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Jacobson's lemma and Cline's formula for weighted generalized inverses in a ring with involution

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Abstract. Let R be a ring with involution and $e,\ f \in R$ be Hermitian and invertible. We first present some equivalent conditions for paq to be $\{1, 3f\}$-invertible, assuming that $p,\ a,\ q \in R$ with $p'pa = a = aqq'$ for some $p',\ q' \in R$ and a is $\{1, 3e\}$-invertible. Then, these results are applied to give the sufficient and necessary conditions under which Jacobson's lemma and Cline's formula for weighted pseudo core inverses hold. Also, Jacobson's lemma for weighted Moore-Penrose inverses is investigated.

1. Introduction

Let R be a unitary ring and $a,\ b \in R$. It is known as Jacobson's lemma that if $1 - ab$ is invertible, then so is $1 - ba$. In this case, $(1 - ba)^{-1} = 1 + b(1 - ab)^{-1}a$. Naturally, many scholars considered whether Jacobson's lemma can work for kinds of generalized inverses and gave quantities of interesting results. For instance, if $1 - ab$ is regular with an inner inverse c, then $1 - ba$ is regular with an inner inverse $1 + bca$. In 2010, Castro-González et al. [3] investigated Jacobson's lemma for reflexive inverses, group inverses and Drazin inverses. Another famous conclusion is Cline's formula. In 1965, Cline [6] proved that if ab is Drazin invertible, then so is ba, in which case, $(ba)^D = b[(ab)^D]^2a$. Cline's formulas for generalized Drazin inverses and pseudo Drazin inverses were established by Liao et al. [16] and Wang et al. [27], respectively. For more details, readers are referred to [10, 14–16, 19, 29, 34].

However, in the case of pseudo core inverses, Shi et al. [26] found that Jacobson's lemma and Cline's formula do not hold, either. In order to investigate under what conditions Jacobson's lemma and Cline's formula for pseudo core inverses hold, they first proved that if $a \in R$ is $\{1, 3\}$-invertible, then paq is $\{1, 3\}$-invertible if and only if $p^*paa^{(1,3)} + 1 - aa^{(1,3)}$ is invertible, where $p,\ q \in R$ with $p'pa = a = aqq'$ for some $p', q' \in R$. Then, from this result, they gave some characterizations of the pseudo core invertibility of $1 - ba$ (resp., ba) by means of a unit, when $1 - ab$ (resp., ab) is pseudo core invertible. Also, Jacobson's lemma for Moore-Penrose inverses was considered.

The theme of this article can be described as the relevant research of Jacobson's lemma and Cline's formula for weighted generalized inverses with weights $e,\ f$, where $e,\ f$ are Hermitian and invertible.

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We first present some sufficient and necessary conditions for paq to be $\{1,3f\}$-invertible when a is $\{1,3e\}$-invertible, from which a new characterization of the $\{1,3\}$-invertibility of paq is given. Then, Cline's formula and Jacobson's lemma for weighted pseudo core inverses are discussed. At last, Jacobson's lemma for weighted Moore-Penrose inverses is studied.

2. Preliminaries

For convenience, R denotes a unitary ring with an involution $*$ throughout this paper. Firstly, recall the definition of the Moore-Penrose inverse.

Definition 2.1. [23] *Let $a \in R$. Then a is said to be Moore-Penrose invertible if there exists $x \in R$ such that the following four equations hold:*

$$(1)\ axa = a,\ \ (2)\ xax = x,\ \ (3)\ (ax)^* = ax,\ \ (4)\ (xa)^* = xa.$$

Such an x is called the Moore-Penrose inverse of a. If such an x exists, then it is unique and denoted by $a^\dagger$.

If the equation (1) holds, then a is called regular and x is called a $\{1\}$-inverse of a (or an inner inverse). If x satisfies equations (1) and (3) (resp., (1) and (4)), then x is called a $\{1,3\}$-inverse (resp., $\{1,4\}$-inverse) of a. We use $a^{(1,3)}$ (resp., $a^{(1,4)}$) to denote a $\{1,3\}$-inverse (resp., $\{1,4\}$-inverse) of a.

Now, we recall the definition of the weighted Moore-Penrose inverse. An element $a \in R$ is called Hermitian if $a^* = a$. Throughout this paper, $e,\ f \in R$ are Hermitian and invertible.

Definition 2.2. [25] *Let $a \in R$. Then a is said to have a weighted Moore-Penrose inverse with weights $e,\ f$ if there exists $x \in R$ such that the following four equations hold:*

$$(1)\ axa = a,\ \ (2)\ xax = x,\ \ (3e)\ (eax)^* = eax,\ \ (4f)\ (fxa)^* = fxa.$$

Such an x is called the weighted Moore-Penrose inverse of a with weights $e,\ f$. If x exists, then it is unique and denoted by $a^\dagger_{e,f}$.

If x satisfies equations (1) and (3e) (resp., (1) and (4f)), then x is called a $\{1,3e\}$-inverse (resp., $\{1,4f\}$-inverse) of a. We use $a^{(1,3e)}$ (resp., $a^{(1,4f)}$) to denote a $\{1,3e\}$-inverse (resp., $\{1,4f\}$-inverse) of a. The set of all $\{1,3e\}$-inverses (resp., $\{1,4f\}$-inverses) of a is denoted by $a\{1,3e\}$ (resp., $a\{1,4f\}$).

In 1958, Drazin [11] introduced the pseudo inverse in rings and semigroups, which was called the Drazin inverse later. For more results of the Drazin inverse, readers are referred to [6–9, 14, 15, 22].

Definition 2.3. [11] *Let $a \in R$. If there exist $x \in R$ and $k \in \mathbb{N}^+$ such that*

$$xa^{k+1} = a^k,\quad ax^2 = x,\quad xa = ax,$$

then x is called the Drazin inverse of a. It is unique and denoted by a^D when the Drazin inverse exists.

If k is the smallest positive integer such that the above equations hold, then k is called the Drazin index of a and denoted by $\mathrm{i}(a)$. In particular, x is called the group inverse of a and denoted by $a^\#$ when $k = 1$. When a is Drazin invertible, the idempotent $1 - aa^D$ is called the spectral idempotent of a, denoted by a^π.

In 2010, Baksalary and Trenkler [1] introduced the core inverse of a complex matrix. In 2014, the core inverse of a complex matrix was extented to the core-EP inverse of a complex matrix by Manjunatha Prasad et al. [17]. In 2018, Gao et al. [13] generalized the core-EP inverse of a complex matrix to an element in a ring with involution. For more results of the pseudo core inverse, readers are referred to [5, 12, 24, 28].

Definition 2.4. [13] *Let $a \in R$. If there exist $x \in R$ and $k \in \mathbb{N}^+$ such that*

$$xa^{k+1} = a^k,\quad ax^2 = x,\quad (ax)^* = ax,$$

then x is called the pseudo core inverse of a. It is unique and denoted by $a^{\textcircled{D}}$ when the pseudo core inverse exists.

The smallest positive integer k satisfying the above equations is called the pseudo core index of a. In particular, x is called the core inverse of a and denoted by $a^{\#\!\!\!\bigcirc}$ when $k = 1$.

In 2018, Mosić et al. [21] introduced the weighted core inverse in a ring with involution. In 2020, Zhu and Wang [32] introduced the notion of the weighted pseudo core inverse.

Definition 2.5. [32] *Let $a \in R$. If there exist $x \in R$ and $k \in \mathbb{N}^+$ such that*

$$xa^{k+1} = a^k, \quad ax^2 = x, \quad (eax)^* = eax,$$

then x is called the pseudo e-core inverse of a. It is unique and denoted by $a^{e,\textcircled{D}}$ when the pseudo e-core inverse exists.

If k is the smallest positive integer such that above equations hold, then k is called the pseudo e-core index of a. In particular, x is called the e-core inverse of a and denoted by $a^{e,\#\!\!\!\bigcirc}$ when $k = 1$. If a is pseudo e-core invertible, then a is Drazin invertible and the pseudo e-core index is equal to the Drazin index. For ease of notations, we still use $\mathrm{i}(a)$ to denote the pseudo e-core index of a. The pseudo f-dual core inverse of an element is defined as follows.

Definition 2.6. [32] *Let $a \in R$. If there exist $x \in R$ and $k \in \mathbb{N}^+$ such that*

$$a^{k+1}x = a^k, \quad x^2a = x, \quad (fxa)^* = fxa,$$

then x is called the pseudo f-dual core inverse of a. It is unique and denoted by $a_{f,\textcircled{D}}$ when the pseudo f-dual core inverse exists.

The symbols $R^{\{1,3\}}$, $R^{\{1,3e\}}$, $R^{\{1,4\}}$, $R^{\{1,4f\}}$, $R^{\dagger}$, $R^{\dagger}_{e,f}$, R^{D}, $R^{\textcircled{D}}$, $R^{e,\textcircled{D}}$, $R_{f,\textcircled{D}}$ denote the sets of all $\{1,3\}$-invertible, $\{1,3e\}$-invertible, $\{1,4\}$-invertible, $\{1,4f\}$-invertible, Moore-Penrose invertible, Moore-Penrose invertible with weights e, f, Drazin invertible, pseudo core invertible, pseudo e-core invertible, pseudo f-dual core invertible elements in R, respectively.

In 2011, Mary [18] introduced the notion of the inverse along an element. In 2016, Zhu [31] defined the one-side inverse along an element.

Definition 2.7. [18] *Let a, $d \in R$. If there exists $y \in R$ such that*

$$y \in dR \cap Rd, \quad yad = d = day,$$

then a is said to be invertible along d. If such y exists, then it is unique and denoted by $a^{\|d}$.

Definition 2.8. [31] *Let a, $d \in R$. If there exists $y \in R$ such that*

$$y \in dR, \ d = day, \quad (resp., \ y \in Rd, \ d = yad,)$$

then a is said to be right (resp., left) invertible along d.

3. The $\{1, 3f\}$-invertibility and $\{1, 4f\}$-invertibility of paq

In this section, we consider the $\{1,3f\}$-invertibility and $\{1,4f\}$-invertibility of paq. At first, we give an auxiliary lemma.

Lemma 3.1. *Let a, $d \in R$ be Hermitian. Then the following statements are equivalent:*

(1) *a is invertible along d;*

(2) *a is left invertible along d;*

(3) *a is right invertible along d.*

In this case, $a^{\|d}$ is Hermitian.

Proof. From [31, Theorems 2.3 and 2.4], we get that a is left (resp., right) invertible along d if and only if $Rd = Rdad$ (resp., $dR = dadR$). Since $a,\ d \in R$ are Hermitian, we can get that $Rd = Rdad$ if and only if $dR = dadR$. The rest of the proof is clear by [20, Theorem 2.2].

In this case, we can verify that $(a^{\| d})^*$ is also the inverse of a along d. Thus, $a^{\| d} = (a^{\| d})^*$. □

Theorem 3.2. *Let $p,\ a,\ q \in R$ with $p'pa = a = aqq'$ for some $p', q' \in R$. If $a \in R^{\{1,3e\}}$, then the following statements are equivalent:*

(1) $paq \in R^{\{1,3f\}}$;

(2) $(p^*fp)^{\| aa^{(1,3e)}e^{-1}}$ *exists;*

(3) $1 - aa^{(1,3e)} + aa^{(1,3e)}e^{-1}p^*fp$ *is invertible.*

In this case,

$$(p^*fp)^{\| aa^{(1,3e)}e^{-1}} = aq(paq)^{(1,3f)}f^{-1}(p')^*,$$
$$q'a^{(1,3e)}(p^*fp)^{\| aa^{(1,3e)}e^{-1}}p^*f \in (paq)\{1,3f\}.$$

Proof. (1) ⇒ (2): Take $z = aq(paq)^{(1,3f)}f^{-1}(p')^*$. It is easy to obtain that $z \in aa^{(1,3e)}e^{-1}R$. Since $paq(paq)^{(1,3f)}f^{-1}$ is Hermitian and $p'pa = a$, we conclude that $z = p'paq(paq)^{(1,3f)}f^{-1}(p')^*$ is Hermitian, which together with $aa^{(1,3e)}z = z$ implies that $z \in R(aa^{(1,3e)})^* = Reaa^{(1,3e)}e^{-1} = Raa^{(1,3e)}e^{-1}$. Then

$$\begin{aligned} zp^*fpaa^{(1,3e)}e^{-1} &= z^*p^*fpaa^{(1,3e)}e^{-1} \\ &= p'(aq(paq)^{(1,3f)}f^{-1})^*p^*fpaa^{(1,3e)}e^{-1} \\ &= p'(paq(paq)^{(1,3f)}f^{-1})^*fpaa^{(1,3e)}e^{-1} \\ &= p'paq(paq)^{(1,3f)}f^{-1}fpaa^{(1,3e)}e^{-1} \\ &= p'paq(paq)^{(1,3f)}paqq'a^{(1,3e)}e^{-1} \\ &= p'paqq'a^{(1,3e)}e^{-1} \\ &= aa^{(1,3e)}e^{-1}. \end{aligned}$$

Because p^*fp, z and $aa^{(1,3e)}e^{-1}$ are all Hermitian, we can get

$$aa^{(1,3e)}e^{-1}p^*fpz = (zp^*fpaa^{(1,3e)}e^{-1})^* = aa^{(1,3e)}e^{-1}.$$

Therefore, $(p^*fp)^{\| aa^{(1,3e)}e^{-1}}$ exists and $(p^*fp)^{\| aa^{(1,3e)}e^{-1}} = aq(paq)^{(1,3f)}f^{-1}(p')^*$.

(2) ⇒ (1): Set $y = q'a^{(1,3e)}(p^*fp)^{\| aa^{(1,3e)}e^{-1}}p^*f$. By Lemma 3.1, it follows that $(p^*fp)^{\| aa^{(1,3e)}e^{-1}}$ is Hermitian. This together with $aa^{(1,3e)}(p^*fp)^{\| aa^{(1,3e)}e^{-1}} = (p^*fp)^{\| aa^{(1,3e)}e^{-1}}$ implies that

$$fpaqy = fpaqq'a^{(1,3e)}(p^*fp)^{\| aa^{(1,3e)}e^{-1}}p^*f = fp(p^*fp)^{\| aa^{(1,3e)}e^{-1}}p^*f$$

is Hermitian. And

$$\begin{aligned} paqypaq &= paqq'a^{(1,3e)}(p^*fp)^{\| aa^{(1,3e)}e^{-1}}p^*fpaq \\ &= p[aa^{(1,3e)}(p^*fp)^{\| aa^{(1,3e)}e^{-1}}]p^*fpaq \\ &= p(p^*fp)^{\| aa^{(1,3e)}e^{-1}}p^*fpaa^{(1,3e)}e^{-1}eaq \\ &= paa^{(1,3e)}e^{-1}eaq = paq. \end{aligned}$$

Therefore, $paq \in R^{\{1,3f\}}$ and $q'a^{(1,3e)}(p^*fp)^{\| aa^{(1,3e)}e^{-1}}p^*f \in (paq)\{1,3f\}$.

(2) ⇔ (3): It is clear by [20, Theorem 3.2]. □

Corollary 3.3. *Let $p,\ a,\ q \in R$ with $p'pa = a = aqq'$ for some $p', q' \in R$. If $a \in R^{\{1,3e\}}$, then the following statements are equivalent:*

(1) $paq \in R^{\{1,3f\}}$;

(2) p^*fp *is left invertible along* $aa^{(1,3e)}e^{-1}$;

(3) p^*fp *is right invertible along* $aa^{(1,3e)}e^{-1}$;

(4) $1 - aa^{(1,3e)} + aa^{(1,3e)}e^{-1}p^*fp$ *is left invertible;*

(5) $1 - aa^{(1,3e)} + aa^{(1,3e)}e^{-1}p^*fp$ *is right invertible.*

Proof. (1) ⇔ (2) ⇔ (3): By Theorem 3.2, we get that (1) holds if and only if p^*fp is invertible along $aa^{(1,3e)}e^{-1}$. Noting that p^*fp and $aa^{(1,3e)}e^{-1}$ are Hermitian, we can complete the proof by Lemma 3.1.

(2) ⇔ (4) and (3) ⇔ (5): They are clear by [31, Corollaries 3.3 and 3.5]. □

From the above results, we can immediately get the relevant result for $\{1,3\}$-invertibility, in which the equivalence between (1) and (3) can be found in [26, Theorem 4.3].

Corollary 3.4. *Let* $p,\ a,\ q \in R$ *with* $p'pa = a = aqq'$ *for some* $p', q' \in R$. *If* $a \in R^{\{1,3\}}$, *then the following statements are equivalent:*

(1) $paq \in R^{\{1,3\}}$;

(2) $(p^*p)^{\|aa^{(1,3)}}$ *exists;*

(3) $1 - aa^{(1,3)} + aa^{(1,3)}p^*p$ *is invertible.*

In this case,

$$(p^*p)^{\|aa^{(1,3)}} = aq(paq)^{(1,3)}(p')^*,$$
$$q'a^{(1,3)}(p^*p)^{\|aa^{(1,3)}}p^* \in (paq)\{1,3\}.$$

Dually, we consider the $\{1,4f\}$-invertibility of paq and get the following theorem whose proof is omitted.

Theorem 3.5. *Let* $p,\ a,\ q \in R$ *with* $p'pa = a = aqq'$ *for some* $p', q' \in R$. *If* $a \in R^{\{1,4e\}}$, *then the following statements are equivalent:*

(1) $paq \in R^{\{1,4f\}}$;

(2) $(qf^{-1}q^*)^{\|ea^{(1,4e)}a}$ *exists;*

(3) $1 - a^{(1,4e)}a + qf^{-1}q^*ea^{(1,4e)}a$ *is invertible.*

In this case,

$$(qf^{-1}q^*)^{\|ea^{(1,4e)}a} = (q')^*f(paq)^{(1,4f)}pa,$$
$$f^{-1}q^*(qf^{-1}q^*)^{\|ea^{(1,4e)}a}a^{(1,4e)}p' \in paq\{1,4f\}.$$

Inspired by [4, Theorem 3.2], we obtain the next result which presents some equivalent conditions for paq to be $\{1,4f\}$-invertible when a is $\{1,3e\}$-invertible.

Proposition 3.6. *Let* $p,\ a,\ q \in R$ *with* $p'pa = a = aqq'$ *for some* $p', q' \in R$. *If* $a \in R^{\{1,3e\}}$, *then the following statements are equivalent:*

(1) $paq \in R^{\{1,4f\}}$;

(2) $aq \in R^{\dagger}_{e,f}$;

(3) $(aq)^{\|f^{-1}(aq)^*e}$ *exists;*

(4) $aqf^{-1}(eaq)^* + 1 - aa^{(1,3e)}$ *is invertible.*

In this case,

$$(aq)^{\dagger}_{e,f} = (paq)^{(1,4f)}paa^{(1,3e)},\ (aq)^{\dagger}_{e,f}p' \in (paq)\{1,4f\}.$$

Proof. (1) ⇒ (2): Take $z = (paq)^{(1,4f)}paa^{(1,3e)}$. By direct computation, we get

$$\begin{aligned}(aq)z(aq) &= aq(paq)^{(1,4f)}paa^{(1,3e)}aq = p'paq(paq)^{(1,4f)}paq\\ &= p'paq = aq,\\ z(aq)z &= (paq)^{(1,4f)}paa^{(1,3e)}aq(paq)^{(1,4f)}paa^{(1,3e)}\\ &= (paq)^{(1,4f)}paq(paq)^{(1,4f)}paqq'a^{(1,3e)}\\ &= (paq)^{(1,4f)}paa^{(1,3e)} = z.\end{aligned}$$

Since

$$\begin{aligned}e(aq)z &= eaq(paq)^{(1,4f)}paa^{(1,3e)} = ep'paq(paq)^{(1,4f)}paqq'a^{(1,3e)}\\ &= ep'paqq'a^{(1,3e)} = eaa^{(1,3e)}\end{aligned}$$

and

$$fz(aq) = f(paq)^{(1,4f)}paa^{(1,3e)}aq = f(paq)^{(1,4f)}paq,$$

we conclude that $e(aq)z$ and $fz(aq)$ are Hermitian. Therefore, $aq \in R^{\dagger}_{e,f}$.

(2) ⇒ (1): Take $y = (aq)^{\dagger}_{e,f}p'$. Then $(paq)y(paq) = paq(aq)^{\dagger}_{e,f}p'paq = paq$ and $fy(paq) = f(aq)^{\dagger}_{e,f}p'paq = f(aq)^{\dagger}_{e,f}aq$ is Hermitian. So, $paq \in R^{\{1,4f\}}$.

(2) ⇔ (3): By a similar proof to [2, Theorem 3.2], we can complete it.

(3) ⇔ (4): It is clear that $e^{-1}(q'a^{(1,3e)})^*f$ is an inner inverse of $f^{-1}(aq)^*e$ and $(e^{-1}(q'a^{(1,3e)})^*f)(f^{-1}(aq)^*e) = aa^{(1,3e)}$. The rest of the proof is obvious by [20, Theorem 3.2]. □

Dually, we get the following proposition.

Proposition 3.7. *Let p, a, $q \in R$ with $p'pa = a = aqq'$ for some $p', q' \in R$. If $a \in R^{\{1,4f\}}$, then the following statements are equivalent:*

(1) $paq \in R^{\{1,3e\}}$;

(2) $pa \in R^{\dagger}_{e,f}$;

(3) $(pa)^{\| f^{-1}(pa)^*e}$ *exists;*

(4) $f^{-1}(pa)^*epa + 1 - a^{(1,4f)}a$ *is invertible.*

In this case,

$$(pa)^{\dagger}_{e,f} = a^{(1,4f)}aq(paq)^{(1,3e)},\ q'(pa)^{\dagger}_{e,f} \in (paq)\{1,3e\}.$$

4. The relation between $xy \in R^{e,\circledS}$ and $yx \in R^{f,\circledS}$

In this section, we investigate Cline's formula for weighted pseudo core inverses. Firstly, some auxiliary lemmas are given. The following lemma can be seen as a generalization of [26, Lemma 3.2].

Lemma 4.1. *Let $t \in R$ be idempotent. Then $t \in R^{\{1,3f\}}$ if and only if $1 - t \in R^{\{1,4f\}}$.*

Proof. It is easy to obtain that $t \in R^{\{1,3f\}}$ if and only if there exists $p = p^2 \in R$ such that $(fp)^* = fp$ and $tR = pR$. Noting that $tR = pR$ is equivalent to $R(1-t) = R(1-p)$, we get that $t \in R^{\{1,3f\}}$ if and only if $1 - t \in R^{\{1,4f\}}$. □

Lemma 4.2. [13, Lemma 2.1] *Let $a \in R$. If there exist $x \in R$ and $k \in \mathbb{N}^+$ such that*

$$xa^{k+1} = a^k, \quad ax^2 = x,$$

then

(1) *$ax = a^m x^m$ for arbitrary positive integer m;*

(2) *$xax = x$;*

(3) *a is Drazin invertible, $a^D = x^{k+1}a^k$ and $\mathrm{i}(a) \leq k$.*

In [30], the authors denote

$$T_l(a) = \{x \in R : \ xa^{k+1} = a^k, \ ax^2 = x \text{ for some positive integer } k\}.$$

Lemma 4.3. [30, Lemma 2.2] *Let $a \in R^D$, $k_1, ..., k_n, s_1, ..., s_n \in \mathbb{N}$ and $x_1, ..., x_n \in T_l(a)$. If $s_n \neq 0$, then*

$$\prod_{i=1}^{n} a^{k_i} x_i^{s_i} = a^k x_n^s, \text{ where } k = \sum_{i=1}^{n} k_i \text{ and } s = \sum_{i=1}^{n} s_i.$$

Lemma 4.4. [6] *Let $x, \ y \in R$. If $\alpha = xy \in R^D$, then $\beta = yx \in R^D$. In this case, $\beta^D = y(\alpha^D)^2 x$.*

The following lemma can be seen as a generalization of [26, Theorem 3.3].

Lemma 4.5. *If $a \in R^D$, then the following statements are equivalent:*

(1) *$a \in R^{f,\textcircled{D}}$;*

(2) *there exists $x \in R^{\{1,3f\}}$ such that $xR = aa^D R$;*

(3) *$a^\pi \in R^{\{1,4f\}}$.*

In this case, $a^{f,\textcircled{D}} = a^D(aa^D)^{(1,3f)} = a^D(1 - (a^\pi)^{(1,4f)}a^\pi)$.

Proof. Let $\mathrm{i}(a) = k$.

(1) ⇒ (2): By [32, Theorem 3.9], it follows that a^k is $\{1, 3f\}$-invertible. Take $x = a^k$. Then $xR = a^k R = aa^D R$.

(2) ⇒ (1): Since $xR = aa^D R = a^k R$, it follows that there exists $t \in R$ such that $a^k t = xx^{(1,3f)}$. We can verify that t is a $\{1, 3f\}$-inverse of a^k. Similarly, aa^D is $\{1, 3f\}$-invertible and $a^k(a^k)^{(1,3f)} = aa^D(aa^D)^{(1,3f)}$. It follows from [32, Theorem 3.9] that $a \in R^{f,\textcircled{D}}$ and $a^{f,\textcircled{D}} = a^D[a^k(a^k)^{(1,3f)}] = a^D[aa^D(aa^D)^{(1,3f)}] = a^D(aa^D)^{(1,3f)}$.

(2) ⇔ (3): It follows from Lemma 4.1. □

Now, we give the main result of this section.

Theorem 4.6. *Let $x, \ y \in R$. If $\alpha = xy \in R^{e,\textcircled{D}}$, then the following statements are equivalent:*

(1) *$\beta = yx \in R^{f,\textcircled{D}}$;*

(2) *$y\alpha^D x \in R^{\{1,3f\}}$;*

(3) *$(y^* f y)^{\|\alpha\alpha^{e,\textcircled{D}} e^{-1}}$ exists.*

In this case, $\beta^{f,\textcircled{D}} = y\alpha^{e,\textcircled{D}}(y^* f y)^{\|\alpha\alpha^{e,\textcircled{D}} e^{-1}} y^* f$.

Proof. (1) ⇔ (2): It follows from Lemma 4.4 that $\beta \in R^D$ and $\beta\beta^D = y\alpha^D x$. Therefore, $\beta \in R^{f,\textcircled{D}}$ if and only if $y\alpha^D x \in R^{\{1,3f\}}$ according to Lemma 4.5.

(2) ⇔ (3): Take $p = y$, $a = \alpha^D$, $q = x$ and $p' = \alpha^D x$, $q' = y\alpha^D$. Then, we can verify that $a = p'pa = aqq'$. It follows from Lemmas 4.2 and 4.3 that $\alpha^D\alpha^2\alpha^{e,\textcircled{D}}\alpha^D = \alpha^2(\alpha^D)^3 = \alpha^D$ and $(e\alpha^D\alpha^2\alpha^{e,\textcircled{D}})^* = (e\alpha\alpha^{e,\textcircled{D}})^* = e\alpha\alpha^{e,\textcircled{D}}$. That is, $a = \alpha^D \in R^{\{1,3e\}}$ with a $\{1,3e\}$-inverse $\alpha^2\alpha^{e,\textcircled{D}}$. Thus, $paq \in R^{\{1,3f\}}$ if and only if $(p^*fp)^{\|aa^{(1,3e)}e^{-1}}$ exists by Theorem 3.2. Since $aa^{(1,3e)} = \alpha^D\alpha^2\alpha^{e,\textcircled{D}} = \alpha\alpha^{e,\textcircled{D}}$, we get that $y\alpha^D x \in R^{\{1,3f\}}$ if and only if $(y^*fy)^{\|\alpha(\alpha)^{e,\textcircled{D}}e^{-1}}$ exists.

In this case, it follows from Theorem 3.2 that $y\alpha^D\alpha^2(\alpha)^{e,\textcircled{D}}(y^*fy)^{\|\alpha\alpha^{e,\textcircled{D}}e^{-1}}y^*f \in (y\alpha^D x)\{1,3f\}$. Then, by Lemmas 4.2, 4.3 and 4.5, we get

$$\begin{aligned}
\beta^{f,\textcircled{D}} &= \beta^D(\beta\beta^D)^{(1,3f)} \\
&= y(\alpha^D)^2xy\alpha^D\alpha^2(\alpha)^{e,\textcircled{D}}(y^*fy)^{\|\alpha\alpha^{e,\textcircled{D}}e^{-1}}y^*f \\
&= y\alpha^3(\alpha^{e,\textcircled{D}})^4(y^*fy)^{\|\alpha\alpha^{e,\textcircled{D}}e^{-1}}y^*f \\
&= y\alpha^{e,\textcircled{D}}(y^*fy)^{\|\alpha\alpha^{e,\textcircled{D}}e^{-1}}y^*f.
\end{aligned}$$

□

Remark 4.7. *In Theorem 4.6, it is easy to verify that the condition* (3) *holds if and only if* $1 - \alpha\alpha^{e,\textcircled{D}} + \alpha\alpha^{e,\textcircled{D}}e^{-1}y^*fy$ *is invertible by [20, Theorem 3.2]. Due to the limited space, we will omit the similar equivalence when studying Jacobson's lemma for weighted generalized inverses.*

The equivalence between (1) and (2) in the following corollary can be found in [26, Theorem 4.5].

Corollary 4.8. *Let* $x,\ y \in R$. *Suppose* $\alpha = xy \in R^{\textcircled{D}}$. *Then the following statements are equivalent:*

(1) $\beta = yx \in R^{\textcircled{D}}$;

(2) $y\alpha^D x \in R^{\{1,3\}}$;

(3) $(y^*y)^{\|\alpha\alpha^{\textcircled{D}}}$ *exists.*

In this case, $\beta^{\textcircled{D}} = y\alpha^{\textcircled{D}}(y^*y)^{\|\alpha\alpha^{\textcircled{D}}}y^*$.

Using Proposition 3.6, we can get the following proposition by an analogous method to Theorem 4.6.

Proposition 4.9. *Let* $x,\ y \in R$. *If* $\alpha = xy \in R^{e,\textcircled{D}}$, *then the following statements are equivalent:*

(1) $\beta = yx \in R_{f,\textcircled{D}}$;

(2) $y\alpha^D x \in R^{\{1,4f\}}$;

(3) $(\alpha^D x)^\dagger_{e,f}$ *exists.*

In this case, $\beta_{f,\textcircled{D}} = (\alpha^D x)^\dagger_{e,f}(\alpha^D)^2 x$.

5. The relation between $1 - xy \in R^{e,\textcircled{D}}$ and $1 - yx \in R^{f,\textcircled{D}}$

In 2009, Patrício et al. [22] asked whether Jacobson's lemma holds for Drazin inverses. Castro-González et al. [3], Cvetković-Ilić and Harte [10] gave a positive answer to this question, respectively. Later, Lam and Nielsen [14] also investigated it.

Lemma 5.1. [14, Theorem 2.4] *Let* $x,\ y \in R$. *If* $\alpha = 1 - xy \in R^D$ *with* $\mathrm{i}(\alpha) = k$, *then* $\beta = 1 - yx \in R^D$ *with* $\mathrm{i}(\beta) = k$. *Moreover,* $\beta^\pi = y\alpha^\pi rx$, *where* $r = 1 + \alpha + \cdots + \alpha^{k-1}$.

The main result of this section is presented as follows.

Theorem 5.2. *Let $x,\ y \in R$. Suppose $\alpha = 1 - xy \in R^{e,\textcircled{D}}$ with $\mathrm{i}(\alpha) = k$. Then the following statements are equivalent:*

(1) $\beta = 1 - yx \in R^{f,\textcircled{D}}$;

(2) $y\alpha^\pi rx \in R^{\{1,4f\}}$, *where* $r = 1 + \alpha + \cdots + \alpha^{k-1}$;

(3) $(xf^{-1}x^*)^{\|e(1-\alpha\alpha^{e,\textcircled{D}})}$ *exists.*

In this case, $\beta^{f,\textcircled{D}} = (1 + y\alpha^D x)(1 - f^{-1}x^*tx)$, *where* $t = (xf^{-1}x^*)^{\|e(1-\alpha\alpha^{e,\textcircled{D}})}$.

Proof. (1) ⇔ (2): It is clear by Lemma 4.5 that $\beta \in R^{f,\textcircled{D}}$ if and only if $\beta^\pi \in R^{\{1,4f\}}$. Then, by Lemma 5.1 we can get $\beta^\pi = yr\alpha^\pi x$.

(2) ⇔ (3): Since $\alpha^D\alpha = \alpha\alpha^D$, it follows $y\alpha^\pi rx = yr\alpha^\pi x$. Then,

$$x(yr\alpha^\pi) = (1-\alpha)r\alpha^\pi = (1-\alpha^k)\alpha^\pi = \alpha^\pi,$$
$$(\alpha^\pi x)yr = \alpha^\pi(1-\alpha)r = \alpha^\pi(1-\alpha^k) = \alpha^\pi.$$

Take $a = \alpha^\pi$, $p = yr$, $q = x$ and $p' = x$, $q' = yr$. It is clear that $a = p'pa = aqq'$. By Lemma 4.5, we get $\alpha^\pi \in R^{\{1,4e\}}$ and $1 - \alpha\alpha^{e,\textcircled{D}} \in \alpha^\pi\{1,4e\}$. Therefore, it follows from Theorem 3.5 that $y\alpha^\pi rx \in R^{\{1,4f\}}$ if and only if $(xf^{-1}x^*)^{\|e(1-\alpha\alpha^{e,\textcircled{D}})}$ exists.

In this case, a $\{1,4f\}$-inverse of β^π is $(\beta^\pi)^{(1,4f)} = f^{-1}x^*(xf^{-1}x^*)^{\|e(1-\alpha\alpha^{e,\textcircled{D}})}(1-\alpha\alpha^{e,\textcircled{D}})x$. Since $(xf^{-1}x^*)^{\|e(1-\alpha\alpha^{e,\textcircled{D}})} \in Re(1-\alpha\alpha^{e,\textcircled{D}}) = R\alpha^\pi$, it follows

$$(\beta^\pi)^{(1,4f)} = f^{-1}x^*(xf^{-1}x^*)^{\|e(1-\alpha\alpha^{e,\textcircled{D}})}x,$$

which implies

$$\begin{aligned}(\beta^\pi)^{(1,4f)}\beta^\pi &= f^{-1}x^*(xf^{-1}x^*)^{\|e(1-\alpha\alpha^{e,\textcircled{D}})}xyr\alpha^\pi x\\ &= f^{-1}x^*(xf^{-1}x^*)^{\|e(1-\alpha\alpha^{e,\textcircled{D}})}\alpha^\pi x\\ &= f^{-1}x^*(xf^{-1}x^*)^{\|e(1-\alpha\alpha^{e,\textcircled{D}})}x.\end{aligned}$$

Then, by Lemma 4.5 and [26, (3.3)], we can get

$$\begin{aligned}\beta^{f,\textcircled{D}} &= \beta^D(1-(\beta^\pi)^{(1,4f)}\beta^\pi)\\ &= (1+y\alpha^D x)(1-\beta^\pi)(1-(\beta^\pi)^{(1,4f)}\beta^\pi)\\ &= (1+y\alpha^D x)(1-(\beta^\pi)^{(1,4f)}\beta^\pi)\\ &= (1+y\alpha^D x)(1-f^{-1}x^*(xf^{-1}x^*)^{\|e(1-\alpha\alpha^{e,\textcircled{D}})}x).\end{aligned}$$

□

The equivalence between (1) and (2) in the following corollary can be found in [26, Theorem 3.10].

Corollary 5.3. *Let $x,\ y \in R$. Suppose $\alpha = 1 - xy \in R^{\textcircled{D}}$ with $\mathrm{i}(\alpha) = k$. Then the following statements are equivalent:*

(1) $\beta = 1 - yx \in R^{\textcircled{D}}$;

(2) $y\alpha^\pi rx \in R^{\{1,4\}}$, *where* $r = 1 + \alpha + \cdots + \alpha^{k-1}$;

(3) $(xx^*)^{\|(1-\alpha\alpha^{\textcircled{D}})}$ *exists.*

In this case, $\beta^{\textcircled{D}} = (1 + y\alpha^D x)(1 - x^*tx)$, *where* $t = (xx^*)^{\|(1-\alpha\alpha^{\textcircled{D}})}$.

By a similar method to Theorem 5.2, we have the following proposition.

Proposition 5.4. *Let $x,\ y \in R$. Suppose $\alpha = 1 - xy \in R^{e,\textcircled{D}}$ with $\mathrm{i}(\alpha) = k$. Then the following statements are equivalent:*

(1) $\beta = 1 - yx \in R_{f,\textcircled{D}}$;

(2) $y\alpha^{\pi}rx \in R^{\{1,3f\}}$, *where* $r = 1 + \alpha + \cdots + \alpha^{k-1}$;

(3) $(y\alpha^{\pi})^{\dagger}_{f,e}$ *exists.*

In this case, $\beta_{f,\textcircled{D}} = [1 - y\alpha^{\pi}(y\alpha^{\pi})^{\dagger}_{f,e}](1 + y\alpha^{D}x)$.

6. The relation between $1 - xy \in R^{\dagger}_{e_1,f_1}$ and $1 - yx \in R^{\dagger}_{e_2,f_2}$

In this section, Jacobson's lemma for weighted Moore-Penrose inverses is discussed. The elements $e_1,\ e_2,\ f_1,\ f_2 \in R$ are always Hermitian and invertible in this section. Firstly, by a similar proof to [26, Lemma 5.1], we give a lemma as follows.

Lemma 6.1. *If* $a \in R$ *is regular with an inner inverse* a^-, *then* $a \in R^{\{1,3e\}}$ *if and only if* $aa^- \in R^{\{1,3e\}}$. *In this case,*

$$aa^{(1,3e)} \in (aa^-)\{1,3e\} \quad \text{and} \quad a^-(aa^-)^{(1,3e)} \in a\{1,3e\},$$

for any $(aa^-)^{(1,3e)} \in (aa^-)\{1,3e\}$.

Let $x,\ y \in R$. It is well known that if $1 - xy$ is regular, then $1 - yx$ is regular. In this case, if $(1 - xy)^-$ is an inner inverse of $1 - xy$, then $1 + y(1 - xy)^-x$ is an inner inverse of $1 - yx$.

Proposition 6.2. *Let* $x,\ y \in R$. *Suppose* $\alpha = 1 - xy \in R^{\{1,3e_1\}}$. *Then the following statements are equivalent:*

(1) $\beta = 1 - yx \in R^{\{1,3e_2\}}$;

(2) $1 - y\alpha^{\pi}_{r}x \in R^{\{1,3e_2\}}$;

(3) $y\alpha^{\pi}_{r}x \in R^{\{1,4e_2\}}$;

(4) $(xe_2^{-1}x^*)^{\parallel e_1\alpha^{\pi}_{r}}$ *exists,*

where $\alpha^{\pi}_{r} = 1 - \alpha\alpha^{(1,3e_1)}$. *In this case,*

$$(1 + y\alpha^{(1,3e_1)}x)(1 - e_2^{-1}x^*(xe_2^{-1}x^*)^{\parallel e_1\alpha^{\pi}_{r}}x) \in \beta\{1,3e_2\}.$$

Proof. Since α is regular, we conclude that β is regular with an inner inverse $\beta^- = (1 + y\alpha^{(1,3e_1)}x)$.

(1) $\Leftrightarrow$ (2): It is clear that $\beta\beta^- = 1 - y\alpha^{\pi}_{r}x$. So, by Lemma 6.1 we can get that $\beta = 1 - yx \in R^{\{1,3e_2\}}$ if and only if $1 - y\alpha^{\pi}_{r}x \in R^{\{1,3e_2\}}$.

(2) $\Leftrightarrow$ (3): It is obvious by Lemma 4.1.

(3) $\Leftrightarrow$ (4): Take $p = y,\ a = \alpha^{\pi}_{r},\ q = x$ and $p' = \alpha^{\pi}_{r}x,\ q' = y$. It is easy to verify that $p'pa = a = aqq'$. Since $e_1\alpha^{\pi}_{r}$ is Hermitian and α^{π}_{r} is idempotent, we get that α^{π}_{r} is $\{1,4e_1\}$-invertible. Then, by Theorem 3.5 we get that $paq \in R^{\{1,4e_2\}}$ if and only if $(qe_2^{-1}q^*)^{\parallel e_1a^{(1,4e_1)}a}$ exists. That is, $y\alpha^{\pi}_{r}x \in R^{\{1,4e_2\}}$ if and only if $(xe_2^{-1}x^*)^{\parallel e_1\alpha^{\pi}_{r}}$ exists.

In this case, by Lemma 6.1 we can get $\beta^-(\beta\beta^-)^{(1,3e_2)} \in \beta\{1,3e_2\}$. According to Theorem 3.5 and $(xe_2^{-1}x^*)^{\parallel e_1\alpha^{\pi}_{r}} \in R\alpha^{\pi}_{r}$, we conclude that

$$e_2^{-1}x^*(xe_2^{-1}x^*)^{\parallel e_1\alpha^{\pi}_{r}}\alpha^{\pi}_{r}\alpha^{\pi}_{r}x = e_2^{-1}x^*(xe_2^{-1}x^*)^{\parallel e_1\alpha^{\pi}_{r}}x$$

is a $\{1,4e_2\}$-inverse of $y\alpha^{\pi}_{r}x$. Therefore, $(y\alpha^{\pi}_{r}x)^{(1,4e_2)}y\alpha^{\pi}_{r}x = e_2^{-1}x^*(xe_2^{-1}x^*)^{\parallel e_1\alpha^{\pi}_{r}}x$, which implies $(\beta\beta^-)^{(1,3e_2)} = 1 - e_2^{-1}x^*(xe_2^{-1}x^*)^{\parallel e_1\alpha^{\pi}_{r}}x$. Thus, $\beta^-(\beta\beta^-)^{(1,3e_2)} = (1 + y\alpha^{(1,3e_1)}x)(1 - e_2^{-1}x^*(xe_2^{-1}x^*)^{\parallel e_1\alpha^{\pi}_{r}}x)$. $\square$

Dually, we have the following result.

Proposition 6.3. *Let* $x,\ y \in R$. *Suppose* $\alpha = 1 - xy \in R^{\{1,4f_1\}}$. *Then the following statements are equivalent:*

(1) $\beta = 1 - yx \in R^{\{1,4f_2\}}$;

(2) $1 - y\alpha_l^\pi x \in R^{\{1,4f_2\}}$;

(3) $y\alpha_l^\pi x \in R^{\{1,3f_2\}}$;

(4) $(y^* f_2 y)^{\|\alpha_l^\pi f_1^{-1}}$ *exists,*

where $\alpha_l^\pi = 1 - \alpha^{(1,4f_1)}\alpha$. *In this case,*

$$(1 - y(y^* f_2 y)^{\|\alpha_l^\pi f_1^{-1}} y^* f_2)(1 + y\alpha^{(1,4f_1)}x) \in \beta\{1, 4f_2\}.$$

It is well known that $a \in R$ is Moore-Penrose invertible if and only if $a \in R^{\{1,3\}} \cap R^{\{1,4\}}$. A similar conclusion also holds for the weighted Moore-Penrose inverse.

Lemma 6.4. [33, Theorem 2.1] *Let $a \in R$. Then a is Moore-Penrose invertible with weights e, f if and only if $a \in R^{\{1,3e\}} \cap R^{\{1,4f\}}$. In this case, $a^\dagger_{e,f} = a^{(1,4f)}aa^{(1,3e)}$, for any $a^{(1,3e)} \in a\{1, 3e\}$ and $a^{(1,4f)} \in a\{1, 4f\}$.*

Theorem 6.5. *Let x, $y \in R$. Suppose $\alpha = 1 - xy \in R^\dagger_{e_1,f_1}$. Then the following statements are equivalent:*

(1) $\beta = 1 - yx \in R^\dagger_{e_2,f_2}$;

(2) $1 - y\alpha_r^\pi x \in R^{\{1,3e_2\}}$ *and* $1 - y\alpha_l^\pi x \in R^{\{1,4f_2\}}$;

(3) $y\alpha_r^\pi x \in R^{\{1,4e_2\}}$ *and* $y\alpha_l^\pi x \in R^{\{1,3f_2\}}$;

(4) $(xe_2^{-1}x^*)^{\|e_1\alpha_r^\pi}$ *and* $(y^* f_2 y)^{\|\alpha_l^\pi f_1^{-1}}$ *exist,*

where $\alpha_r^\pi = 1 - \alpha\alpha^\dagger_{e_1,f_1}$ *and* $\alpha_l^\pi = 1 - \alpha^\dagger_{e_1,f_1}\alpha$. *In this case,*

$$\beta^\dagger_{e_2,f_2} = (1 - y(y^* f_2 y)^{\|\alpha_l^\pi f_1^{-1}} y^* f_2)(1 + y\alpha^\dagger_{e_1,f_1}x)(1 - e_2^{-1}x^*(xe_2^{-1}x^*)^{\|e_1\alpha_r^\pi} x).$$

Proof. The equivalence of the conditions (1) – (4) clearly follows from Propositions 6.2 and 6.3, Lemma 6.4. Next, we give a formula of $\beta^\dagger_{e_2,f_2}$.

It is clear that $1 + y\alpha^\dagger_{e_1,f_1}x$ is an inner inverse of β. If (4) holds, from the proof to Proposition 6.2, we get

$$\begin{aligned}
\beta\beta^-(\beta\beta^-)^{(1,3e_2)} &= (1 - y\alpha_r^\pi x)(1 - (y\alpha_r^\pi x)^{(1,4e_2)} y\alpha_r^\pi x) \\
&= 1 - (y\alpha_r^\pi x)^{(1,4e_2)} y\alpha_r^\pi x \\
&= = 1 - e_2^{-1}x^*(xe_2^{-1}x^*)^{\|e_1\alpha_r^\pi} x.
\end{aligned}$$

Also, from Proposition 6.3, we get $\beta^{(1,4f_2)} = (1 - y(y^* f_2 y)^{\|\alpha_l^\pi f_1^{-1}} y^* f_2)(1 + y\alpha^\dagger_{e_1,f_1}x)$. Therefore,

$$\begin{aligned}
\beta^\dagger_{e_2,f_2} &= \beta^{(1,4f_2)}\beta\beta^{(1,3e_2)} \\
&= \beta^{(1,4f_2)}\beta\beta^-(\beta\beta^-)^{(1,3e_2)} \\
&= (1 - y(y^* f_2 y)^{\|\alpha_l^\pi f_1^{-1}} y^* f_2)(1 + y\alpha^\dagger_{e_1,f_1}x)(1 - e_2^{-1}x^*(xe_2^{-1}x^*)^{\|e_1\alpha_r^\pi} x).
\end{aligned}$$

□

The equivalence among (1) – (3) in the following corollary can be found in [26, Theorem 5.8].

Corollary 6.6. *Let x, $y \in R$. Suppose $\alpha = 1 - xy \in R^\dagger$. Then the following statements are equivalent:*

(1) $\beta = 1 - yx \in R^\dagger$;

(2) $1 - y\alpha_r^\pi x \in R^{\{1,3\}}$ *and* $1 - y\alpha_l^\pi x \in R^{\{1,4\}}$;

(3) $y\alpha_r^\pi x \in R^{\{1,4\}}$ *and* $y\alpha_l^\pi x \in R^{\{1,3\}}$;

(4) $(xx^*)^{\|\alpha_r^\pi}$ *and* $(y^*y)^{\|\alpha_l^\pi}$ *exist,*

where $\alpha_r^\pi = 1 - \alpha\alpha^\dagger$ *and* $\alpha_l^\pi = 1 - \alpha^\dagger\alpha$. *In this case,*

$$\beta^\dagger = (1 - y(y^*y)^{\|\alpha_l^\pi}y^*)(1 + y\alpha^\dagger x)(1 - x^*(xx^*)^{\|\alpha_r^\pi}x).$$

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