrho^{\hbar}-s^{\hbar}}{\hbar}\right)\Im(s, y(s))d\omega(s) + \int_{s_i}^{\varrho} s^{\hbar-1}\Delta\left(\frac{\varrho^{\hbar}-s^{\hbar}}{\hbar}\right)\sigma(s, y(s))dZ_H(s) \\ + \int_{s_i}^{\varrho} s^{\hbar-1}\Delta\left(\frac{\varrho^{\hbar}-s^{\hbar}}{\hbar}\right)\int_V h(s, y(s), v)\tilde{N}(ds, dv), \ \varrho \in (s_i, \varrho_{i+1}], \ i = 0, 1, \ldots, m. \\ 0, \qquad\qquad otherwise. \end{cases}$$

For $\varrho \in J$, we will show that $\mho_1$ verifies a contraction condition while $\mho_2$ is a compact operator.
To prove that $\mho_1$ satisfies a contraction condition, we take $y_1, \ y_2 \in \wp_d$, then for each $\varrho \in J$ and by conditions $(H1)$, $(H5)$ and $(H6)$
we have for $\varrho \in (0, \varrho_1]$

$$\begin{aligned} E \parallel (\mho_1 y_1)(\varrho) - (\mho_1 y_2)(\varrho) \parallel^2 &\leq 4\Big\{E \parallel \psi(y_1) - \psi(y_2) \parallel \\ &\quad + E \parallel \int_0^{\varrho} s^{\hbar-1}\Delta\left(\frac{\varrho^{\hbar} - s^{\hbar}}{\hbar}\right)[\Re(s, y_1(s)) - \Re(s, y_2(s))]ds \parallel^2 \Big\} \\ &\leq \Big\{4M_6 + \frac{4M^2 M_1 G^{2\hbar-1}}{2\hbar - 1}\Big\} E \parallel y_1(\varrho) - y_2(\varrho) \parallel^2, \end{aligned} \tag{8}$$

for $\varrho \in (\varrho_i, s_i]$

$$\begin{aligned} E \parallel (\mho_1 y_1)(\varrho) - (\mho_1 y_2)(\varrho) \parallel^2 &\leq E \parallel g_i(\varrho, y_1(\varrho)) - g_i(\varrho, y_2(\varrho)) \parallel^2 \\ &\leq M_4 \, E \parallel y_1(\varrho) - y_2(\varrho) \parallel^2 \end{aligned} \tag{9}$$

and for $\varrho \in (s_i, \varrho_{i+1}]$

$$\begin{aligned} E \parallel (\mho_1 y_1)(\varrho) - (\mho_1 y_2)(\varrho) \parallel^2 &\leq 4E \parallel \Delta\left(\frac{\varrho^{\hbar} - s_i^{\hbar}}{\hbar}\right)(g_i(s_i, y_1(s_i)) - g_i(s_i, y_2(s_i))) \parallel^2 \\ &\quad + 4E \parallel \int_{s_i}^{\varrho} s^{\hbar-1}\Delta\left(\frac{\varrho^{\hbar} - s^{\hbar}}{\hbar}\right)[\Re(s, y_1(s)) - \Re(s, y_2(s))]ds \parallel^2 \\ &\leq \left[4M^2 M_4 + \frac{4M^2 M_1 G^{2\hbar-1}}{2\hbar - 1}\right] E \parallel y_1(\varrho) - y_2(\varrho) \parallel^2 . \end{aligned} \tag{10}$$

Combining (8), (9) and (10) , we get

$$\begin{aligned} E \parallel (\mho_1 y_1)(\varrho) - (\mho_1 y_2)(\varrho) \parallel^2 &\leq \left[4M_6 + \frac{4M^2 M_1 G^{2\hbar-1}}{2\hbar - 1} + M_4(1 + 4M^2)\right] E \parallel y_1(\varrho) - y_2(\varrho) \parallel^2 \\ &\leq \ell_1 E \parallel y_1(\varrho) - y_2(\varrho) \parallel^2, \end{aligned}$$

therefore,

$$\sup_{\varrho \in J} E \parallel (\mho_1 y_1)(\varrho) - (\mho_1 y_2)(\varrho) \parallel^2 \leq \ell_1 \sup_{\varrho \in J} E \parallel y_1(\varrho) - y_2(\varrho) \parallel^2$$

hence,

$$\| \mho_1 y_1 - \mho_1 y_2 \|_{\bar{C}}^2 \leq \ell_1 \| y_1 - y_2 \|_{\bar{C}}^2 .$$

Thus, $\mho_1$ is a contraction.
To prove that $\mho_2$ is compact, first we prove that $\mho_2$ is continuous on $\wp_d$.
Let $\{y_n\} \subset \wp_d$ with $y_n \to \Lambda$ in $\wp_q$ and rewrite $u(t) = u(t, y)$ the control function defined above. Then for each $s \in J,\ y_n(s) \to y(s)$, and by $H2(i)$, $H3(i)$ and $H4(i)$, we have
$\Im(s, y_n(s)) \to \Im(s, y(s))$, as $n \to \infty$,
$\sigma(s, y_n(s)) \to \sigma(s, y(s))$, as $n \to \infty$ and
$h(s, y_n(s), v) \to h(s, y(s), v)$, as $n \to \infty$.
By the dominated convergence theorem, we have,

$$\begin{aligned}
&\| \mho_2 y_n - \mho_2 y \|_{\bar{C}}^2 = \sup_{\varrho \in J} \Bigg\{ E \| \int_{s_i}^{\varrho} s^{\hbar-1} \Delta\left(\frac{\varrho^{\hbar} - s^{\hbar}}{\hbar}\right) Q(u(s, y_n) - u(s, y))ds \\
&+ \int_{s_i}^{\varrho} s^{\hbar-1} \Delta\left(\frac{\varrho^{\hbar} - s^{\hbar}}{\hbar}\right) [\Im(s, y_n(s)) - \Im(s, y(s))] d\omega(s) \\
&+ \int_{s_i}^{\varrho} s^{\hbar-1} \Delta\left(\frac{\varrho^{\hbar} - s^{\hbar}}{\hbar}\right) [\sigma(s, y_n(s)) - \sigma(s, y(s))] dZ_H(s) \\
&+ \int_{s_i}^{\varrho} s^{\hbar-1} \Delta\left(\frac{\varrho^{\hbar} - s^{\hbar}}{\hbar}\right) \int_V [h(s, y_n(s), v) - h(s, y(s), v)] \tilde{N}(ds, dv) \|^2 \Bigg\} \to 0,
\end{aligned}$$

as $n \to \infty$, that is continuous.
Next we prove that the family $\{\mho_2 y : y \in \wp_d\}$ is an equicontinuous family of functions.
Let $\epsilon > 0$ small, $s_i < \varrho_\alpha < \varrho_\beta \leq \varrho_{i+1}$, then

$$\begin{aligned}
&E \| (\mho_2 y)(\varrho_\beta) - (\mho_2 y)(\varrho_\alpha) \|^2 \\
&\leq E \| \int_{s_i}^{\varrho_\alpha - \epsilon} s^{\hbar-1} \left(\Delta\left(\frac{\varrho_\beta^{\hbar} - s^{\hbar}}{\hbar}\right) - \Delta\left(\frac{\varrho_\alpha^{\hbar} - s^{\hbar}}{\hbar}\right) \right) Qu(s)ds \|^2 \\
&+ \| \int_{\varrho_\alpha - \epsilon}^{\varrho_\alpha} s^{\hbar-1} \left(\Delta\left(\frac{\varrho_\beta^{\hbar} - s^{\hbar}}{\hbar}\right) - \Delta\left(\frac{\varrho_\alpha^{\hbar} - s^{\hbar}}{\hbar}\right) \right) Qu(s)ds \|^2 \\
&+ \| \int_{\varrho_\alpha}^{\varrho_\beta} s^{\hbar-1} \Delta\left(\frac{\varrho_\beta^{\hbar} - s^{\hbar}}{\hbar}\right) Qu(s)ds \|^2 \\
&+ E \| \int_{s_i}^{\varrho_\alpha - \epsilon} s^{\hbar-1} \left(\Delta\left(\frac{\varrho_\beta^{\hbar} - s^{\hbar}}{\hbar}\right) - \Delta\left(\frac{\varrho_\alpha^{\hbar} - s^{\hbar}}{\hbar}\right) \right) \Im(s, y(s)) d\omega(s) \|^2 \\
&+ \| \int_{\varrho_\alpha - \epsilon}^{\varrho_\alpha} s^{\hbar-1} \left(\Delta\left(\frac{\varrho_\beta^{\hbar} - s^{\hbar}}{\hbar}\right) - \Delta\left(\frac{\varrho_\alpha^{\hbar} - s^{\hbar}}{\hbar}\right) \right) \Im(s, y(s)) d\omega(s) \|^2 \\
&+ \| \int_{\varrho_\alpha}^{\varrho_\beta} s^{\hbar-1} \Delta\left(\frac{\varrho_\beta^{\hbar} - s^{\hbar}}{\hbar}\right) \Im(s, y(s)) d\omega(s) \|^2
\end{aligned}$$

$$+E \parallel \int_{s_i}^{\varrho_\alpha-\epsilon} s^{\hbar-1}\Big(\Delta\left(\frac{\varrho_\beta^\hbar-s^\hbar}{\hbar}\right)-\Delta\left(\frac{\varrho_\alpha^\hbar-s^\hbar}{\hbar}\right)\Big)\sigma(s,y(s))dZ_H(s) \parallel^2$$

$$+ \parallel \int_{\varrho_\alpha-\epsilon}^{\varrho_\alpha} s^{\hbar-1}\Big(\Delta\left(\frac{\varrho_\beta^\hbar-s^\hbar}{\hbar}\right)-\Delta\left(\frac{\varrho_\alpha^\hbar-s^\hbar}{\hbar}\right)\Big)\sigma(s,y(s))dZ_H(s) \parallel^2$$

$$+ \parallel \int_{\varrho_\alpha}^{\varrho_\beta} s^{\hbar-1}\Delta\left(\frac{\varrho_\beta^\hbar-s^\hbar}{\hbar}\right)\sigma(s,y(s))dZ_H(s) \parallel^2$$

$$+E \parallel \int_{s_i}^{\varrho_\alpha-\epsilon} s^{\hbar-1}\Big(\Delta\left(\frac{\varrho_\beta^\hbar-s^\hbar}{\hbar}\right)-\Delta\left(\frac{\varrho_\alpha^\hbar-s^\hbar}{\hbar}\right)\Big)\int_V h(s,y(s),v)\tilde{N}(ds,dv) \parallel^2$$

$$+ \parallel \int_{\varrho_\alpha-\epsilon}^{\varrho_\alpha} s^{\hbar-1}\Big(\Delta\left(\frac{\varrho_\beta^\hbar-s^\hbar}{\hbar}\right)-\Delta\left(\frac{\varrho_\alpha^\hbar-s^\hbar}{\hbar}\right)\Big)\int_V h(s,y(s),v)\tilde{N}(ds,dv) \parallel^2$$

$$+ \parallel \int_{\varrho_\alpha}^{\varrho_\beta} s^{\hbar-1}\Delta\left(\frac{\varrho_\beta^\hbar-s^\hbar}{\hbar}\right)\int_V h(s,y(s),v)\tilde{N}(ds,dv) \parallel^2 .$$

We see that $E \parallel (\mho_2 y)(\varrho_\beta) - (\mho_2 y)(\varrho_\alpha) \parallel^2$ tends to zero independently of $y \in \wp_d$ as $\varrho_\beta \to \varrho_\alpha$, and with ϵ sufficiently small, since the compactness of $\Delta(\varrho)$ for $\varrho > 0$ implies the continuity in the uniform operator topology (see [28]). Similarly, we can prove that the function $\mho_2 y,\ y \in \wp_d$ are equicontinuous at $\varrho = 0$. Hence $\mho_2$ maps $\wp_d$ into a family of equicontinuous functions.

It remains to prove that $\mu(\varrho) = \{(\mho_2 y)(\varrho) : y \in \wp_d\}$ is relatively compact in $\wp_d$. Obviously, $\mu(0)$ is relatively compact in $\wp_d$.

Let $s_i < \varrho \le \varrho_{i+1}$ be fixed and let ϵ be a given real number satisfying $s_i < \epsilon < \varrho$, for $y \in \wp_d$, we define

$$\begin{aligned}(\mho_2^\epsilon y)(\varrho) \ = \ & \int_{s_i}^{\varrho-\epsilon} s^{\hbar-1}\Delta\left(\frac{\varrho^\hbar-s^\hbar}{\hbar}\right)Qu(s)ds + \int_{s_i}^{\varrho-\epsilon} s^{\hbar-1}\Delta\left(\frac{\varrho^\hbar-s^\hbar}{\hbar}\right)\Im(s,y(s))d\omega(s) \\ & + \int_{s_i}^{\varrho-\epsilon} s^{\hbar-1}\Delta\left(\frac{\varrho^\hbar-s^\hbar}{\hbar}\right)\sigma(s,y(s))dZ_H(s) \\ & + \int_{s_i}^{\varrho-\epsilon} s^{\hbar-1}\Delta\left(\frac{\varrho^\hbar-s^\hbar}{\hbar}\right)\int_V h(s,y(s),v)\tilde{N}(ds,dv),\ \varrho \in (s_i,\varrho_{i+1}],\ i=1,2,\dots,m.\end{aligned}$$

Since $u(s)$ is bounded and $\Delta(\varrho)$ is a compact operator, then the set
$\mu^\epsilon(\varrho) = \{(\mho_2^\epsilon y)(\varrho) : y \in \wp_d\}$ is relatively compact in Λ for every ϵ, $s_i < \epsilon < \varrho$.
Moreover, for every $y \in \wp_d$, we have

$$E \parallel \mho_2 y - \mho_2^\epsilon y \parallel_{\mathcal{C}}^2 \le 16 \sup_{\varrho\in J}\Big\{E \parallel \int_{\varrho-\epsilon}^{\varrho} s^{\hbar-1}\Delta\left(\frac{\varrho^\hbar-s^\hbar}{\hbar}\right)Qu(s)ds \parallel^2$$

$$+E \parallel \int_{\varrho-\epsilon}^{\varrho} s^{\hbar-1}\Delta\left(\frac{\varrho^\hbar-s^\hbar}{\hbar}\right)\Im(s,y(s))d\omega(s) \parallel^2$$

$$+E \parallel \int_{\varrho-\epsilon}^{\varrho} s^{\hbar-1}\Delta\left(\frac{\varrho^\hbar-s^\hbar}{\hbar}\right)\sigma(s,y(s))dZ_H(s) \parallel^2$$

$$+E \parallel \int_{\varrho-\epsilon}^{\varrho} s^{\hbar-1}\Delta\left(\frac{\varrho^\hbar-s^\hbar}{\hbar}\right)\int_V h(s,y(s),v)\tilde{N}(ds,dv) \parallel^2 \Big\}.$$

We see that for each $y \in \wp_d$, $\parallel \mho_2 y - \mho_2^\epsilon \parallel_{\mathcal{C}}^2 \to 0$ as $\epsilon \to 0^+$. Therefore, there are relative compact sets arbitrary close to the set $\mu(\varrho) = \{(\mho_2 y)(\varrho) : y \in \wp_d\}$, hence the set $\mu(\varrho)$ is also relatively compact in $\wp_d$.

Thus, by Arzela-Ascoli theorem $\mho_2$ is a compact operator. These arguments enable us to conclude that $\mho = \mho_1 + \mho_2$ is a condensing map on $\wp_d$, and by the fixed point theorem of Sadovskii there exists a fixed point $y(\cdot)$ for $\mho$ on $\wp_d$. Thus, the system (1) is controllable on J.

4. Example

In this section, we present an example to illustrate our main result.
Let us consider the following noninstantaneous impulsive conformable fractional stochastic differential equation with Rosenblatt process and nonlocal condition:

$$\begin{cases} D_{0+}^{\frac{3}{4}}y(\varrho,z)+\frac{\partial^2}{\partial z^2}y(\varrho,z)=3^{-\varrho}y(\varrho,z)+\eta(\varrho,z)+e^{-\varrho}y(\varrho,z)\frac{d\omega(\varrho)}{dt} \\ +\frac{\sin\varrho}{1+\sin\varrho}y(\varrho,z)\frac{dZ_H(\varrho)}{d\varrho}+\int_V\bar{h}(\varrho,y(\varrho,z),v)\tilde{N}(d\varrho,dv),\ \ \varrho\in(0,\frac{2}{3}]\cup(\frac{4}{3},2],\ 0\le z\le\pi, \\ y(\varrho,0)=y(\varrho,\pi)=0,\ \varrho\in(0,2], \\ y(\varrho,z)=\frac{2}{5}e^{-(\varrho-\frac{1}{4})}\frac{|y(\varrho,z)|}{1+|y(\varrho,z)|},\ \ \varrho\in(\frac{2}{3},\frac{4}{3}],\ 0\le z\le\pi, \\ y(0,z)+\sum_{i=1}^{2}c_iy(\varrho_i,z)=y_0(z),\ \ 0\le z\le\pi, \end{cases} \tag{11}$$

where $D_{0+}^{\frac{3}{4}}$ is conformable fractional derivative of order $\hbar=\frac{3}{4}$, ω is a Wiener process and Z_H is a Rosenblatt process with Hurst parameter $H\in(\frac{1}{2},1)$.
Let $\Lambda=\Phi=\Xi=U=L_2([0,\pi])$ and T be defined by $Tm=-(\frac{\partial^2}{\partial z^2})m$ with domain $D(T)=\{\xi\in\Lambda:\xi,\ \frac{d\xi}{dz}$ are absolutely continuous, and $(\frac{d^2}{dz^2})\xi\in\Lambda,\ \xi(0)=\xi(\pi)=0\}$.
Then $-T$ generates a strongly continuous semigroup $\Delta(\cdot)$ which is compact, analytic, and self-adjoint. Furthermore, $-T$ has discrete spectrum with eigenvalues $n^2,\ n\in N$ and the corresponding normalized eigenfunctions given by

$$e_n=\sqrt{\frac{2}{\pi}}\sin ny,\ \ n=1,2,...$$

In addition $(e_n)_{n\in N}$ is a complete orthonormal basis in Λ. Then

$$-T\xi=\sum_{n=1}^{\infty}n^2\langle\xi,e_n\rangle e_n,\ \xi\in D(T).$$

Furthermore, $-T$ is the infinitesimal generator of an analytic semigroup of bounded linear operator, $\{\Delta(\varrho)\}_{\varrho\ge0}$ on Λ and is given by

$$\Delta(\varrho)\xi=\sum_{n=1}^{\infty}e^{-n^2\varrho}\langle\xi,e_n\rangle_{e_n},\ \xi\in\Lambda,\ \varrho\ge0.$$

with $\|\Delta(\varrho)\|\le e^{-\varrho}\le1$.
We define the bounded operator $Q:U\to\Lambda$ by $Q=I$.
In order to define the operator $S:\Phi\to\Phi$, we choose a sequence $\{\lambda_n\}_{n\in N}\subset R^+$, set $Se_n=\lambda_ne_n$, and assume that

$$Tr(S)=\sum_{n=1}^{\infty}\sqrt{\lambda_n}<\infty.$$

We define $\mathfrak{R}:J\times\Lambda\to\Lambda$, $\mathfrak{I}:J\times\Lambda\to L(\Xi,\Lambda)$, $\sigma:J\times\Lambda\to L_2^0(Y,X)$,$h:J\times\Lambda\times V\to\Lambda$, $g_1:(\frac{2}{3},\frac{4}{3}]\times\Lambda\to\Lambda$ and $\psi:C(J,\Lambda)\to\Lambda$ by
$\mathfrak{R}(\varrho,y)=e^{-\varrho}y(\varrho,z)$, $\mathfrak{I}(\varrho,y)(z)=e^{-\varrho}y(\varrho,z)$, $\sigma(\varrho,y)(z)=\frac{\sin\varrho}{1+\sin\varrho}y(\varrho,z)$, $h=\bar{h}(\varrho,y(\varrho,z),v)$, $g_1(\varrho,y(\varrho))=\frac{2}{5}e^{-(\varrho-\frac{1}{4})}\frac{|x(\varrho,\cdot)|}{1+|x(\varrho,\cdot)|}$ and $\psi=\sum_{i=1}^{2}c_iy(\varrho_i,z)$, respectively and $J=(0,2]$. Then $\mathfrak{R}$, $\mathfrak{I}$, σ, h, g_1 and ψ satisfy $(H1)$-$(H6)$,

$$36\left\{1+\frac{M_\Pi M_QM^2G^{2\hbar-1}}{2\hbar-1}\right\}\left\{M_5+M_3(M^2+1)+\frac{M^2G^{2\hbar-1}M_2}{2\hbar-1}+\frac{Tr(S)M^2G^{2\hbar-1}}{2\hbar-1}\Lambda_2\right.$$
$$\left.+\frac{2HM^2G^{2H+2\hbar-2}}{2\hbar-1}\Lambda_3+\frac{M^2G^{2\hbar-1}}{2\hbar-1}\Lambda_4\right\}<1$$

and

$$4M_6 + \frac{4M^2 M_1 G^{2\hbar-1}}{2\hbar - 1} + M_4(1 + 4M^2) < 1.$$

Hence, according to Theorem 3.1, the system (11) is controllable on $J = (0, 2]$.

Conclusion

In this article, By using Sadovskii's fixed point theorem, fractional calculus and stochastic analysis, we studied the controllability of noninstantaneous impulsive conformable fractional stochastic differential equation with nonlinear noise and nonlocal condition via Rosenblatt process and Poisson jump. In the end paper, an example was provided to illustrate the applicability of the results.

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References

[1] Weiming Wang, Yongli Cai, Zuqin Ding, and Zhanji Gui, A stochastic differential equation SIS epidemic model incorporating Ornstein–Uhlenbeck process, Physica A: Statistical Mechanics and its Applications 509 (2018), 921–936.
[2] Othman A.M. Omar, Reda A. Elbarkouky, Hamdy M. Ahmed, Fractional stochastic models for COVID-19: Case study of Egypt, Results in Physics 23 (2021), 104018.
[3] Humberto Verdejo, Almendra Awerkin, Wolfgang Kliemann, Cristhian Becker, Modelling uncertainties in electrical power systems with stochastic differential equations, International Journal of Electrical Power and Energy Systems 113 (2019), 322–332.
[4] Sha He, Sanyi Tang, Yanni Xiao, Robert A. Cheke, Stochastic modelling of air pollution impacts on respiratory infection risk, Bulletin of Mathematical Biology 80 (2018), 3127–3153.
[5] Mark M. Meerschaert, Alla Sikorskii, Stochastic models for fractional calculus, de Gruyter, (2019).
[6] Hamdy M. Ahmed, Mahmoud M. El-Borai, Hilfer fractional stochastic integro-differential equations, Applied Mathematics and computation 331 (2018), 182–189.
[7] Hyunsoo Kim, Rathinasamy Sakthivel, Amar Debbouche, Delfim FM Torres, Traveling wave solutions of some important Wick-type fractional stochastic nonlinear partial differential equations, Chaos, Solitons and Fractals 131 (2020), 109542.
[8] A. Atangana, E. Bonyah, Fractional stochastic modeling: new approach to capture more heterogeneity, Chaos: An Interdisciplinary Journal of Nonlinear Science 29 (2019), 013118.
[9] Guang-an Zou, Guangying Lv, Jiang-Lun Wu, Stochastic Navier-Stokes equations with Caputo derivative driven by fractional noises, Journal of Mathematical Analysis and Applications 461 (2018), 595–609.
[10] Hamdy M. Ahmed, Quanxin Zhu, The averaging principle of Hilfer fractional stochastic delay differential equations with Poisson jumps, Applied Mathematics Letters 112 (2021), 106755.
[11] Xinzhu Meng, Lu Wang, and Tonghua Zhang, Global dynamics analysis of a nonlinear impulsive stochastic chemostat system in a polluted environment, J. Appl. Anal. Comput 6 (2016), 865–875.
[12] Chao Wang, Ravi P. Agarwal, Sakthivel Rathinasamy, Almost periodic oscillations for delay impulsive stochastic Nicholson's blowflies timescale model, Computational and Applied Mathematics 37 (2018), 3005–3026.
[13] Ruihua Wu, Xiaoling Zou, Ke Wang, Asymptotic behavior of a stochastic non-autonomous predator–prey model with impulsive perturbations, Communications in Nonlinear Science and Numerical Simulation 20 (2015), 965–974.
[14] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, A new definition of fractional derivative, Journal of Computational and Applied Mathematics 264 (2014) 65-–70.
[15] Won Sang Chung, Fractional Newton mechanics with conformable fractional derivative, Journal of computational and applied mathematics 290 (2015) 150–158.
[16] J. Rosales-García, J. A. Andrade-Lucio, O. Shulika, Conformable derivative applied to experimental Newton's law of cooling, Revista mexicana de física 66, no. 2 (2020) 224–227.
[17] Abdellatif Ben Makhlouf, Lassaad Mchiri, Mohamed Rhaima, Mohamed Ali Hammami, Stability of conformable stochastic systems depending on a parameter, Asian Journal of Control (2022) 1–10.
[18] Abdellatif Ben Makhlouf, Lassaad Mchiri, Partial Stability of Conformable Stochastic Systems, Mathematical Problems in Engineering 2022 (2022)1–9.
[19] Hamdy M. Ahmed, Conformable fractional stochastic differential equations with control function, Systems and Control Letters 158 (2021), 105062.
[20] Jerzy Klamka, Stochastic controllability and minimum energy control of systems with multiple delays in control, Applied Mathematics and Computation 206 (2008), 704–715.

[21] Yanqing Wang, Xiuxiang Zhou, Exact controllability of stochastic differential equations with memory, Systems and Control Letters 142 (2020), 104732.

[22] Hamdy M. Ahmed, Controllability of impulsive neutral stochastic differential equations with fractional Brownian motion, IMA Journal of Mathematical Control and Information 32 (2015), 781–794.

[23] Viorel Barbu, Luciano Tubaro, Exact controllability of stochastic differential equations with multiplicative noise, Systems and Control Letters 122 (2018), 19–23.

[24] Renren Zhang, Lei Guo, Controllability of stochastic game-based control systems, SIAM Journal on Control and Optimization 57 (2019), 3799–3826.

[25] A. Afreen, A. Raheem, A. Khatoon, Controllability of a second-order non-autonomous stochastic semilinear system with several delays in control, Chaos, Solitons and Fractals 155 (2022), 111763.

[26] Zuomao Yan, Qiong Yang, Optimal controllability of non-instantaneous impulsive partial stochastic differential systems with fractional sectorial operators, Bulletin des Sciences Mathématiques 159 (2020), 102828.

[27] P. Balasubramaniam, Controllability of semilinear noninstantaneous impulsive ABC neutral fractional differential equations, Chaos, Solitons and Fractals 152 (2021), 111276.

[28] Muslim Malik, Anjali Rose, Anil Kumar, Controllability of Sobolev type fuzzy differential equation with non-instantaneous impulsive condition, Discrete and Continuous Dynamical Systems-S 15 2 (2022), 387.

[29] Zhen Xin, Yuhe Yang, Qiaoxia Li, Controllability of nonlinear ordinary differential equations with non-instantaneous impulses, Mathematical Modelling and Control 2 (2022), 1–6.

[30] C. A. Tudor, Analysis of the Rosenblatt process, ESAIM: Probability and Statistics, 12 (2008), 230–257.

[31] El Hassan Lakhel and M. McKibben, Controllability for time-dependent neutral stochastic functional differential equations with Rosenblatt process and impulses, International Journal of Control, Automation and Systems 17(2019), 286–297.