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Further characterizations of k-generalized projectors and k-hypergeneralized projectors

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Abstract. The paper focuses on the classes of the k-generalized and k-hypergeneralized projectors. Several original features of these classes are identified and new properties are characterized. We present some relations between k-generalized and k-hypergeneralized projectors that generalize appropriate relations between generalized and hypergeneralized projectors given in [Further properties of generalized and hypergeneralized projectors, Linear Algebra and its Applications, 389 (2004) 295–303] and [Further results on generalized and hypergeneralized projectors, Linear Algebra and its Applications, 429 (2008) 1038–1050].

1. Introduction

Let $\mathbb{N}^+$ denote the set of all positive integers. For $n \in \mathbb{N}^+$, let $\overline{1,n} = \{1, \cdots, n\}$. The symbols $\mathbb{C}^{m\times n}$ and $\mathbb{C}^n$ will denote the set of complex $m \times n$ matrices and n-dimensional complex vector spaces. For a matrix $A \in \mathbb{C}^{m\times n}$, the symbols A^*, $\mathcal{R}(A)$, $\mathcal{N}(A)$ and $r(A)$ will stand for the conjugate transpose, range, nullspace and rank of A, respectively. For a matrix $A \in \mathbb{C}^{n\times n}$, we denote by $\delta(A)$ and $\mathrm{tr}(A)$, the spectrum and the trace of A, respectively. By I_n we will represent the identity matrix of order n. Henceforth, the symbol Φ_n will stand for the set of all complex numbers such that $z^n = 1$, i.e.

$$\Phi_n = \{z \in \mathbb{C} : z^n = 1\}.$$

We define $A^0 = I_n$, for $A \in \mathbb{C}^{n\times n}$.

The symbol $A^\dagger$ will mean the unique generalized inverse of A which verifies

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (AA^\dagger)^* = AA^\dagger, \quad (A^\dagger A)^* = A^\dagger A,$$

called the Moore-Penrose inverse of A.

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The index of a matrix $A \in \mathbb{C}^{n\times n}$, is the smallest nonnegative integer k such that $r(A^{k+1}) = r(A^k)$, denoted by Ind(A). The symbol $\mathbb{C}_n^{\mathrm{CM}}$ will stand for a set of all matrices of order n with the index at most one, i.e.

$$\mathbb{C}_n^{\mathrm{CM}} = \{A \in \mathbb{C}^{n\times n} : \mathrm{Ind}(A) \leq 1)\}.$$

The group inverse of $A \in \mathbb{C}_n^{\mathrm{CM}}$, introduced in [11], is the unique matrix $G \in \mathbb{C}^{n\times n}$ such that

$$(1)\ AGA = A, \quad (2)\ GAG = G, \quad (5)\ GA = AG,$$

denoted by $A^{\#}$. Based on the matrices with the index at most one, Baksalary and Trenkler [6] proposed a new generalized inverse, known as core inverse. For a matrix $A \in \mathbb{C}_n^{\mathrm{CM}}$, the unique matrix $G \in \mathbb{C}^{n\times n}$ with

$$AG = AA^{\dagger} \text{ and } \mathcal{R}(G) \subseteq \mathcal{R}(A),$$

is called the core inverse of A and denoted by $A^{\circledSharp}$. Replacing A by A^*, the dual core inverse of $A \in \mathbb{C}_n^{\mathrm{CM}}$ is defined in the same paper [6], as the unique matrix $G \in \mathbb{C}^{n\times n}$ such that

$$GA = A^{\dagger}A \text{ and } \mathcal{R}(G) \subseteq \mathcal{R}(A^*),$$

denoted by $A_{\circledSharp}$.

The symbols, $\mathbb{C}_{m,n}^{\mathrm{PI}}$ and $\mathbb{C}_{m,n}^{\mathrm{CA}}$ stand for the sets consisted of partial isometries and contractions, respectively, i.e.,

$$\mathbb{C}_{m,n}^{\mathrm{PI}} = \{A \in \mathbb{C}^{m\times n} : AA^*A = A\} = \{A \in \mathbb{C}^{m\times n} : A^{\dagger} = A^*\}, \tag{1.1}$$

$$\mathbb{C}_{m,n}^{\mathrm{CA}} = \{A \in \mathbb{C}^{m\times n} : \|Ax\| \leq \|x\| \text{ for all } x \in \mathbb{C}^n\}, \tag{1.2}$$

where $\|\cdot\|$ denotes the 2-norm of a vector. Also, $\mathbb{C}_n^{\mathrm{N}}$, $\mathbb{C}_n^{(k+2)\text{-P}}$, $\mathbb{C}_n^{\mathrm{SD}}$, $\mathbb{C}_n^{\mathrm{EP}}$ and $\mathbb{C}_n^{\text{bi-EP}}$ stand for the sets consisting of normal, $(k + 2)$-potent, star-dagger, EP and bi-EP matrices, respectively, i.e.,

$$\mathbb{C}_n^{\mathrm{N}} = \{A \in \mathbb{C}^{n\times n} : AA^* = A^*A\}, \tag{1.3}$$

$$\mathbb{C}_n^{(k+2)\text{-P}} = \{A \in \mathbb{C}^{n\times n} : A^{k+2} = A\}, \text{where } k \text{ is a nonnegative integer}, \tag{1.4}$$

$$\mathbb{C}_n^{\mathrm{SD}} = \{A \in \mathbb{C}^{n\times n} : A^{\dagger}A^* = A^*A^{\dagger}\}, \tag{1.5}$$

$$\mathbb{C}_n^{\mathrm{EP}} = \{A \in \mathbb{C}^{n\times n} : AA^{\dagger} = A^{\dagger}A\} = \{A \in \mathbb{C}^{n\times n} : \mathcal{R}(A) = \mathcal{R}(A^*)\}, \tag{1.6}$$

$$\mathbb{C}_n^{\text{bi-EP}} = \{A \in \mathbb{C}^{n\times n} : AA^{\dagger}A^{\dagger}A = A^{\dagger}AAA^{\dagger}\}. \tag{1.7}$$

For $m \in \mathbb{N}^+$, the sets of all m-EP matrices and m-normal matrices are defined by the following:

$$\mathbb{C}_n^{m\text{-EP}} = \{A \in \mathbb{C}^{n\times n} : A^mA^{\dagger} = A^{\dagger}A^m\} \text{ and } \mathbb{C}_n^{m\text{-N}} = \{A \in \mathbb{C}^{n\times n} : A^mA^* = A^*A^m\}. \tag{1.8}$$

In 1997, Groß and Trenkler [13] introduced generalized and hypergeneralized projectors: a generalized projector is a square matrix A such that $A^2 = A^*$, while a hypergeneralized projector is a square matrix A such that $A^2 = A^{\dagger}$. Later, in [1–4], different properties and characterizations of generalized and hypergeneralized projectors are given and finally generalized by Benítez and Tošić [8, 18] who introduced k-generalized and k-hypergeneralized projectors defined by the following:

$$\mathbb{C}_n^{k\text{-GP}} = \{A \in \mathbb{C}^{n\times n} : A^k = A^*\} \text{ and } \mathbb{C}_n^{k\text{-HGP}} = \{A \in \mathbb{C}^{n\times n} : A^k = A^{\dagger}\}, \tag{1.9}$$

where $k \in \mathbb{N}^+$ and $k \geq 2$.

Different topics related to k-generalized and k-hypergeneralized projectors have been investigated extensively in the past two decades. Deng, Li and Du [10] introduced a k-generalized and k-hypergeneralized projector on a Hilbert space and presented their several characterizations. Zhu and Liu [19] proved that a linear combination of two k-hypergeneralized projectors is still a k-hypergeneralized projector under given certain conditions while Fu and Liu [12] presented the group inverse in terms of a linear combination of k-hypergeneralized projectors. Using the spectral theorem for normal operators on Hilbert spaces, some interesting characterizations of k-generalized projectors were given in [15].

Inspired by the above mentioned results of generalized, hypergeneralized, k-generalized, and k-hypergeneralized projectors, we will present some new results:

- Certain characterizations of k-generalized and k-hypergeneralized projectors are given in terms of the Moore-Penrose, group, and core inverse of a matrix A, as well as appropriate matrix expressions.
- Several characterizations of the classes of k-generalized and k-hypergeneralized projectors are captured using various matrix classes, such as normal, EP, bi-EP, m-EP and m-normal matrices, etc..
- Relationships between k-generalized and k-hypergeneralized projectors are discussed.

2. Preliminaries

In this section, we will recall some useful results to study characterizations of k-generalized and k-hypergeneralized projectors. We begin with a well-known decomposition of square matrices.

Lemma 2.1. [14] *(H-S decomposition) Let $A \in \mathbb{C}^{n\times n}$ and $r(A) = r$. Then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that*

$$A = U\begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^*, \tag{2.1}$$

where $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r)$ is the diagonal matrix of singular values of A, $\sigma_i > 0$, $i = \overline{1, r}$, $K \in \mathbb{C}^{r\times r}$, $L \in \mathbb{C}^{r\times(n-r)}$ and

$$KK^* + LL^* = I_r. \tag{2.2}$$

Using the above mentioned H-S decomposition, the Moore-Penrose and group inverse can be represented as follows.

Lemma 2.2. [5] *Let A be given by* (2.1)*. The following statements hold:*

(1) *The Moore-Penrose inverse of A is given by*

$$A^\dagger = U\begin{bmatrix} K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1} & 0 \end{bmatrix} U^*. \tag{2.3}$$

(2) *The group inverse of A exists if and only if K is nonsingular. In this case*

$$A^\# = U\begin{bmatrix} K^{-1}\Sigma^{-1} & K^{-1}\Sigma^{-1}K^{-1}L \\ 0 & 0 \end{bmatrix} U^*.$$

Using the H-S decomposition, the following seven classes of matrices can be characterized:

Lemma 2.3. [3] *Let A be given by* (2.1)*. Then*
(a) $A \in \mathbb{C}_{n,n}^{\mathrm{PI}} \Leftrightarrow \Sigma = I_r$.
(b) $A \in \mathbb{C}_{n,n}^{\mathrm{CA}} \Leftrightarrow I_r - \Sigma^2 = CC^*$ *for some* $C \in \mathbb{C}^{r\times r}$.
(c) $A \in \mathbb{C}_n^{\mathrm{N}} \Leftrightarrow L = 0$ *and* $K\Sigma = \Sigma K$.
(d) $A \in \mathbb{C}_n^{(k+2)\text{-P}} \Leftrightarrow (\Sigma K)^{k+1} = I_r$.
(e) $A \in \mathbb{C}_n^{\mathrm{SD}} \Leftrightarrow K\Sigma = \Sigma K$.
(f) $A \in \mathbb{C}_n^{\mathrm{EP}} \Leftrightarrow L = 0$.
(g) $A \in \mathbb{C}_n^{\text{bi-EP}} \Leftrightarrow L^*K = 0$.

The following two lemmas provide characterizations of $A \in \mathbb{C}^{n\times n}$ being a k-generalized and a k-hypergeneralized projector.

Lemma 2.4. [8] *Let $A \in \mathbb{C}^{n\times n}$ and $r(A) = r$. Then the following statements are equivalent:*
(*a*) *A is a k-generalized projector.*
(*b*) *A is a normal matrix and $\delta(A) \subseteq \{0\} \cup \Phi_{k+1}$.*
(*c*) *A is a normal matrix and $A^{k+2} = A$.*
(*d*) *A can be expressed as*

$$A = U\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}U^*,$$

where U is unitary and $D \in \mathbb{C}^{r\times r}$ is a diagonal matrix such that $D^{k+1} = I_r$.

Lemma 2.5. [18] *Let $A \in \mathbb{C}^{n\times n}$ and $r(A) = r$. Then the following statements are equivalent:*
(*a*) *A is a k-hypergeneralized projector.*
(*b*) *A is a* EP *matrix, $\delta(A) \subseteq \{0\} \cup \Phi_{k+1}$ and A is diagonalizable.*
(*c*) *A is a* EP *matrix and $A^{k+2} = A$.*
(*d*) *A has the following representation*

$$A = U\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}U^*,$$

where U is unitary and $D \in \mathbb{C}^{r\times r}$ is a nonsingular matrix such that $D^{k+1} = I_r$.

It follows from [8] that

$$A^{\#} = A^{\dagger} = A^* = A^k, \tag{2.4}$$

whenever A is a k-generalized projector, and as we will see in the next lemma, the condition (2.4) is sufficient for a matrix $A \in \mathbb{C}_n^{\mathrm{CM}}$ to be a k-generalized projector.

Lemma 2.6. *Let $A \in \mathbb{C}^{n\times n}$. Then $A \in \mathbb{C}_n^{k\text{-GP}}$ if and only if $A^{\#} = A^{\dagger} = A^* = A^k$ and $A \in \mathbb{C}_n^{\mathrm{CM}}$.*

An analogous result for k-hypergeneralized projectors is provided as follows.

Lemma 2.7. [18] *Let $A \in \mathbb{C}^{n\times n}$. Then $A \in \mathbb{C}_n^{k\text{-HGP}}$ if and only if $A^{\#} = A^{\dagger} = A^k$ and $A \in \mathbb{C}_n^{\mathrm{CM}}$.*

The following auxiliary lemma will be exploited to establish some characterizations of the classes of k-generalized and k-hypergeneralized projectors in terms of m-EP and m-normal matrices.

Lemma 2.8. [16] *Let m be a positive integer and A be given by* (2.1)*. Then*
(*a*) *$A \in \mathbb{C}_n^{m\text{-EP}}$ if and only if*

$$K^*K(\Sigma K)^{m-1} = (\Sigma K)^{m-1},\ L^*\Sigma^{-1}(\Sigma K)^{m-1} = 0 \text{ and } (\Sigma K)^{m-1}\Sigma L = 0. \tag{2.5}$$

(*b*) *$A \in \mathbb{C}_n^{m\text{-N}}$ if and only if*

$$(\Sigma L)^*(\Sigma K)^{m-1} = 0,\ (\Sigma K)^{m-1}\Sigma L \text{ and } (\Sigma K)^{m-1}\Sigma^2 = (\Sigma K)^*(\Sigma K)^m. \tag{2.6}$$

3. Characterizations of *k*-generalized projectors

In this section, we will represent certain new characterizations of k-generalized projectors. The following auxiliary result is a particular version of the H-S decomposition for a k-generalized projector and will be exploited to establish some of the assertions to come.

Theorem 3.1. *Let $A \in \mathbb{C}^{n\times n}$ be given by* (2.1)*. Then $A \in \mathbb{C}_n^{k\text{-GP}}$ if and only if $L = 0$, $\Sigma = I_r$ and $K^{k+1} = I_r$.*

Proof. (⇐) : *It follows by Lemma 2.3 and Lemma 2.4.*
(⇒) : *By Lemma 2.3 and Lemma 2.4, we have*

$$L = 0,\ \Sigma K = K\Sigma \ \text{ and } \ (\Sigma K)^{k+1} = I_r. \tag{3.1}$$

From $L = 0$ *and (2.2), we get* $K^* = K^{-1}$. *Also, by (3.1), we have that*

$$\Sigma^{k+1} = K^{-(k+1)} = (K^*)^{k+1}. \tag{3.2}$$

By taking the conjugate transpose of (3.2), we obtain

$$\Sigma^{k+1} = (\Sigma^{k+1})^* = (K^{-(k+1)})^* = K^{k+1} = \Sigma^{-(k+1)},$$

which implies $\Sigma = I_r$. *Hence* $L = 0,\ \Sigma = I_r$ *and* $K^{k+1} = I_r$. □

Theorem 5 in [2] and (2.18) in [2] as well as Theorem 2 in [3] established some necessary and sufficient conditions for a matrix $A \in \mathbb{C}^{n\times n}$ to be a generalized projector in terms of its conjugate transpose, Moore-Penrose inverse and group inverse. The next theorem shows that the corresponding equivalences remain valid also in the case when A is a k-generalized projector.

Theorem 3.2. *Let* $A \in \mathbb{C}^{n\times n}$. *The following statements are equivalent:*
(a) $A \in \mathbb{C}_n^{k\text{-GP}}$.
(b) $A^* \in \mathbb{C}_n^{k\text{-GP}}$.
(c) $A^\dagger \in \mathbb{C}_n^{k\text{-GP}}$.
(d) $A^\# \in \mathbb{C}_n^{k\text{-GP}}$ *and* $A \in \mathbb{C}_n^{\text{CM}}$.

Proof. $(a) \Leftrightarrow (b)$: *The proof follows from the equality* $(A^*)^k = (A^k)^*$.
$(a) \Rightarrow (c)$: *According to* (d) *of Lemma 2.4, we have*

$$A^\dagger = U\begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*,$$

where D^{-1} *is a diagonal matrix and* $(D^{-1})^{k+1} = I_r$. *Now, the implication follows straightforwardly from* $(d) \Rightarrow (a)$ *of Lemma 2.4.*
$(c) \Rightarrow (a)$: *The implication follows if we replace* A *by* $A^\dagger$ *in the proof of* $(a) \Rightarrow (c)$.
$(a) \Leftrightarrow (d)$: *This follows similarly as in the part* $(a) \Leftrightarrow (c)$. □

Remark 3.3. *If we take* $k = 2$ *in Theorem 3.2, we will obtain* (2.18) *from* [2] *and Theorem* 5 *from* [2], *as well as Theorem* 2 *from* [3].

The following theorem provides characterizations of $A \in \mathbb{C}^{n\times n}$ being a k-generalized projector in terms of the following equalities: $A^{k+1} = AA^*$, $A^{k+1} = A^*A$, $A^*A^{k+1} = A^*AA^*$ and $A^{k+1}A^* = A^*AA^*$.

Theorem 3.4. *Let* $A \in \mathbb{C}^{n\times n}$ *with* $k \in \mathbb{N}^+$ *and* $k \geq 2$. *The following statements are equivalent:*
(a) $A \in \mathbb{C}_n^{k\text{-GP}}$.
(b) $A^{k+1} = AA^*$.
(c) $A^{k+1} = A^*A$.
(d) $A^*A^{k+1} = A^*AA^*$.
(e) $A^{k+1}A^* = A^*AA^*$.

Proof. The implications $(a) \Rightarrow (b)$, $(a) \Rightarrow (c)$, $(a) \Rightarrow (d)$ *and* $(a) \Rightarrow (e)$ *follow by direct verification.*
$(b) \Rightarrow (a)$: *Suppose that* $A^{k+1} = AA^*$. *Then by* (2.1), *we have that*

$$(\Sigma K)^{k+1} = \Sigma K(\Sigma K)^* + \Sigma L(\Sigma L)^* \ \text{ and } \ (\Sigma K)^k(\Sigma L) = 0.$$

By simple computations, we obtain $\Sigma^2 = (\Sigma K)^{k+1}$. *It can be deduced that* K *is nonsingular, hence (recall that* Σ *is always nonsingular) from* $(\Sigma K)^k(\Sigma L) = 0$ *we infer* $L = 0$. *From (2.2) and the fact that* $L = 0$, *we get* $K^* = K^{-1}$. *Since*

$\Sigma^2 = (\Sigma K)^{k+1} = \Sigma K(\Sigma K)^k$, *it follows that* $K^{-1}\Sigma = (\Sigma K)^k$. *Thus,* $(\Sigma K)^* = (\Sigma K)^k$. *Now,* $L = 0$ *and* $(\Sigma K)^* = (\Sigma K)^k$ *imply that* $A^k = A^*$, *i.e.* $A \in \mathbb{C}_n^{k\text{-GP}}$.

$(c) \Rightarrow (a)$: *Suppose that* $A^{k+1} = A^*A$. *Taking the conjugate of* $A^{k+1} = A^*A$, *we obtain* $(A^*)^{k+1} = A^*A$ *which imples by the implication* $(b) \Rightarrow (a)$, *that* $A^* \in \mathbb{C}_n^{k\text{-GP}}$. *Now, by Theorem 3.2, we get that* $A \in \mathbb{C}_n^{k\text{-GP}}$.

$(d) \Rightarrow (a)$: *Suppose that* $A^*A^{k+1} = A^*AA^*$. *From* $A^*A^{k+1} = A^*AA^*$ *and (2.1), we have*

$$K^*\Sigma^3 = K^*\Sigma(\Sigma K)^{k+1}, \quad L^*\Sigma^3 = L^*\Sigma(\Sigma K)^{k+1}, \tag{3.3}$$

$$K^*\Sigma(\Sigma K)^k\Sigma L = 0 \ \text{ and } \ L^*\Sigma(\Sigma K)^k\Sigma L = 0. \tag{3.4}$$

Now, by multiplying the first and the second equalities of (3.3)*, from the left side by K and L, respectively, we get*

$$KK^*\Sigma^3 = KK^*\Sigma(\Sigma K)^{k+1} \ \text{ and } \ LL^*\Sigma^3 = LL^*\Sigma(\Sigma K)^{k+1},$$

which by (2.2)*, implies that* $\Sigma^2 = (\Sigma K)^{k+1}$. *Thus K is nonsingular and by* (3.4) *we get* $L = 0$. *The rest of the proof follows as in the part* $(b) \Rightarrow (a)$.

$(e) \Rightarrow (a)$: *Suppose that* $A^{k+1}A^* = A^*AA^*$. *By taking the conjugate of* $A^{k+1}A^* = A^*AA^*$, *we get* $(A^*)^*(A^*)^{k+1} = AA^*A$, *which implies by* $(d) \Rightarrow (a)$ *that* $A^* \in \mathbb{C}_n^{k\text{-GP}}$. *Now, from Theorem 3.2 it follows that* $A \in \mathbb{C}_n^{k\text{-GP}}$. □

The following theorem provides characterizations of $A \in \mathbb{C}^{n\times n}$ being a k-generalized projector in terms of the powers of the Moore-Penrose inverse and the group inverse of A.

Theorem 3.5. *Let* $A \in \mathbb{C}^{n\times n}$ *and let* m, l, k *be nonnegative integers such that* $l \geq k - m + 1$. *Then the following statements are equivalent:*

(a) $A \in \mathbb{C}_n^{k\text{-GP}}$.

(b) $A^m = A^*(A^\dagger)^l A^{m+l-k}$ *and* $A \in \mathbb{C}_n^{\mathrm{CM}}$.

(c) $A^m = A^*(A^\#)^l A^{m+l-k}$ *and* $A \in \mathbb{C}_n^{\mathrm{CM}}$.

Proof. *The implications* $(a) \Rightarrow (b)$ *and* $(a) \Rightarrow (c)$ *follow by calculations from Lemma 2.2.*

$(b) \Rightarrow (a)$: *Suppose that* $A^m = A^*(A^\dagger)^l A^{m+l-k}$. *Evidently,* $\mathcal{R}(A^m) \subseteq \mathcal{R}(A^*)$ *which together with* $r(A) = r(A^2)$, *gives* $\mathcal{R}(A) = \mathcal{R}(A^m) \subseteq \mathcal{R}(A^*)$. *Thus* $\mathcal{R}(A) = \mathcal{R}(A^*)$. *From* (f) *of Lemma 2.3, we have* $L = 0$, *which implies that* $K^* = K^{-1}$ *and* ΣK *is nonsingular. Hence, the assumption* $A^m = A^*(A^\dagger)^l A^{m+l+k}$ *gives*

$$(\Sigma K)^m = (\Sigma K)^*(\Sigma K)^{-l}(\Sigma K)^{m+l-k}.$$

Therefore, we have $(\Sigma K)^* = (\Sigma K)^k$, *which together with* $L = 0$ *yields* $A^k = A^*$.

$(c) \Rightarrow (a)$: *Note that the condition* $A \in \mathbb{C}_n^{\mathrm{CM}}$ *implies the existence of* $A^\#$. *This follows similarly as in the part* $(b) \Rightarrow (a)$. □

The example provided below shows that Theorem 3.5 is not valid without the assumption that $A \in \mathbb{C}_n^{\mathrm{CM}}$ in its items (b).

Example 3.6. *Let* $m = k = l = 2$ *and let*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is easy to verify that $A^2 = A^*(A^\dagger)^2A^2$, $A \notin \mathbb{C}_n^{\mathrm{CM}}$ *and* $A \notin \mathbb{C}_n^{2\text{-GP}}$.

Theorem 2 in [1] provides certain characterizations of a generalized projector in terms of its conjugate transpose, Moore-Penrose inverse, and group inverse. The generalization of this result for the case of a k-generalized projector is given in the following theorem.

Theorem 3.7. *Let* $A \in \mathbb{C}^{n\times n}$ *with* $k \in \mathbb{N}^+$ *and* $k \geq 2$. *Then the following statements are equivalent:*

(a) $A \in \mathbb{C}_n^{k\text{-GP}}$.

(b) $A^{k-1} = A^*A^\dagger$ *and* $A \in \mathbb{C}_n^{\mathrm{CM}}$.

(c) $A^{k-1} = A^\dagger A^*$ *and* $A \in \mathbb{C}_n^{\mathrm{CM}}$.

(d) $A^{k-1} = A^*A^\#$ *and* $A \in \mathbb{C}_n^{\mathrm{CM}}$.

(e) $A^{k-1} = A^\#A^*$ *and* $A \in \mathbb{C}_n^{\mathrm{CM}}$.

Proof. $(a) \Rightarrow (b)$: *Suppose that* $A \in \mathbb{C}_n^{k\text{-GP}}$. *From Lemma 2.6 we get that* $A \in \mathbb{C}_n^{\mathrm{CM}}$ *and* $A^*A^\dagger = A^{2k} = A^{k-1}A^\dagger A = A^{k-1}$.

$(b) \Rightarrow (a)$: *Suppose that* $A^{k-1} = A^*A^\dagger$ *and* $A \in \mathbb{C}_n^{\mathrm{CM}}$. *From* $A^{k-1} = A^*A^\dagger$ *and* $Ind(A) \leq 1$, *we get that*

$$\mathcal{R}(A) = \mathcal{R}(A^{k-1}) = \mathcal{R}(A^*A^\dagger) \subseteq \mathcal{R}(A^*).$$

Hence $\mathcal{R}(A) = \mathcal{R}(A^*)$, *i.e.,* $AA^\dagger = A^\dagger A$. *Multiplying* $A^{k-1} = A^*A^\dagger$ *by* A^2 *from the right, gives* $A^{k+1} = A^*A$. *Now by Theorem 3.4, we have* $A \in \mathbb{C}_n^{k\text{-GP}}$.

$(a) \Leftrightarrow (c)$: *This follows similarly as in the part* $(a) \Leftrightarrow (b)$.

$(a) \Rightarrow (d)$: *Suppose that* $A \in \mathbb{C}_n^{k\text{-GP}}$. *By Lemma 2.6 we get* $A^*A^\# = A^{2k} = A^{k-1}$.

$(d) \Rightarrow (a)$: *Multiplying* $A^{k-1} = A^*A^\#$ *by* A^2 *from the right, we obtain* $A^{k+1} = A^*A$. *Now, from Theorem 3.4 we get* $A \in \mathbb{C}_n^{k\text{-GP}}$.

$(a) \Leftrightarrow (e)$: *This follows similarly as in the part* $(a) \Leftrightarrow (d)$. □

Remark 3.8. *The case* $k = 2$ *in Theorem 3.7, is exactly Theorem* 2 *given in* [1].

The example below shows that the equivalences established in Theorem 3.7, are not valid if we remove the assumption that $A \in \mathbb{C}_n^{\mathrm{CM}}$ in items $(b) - (e)$.

Example 3.9. *Let* $k = 3$ *and*

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

We can verify that $A^2 = A^*A^\dagger$, $A^2 = A^\dagger A^*$, $A \notin \mathbb{C}_n^{\mathrm{CM}}$ *and* $A \notin \mathbb{C}_n^{3\text{-GP}}$.

The properties of the class $\mathbb{C}_n^{2\text{-GP}}$ in terms of the matrix classes $\mathbb{C}_n^{\mathrm{PI}}$, $\mathbb{C}_n^{\mathrm{CA}}$, $\mathbb{C}_n^{4\text{-P}}$, $\mathbb{C}_n^{\mathrm{SD}}$ and $\mathbb{C}_n^{\text{bi-EP}}$ are given in [3]. In the next theorem we show a similar result for the class $\mathbb{C}_n^{k\text{-GP}}$.

Theorem 3.10. *The following statements hold:*

(a) $\mathbb{C}_n^{k\text{-GP}} = \mathbb{C}_n^{\mathrm{PI}} \cap \mathbb{C}_n^{(k+2)\text{-P}} \cap \mathbb{C}_n^{\text{bi-EP}}$.

(b) $\mathbb{C}_n^{k\text{-GP}} = \mathbb{C}_n^{\mathrm{SD}} \cap \mathbb{C}_n^{(k+2)\text{-P}} \cap \mathbb{C}_n^{\text{bi-EP}}$.

(c) $\mathbb{C}_n^{k\text{-GP}} = \mathbb{C}_n^{\mathrm{CA}} \cap \mathbb{C}_n^{(k+2)\text{-P}} \cap \mathbb{C}_n^{\text{bi-EP}}$.

Proof. *By Theorem 3.1 and Lemma 2.3 we have that* $A \in \mathbb{C}_n^{k\text{-GP}}$ *is a subset of the following sets:*

$$\mathbb{C}_n^{\mathrm{PI}} \cap \mathbb{C}_n^{(k+2)\text{-P}} \cap \mathbb{C}_n^{\text{bi-EP}},\ \mathbb{C}_n^{\mathrm{SD}} \cap \mathbb{C}_n^{(k+2)\text{-P}} \cap \mathbb{C}_n^{\text{bi-EP}}\ \text{ and } \mathbb{C}_n^{\mathrm{CA}} \cap \mathbb{C}_n^{(k+2)\text{-P}} \cap \mathbb{C}_n^{\text{bi-EP}}.$$

So, we need to prove the reverse inclusion in the three items.

(a) *Let* $A \in \mathbb{C}_n^{\mathrm{PI}} \cap \mathbb{C}_n^{(k+2)\text{-P}} \cap \mathbb{C}_n^{\text{bi-EP}}$. *By (a), (d) and (g) of Lemma 2.3, we get that*

$$\Sigma = I_r,\ (\Sigma K)^{k+1} = I_r\ \text{ and }\ L^*K = 0.$$

By the first and the second equality above, it follows that K *is nonsingular and* $K^{k+1} = I_r$. *Also, by the third one and the nonsingularity of* K, *it follows that* $L = 0$. *Now, by Theorem 3.1 we have* $A \in \mathbb{C}_n^{k\text{-GP}}$.

(b) *Let* $A \in \mathbb{C}_n^{\mathrm{SD}} \cap \mathbb{C}_n^{(k+2)\text{-P}} \cap \mathbb{C}_n^{\text{bi-EP}}$. *By (e), (d) and (g) of Lemma 2.3, it follows that*

$$\Sigma K = K\Sigma,\ (\Sigma K)^{k+1} = I_r\ \text{ and }\ L^*K = 0. \tag{3.5}$$

Hence K *is nonsingular and* $L = 0$. *Now by Lemma 2.4 and Theorem 3.1, it follows that* $A \in \mathbb{C}_n^{k\text{-GP}}$.

(c) *Let* $A \in \mathbb{C}_n^{\mathrm{CA}} \cap \mathbb{C}_n^{(k+2)\text{-P}} \cap \mathbb{C}_n^{\text{bi-EP}}$. *By (d) and (g) of Lemma 2.3, we have*

$$(\Sigma K)^{k+1} = I_r\ \text{ and }\ L^*K = 0,$$

which implies that $L = 0$. Also, by (2.2) we have that $K^* = K^{-1}$ and by (b) of Lemma 2.3 it follows that $I_r - \Sigma^2 = I_r - \Sigma K(\Sigma K)^*$ is positive semi-definite. Thus $tr(I_r - \Sigma K(\Sigma K)^*) \geq 0$, i.e.

$$tr(\Sigma K(\Sigma K)^*) \leq r. \tag{3.6}$$

Let $\lambda_1, \lambda_2 \cdots \lambda_r$ be the eigenvalues of ΣK. Since $(\Sigma K)^{k+1} = I_r$ we have that $|\lambda_i| = 1$, $i = \overline{1, r}$. Now, by Schur's lemma, ΣK can be expressed as

$$\Sigma K = V \begin{bmatrix} \lambda_1 & t_{12} & \cdots & t_{1r} \\ 0 & \lambda_2 & \cdots & t_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_r \end{bmatrix} V^*, \tag{3.7}$$

for some unitary matrix V. From (3.6) and (3.7), we have

$$tr(\Sigma K(\Sigma K)^*) = |\lambda_1|^2 + |\lambda_2|^2 + \cdots + |\lambda_r|^2 + \sum_{1 \leq i < j \leq r} |t_{ij}|^2 \leq r. \tag{3.8}$$

By (3.8) and $|\lambda_i| = 1, i = \overline{1, r}$, we obtain that

$$\sum_{1 \leq i < j \leq r} |t_{ij}|^2 = 0 \Rightarrow t_{ij} = 0,\ i, j = \overline{1, r}, i \neq j.$$

Using (3.7), we get

$$\Sigma^2 = \Sigma K(\Sigma K)^* = V \begin{bmatrix} \lambda_1\overline{\lambda}_1 & 0 & \cdots & 0 \\ 0 & \lambda_2\overline{\lambda}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_r\overline{\lambda}_r \end{bmatrix} V^* = I_r.$$

Substituting $\Sigma = I_r$ into $(\Sigma K)^{k+1} = I_r$ gives $K^{k+1} = I_r$. Now, by Theorem 3.1 we have that $\mathbb{C}_n^{k\text{-GP}}$. □

Remark 3.11. *The case $k = 2$ in Theorem 3.10 contains the results from Theorem 3, Theorem 4 and (2.7) in [3].*

Certain descriptions of $\mathbb{C}_n^{k\text{-GP}}$ related to different classes of matrices can be found in the following theorem.

Theorem 3.12. *The following statements hold:*
(a) $\mathbb{C}_n^{k\text{-GP}} = \mathbb{C}_n^{\text{PI}} \cap \mathbb{C}_n^{(k+2)\text{-P}} \cap \mathbb{C}_n^{m\text{-EP}}$.
(b) $\mathbb{C}_n^{k\text{-GP}} = \mathbb{C}_n^{\text{SD}} \cap \mathbb{C}_n^{(k+2)\text{-P}} \cap \mathbb{C}_n^{m\text{-EP}}$.
(c) $\mathbb{C}_n^{k\text{-GP}} = \mathbb{C}_n^{\text{CA}} \cap \mathbb{C}_n^{(k+2)\text{-P}} \cap \mathbb{C}_n^{m\text{-EP}}$.
(d) $\mathbb{C}_n^{k\text{-GP}} = \mathbb{C}_n^{\text{PI}} \cap \mathbb{C}_n^{(k+2)\text{-P}} \cap \mathbb{C}_n^{m\text{-N}}$.
(e) $\mathbb{C}_n^{k\text{-GP}} = \mathbb{C}_n^{\text{SD}} \cap \mathbb{C}_n^{(k+2)\text{-P}} \cap \mathbb{C}_n^{m\text{-N}}$.
(f) $\mathbb{C}_n^{k\text{-GP}} = \mathbb{C}_n^{\text{CA}} \cap \mathbb{C}_n^{(k+2)\text{-P}} \cap \mathbb{C}_n^{m\text{-N}}$.

Proof. By Theorem 3.1 and Lemma 2.8, it follows that

$$\mathbb{C}_n^{k\text{-GP}} \subseteq \mathbb{C}_n^{m\text{-EP}} \quad \text{and} \quad \mathbb{C}_n^{k\text{-GP}} \subseteq \mathbb{C}_n^{m\text{-N}}. \tag{3.9}$$

(a) By Theorem 3.1, Lemma 2.3 and (3.9), we have that $\mathbb{C}_n^{\text{PI}} \cap \mathbb{C}_n^{(k+2)\text{-P}} \cap \mathbb{C}_n^{m\text{-EP}}$. To show the converse inclusion, let us suppose that $A \in \mathbb{C}_n^{\text{PI}} \cap \mathbb{C}_n^{(k+2)\text{-P}} \cap \mathbb{C}_n^{m\text{-EP}}$ and that A is given by (2.1). By (a), (d) of Lemma 2.3 and (2.5), it follows that

$$\Sigma = I_r,\ (\Sigma K)^{k+1} = I_r \ \text{ and } \ L = 0. \tag{3.10}$$

Hence by Theorem 3.1 we have that $A \in \mathbb{C}_n^{k\text{-GP}}$.

(b) *The inclusion* $\mathbb{C}_n^{k\text{-GP}} \subseteq \mathbb{C}_n^{\rm SD} \cap \mathbb{C}_n^{(k+2)\text{-P}} \cap \mathbb{C}_n^{m\text{-EP}}$ *follows from Theorem 3.1, Lemma 2.3 and (3.9). To show the converse inclusion, let us suppose that* $A \in \mathbb{C}_n^{\rm SD} \cap \mathbb{C}_n^{(k+2)\text{-P}} \cap \mathbb{C}_n^{m\text{-EP}}$ *and that A is given by* (2.1). *By Lemma 2.3 and (2.5), we have*

$$\Sigma K = K\Sigma,\ (\Sigma K)^{k+1} = I_r \ \text{ and } \ L = 0.$$

Hence, by Lemma 2.3 and Lemma 2.4 it follows that $A \in \mathbb{C}_n^{k\text{-GP}}$.

(f) *Evidently, by Theorem 3.1, Lemma 2.3 and (2.6), we have that* $\mathbb{C}_n^{k\text{-GP}} \subseteq \mathbb{C}_n^{\rm CA} \cap \mathbb{C}_n^{(k+2)\text{-P}} \cap \mathbb{C}_n^{m\text{-N}}$. *Suppose that* $A \in \mathbb{C}_n^{\rm CA} \cap \mathbb{C}_n^{(k+2)\text{-P}} \cap \mathbb{C}_n^{m\text{-N}}$. *From Lemma 2.3 and (2.6), we get that*

$$(\Sigma K)^{k+1} = I_r, (\Sigma L)^*(\Sigma K)^{m-1} = 0,$$

$$(\Sigma K)^{m-1}\Sigma L = 0 \ \text{ and } \ (\Sigma K)^{m-1}\Sigma^2 = (\Sigma K)^*(\Sigma K)^m.$$

By $(\Sigma K)^{k+1} = I_r$, *we have that* ΣK *is nonsingular. Since* $(\Sigma K)^{m-1}\Sigma L = 0$, *we have* $L = 0$, *which implies* $A \in \mathbb{C}_n^{\text{bi-EP}}$. *According to* ($c$) *of Theorem 3.10, we have* $A \in \mathbb{C}_n^{k\text{-GP}}$.

The proofs of (c), (d) *and* (e) *follow similarly.* □

Theorem 5 [3] represents necessary and sufficient conditions for the product of two generalized projectors to be a generalized projector in the case when either one of them is idempotent. In the following theorem, we will prove that the same result is valid in the case of k-generalized projectors.

Theorem 3.13. *Let* $A, B \in \mathbb{C}_n^{k\text{-GP}}$ *and let either A or B be idempotent. Then the following statements are equivalent:*
(a) $AB \in \mathbb{C}_n^{k\text{-GP}}$.
(b) $AB \in \mathbb{C}_n^{\rm N}$.
(c) $AB = BA$.

Proof. *We will assume that A is an idempotent.*

(a) ⇒ (b) : *Evidently follows.*

(b) ⇒ (c) : *Suppose that* $AB \in \mathbb{C}_n^{\rm N}$. *Since A is a k-generalized projector and idempotent, by* (h) *of Lemma 2.3 and Theorem 3.1, A can be represented as*

$$A = U\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}U^*,$$

where U is unitary. Hence B can be expressed as

$$B = U\begin{bmatrix} D & E \\ F & G \end{bmatrix}U^*,$$

where $D \in \mathbb{C}^{r\times r}$, $E \in \mathbb{C}^{r\times(n-r)}$, $F \in \mathbb{C}^{(n-r)\times r}$ *and* $G \in \mathbb{C}^{(n-r)\times(n-r)}$. *Since* $AB \in \mathbb{C}_n^{\rm N}$, *we get that*

$$DD^* + EE^* = D^*D,\ D^*E = 0,\ E^*D = 0 \ \text{ and } \ E^*E = 0.$$

From $E^*E = 0$, *we have* $r(E) = r(E^*E) = 0$, *i.e.* $E = 0$. *By* $B^* = B^k$ *we have* $F = 0$. *Hence B can be expressed as*

$$B = U\begin{bmatrix} D & 0 \\ 0 & G \end{bmatrix}U^*.$$

Evidently, $AB = BA$.

(c) ⇒ (a) : *Suppose that* $AB = BA$. *by* $A^k = A^*$, $B^k = B^*$ *and* $AB = BA$, *we have that*

$$(AB)^k = A^kB^k = A^*B^* = (BA)^* = (AB)^*.$$

Thus, $AB \in \mathbb{C}_n^{k\text{-GP}}$. □

4. Characterizations of k-hypergeneralized projectors

In this section, the H-S decomposition will be exploited to establish some characterizations of k-hypergeneralized projectors. Observe that k-generalized projectors and k-hypergeneralized projectors have some similar properties.

Theorem 4.1. *Let $A \in \mathbb{C}^{n\times n}$ be given by (2.1) and $k \in \mathbb{N}^+$ and $k \geq 2$. Then $A \in \mathbb{C}_n^{k\text{-HGP}}$ if and only if $L = 0$ and $(\Sigma K)^{k+1} = I_r$.*

Proof. ($\Leftarrow$) : *It is evident.*

($\Rightarrow$) : *Suppose that $A \in \mathbb{C}_n^{k\text{-HGP}}$. By (2.3) we get*

$$L = 0 \text{ and } (\Sigma K)^k = K^*\Sigma^{-1}.$$

Combining $L = 0$ with (2.2), we get $K^* = K^{-1}$. Hence $(\Sigma K)^{k+1} = I_r$. □

Remark 4.2. *According to Theorem 3.1 and Theorem 4.1, we have that $\mathbb{C}_n^{k\text{-GP}} \subseteq \mathbb{C}_n^{k\text{-HGP}}$.*

In Theorem 2.6 in [18], we have the following equivalences:

$$A \in \mathbb{C}_n^{k\text{-HGP}} \Leftrightarrow A^* \in \mathbb{C}_n^{k\text{-HGP}} \Leftrightarrow A^\dagger \in \mathbb{C}_n^{k\text{-HGP}}. \tag{4.1}$$

In the following theorem, we present a similar equivalence related with the group inverse of A.

Theorem 4.3. *Let $A \in \mathbb{C}^{n\times n}$. The following statements are equivalent:*

(a) $A \in \mathbb{C}_n^{k\text{-HGP}}$.

(b) $A^\# \in \mathbb{C}_n^{k\text{-HGP}}$ *and* $A \in \mathbb{C}_n^{\mathrm{CM}}$.

Proof. (a) $\Rightarrow$ (b) : *From Lemma 2.7, we get $A^\# = A^\dagger$. Hence by* (4.1) *we have $A^\# \in \mathbb{C}_n^{k\text{-HGP}}$.*

(b) $\Rightarrow$ (a) : *Using that $(A^\#)^\# = A$ and the implication (a) $\Rightarrow$ (b) we have that*

$$A^\# \in \mathbb{C}_n^{k\text{-HGP}} \Rightarrow (A^\#)^\# \in \mathbb{C}_n^{k\text{-HGP}} \Rightarrow A \in \mathbb{C}_n^{k\text{-HGP}}.$$

□

Analogously as in Theorem 3.4, we give several characterizations of k-hypergeneralized projectors in terms of the following equalities: $A^{k+1} = A^\dagger A$, $A^{k+1} = AA^\dagger$, $A^{k+1}A^\dagger = A^\dagger$ and $A^\dagger A^{k+1} = A^\dagger$.

Theorem 4.4. *Let $A \in \mathbb{C}^{n\times n}$ with $k \in \mathbb{N}^+$ and $k \geq 2$. The following statements are equivalent:*

(a) $A \in \mathbb{C}_n^{k\text{-HGP}}$.

(b) $A^{k+1} = A^\dagger A$.

(c) $A^{k+1} = AA^\dagger$.

(d) $A^{k+1}A^\dagger = A^\dagger$.

(e) $A^\dagger A^{k+1} = A^\dagger$.

Proof. The implications (a) $\Rightarrow$ (b), (a) $\Rightarrow$ (c), *as well as equivalences* (b) $\Leftrightarrow$ (d) *and* (c) $\Leftrightarrow$ (e) *follow evidently.*

(b) $\Rightarrow$ (a) : *Suppose that $A^{k+1} = A^\dagger A$. Then*

$$AA^kA = AA^{k+1} = AA^\dagger A = A,$$
$$A^kAA^k = A^kA^{k+1} = A^kA^\dagger A = A^{k-1}AA^\dagger A = A^k.$$

Also, AA^k and A^kA are Hermitian. Thus $A^k = A^\dagger$, i.e. $A \in \mathbb{C}_n^{k\text{-HGP}}$.

(c) $\Rightarrow$ (a) : *This follows similarly as in the part* (b) $\Rightarrow$ (a).

□

The next theorem represents several characterizations of k-hypergenerali-zed projectors in terms of certain equalities related to the Moore-Penrose and group inverse of a matrix $A \in \mathbb{C}^{n\times n}$.

Theorem 4.5. *Let $A \in \mathbb{C}^{n\times n}$ with $k \in \mathbb{N}^+$ and $k \geq 2$. Then the following statements are equivalent:*
(a) $A \in \mathbb{C}_n^{k\text{-HGP}}$.
(b) $A^{k-1} = A^\dagger A^\#$ *and* $A \in \mathbb{C}_n^{\mathrm{CM}}$.
(c) $A^{k-1} = A^\# A^\dagger$ *and* $A \in \mathbb{C}_n^{\mathrm{CM}}$.
(d) $A^k = A^\dagger A A^\#$ *and* $A \in \mathbb{C}_n^{\mathrm{CM}}$.
(e) $A^k = A^\# A A^\dagger$ *and* $A \in \mathbb{C}_n^{\mathrm{CM}}$.
(f) $A = (A^\dagger)^k$.

Proof. (a) ⇒ (b) : *Let $A \in \mathbb{C}_n^{k\text{-HGP}}$. Then $A^{k+2} = A$, so by Lemma 2.7, we have*

$$A^\dagger A^\# = A^{2k} = A^{k-2}A^{k+2} = A^{k-2}A = A^{k-1}.$$

(b) ⇒ (a) : *Multiplying $A^{k-1} = A^\dagger A^\#$ by A^2 from the right, we get $A^{k+1} = A^\dagger A$. Now by Theorem 4.4, it follows that $A \in \mathbb{C}_n^{k\text{-HGP}}$.*
(a) ⇔ (c) : *This follows similarly as in the part (a) ⇔ (b).*
(b) ⇒ (d) : *It is evident.*
(d) ⇒ (a) : *From $A^k = A^\dagger A A^\#$, we obtain that $A^{k+1} = A^\dagger A$. Thus, $A \in \mathbb{C}_n^{k\text{-HGP}}$ according to Theorem 4.4.*
(a) ⇔ (e): *This follows similarly as in the part (a) ⇔ (d).*
(a) ⇒ (f) : *Evidently, by $A^\dagger = A^k$ we have*

$$(A^\dagger)^k = A^{k^2} = (A^{k+2})^{k-2}A^4 = A^{k-2}A^4 = A^{k+2} = A.$$

(f) ⇒ (a) : *Since $A = (A^\dagger)^k$, it follows that $A \in \mathbb{C}_n^{\mathrm{EP}}$. Multiplying $A = (A^\dagger)^k$ by A^{k+1} from the right, we get $A^{k+2} = (A^\dagger)^k A^{k+1} = A$. Now, by Lemma 2.5 we have that $A \in \mathbb{C}_n^{k\text{-HGP}}$.* □

Remark 4.6. *If we take $k = 2$ in Theorem 4.5, we get Theorem 3 in* [1].

Remark 4.7. *According to [6], we have the following representations of the core and dual core inverses of a matrix $A \in \mathbb{C}^{n\times n}$:*

$$A^{\#\!\!\!\!\bigcirc} = A^\# A A^\dagger \text{ and } A_{\#\!\!\!\!\bigcirc} = A^\dagger A A^\#.$$

Evidently, by (e) and (d) of Theorem 4.5, we have that for a k-hypergeneralized projector A, A^k is the core and the dual core inverse of A, i.e.

$$A^k = A_{\#\!\!\!\!\bigcirc} = A^{\#\!\!\!\!\bigcirc}.$$

The next theorem gives several characterizations of k-hypergeneralized projectors in terms of the powers of the Moore-Penrose and group inverses.

Theorem 4.8. *Let $A \in \mathbb{C}^{n\times n}$ and let m, l, k be nonnegative integers such that $m + l - k \geq 1$. Then the following statements are equivalent:*
(a) $A \in \mathbb{C}_n^{k\text{-HGP}}$.
(b) $A^m = A^\dagger (A^\#)^l A^{m+l-k}$ *and* $A \in \mathbb{C}_n^{\mathrm{CM}}$.
(c) $A^m = (A^\dagger)^l A^\# A^{m+l-k}$ *and* $A \in \mathbb{C}_n^{\mathrm{CM}}$.

Proof. The implications (a) ⇒ (b) and (a) ⇒ (c) follow straightforwardly from Lemma 2.7.
(b) ⇒ (a) : *Suppose that $A^m = A^\dagger (A^\#)^l A^{m+l-k}$ and $A \in \mathbb{C}_n^{\mathrm{CM}}$. Evidently $\mathcal{R}(A^m) \subseteq \mathcal{R}(A^*)$. Since $r(A) = r(A^2)$ we have that $\mathcal{R}(A) = \mathcal{R}(A^m) \subseteq \mathcal{R}(A^*)$. Hence, $\mathcal{R}(A) = \mathcal{R}(A^*)$, i.e. $A \in \mathbb{C}_n^{\mathrm{EP}}$. Thus A can be represented by*

$$A = U \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} U^*,$$

where U is unitary and $D \in \mathbb{C}^{r\times r}$ is nonsingular. By $A^m = A^\dagger (A^\#)^l A^{m+l-k}$, we have $D^{k+1} = I_r$. Hence, from Lemma 2.5 we get $A \in \mathbb{C}_n^{k\text{-HGP}}$.
(c) ⇒ (a) : *This follows similarly as in the part (b) ⇒ (a).* □

The next theorem characterizes the class $\mathbb{C}_n^{k\text{-HGP}}$ in terms of the classes $\mathbb{C}_n^{(k+2)\text{-P}}$, $\mathbb{C}_n^{m\text{-EP}}$ and $\mathbb{C}_n^{\text{bi-EP}}$.

Theorem 4.9. *The following statements hold:*
(a) $\mathbb{C}_n^{k\text{-HGP}} = \mathbb{C}_n^{(k+2)\text{-P}} \cap \mathbb{C}_n^{m\text{-EP}}$.
(b) $\mathbb{C}_n^{k\text{-HGP}} = \mathbb{C}_n^{(k+2)\text{-P}} \cap \mathbb{C}_n^{\text{bi-EP}}$.

Proof. *(a) Using Theorem 4.1, Lemma 2.3 and (2.5), we can verify that* $\mathbb{C}_n^{k\text{-HGP}} \subseteq \mathbb{C}_n^{(k+2)\text{-P}} \cap \mathbb{C}_n^{m\text{-EP}}$. *Conversely, by* ($d$) *of Lemma 2.3 and (2.5), we have that* $A \in \mathbb{C}_n^{(k+2)\text{-P}} \cap \mathbb{C}_n^{m\text{-EP}}$ *if and only if*

$$(\Sigma K)^{k+1} = I_r, \ K^*K(\Sigma K)^{m-1} = (\Sigma K)^{m-1},$$
$$L^*\Sigma^{-1}(\Sigma K)^{m-1} = 0 \ \text{and} \ (\Sigma K)^{m-1}\Sigma L = 0.$$

Since $(\Sigma K)^{k+1} = I_r$, *it follows that* ΣK *is nonsingular. Now, by* $(\Sigma K)^{m-1}\Sigma L = 0$ *we have that* $L = 0$. *Now, from Theorem 4.1, we get that* $\mathbb{C}_n^{(k+2)\text{-P}} \cap \mathbb{C}_n^{m\text{-EP}} \subseteq \mathbb{C}_n^{k\text{-HGP}}$.

(b) *By Theorem 4.1 and Lemma 2.3, we have that* $\mathbb{C}_n^{k\text{-HGP}} \subseteq \mathbb{C}_n^{(k+2)\text{-P}} \cap \mathbb{C}_n^{\text{bi-EP}}$. *Conversely, according to* (d) *and* (g) *of Lemma 2.3, we obtain that*

$$A \in \mathbb{C}_n^{(k+2)\text{-P}} \cap \mathbb{C}_n^{\text{bi-EP}} \Rightarrow (\Sigma K)^{k+1} = I_r \ \text{and} \ L^*K = 0,$$

which implies $(\Sigma K)^{k+1} = I_r$ *and* $L = 0$. *Hence, it follows from Theorem 4.1 that* $\mathbb{C}_n^{(k+2)\text{-P}} \cap \mathbb{C}_n^{\text{bi-EP}} \subseteq \mathbb{C}_n^{k\text{-HGP}}$. □

Remark 4.10. *If we take* $k = 2$ *in* (b) *of Theorem 4.9, we obtain Theorem* 3 *from* [2].

Next theorem represents certain relations between different classes of matrices among which are classes of k-generalized and k-hypergeneralized projectors.

Theorem 4.11. *The following stetements hold:*
(a) $\mathbb{C}_n^{k\text{-GP}} = \mathbb{C}_n^{\text{PI}} \cap \mathbb{C}_n^{k\text{-HGP}}$.
(b) $\mathbb{C}_n^{k\text{-GP}} = \mathbb{C}_n^{\text{SD}} \cap \mathbb{C}_n^{k\text{-HGP}}$.
(c) $\mathbb{C}_n^{k\text{-GP}} = \mathbb{C}_n^{\text{CA}} \cap \mathbb{C}_n^{k\text{-HGP}}$.
(d) $\mathbb{C}_n^{k\text{-GP}} = \mathbb{C}_n^{\text{N}} \cap \mathbb{C}_n^{k\text{-HGP}}$.

Proof. (a) *By Theorem 3.1,* (a) *of Lemma 2.3 and Remark 4.2, it is clear that* $\mathbb{C}_n^{k\text{-GP}} \subseteq \mathbb{C}_n^{\text{PI}} \cap \mathbb{C}_n^{k\text{-HGP}}$.
Conversely, according to (a) *of Lemma 2.3 and Theorem 4.1, we have*

$$\begin{aligned} A \in \mathbb{C}_n^{\text{PI}} \cap \mathbb{C}_n^{k\text{-HGP}} &\Rightarrow \Sigma = I_r, \ L = 0 \ \text{and} \ (\Sigma K)^{k+1} = I_r \\ &\Rightarrow \Sigma = I_r, \ L = 0 \ \text{and} \ K^{k+1} = I_r. \end{aligned}$$

Then it follows from Theorem 3.1 that $\mathbb{C}_n^{\text{PI}} \cap \mathbb{C}_n^{k\text{-HGP}} \subseteq \mathbb{C}_n^{k\text{-GP}}$.

(b) *By Theorem 3.1,* (e) *of Lemma 2.3 and Remark 4.2, we have that* $\mathbb{C}_n^{k\text{-GP}} \subseteq \mathbb{C}_n^{\text{SD}} \cap \mathbb{C}_n^{k\text{-HGP}}$.
Conversely, by item (e) *of Lemma 2.3 and Theorem 4.1, it follows that*

$$A \in \mathbb{C}_n^{\text{SD}} \cap \mathbb{C}_n^{k\text{-HGP}} \Rightarrow K\Sigma = \Sigma K, \ L = 0 \ \text{and} \ (\Sigma K)^{k+1} = I_r.$$

Evidently, by Lemma 2.3 and Lemma 2.4 it follows that $A \in \mathbb{C}_n^{k\text{-GP}}$. *Hence,* $\mathbb{C}_n^{\text{SD}} \cap \mathbb{C}_n^{k\text{-HGP}} \subseteq \mathbb{C}_n^{k\text{-GP}}$.

(c) *By Theorem 3.1,* (b) *of Lemma 2.3 and Remark 4.2, it is easy to check that* $\mathbb{C}_n^{k\text{-GP}} \subseteq \mathbb{C}_n^{\text{CA}} \cap \mathbb{C}_n^{k\text{-HGP}}$.
Conversely, from (b) *of Theorem 4.9, we get that*

$$A \in \mathbb{C}_n^{\text{CA}} \cap \mathbb{C}_n^{k\text{-HGP}} \Rightarrow A \in \mathbb{C}_n^{\text{CA}} \cap \mathbb{C}_n^{(k+2)\text{-P}} \cap \mathbb{C}_n^{\text{bi-EP}}.$$

Hence, by (c) *of Theorem 3.10 we have* $A \in \mathbb{C}_n^{k\text{-GP}}$. *Therefore,* $\mathbb{C}_n^{\text{CA}} \cap \mathbb{C}_n^{k\text{-HGP}} \subseteq \mathbb{C}_n^{k\text{-GP}}$.

(d) *By Remark 4.2 and Lemma 2.4, it follows that* $\mathbb{C}_n^{k\text{-GP}} \subseteq \mathbb{C}_n^{\text{N}} \cap \mathbb{C}_n^{k\text{-HGP}}$.
Conversely, in view of (c) *of Lemma 2.3 and Theorem 4.1, we have*

$$A \in \mathbb{C}_n^{\text{N}} \cap \mathbb{C}_n^{k\text{-HGP}} \Rightarrow K\Sigma = \Sigma K, \ L = 0 \ \text{and} \ (\Sigma K)^{k+1} = I_r,$$

which implies $A \in \mathbb{C}_n^{k\text{-GP}}$ *by Lemma 2.3 and Lemma 2.4. Hence,* $\mathbb{C}_n^{\text{N}} \cap \mathbb{C}_n^{k\text{-HGP}} \subseteq \mathbb{C}_n^{k\text{-GP}}$. □

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