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Characterizations and properties of the matrices A such that $AA_{(L)}^{(-1)} - A_{(L)}^{(-1)}A$ are nonsingular

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Abstract. In this paper, we consider the co-BD matrices, a class of matrices characterized by the invertibility of $AA_{(L)}^{(-1)} - A_{(L)}^{(-1)}A$, where $A_{(L)}^{(-1)}$ is the Bott-Duffin inverse of A with respect to a subspace L. Different characterizations and properties of this class of matrices are given. Also, we consider some characterizations of the nonsingularity of $AA_{(L)}^{(-1)} + A_{(L)}^{(-1)}A$ and $I_n - A(A_{(L)}^{(-1)})^2A$.

1. Introduction

The symbol $\mathbb{C}^{m\times n}$ will denote the set of all complex $m \times n$ matrices. The symbols $\mathcal{R}(A)$, $\mathcal{N}(A)$, A^* and $\text{rank}(A)$ represent the range space, null space, conjugate transpose and rank of $A \in \mathbb{C}^{m\times n}$ respectively. The symbol I_n means the identity matrix in $\mathbb{C}^{n\times n}$. The symbol O means the null matrix. If L is a subspace of $\mathbb{C}^n$, we use the notation $L \leqslant \mathbb{C}^n$ while $L^\perp$ means the orthogonal complement subspace of L. The dimension of L we denote by $\dim(L)$. $P_{L,M}$ stands for the oblique projector onto L along M, where $L, M \leqslant \mathbb{C}^n$ and $L \oplus M = \mathbb{C}^n$. P_L is the orthogonal projector onto L. Additionally, the Moore–Penrose inverse $A^\dagger \in \mathbb{C}^{n\times m}$ of $A \in \mathbb{C}^{m\times n}$ is the unique matrix verifying $AA^\dagger A = A$, $A^\dagger AA^\dagger = A^\dagger$, $(AA^\dagger)^* = AA^\dagger$, $(A^\dagger A)^* = A^\dagger A$ (see [3, 9, 20, 23]).

A matrix $X \in \mathbb{C}^{n\times m}$ that satisfies $XAX = X$ is called an outer inverse of A and is denoted by $A^{(2)}$. Let $L \leqslant \mathbb{C}^n$, $\dim L = l \leqslant \text{rank}(A)$ and $S \leqslant \mathbb{C}^m$, $\dim S = m - l$. There exists an unique outer inverse X of A such that $\mathcal{R}(X) = T$ and $\mathcal{N}(X) = S$ if and only if $AT \oplus S = \mathbb{C}^m$. In case that exists such X we call an outer inverse with prescribed range and null space and denote by $A_{T,S}^{(2)}$ (see [3, 23]).

Bott and Duffin, in their famous paper [6], introduced "constrained inverse" of a square matrix as an important tool in the electrical network theory. In [3], this inverse is called in their honour as the Bott-Duffin inverse (in short, BD-inverse [8]). Let $A \in \mathbb{C}^{n\times n}$ and $L \leqslant \mathbb{C}^n$. If $AP_L + P_{L^\perp}$ is nonsingular, then the BD-inverse of A with respect to L, denoted by $A_{(L)}^{(-1)}$, is defined by $A_{(L)}^{(-1)} = P_L(AP_L + P_{L^\perp})^{-1}$(see [6]). There are huge literatures on the BD-inverse and here we will mention only the part. Some important applications of the BD-inverse can be founded in Ben-Israel's and Greville's book [3]. Chen presented several properties and

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different representations of the BD-inverse in [8]. Also certain relationship between the nonsingularity of bordered matrices and the BD-inverse are given in [7]. Wei in [24] studied the various normwise relative condition numbers that measure the sensitivity of the BD-inverse and the solution of constrained linear systems. The perturbation theory for the BD-inverse was discussed in [21].

The nonsingularity of the difference and the sum of two idempotent has been considered first in the matrix and operatorseetings (see [1, 10–12, 14, 18, 22, 25–27]), and later in the ring case (see [15, 16]) and C^*-algebras (see [17]). Benítez and Rakočević characterized the class of co-EP matrices by the invertibility of $AA^\dagger - A^\dagger A$ (see [4]) which has been later further investigate in the papers [2] and [5]. Zuo, Baksalary and Cvetković-Ilić [28] further characterized the co-EP matrices and investigated the problem of completion of an upper triangular matrix to a co-EP matrix.

Motivated by the class of co-EP matrices, we introduce the following definition:

Definition 1.1. *Let $A \in \mathbb{C}^{n\times n}$ and $L \leqslant \mathbb{C}^n$ be such that $A^{(-1)}_{(L)}$ exists. A matrix A is a co-BD matrix if $AA^{(-1)}_{(L)} - A^{(-1)}_{(L)}A$ is nonsingular.*

The main contributions of this paper are the following:

(1) Different characterizations of the nonsingularity of $AA^{(-1)}_{(L)} - A^{(-1)}_{(L)}A$, $AA^{(-1)}_{(L)} + A^{(-1)}_{(L)}A$ and $I_n - A(A^{(-1)}_{(L)})^2A$ will be given;

(2) Using appropriate matrix decomposition and certain properties of oblique projectors, the explicit representation of the inverses of $AA^{(-1)}_{(L)} - A^{(-1)}_{(L)}A$, $AA^{(-1)}_{(L)} + A^{(-1)}_{(L)}A$, $I_n - A(A^{(-1)}_{(L)})^2A$ and $aAA^{(-1)}_{(L)} + bA^{(-1)}_{(L)}A - cA(A^{(-1)}_{(L)})^2A$, where $a, b \neq 0$ and $c \in \mathbb{C}$, will be presented.

The paper is organized as follow: In Section 2, we give some auxilliary lemmas. In Section 3, some equivalent conditions of the nonsingularity of $AA^{(-1)}_{(L)} - A^{(-1)}_{(L)}A$ and $AA^{(-1)}_{(L)} + A^{(-1)}_{(L)}A$ are considered. In Section 4, we discuss the nonsingularity of $I_n - A(A^{(-1)}_{(L)})^2A$ and $aAA^{(-1)}_{(L)} + bA^{(-1)}_{(L)}A - cA(A^{(-1)}_{(L)})^2A$, where $a, b \neq 0$ and $c \in \mathbb{C}$. In particular, we give an explicit formulae for the inverses of $AA^{(-1)}_{(L)} - A^{(-1)}_{(L)}A$, $AA^{(-1)}_{(L)} + A^{(-1)}_{(L)}A$, $I_n - A(A^{(-1)}_{(L)})^2A$ and $aAA^{(-1)}_{(L)} + bA^{(-1)}_{(L)}A - cA(A^{(-1)}_{(L)})^2A$, where $a, b \neq 0$ and $c \in \mathbb{C}$.

2. Preliminaries

For a given matrix $A \in \mathbb{C}^{n\times n}$, we will consider a matrix decomposition with respect to a given subspace $L \leqslant \mathbb{C}^n$. Note that for the orthogonal projector P_L, there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that

$$P_L = U\begin{bmatrix} I_l & O \\ O & O \end{bmatrix}U^*, \tag{1}$$

where $l = \dim(L)$ and

$$A = U\begin{bmatrix} A_L & B_L \\ C_L & D_L \end{bmatrix}U^*, \tag{2}$$

for some $A_L \in \mathbb{C}^{l\times l}$, $B_L \in \mathbb{C}^{l\times(n-l)}$, $C_L \in \mathbb{C}^{(n-l)\times l}$, $D_L \in \mathbb{C}^{(n-l)\times(n-l)}$.

Next lemma gives the necessary and sufficient condition for the existence of $A^{(-1)}_{(L)}$ as well as the representation of $A^{(-1)}_{(L)}$ using (1) and (2).

Lemma 2.1. *Let P_L and A be given by* (1) *and* (2)*, respectively. $A^{(-1)}_{(L)}$ exists if and only if A_L is invertible. In this case,*

$$A^{(-1)}_{(L)} = U\begin{bmatrix} A_L{}^{-1} & O \\ O & O \end{bmatrix}U^*. \tag{3}$$

Proof. By (1) and (2), we have that

$$AP_L + P_{L^\perp} = U\begin{bmatrix} A_L & O \\ C_L & I_{n-l} \end{bmatrix}U^*.$$

Evidently, $AP_L + P_{L^\perp}$ is invertible if and only if A_L is invertible. In this case, since $A^{(-1)}_{(L)} = P_L(AP_L + P_{L^\perp})^{-1}$, it follows that (3) is satisfied. □

Lemma 2.2. [8] *Let $A \in \mathbb{C}^{n\times n}$ and $L \leqslant \mathbb{C}^n$. If $AP_L + P_{L^\perp}$ is invertible, then the followings hold:*

$$P_L = A^{(-1)}_{(L)}AP_L = P_LAA^{(-1)}_{(L)}; \tag{4}$$

$$A^{(-1)}_{(L)} = P_LA^{(-1)}_{(L)} = A^{(-1)}_{(L)}P_L; \tag{5}$$

$$AA^{(-1)}_{(L)} = P_{AL,L^\perp}; \tag{6}$$

$$A^{(-1)}_{(L)}A = P_{L,(A^*L)^\perp}; \tag{7}$$

$$A^{(-1)}_{(L)} = A^{(2)}_{L,L^\perp}. \tag{8}$$

The following lemmas will be useful throughout the paper.

Lemma 2.3. [19] *Let $G \in \mathbb{C}^{m\times n}$ and $F \in \mathbb{C}^{n\times p}$. Then*

$$\mathrm{rank}(GF) = \mathrm{rank}(F) - \dim(\mathcal{R}(F) \cap \mathcal{N}(G)).$$

Lemma 2.4. [18] *Let $P, Q \in \mathbb{C}^{n\times n}$ be idempotent matrices. Then the following conditions are equivalent:*

(*a*) *$P - Q$ is nonsingular;*

(*b*) *$I_n - PQ$ and $P + Q$ are nonsingular.*

3. Nonsingularity of the difference and the sum of $AA^{(-1)}_{(L)}$ and $A^{(-1)}_{(L)}A$

The next theorem gives characterization of co-BD matrices. Using the decomposition of (1) and (2), the representation of $(AA^{(-1)}_{(L)} - A^{(-1)}_{(L)}A)^{-1}$ will be given.

Theorem 3.1. *Let $A \in \mathbb{C}^{n\times n}$, $L \leqslant \mathbb{C}^n$ and $\dim(L) = l$ be such that $A^{(-1)}_{(L)}$ exists. Let A be given by (2). Then the following statements are equivalent:*

(*a*) *A is a co-BD matrix;*

(*b*) $\mathrm{rank}(B_L) + \mathrm{rank}(C_L) = n$*;*

(*c*) $\mathrm{rank}(B_L) = \frac{n}{2}$ *and* $\mathrm{rank}(C_L) = \frac{n}{2}$*;*

(*d*) *$AP_L - P_LA$ is nonsingular,*

in which case,

$$(AA^{(-1)}_{(L)} - A^{(-1)}_{(L)}A)^{-1} = U\begin{bmatrix} O & A_L{C_L}^{-1} \\ -{B_L}^{-1}A_L & O \end{bmatrix}U^*. \tag{9}$$

Proof. (a) $\Rightarrow$ (b). From (2) and (3), we have

$$AA_{(L)}^{(-1)} = U\begin{bmatrix} I_l & O \\ C_L {A_L}^{-1} & O \end{bmatrix}U^*, \tag{10}$$

$$A_{(L)}^{(-1)}A = U\begin{bmatrix} I_l & {A_L}^{-1}B_L \\ O & O \end{bmatrix}U^*. \tag{11}$$

Then,

$$\operatorname{rank}(AA_{(L)}^{(-1)} - A_{(L)}^{(-1)}A) = \operatorname{rank}\left(\begin{bmatrix} O & B_L \\ C_L & O \end{bmatrix}\right). \tag{12}$$

Therefore, it is clear by (12) that if (a) holds then $\operatorname{rank}(B_L) + \operatorname{rank}(C_L) = n$.

(b) $\Rightarrow$ (c). Since $B_L \in \mathbb{C}^{l\times(n-l)}$ and $C_L \in \mathbb{C}^{(n-l)\times l}$, it follows that $\operatorname{rank}(B_L) + \operatorname{rank}(C_L) \leqslant 2l$ and $\operatorname{rank}(B_L) + \operatorname{rank}(C_L) \leqslant 2(n-l)$. If ($b$) holds, then $n = 2l$. Therefore, $\operatorname{rank}(B_L) = \frac{n}{2}$ and $\operatorname{rank}(C_L) = \frac{n}{2}$.

(c) $\Rightarrow$ (d). From (1) and (2), we have $\operatorname{rank}(AP_L - P_LA) = \operatorname{rank}(B_L) + \operatorname{rank}(C_L)$. Thus, if $\operatorname{rank}(B_L) = \operatorname{rank}(C_L) = \frac{n}{2}$, then ($d$) holds.

(d) $\Rightarrow$ (a). In term of (1) and (2), it is clear that $\operatorname{rank}(AP_L - P_LA) = \operatorname{rank}(AA_{(L)}^{(-1)} - A_{(L)}^{(-1)}A)$. Thus, if ($d$) holds, then $\operatorname{rank}(AA_{(L)}^{(-1)} - A_{(L)}^{(-1)}A) = n$, which implies A is a co-BD matrix.

From (10) and (11), we know

$$AA_{(L)}^{(-1)} - A_{(L)}^{(-1)}A = U\begin{bmatrix} O & -{A_L}^{-1}B_L \\ C_L{A_L}^{-1} & O \end{bmatrix}U^*. \tag{13}$$

Then (9) can be easily verified by (13). □

Example 3.2. *Let*

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{bmatrix} \text{ and } L = \mathcal{R}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\right).$$

From Lemma 2.1, we can verify $A_{(L)}^{(-1)}$ exists. By simple calculation, we have

$$AA_{(L)}^{(-1)} - A_{(L)}^{(-1)}A = \begin{bmatrix} 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

Using (9), we have

$$(AA_{(L)}^{(-1)} - A_{(L)}^{(-1)}A)^{-1} = \begin{bmatrix} 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{bmatrix}.$$

Motivated by [14, 18, 27], which show some necessary and sufficient conditions for the nonsingularity of the difference of two idempotent matrices, we present characterizations of co-BD matrices in terms of subspace operations.

Remark 3.3. *Note (6) and (7), we have $\mathcal{R}(AA_{(L)}^{(-1)}) = AL$, $\mathcal{N}(AA_{(L)}^{(-1)}) = L^{\perp}$, $\mathcal{R}(A_{(L)}^{(-1)}A) = L$ and $\mathcal{N}(A_{(L)}^{(-1)}A) = (A^*L)^{\perp}$. From [14, Corollary 1], then the following statements are equivalent:*

(*a*) *A is a co-BD matrix;*

(*b*) $AL \cap L = \{0\}$ *and* $(A^*L)^\perp \cap L^\perp = \{0\}$*;*

(*c*) $AL \oplus L = \mathbb{C}^n$ *and* $A^*L \oplus L = \mathbb{C}^n$*.*

Remark 3.4. *From the equivalence of* (*a*) *and* (*c*) *in Theorem 3.1, it is easy to verify that A is a co-BD matrix is equivalent to* A^* *is a co-BD matrix. This means that replacing A with* A^**, the conclusions in Theorem 3.1 and Theorem 3.3 are still valid. Under the hypotheses of Theorem 3.1, then the following statements are equivalent:*

(*a*) *A is a co-BD matrix;*

(*b*) A^* *is a co-BD matrix;*

(*c*) $A^*P_L - P_LA^*$ *is nonsingular;*

(*d*) $A^*(A^{*(-1)}_{(L)})^2A^* - P_L$ *is nonsingular;*

(*e*) $P_LA^*P_{L^\perp} + P_{L^\perp}A^*P_L$ *is nonsingular;*

(*f*) $P_LA^*L^\perp \oplus P_{L^\perp}A^*L = \mathbb{C}^n$*;*

(*g*) $A^{*(-1)}_{(L)}A^*L^\perp \oplus P_{L^\perp}A^*L = \mathbb{C}^n$*.*

The following theorem provides characterizations of $A \in \mathbb{C}^{n\times n}$ being a co-BD matrix in terms of ranges, null spaces and ranks of selected functions of $AA^{(-1)}_{(L)}$ and $A^{(-1)}_{(L)}A$.

Theorem 3.5. *Let* $A \in \mathbb{C}^{n\times n}$*,* $L \leqslant \mathbb{C}^n$ *and* $\dim(L) = l$ *be such that* $A^{(-1)}_{(L)}$ *exists. Let* $P = AA^{(-1)}_{(L)}$*,* $Q = A^{(-1)}_{(L)}A$*,* $\bar{P} = I_n - AA^{(-1)}_{(L)}$ *and* $\bar{Q} = I_n - A^{(-1)}_{(L)}A$*. Then the following statements are equivalent:*

(*a*) *A is a co-BD matrix;*

(*b*) $\mathcal{N}(PQ - QP) = \mathcal{N}(I_n - P - Q)$*;*

(*c*) $\mathcal{R}(PQ - QP) = \mathcal{R}(I_n - P - Q)$*;*

(*d*) $\mathrm{rank}(PQ - QP) = \mathrm{rank}(\bar{P}\bar{Q}) + \mathrm{rank}(PQ) = n$*;*

(*e*) $\mathrm{rank}(PQ - QP) = \mathrm{rank}(I_n - P - Q) = n$*.*

Proof. (*a*) ⇒ (*b*). From Remark 3.3, (6) and (7), if A is a co-BD matrix, we have $\mathcal{R}(P) \cap \mathcal{R}(Q) = \{0\}$ and $\mathcal{N}(P) \cap \mathcal{N}(Q) = \{0\}$. For any $a \in N(PQ - QP)$, we have $PQa = QPa \in \mathcal{R}(P) \cap \mathcal{R}(Q) = \{0\}$. Thus, $PQa = QPa = 0$. Then, note the fact that $I_na - Pa - Qa \in N(P) \cap N(Q) = \{0\}$. Therefore, $\mathcal{N}(PQ - QP) \subset \mathcal{N}(I_n - P - Q)$. It is easy to prove $\mathcal{N}(I_n - P - Q) \subset \mathcal{N}(PQ - QP)$. Hence $\mathcal{N}(I_n - P - Q) = \mathcal{N}(PQ - QP)$.

(*b*) ⇒ (*a*). If $a \in R(P) \cap R(Q)$, then $a = Pa = Qa$, which implies $(PQ - QP)a = 0$. Since $\mathcal{N}(PQ - QP) = \mathcal{N}(I_n - P - Q)$, it follows that $I_na - Pa - Qa = 0$, which implies $a = 0$. Thus $\mathcal{R}(P) \cap \mathcal{R}(Q) = \{0\}$. Similarly, we can also prove $\mathcal{N}(P) \cap \mathcal{N}(Q) = \{0\}$. From Remark 3.3, we have A is a co-BD matrix.

(*b*) ⇔ (*c*). Since the equivalence between (*a*) and (*b*) in Remark 3.4, we have $\mathcal{N}(PQ-QP) = \mathcal{N}(I_n-P-Q) \Leftrightarrow \mathcal{N}(P^*Q^* - Q^*P^*) = \mathcal{N}(I_n - P^* - Q^*) \Leftrightarrow \mathcal{R}(PQ - QP) = \mathcal{R}(I_n - P - Q)$.

(*a*) ⇔ (*d*). Using (10) and (11), by calculating we can obtain (*d*) holds if and only if $\mathrm{rank}(I_{n-l}+C_L{A_L}^{-2}B_L) = n-l$ and $\mathrm{rank}(B_L)+\mathrm{rank}(C_L{A_L}^{-1}(I_{n-l}+C_L{A_L}^{-2}B_L)) = n$, which implies $\mathrm{rank}(B_L)+\mathrm{rank}(C_L) = n$. From Theorem 3.1, we can complete the proof.

(*a*) ⇔ (*e*). Similar to (*a*) ⇔ (*d*). □

In the following theorem, we consider the nonsingularity of the sum of $AA^{(-1)}_{(L)}$ and $A^{(-1)}_{(L)}A$. Moreover, we give the representation of $(AA^{(-1)}_{(L)} + A^{(-1)}_{(L)}A)^{-1}$, when $AA^{(-1)}_{(L)} + A^{(-1)}_{(L)}A$ is nonsingular.

Theorem 3.6. *Let $A \in \mathbb{C}^{n\times n}$ be given in (2), $L \leqslant \mathbb{C}^n$ and $\dim(L) = l$ be such that $A^{(-1)}_{(L)}$ exists. Then the following statements are equivalent:*

(*a*) $AA^{(-1)}_{(L)} + A^{(-1)}_{(L)}A$ *is nonsingular;*

(*b*) $\operatorname{rank}(C_L A_L{}^{-2} B_L) = n - l$;

(*c*) $\operatorname{rank}(A(A^{(-1)}_{(L)})^2 A + P_L) = n$,

in which case,

$$(AA^{(-1)}_{(L)} + A^{(-1)}_{(L)}A)^{-1} = U\begin{bmatrix} \frac{1}{2}(I_l - A_L{}^{-1}B_L N^{-1} C_L A_L{}^{-1}) & A_L{}^{-1}B_L N^{-1} \\ N^{-1}C_L A_L{}^{-1} & -2N^{-1} \end{bmatrix}U^*, \tag{14}$$

where $N = C_L A_L{}^{-2} B_L$.

Proof. (*a*) ⇔ (*b*). By (10) and (11), we have

$$\begin{aligned} \operatorname{rank}(AA^{(-1)}_{(L)} + A^{(-1)}_{(L)}A) &= \operatorname{rank}\left(\begin{bmatrix} 2I_l & A_L{}^{-1}B_L \\ C_L A_L{}^{-1} & O \end{bmatrix}\right) \\ &= \operatorname{rank}\left(\begin{bmatrix} 2I_l & O \\ O & -\frac{1}{2}C_L A_L{}^{-2}B_L \end{bmatrix}\right) \\ &= l + \operatorname{rank}(C_L A_L{}^{-2}B_L). \end{aligned}$$

Thus, $\operatorname{rank}(AA^{(-1)}_{(L)} + A^{(-1)}_{(L)}A) = n$ is equivalent to $\operatorname{rank}(C_L A_L{}^{-2}B_L) = n - l$.

(*b*) ⇔ (*c*). Using (1), (10) and (11),

$$\begin{aligned} \operatorname{rank}(A(A^{(-1)}_{(L)})^2 A + P_L) &= \operatorname{rank}\left(\begin{bmatrix} 2I_l & A_L{}^{-1}B_L \\ C_L A_L{}^{-1} & C_L A_L{}^{-2}B_L \end{bmatrix}\right) \\ &= l + \operatorname{rank}(C_L A_L{}^{-2}B_L). \end{aligned}$$

Thus, $\operatorname{rank}(C_L A_L{}^{-2}B_L) = n - l$ is equivalent to $\operatorname{rank}(A(A^{(-1)}_{(L)})^2 A + P_L) = n$.

From (10) and (11), we know

$$AA^{(-1)}_{(L)} + A^{(-1)}_{(L)}A = U\begin{bmatrix} 2I_l & A_L{}^{-1}B_L \\ C_L A_L{}^{-1} & O \end{bmatrix}U^*. \tag{15}$$

Then (14) can be verified by (15). □

Item (*c*) can be recaptured in a more general form.

Remark 3.7. *Let $a, b \in \mathbb{C}$ and $ab(a + b) \neq 0$. Then the following statements are equivalent:*

(*a*) $AA^{(-1)}_{(L)} + A^{(-1)}_{(L)}A$ *is nonsingular;*

(*b*) $aA(A^{(-1)}_{(L)})^2 A + bP_L$ *is nonsingular.*

Example 3.8. *Let*

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & 0 & 0 \end{bmatrix} \text{ and } L = \mathcal{R}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right).$$

From Lemma 2.1, we can verify $A_{(L)}^{(-1)}$ exists. By calculation, we have

$$A_{(L)}^{(-1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$N = C_L {A_L}^{-2} B_L = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}^2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 10.$$

Using (14), we can obtain

$$(AA_{(L)}^{(-1)} + A_{(L)}^{(-1)}A)^{-1} = \begin{bmatrix} \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{3}{20} & \frac{2}{5} & \frac{1}{10} & \frac{1}{10} \\ \frac{3}{20} & \frac{1}{10} & \frac{2}{5} & -\frac{1}{10} \\ \frac{3}{10} & \frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} \end{bmatrix}.$$

Theorem 3.9. *Let $A \in \mathbb{C}^{n\times n}$, $L \leqslant \mathbb{C}^n$ and $\dim(L) = l$ be such that $A_{(L)}^{(-1)}$ exists. Then the following statements are equivalent:*

(*a*) $AA_{(L)}^{(-1)} + A_{(L)}^{(-1)}A$ *is nonsingular;*

(*b*) $A(A_{(L)}^{(-1)})^2AL^{\perp} \cap L = \{0\}$ *and* $(A^*L)^{\perp} \cap L^{\perp} = \{0\}$*;*

(*c*) $A(A_{(L)}^{(-1)})^2AL^{\perp} + L = \mathbb{C}^n$*;*

(*d*) $A(A_{(L)}^{(-1)})^2AL^{\perp} \oplus L = \mathbb{C}^n$*.*

Proof. (*a*) $\Rightarrow$ (*b*). Since (*a*) is equivalent to (*b*), we only need to prove $A(A_{(L)}^{(-1)})^2AL^{\perp} \cap L = \{0\}$. From the equivalence between (*a*) and (*b*) in Theorem 3.6, we can get $\operatorname{rank}(C_L{A_L}^{-2}B_L) = n - l$, which means $\operatorname{rank}(B_L) = n - l$. Using (1), (2), (3) and Lemma 2.3, we have

$$\begin{aligned} \operatorname{rank}(C_L{A_L}^{-2}B_L) &= \operatorname{rank}(P_{L^{\perp}}A(A_{(L)}^{(-1)})^2AP_{L^{\perp}}) \quad (G = P_{L^{\perp}}, F = A(A_{(L)}^{(-1)})^2AP_{L^{\perp}}) \\ &= \operatorname{rank}(A(A_{(L)}^{(-1)})^2AP_{L^{\perp}}) - \dim(A(A_{(L)}^{(-1)})^2AL^{\perp} \cap L) \\ &= \operatorname{rank}(B_L) - \dim(A(A_{(L)}^{(-1)})^2AL^{\perp} \cap L). \end{aligned} \tag{16}$$

It follows from (16) that (*d*) holds.

(*b*) $\Rightarrow$ (*c*). Using (7) and Lemma 2.3, we have

$$\begin{aligned} \operatorname{rank}(B_L) &= \operatorname{rank}(A_{(L)}^{(-1)}AP_{L^{\perp}}) \quad (G = A_{(L)}^{(-1)}A, F = P_{L^{\perp}}) \\ &= \operatorname{rank}(P_{L^{\perp}}) - \dim(L^{\perp} \cap \mathcal{N}(A_{(L)}^{(-1)}A)) \\ &= n - l - \dim(L^{\perp} \cap (A^*L)^{\perp}). \end{aligned} \tag{17}$$

By Lemma 2.3, we have

$$\begin{aligned} & \operatorname{rank}(P_{L^{\perp}}A^*((A_{(L)}^{(-1)})^*)^2A^*P_{L^{\perp}}) \quad (G = P_{L^{\perp}}A^*((A_{(L)}^{(-1)})^*)^2A^*, F = P_{L^{\perp}}) \\ = {} & \operatorname{rank}(P_{L^{\perp}}) - \dim(L^{\perp} \cap \mathcal{N}(P_{L^{\perp}}A^*((A_{(L)}^{(-1)})^*)^2A^*)) \\ = {} & n - l - \dim(L^{\perp} \cap \mathcal{N}(P_{L^{\perp}}A^*((A_{(L)}^{(-1)})^*)^2A^*)). \end{aligned} \tag{18}$$

If (d) holds, from (16) and (17), we have $\operatorname{rank}(P_{L^\perp}A(A_{(L)}^{(-1)})^2AP_{L^\perp}) = n - l$. Therefore, by (18), we can obtain $\mathcal{N}(P_{L^\perp}A^*((A_{(L)}^{(-1)})^*)^2A^*) \cap L^\perp = 0$, which implies $A(A_{(L)}^{(-1)})^2AL^\perp + L = \mathbb{C}^n$.

$(c) \Rightarrow (a)$. If (e) holds, using (1), (2), (3) and (18), we can obtain $\operatorname{rank}(P_{L^\perp}A^*((A_{(L)}^{(-1)})^*)^2A^*P_{L^\perp}) = \operatorname{rank}(C_L{A_L}^{-2}B_L) = n - l$.

$(c) \Leftrightarrow (d)$. From $(e) \Rightarrow (a)$, we have $\operatorname{rank}(C_L{A_L}^{-2}B_L) = n-l$, which implies $\operatorname{rank}(B_L) = n-l$. From (16), we have $A(A_{(L)}^{(-1)})^2AL^\perp \cap L = 0$. Thus, (f) holds. On the other hand, if (f) holds, it is obvious that (e) holds. □

Remark 3.10. *By (6) and (7), apply* $\mathcal{R}(AA_{(L)}^{(-1)}) = AL$, $\mathcal{N}(AA_{(L)}^{(-1)}) = L^\perp$, $\mathcal{R}(A_{(L)}^{(-1)}A) = L$ *and* $\mathcal{N}(A_{(L)}^{(-1)}A) = (A^*L)^\perp$ *to [14, Corollary 4] and [27, Theorem 2.5], respectively. Then the following statements are equivalent:*

(a) $AA_{(L)}^{(-1)} + A_{(L)}^{(-1)}A$ *is nonsingular;*

(b) $A_{(L)}^{(-1)}AL^\perp \cap AL = \{0\}$ *and* $(A^*L)^\perp \cap L^\perp = \{0\}$*;*

(c) $A_{(L)}^{(-1)}AL^\perp \oplus (AL \cap L) \oplus (A^*L)^\perp = \mathbb{C}^n$ *and* $(A^*L)^\perp \cap L^\perp = \{0\}$.

4. Nonsingularity of the combination of I_n, $AA_{(L)}^{(-1)}$ and $A_{(L)}^{(-1)}A$

In this section, we will consider the nonsingularity of the combinations of I_n, $AA_{(L)}^{(-1)}$ and $A_{(L)}^{(-1)}A$. First, we consider the nonsingularity of $I_n - A(A_{(L)}^{(-1)})^2A$ in terms of rank equalities. Furthermore, we give the representation of the inverse of $I_n - A(A_{(L)}^{(-1)})^2A$.

Theorem 4.1. *Let* $A \in \mathbb{C}^{n\times n}$ *be given in (2),* $L \leqslant \mathbb{C}^n$ *and* $\dim(L) = l$ *be such that* $A_{(L)}^{(-1)}$ *exists. Then the following statements are equivalent:*

(a) $I_n - A(A_{(L)}^{(-1)})^2A$ *is nonsingular;*

(b) $\operatorname{rank}(B_LC_L) = l$*;*

(c) $\operatorname{rank}(2I_n - AA_{(L)}^{(-1)} - A_{(L)}^{(-1)}A) = n$*,*

in which case,

$$(I_n - A(A_{(L)}^{(-1)})^2A)^{-1} = U\begin{bmatrix} I_l - A_LM^{-1}A_L & -A_LM^{-1}B_L \\ -C_LM^{-1}A_L & I_{n-l} - C_LM^{-1}B_L \end{bmatrix}U^*, \tag{19}$$

where $M = B_LC_L$.

Proof. $(a) \Leftrightarrow (b)$. By (10) and (11), we have

$$\begin{aligned}\operatorname{rank}(I_n - A(A_{(L)}^{(-1)})^2A) &= \operatorname{rank}\left(\begin{bmatrix} O & -{A_L}^{-1}B_L \\ -C_L{A_L}^{-1} & I_{n-l} - C_L{A_L}^{-2}B_L \end{bmatrix}\right) \\ &= \operatorname{rank}\left(\begin{bmatrix} -{A_L}^{-1}B_LC_L{A_L}^{-1} & O \\ O & I_{n-l} \end{bmatrix}\right) \\ &= n - l + \operatorname{rank}(B_LC_L).\end{aligned}$$

Thus, $\operatorname{rank}(I_n - A(A_{(L)}^{(-1)})^2A) = n$ is equivalent to $\operatorname{rank}(B_LC_L) = l$.

$(a) \Leftrightarrow (c)$. Apply $P = AA_{(L)}^{(-1)}$ and $Q = A_{(L)}^{(-1)}A$ to [14, Theorem 3].

From (10) and (11), we have

$$I_n - A(A_{(L)}^{(-1)})^2 A = U \begin{bmatrix} O & -A_L{}^{-1}B_L \\ -C_L A_L{}^{-1} & I_{n-l} - C_L A_L{}^{-2} B_L \end{bmatrix} U^*. \tag{20}$$

Then (19) can be verified by (20). □

Item (c) can be recaptured in a more general form.

Remark 4.2. *Let $a, b \in \mathbb{C}$ and $ab(a+b) \neq 0$. Then the following statements are equivalent:*

(a) *$I_n - A(A_{(L)}^{(-1)})^2 A$ is nonsingular;*

(b) *$(a+b)I_n - aAA_{(L)}^{(-1)} - bA_{(L)}^{(-1)}A$ is nonsingular.*

Example 4.3. *Let*

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 3 & 2 & 0 & 3 \\ 1 & 0 & 0 & 1 & 4 \\ 0 & 1 & 3 & 1 & 2 \end{bmatrix} \text{ and } L = \mathcal{R}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right).$$

Using Lemma 2.1, we can verify $A_{(L)}^{(-1)}$ exists. By calculation, we have

$$M = B_L C_L = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 0 & 4 \end{bmatrix}, \text{ and } M^{-1} = \begin{bmatrix} 1 & -\frac{3}{2} \\ 0 & \frac{1}{4} \end{bmatrix}.$$

From (19), we can obtain

$$(I - A(A_{(L)}^{(-1)})^2 A)^{-1} = \begin{bmatrix} 0 & \frac{3}{2} & -\frac{1}{2} & -1 & \frac{3}{2} \\ 0 & \frac{3}{4} & -\frac{1}{4} & 0 & -\frac{1}{4} \\ 0 & -\frac{3}{4} & \frac{1}{4} & 0 & -\frac{3}{4} \\ -1 & \frac{3}{2} & -\frac{1}{2} & 0 & \frac{3}{2} \\ 0 & -\frac{1}{4} & -\frac{1}{4} & 0 & \frac{3}{4} \end{bmatrix}.$$

Theorem 4.4. *Let $A \in \mathbb{C}^{n\times n}$, $L \leqslant \mathbb{C}^n$ and $\dim(L) = l$ be such that $A_{(L)}^{(-1)}$ exists. Then the following statements are equivalent:*

(a) *$I_n - A(A_{(L)}^{(-1)})^2 A$ is nonsingular;*

(b) *$P_{L^\perp}AL \cap (A^*L)^\perp = \{0\}$ and $AL \cap L = \{0\}$;*

(c) *$P_{L^\perp}AL + (A^*L)^\perp = \mathbb{C}^n$;*

(d) *$P_{L^\perp}AL \oplus (A^*L)^\perp = \mathbb{C}^n$.*

Proof. $(a) \Rightarrow (b)$. From the equivalence between (a) and (b), we only need to prove $P_{L^\perp}AL \cap (A^*L)^\perp = \{0\}$. In Theorem 4.1, since (a) is equivalent to (b) , we get $\text{rank}(B_L C_L) = l$, which implies $\text{rank}(C_L) = l$. Using (1), (2), (7) and Lemma 2.3, we have

$$\begin{aligned} \text{rank}(B_L C_L) &= \text{rank}(A_{(L)}^{(-1)} A P_{L^\perp} A P_L) \quad (G = A_{(L)}^{(-1)} A, F = P_{L^\perp} A P_L) \\ &= \text{rank}(P_{L^\perp} A P_L) - \dim(P_{L^\perp} AL \cap \mathcal{N}(A_{(L)}^{(-1)} A)) \\ &= \text{rank}(C_L) - \dim(P_{L^\perp} AL \cap (A^*L)^\perp). \end{aligned} \tag{21}$$

Thus, (e) holds.

$(b) \Rightarrow (c)$. From (1), (2) and Lemma 2.3, we have

$$\begin{aligned} \mathrm{rank}(C_L) &= \mathrm{rank}(P_{L^\perp} A P_L) \quad (G = P_{L^\perp}, F = AP_L) \\ &= l - \dim(AL \cap L), \end{aligned} \tag{22}$$

$$\begin{aligned} \mathrm{rank}(B_L C_L) &= \mathrm{rank}(P_L A^* P_{L^\perp} A^* P_L) \quad (G = P_L A^* P_{L^\perp}, F = A^* P_L) \\ &= \mathrm{rank}(A^* P_L) - \dim(A^* L \cap \mathcal{N}(P_L A^* P_{L^\perp})) \\ &= l - \dim(A^* L \cap \mathcal{N}(P_L A^* P_{L^\perp})). \end{aligned} \tag{23}$$

If (e) holds, using (21) and (22), we can get $\mathrm{rank}(B_L C_L) = l$. Thus, from (23), $\dim(A^*L \cap \mathcal{N}(P_L A^* P_{L^\perp})) = 0$, which implies $P_{L^\perp} AL + (A^*L)^\perp = \mathbb{C}^n$.

$(c) \Rightarrow (a)$. In terms of (23), it easy to have $\mathrm{rank}(B_L C_L) = l$. From Theorem 4.1, we can obtain $I_n - A(A^{(-1)}_{(L)})^2 A$ is nonsingular.

$(c) \Leftrightarrow (d)$. From $(f) \Rightarrow (a)$, we have $\mathrm{rank}(B_L C_L) = \mathrm{rank}(C_L) = l$. Clearly, using (21), we can conclude $P_{L^\perp} AL \cap (A^*L)^\perp = \{0\}$. Thus, from $P_{L^\perp} AL + (A^*L)^\perp = \mathbb{C}^n$ and $P_{L^\perp} AL \cap (A^*L)^\perp = \{0\}$, item (g) holds. On the other hand, if (g) holds, it is obvious that (f) holds. □

Remark 4.5. *From the equivalence between* (a) *and* (c) *in Theorem 4.1,* $\mathrm{rank}(I_n - A(A^{(-1)}_{(L)})^2 A) = n$ *is equivalent to* $\mathrm{rank}((I_n - AA^{(-1)}_{(L)}) + (I_n - A^{(-1)}_{(L)}A)) = n$*. By (6) and (7), applying* $\mathcal{R}(I_n - AA^{(-1)}_{(L)}) = L^\perp$*,* $\mathcal{N}(I_n - AA^{(-1)}_{(L)}) = AL$*,* $\mathcal{R}(I_n - A^{(-1)}_{(L)}A) = (A^*L)^\perp$ *and* $\mathcal{N}(I_n - A^{(-1)}_{(L)}A) = L$ *to [14, Corollary 4 (ii)] and [27, Theorem 2.5 (c),(d)],respectively. Then the following statements are equivalent:*

- (a) $I_n - A(A^{(-1)}_{(L)})^2 A$ *is nonsingular;*
- (b) $(I_n - A^{(-1)}_{(L)}A)AL \cap L^\perp = \{0\}$ *and* $AL \cap L = \{0\}$*;*
- (c) $(I_n - A^{(-1)}_{(L)}A)AL \oplus ((I_n - A^{(-1)}_{(L)}A)^*L)^\perp = \mathbb{C}^n$*;*
- (d) $(I_n - A^{(-1)}_{(L)}A)AL \oplus ((A^*L)^\perp \cap L^\perp) \oplus L = \mathbb{C}^n$ *and* $AL \cap L = \{0\}$*.*

From Lemma 2.4, it is well known that A is co-BD can be characterized by the nonsingularity of $AA^{(-1)}_{(L)} + A^{(-1)}_{(L)}A$ and $I_n - A(A^{(-1)}_{(L)})^2 A$. In the following theorem, we consider whether we can use one of the above conditions to characterize that A is co-BD.

Theorem 4.6. *Let* $A \in \mathbb{C}^{n\times n}$ *be given in (2),* $L \leqslant \mathbb{C}^n$ *and* $\dim(L) = l$ *be such that* $A^{(-1)}_{(L)}$ *exists. Then the following statements are equivalent:*

- (a) *A is co-BD;*
- (b) $AA^{(-1)}_{(L)} + A^{(-1)}_{(L)}A$ *is nonsingular, and* $AL \cap L = \{0\}$*;*
- (c) $AA^{(-1)}_{(L)} + A^{(-1)}_{(L)}A$ *is nonsingular, and* $A^*L \cap L = \{0\}$*;*
- (d) $I_n - A(A^{(-1)}_{(L)})^2 A$ *is nonsingular, and* $AL + L = \mathbb{C}^n$*;*
- (e) $I_n - A(A^{(-1)}_{(L)})^2 A$ *is nonsingular, and* $A^*L + L = \mathbb{C}^n$*,*

in which case, we have

$$(AA^{(-1)}_{(L)} - A^{(-1)}_{(L)}A)^{-1} = P_{AL,L} - (P_{A^*L,L})^*; \tag{24}$$

$$(AA^{(-1)}_{(L)} + A^{(-1)}_{(L)}A)^{-1} = (P_{A^*L,L})^* P_{L,AL} + (P_{L,A^*L})^* P_{AL,L}; \tag{25}$$

$$(I_l - A(A^{(-1)}_{(L)})^2 A)^{-1} = I_n - P_{AL,L}(P_{A^*L,L})^*. \tag{26}$$

Proof. $(a) \Leftrightarrow (b)$. From Lemma 2.4 and the equivalence between (a) and (b) in Remark 3.3, it follows that item (b) holds. On the other hand, since (b) in Theorem 3.9 and Remark 3.3, it follows that (a) holds.

$(a) \Rightarrow (c)$. Similar to $(a) \Leftrightarrow (b)$. From the equivalence between (a) and (c) in Remark 3.3, it is clear that $A^*L \cap L = \{0\}$.

$(c) \Rightarrow (a)$. From (1), (7), (11) and Lemma 2.3, we have

$$\begin{aligned}
\mathrm{rank}(B_L) &= \mathrm{rank}(P_{L^\perp}(A_{(L)}^{(-1)}A)^*) \quad (G = P_{L^\perp}, F = (A_{(L)}^{(-1)}A)^*) \\
&= \mathrm{rank}((A_{(L)}^{(-1)}A)^*) - \dim(A^*L \cap L) \\
&= l - \dim(A^*L \cap L),
\end{aligned}$$

which implies $\mathrm{rank}(B_L) = l$. If (c) holds, we know $\mathrm{rank}(C_L {A_L}^{-2} B_L) = \mathrm{rank}(C_L) = n - l$. By Theorem 3.1, $\mathrm{rank}(B_L) + \mathrm{rank}(C_L) = n$, that means (a) holds.

$(a) \Rightarrow (d)$. From Lemma 2.4, if (a) holds, we can obtain $I_n - A(A_{(L)}^{(-1)})^2A$ is nonsingular. By (c) in Remark 3.3, we can obtain $AL + L = \mathbb{C}^n$.

$(d) \Rightarrow (a)$. From (1), (7), (10) and Lemma 2.3, we have

$$\begin{aligned}
\mathrm{rank}(C_L) &= \mathrm{rank}((AA_{(L)}^{(-1)})^* P_{L^\perp}) \quad (G = (AA_{(L)}^{(-1)})^*, F = P_{L^\perp}) \\
&= n - l - \dim(L^\perp \cap (AL)^\perp).
\end{aligned}$$

Note the fact that $AL + L = \mathbb{C}^n$ means that $(AL)^\perp \cap L^\perp = \{0\}$. Hence, $\mathrm{rank}(C_L) = n - l$. If (d) holds, we have $\mathrm{rank}(B_L C_L) = \mathrm{rank}(B_L) = l$. Using Theorem 3.1, since $\mathrm{rank}(B_L) + \mathrm{rank}(C_L) = n$, it follows that (a) holds.

$(a) \Rightarrow (e)$. Similar to $(a) \Rightarrow (d)$. By (c) in Remark 3.3, we can obtain $A^*L + L = \mathbb{C}^n$.

$(e) \Rightarrow (a)$. From Theorem 4.4, if (e) holds, we have $AL \cap L = \{0\}$. It is well known $A^*L + L = \mathbb{C}^n$ means $(A^*L)^\perp \cap L^\perp = \{0\}$. Similar to $(b) \Rightarrow (a)$, item (a) holds.

Since A is co-BD, it follows from $(a) \Rightarrow (b), (d)$ and $(a) \Rightarrow (c), (e)$ that $AL \oplus L = \mathbb{C}^{n\times n}$ and $A^*L \oplus L = \mathbb{C}^{n\times n}$, respectively. This means $P_{AL,L}$ and $P_{A^*L,L}$ exist. If A is co-BD, we have B_L and C_L are nonsingular. Partitioning $P_{AL,L}$ in conformation with partition A, say

$$P_{AL,L} = U \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} U^*,$$

where U is the same as U in (1). Since $\mathcal{N}(P_{AL,L}) = \mathcal{N}(P_{L^\perp})$ and $\mathcal{R}(AP_L) = \mathcal{R}(P_{AL,L})$, then $P_{AL,L}P_{L^\perp} = P_{AL,L}$ and $P_{AL,L}AP_L = AP_L$. From (1) and (2), we have

$$P_{AL,L} = U \begin{bmatrix} O & A_L {C_L}^{-1} \\ O & I_{\frac{n}{2}} \end{bmatrix} U^*. \tag{27}$$

Similarly, since $\mathcal{N}(P_{A^*L,L}) = \mathcal{N}(P_{L^\perp})$ and $\mathcal{R}(A^*P_L) = \mathcal{R}(P_{A^*L,L})$, then $P_{A^*L,L}P_{L^\perp} = P_{A^*L,L}$ and $P_{A^*L,L}A^*P_L = A^*P_L$. From (1) and (2), we can obtain

$$P_{A^*L,L} = U \begin{bmatrix} O & ({B_L}^{-1}A_L)^* \\ O & I_{\frac{n}{2}} \end{bmatrix} U^*. \tag{28}$$

The proof of the formulae (24), (25) and (26) are similar, therefore, we will only prove (25). Since A is co-BD, it follows from (14), (27), (28) and the invertibility of B_L and C_L that

$$\begin{aligned}
(AA_{(L)}^{(-1)} + A_{(L)}^{(-1)}A)^{-1} &= U \begin{bmatrix} O & A_L {C_L}^{-1} \\ {B_L}^{-1}A_L & -2{B_L}^{-1}{A_L}^2{C_L}^{-1} \end{bmatrix} U^* \\
&= U \begin{bmatrix} O & O \\ {B_L}^{-1}A_L & I_{\frac{n}{2}} \end{bmatrix} \begin{bmatrix} I_{\frac{n}{2}} & -A_L{C_L}^{-1} \\ O & O \end{bmatrix} U^* + U \begin{bmatrix} I_{\frac{n}{2}} & O \\ -{B_L}^{-1}A_L & O \end{bmatrix} \begin{bmatrix} O & A_L{C_L}^{-1} \\ O & I_{\frac{n}{2}} \end{bmatrix} U^* \\
&= (P_{A^*L,L})^* P_{L,AL} + (P_{L,A^*L})^* P_{AL,L}.
\end{aligned}$$

Hence, we can obtain (25). □

Example 4.7. *Let matrix A and subspace L are given by Example 3.2. From (27) and (28), we can obtain*

$$P_{AL,L} = \begin{bmatrix} 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } P_{A^*L,L} = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Using (24), (25) and (26), we have

$$(AA_{(L)}^{(-1)} - A_{(L)}^{(-1)}A)^{-1} = \begin{bmatrix} 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{bmatrix}, \quad (AA_{(L)}^{(-1)} + A_{(L)}^{(-1)}A)^{-1} = \begin{bmatrix} 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ -1 & -2 & 1 & 3 \\ 1 & 1 & 0 & -2 \end{bmatrix} \text{ and}$$

$$(I - A(A_{(L)}^{(-1)})^2A)^{-1} = \begin{bmatrix} 0 & -\frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{bmatrix}.$$

In [27], let $P,\ Q \in \mathbb{C}^{n\times n}$ are idempotent and let $a, b \neq 0$ and $c \in \mathbb{C}$. Then

$$\operatorname{rank}(aP + bQ - cPQ) = \begin{cases} \operatorname{rank}(P - Q), if\ a + b = c \\ \operatorname{rank}(P + Q), if\ a + b \neq c \end{cases}.$$

Motivated by this result, we can get some similar results when $P = AA_{(L)}^{(-1)}$ and $Q = A_{(L)}^{(-1)}A$.

Theorem 4.8. *Let $A \in \mathbb{C}^{n\times n}$ be given in (2), $L \leqslant \mathbb{C}^n$ and $\dim(L) = l$ be such that $A_{(L)}^{(-1)}$ exists. Let $a, b \neq 0$ and $c \in \mathbb{C}$. Then the following statements are hold:*

(*a*) *If $a + b = c$, $aAA_{(L)}^{(-1)} + bA_{(L)}^{(-1)}A - cA(A_{(L)}^{(-1)})^2A$ is nonsingular if and only if $AA_{(L)}^{(-1)} - A_{(L)}^{(-1)}A$ is nonsingular, in which case*

$$(aAA_{(L)}^{(-1)} + bA_{(L)}^{(-1)}A - cA(A_{(L)}^{(-1)})^2A)^{-1} = U\begin{bmatrix} \frac{c}{ab}I_{\frac{n}{2}} & -\frac{1}{b}A_LC_L{}^{-1} \\ -\frac{1}{a}B_L{}^{-1}A_L & O \end{bmatrix}U^*. \tag{29}$$

(*b*) *If $a + b \neq c$, $aAA_{(L)}^{(-1)} + bA_{(L)}^{(-1)}A - cA(A_{(L)}^{(-1)})^2A$ is nonsingular if and only if $AA_{(L)}^{(-1)} + A_{(L)}^{(-1)}A$ is nonsingular, in which case*

$$\begin{aligned}&(aAA_{(L)}^{(-1)} + bA_{(L)}^{(-1)}A - cA(A_{(L)}^{(-1)})^2A)^{-1} \\ = \ & U\begin{bmatrix} \frac{1}{a+b-c}(I_l - \frac{(a-c)(b-c)}{ab}A_L{}^{-1}B_LN^{-1}C_LA_L{}^{-1}) & \frac{b-c}{ab}A_L{}^{-1}B_LN^{-1} \\ \frac{a-c}{ab}N^{-1}C_LA_L{}^{-1} & \frac{c-a-b}{ab}N^{-1} \end{bmatrix}U^*,\end{aligned} \tag{30}$$

where $N = C_LA_L{}^{-2}B_L$.

Proof. (*a*). It is obvious by applying $AA_{(L)}^{(-1)}$ and $A_{(L)}^{(-1)}A$ to [27, Theorem 2.1]. From (10) and (11), we have

$$aAA_{(L)}^{(-1)} + bA_{(L)}^{(-1)}A - cA(A_{(L)}^{(-1)})^2A = U\begin{bmatrix} O & -aA_L{}^{-1}B_L \\ -bC_LA_L{}^{-1} & -cC_LA_L{}^{-2}B_L \end{bmatrix}U^*. \tag{31}$$

Therefore, (29) can be verified by (31).

(*b*). By applying $AA_{(L)}^{(-1)}$ and $A_{(L)}^{(-1)}A$ to [27, Theorem 2.1], we can prove (*b*). From (10) and (11), we have

$$aAA_{(L)}^{(-1)} + bA_{(L)}^{(-1)}A - cA(A_{(L)}^{(-1)})^2A = U\begin{bmatrix} (a+b-c)I_l & (b-c)A_L{}^{-1}B_L \\ (a-c)C_LA_L{}^{-1} & -cC_LA_L{}^{-2}B_L \end{bmatrix}U^*. \tag{32}$$

Therefore, (30) can be verified by (32). □

Remark 4.9. *By (29) and (30), if A is co-BD and $a, b \neq 0$ and $c \in \mathbb{C}$, whether $a + b = c$ or $a + b \neq c$, we have*

$$(aAA_{(L)}^{(-1)} + bA_{(L)}^{(-1)}A - cA(A_{(L)}^{(-1)})^2A)^{-1} = U\begin{bmatrix} \frac{c}{ab}I_{\frac{n}{2}} & \frac{b-c}{ab}A_L{C_L}^{-1} \\ \frac{a-c}{ab}{B_L}^{-1}A_L & \frac{c-a-b}{ab}{B_L}^{-1}{A_L}^2{C_L}^{-1} \end{bmatrix}U^*.$$

Similar to Theorem 4.6, we now give the explicit formulae for $(aAA_{(L)}^{(-1)} + bA_{(L)}^{(-1)}A - cA(A_{(L)}^{(-1)})^2A)^{-1}$ by using $P_{AL,L}$ and $P_{A^*L,L^\perp}$ when A is co-BD.

Corollary 4.10. *Let $A \in \mathbb{C}^{n\times n}$ be given in (2) and $L \leqslant \mathbb{C}^n$ be such that $A_{(L)}^{(-1)}$ exists. Let $a, b \neq 0$ and $c \in \mathbb{C}$. If A is co-BD, then*

$$(aAA_{(L)}^{(-1)} + bA_{(L)}^{(-1)}A - cA(A_{(L)}^{(-1)})^2A)^{-1} = b^{-1}P_{L,AL} + (ab)^{-1}(c-a)(P_{L,A^*L})^* + (ab)^{-1}(a+b-c)(P_{L,A^*L})^*P_{AL,L}. \quad (33)$$

Proof. Since A is co-BD, it follows from (30), (27) and (28) that

$$\begin{aligned}(aAA_{(L)}^{(-1)} + bA_{(L)}^{(-1)}A - cA(A_{(L)}^{(-1)})^2A)^{-1} &= U\begin{bmatrix} \frac{c}{ab}I_{\frac{n}{2}} & \frac{b-c}{ab}A_L{C_L}^{-1} \\ \frac{a-c}{ab}{B_L}^{-1}A_L & \frac{c-a-b}{ab}{B_L}^{-1}{A_L}^2{C_L}^{-1} \end{bmatrix}U^* \\ &= b^{-1}U\begin{bmatrix} I_{\frac{n}{2}} & -A_L{C_L}^{-1} \\ O & O \end{bmatrix}U^* + (ab)^{-1}(c-a)U\begin{bmatrix} I_{\frac{n}{2}} & O \\ -{B_L}^{-1}A_L & O \end{bmatrix}U^* \\ &\quad +(ab)^{-1}(a+b-c)U\begin{bmatrix} I_{\frac{n}{2}} & O \\ -{B_L}^{-1}A_L & O \end{bmatrix}\begin{bmatrix} O & A_L{C_L}^{-1} \\ O & I_{\frac{n}{2}} \end{bmatrix}U^* \\ &= b^{-1}P_{L,AL} + (ab)^{-1}(c-a)(P_{L,A^*L})^* + (ab)^{-1}(a+b-c)(P_{L,A^*L})^*P_{AL,L}.\end{aligned}$$

Hence, we can obtain (33). □

Example 4.11. *Let the matrix A and the subspace L be given as in the Example 3.2. Using simple calculation, we have the following results:*

Using (29), if $a + b = c$, we have

$$(aAA_{(L)}^{(-1)} + bA_{(L)}^{(-1)}A - cA(A_{(L)}^{(-1)})^2A)^{-1} = \begin{bmatrix} \frac{c}{ab} & 0 & \frac{1}{2b} & -\frac{1}{2b} \\ 0 & \frac{c}{ab} & -\frac{1}{2b} & -\frac{1}{2b} \\ \frac{1}{a} & \frac{2}{a} & 0 & 0 \\ -\frac{1}{a} & -\frac{1}{a} & 0 & 0 \end{bmatrix}.$$

Example 4.12. *Let the matrix A and the subspace L be given as in the Example 3.8. Using simple calculation, we have the following results:*

Using (30), if $a + b \neq c$, we have

$$(aAA_{(L)}^{(-1)} + bA_{(L)}^{(-1)}A - cA(A_{(L)}^{(-1)})^2A)^{-1} = \begin{bmatrix} \frac{5ab-3(a-c)(b-c)}{5ab(a+b-c)} & \frac{-2(a-c)(b-c)}{5ab(a+b-c)} & \frac{2(a-c)(b-c)}{5ab(a+b-c)} & \frac{b-c}{5ab} \\ \frac{-3(a-c)(b-c)}{10ab(a+b-c)} & \frac{5ab-(a-c)(b-c)}{5ab(a+b-c)} & \frac{(a-c)(b-c)}{5ab(a+b-c)} & \frac{b-c}{10ab} \\ \frac{3(a-c)(b-c)}{10ab(a+b-c)} & \frac{(a-c)(b-c)}{5ab(a+b-c)} & \frac{5ab-(a-c)(b-c)}{5ab(a+b-c)} & \frac{c-b}{10ab} \\ \frac{3(a-c)}{10ab} & \frac{a-c}{5ab} & \frac{c-a}{5ab} & \frac{c-a-b}{10ab} \end{bmatrix}.$$

5. Conclusion

Throughout this paper, we discuss some different characterizations of the co-BD matrices and we consider the sufficient and necessary conditions for the nonsingularity of $AA_{(L)}^{(-1)} + A_{(L)}^{(-1)}A$ and $I_n - A(A_{(L)}^{(-1)})^2A$, respectively. In case of invertibility, the inverses of these matrices are given in terms of the matrix decomposition of A with respect to a given subspace L.

According to the current research background, we suggest the following topics that can be discussed:

(1) Consider the properties and representation of the co-BD matrices when A is a bounded linear operator in Hilbert space, according to [13].

(2) Using the discussion on continuity of co-BD matrices, discuss the perturbation theory of co-BD inverse.

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