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Weighted Schrödinger- Kirchhoff type problem in dimension 2 with non-linear double exponential growth

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Abstract. In this work, we study the weighted Kirchhoff problem

$$\begin{cases} g\Big(\int_B(\sigma(x)|\nabla u|^2+V(x)u^2)\,dx\Big)\Big[-\operatorname{div}(\sigma(x)\nabla u)+V(x)u\Big] &= f(x,u) & \text{in} & B\\ u &> 0 & \text{in} & B\\ u &= 0 & \text{on} & \partial B, \end{cases}$$

where B is the unit ball in $\mathbb{R}^2$, $\sigma(x)=\log\dfrac{e}{|x|}$, the singular logarithm weight in the Trudinger-Moser embedding, g is a continuous positive function on $\mathbb{R}^+$ and the potential V is a continuous positve function. The nonlinearities are critical or subcritical growth in view of Trudinger-Moser inequalities. We prove the existence of non-trivial solutions via the critical point theory. In the critical case, the associated energy function does not satisfy the condition of compactness. We provide a new condition for growth and we stress its importance to check the min-max compactness level.

1. Introduction

In this paper, we consider the following elliptic problem:

$$\begin{cases} g\Big(\int_B(\sigma(x)|\nabla u|^2+V(x)u^2)\,dx\Big)\Big[-\operatorname{div}(\sigma(x)\nabla u)+V(x)u\Big] &= f(x,u) & \text{in} & B\\ u &> 0 & \text{in} & B\\ u &= 0 & \text{on} & \partial B \end{cases} \tag{1}$$

where $B=B(0,1)$ is the unit open ball in $\mathbb{R}^2$, $f(x,t)$ is a radial function with respect to x, the weight $\sigma(x)$ is given by

$$\sigma(x)=\log\frac{e}{|x|}. \tag{2}$$

The function $g:\mathbb{R}^+\to\mathbb{R}^+$ is a positive continuous function wich will be specified later. The potential $V:\overline{B}\to\mathbb{R}$ is a positive continuous function satisfying some conditions.

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In 1883 Kirchhoff studied the following parabolic problem

$$\rho\frac{\partial^2 u}{\partial t^2}-\Big(\frac{P_0}{h}+\frac{E}{2L}\int_0^L|\frac{\partial u}{\partial x}|^2\,dx\Big)\frac{\partial^2 u}{\partial x^2}. \tag{3}$$

The parameters in equation (3) have the following meanings: L is the length of the string, h is the area of cross-section, E is the Young modulus of the material, ρ is the mass density and P_0 is the initial tension. These kinds of problems have physical motivations. Indeed, the Kirchhoff operator $G(\Big(\int_B|\nabla u|^2\,dx\Big))\Delta u$ also appears in the nonlinear vibration equation namely

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}-G\Big(\displaystyle\int_B|\nabla u|^2\,dx\Big)\mathrm{div}(\nabla u) & = & f(x,u) & \text{in} & B\times(0,T)\\ u & > & 0 & \text{in} & B\times(0,T)\\ u & = & 0 & \text{on} & \partial B\\ u(x,0) & = & u_0(x) & \text{in} & B\\ \dfrac{\partial u}{\partial t}(x,0) & = & u_1(x) & \text{in} & B \end{cases} \tag{4}$$

which have focused the attention of several researchers, mainly following the pioneering work of Lions [27]. We mention that non-local problems also arise in other areas, e.g. biological systems where the function u describes a process that depends on the average of itself (for example, population density), see e.g. [3, 4] and its references. Second order Kirchhoff's classical equation has been extensively studied. We refer to the work of Chipot [17, 18], Corrêa et al [24] and their references.

In the non weighted case, ie, when $\sigma(x)=1$ and $V(x)=0$, problem (1) can be seen as a stationary version of the evolution problem (4). For instance, in (1) if we set $\sigma(x)=1$ and $g(t)=1$, then we find the classical Schrödinger equation $-\triangle u+V(x)u=f(x,u)$. Also, if we take $\sigma(x)=1$, $V(x)=0$ and $g(t)=\bar{a}+\bar{b}t$, with $\bar{a},\bar{b}>0$, we find Kirchhoff's classical equation which has been extensively studied.

We point out that recently, in the case $g(t)=1$ and $V=0$ or $V\neq 0$, Baraket and Jaidane [6, 22] and Calanchi et al. [12], have proved the existence of a nontrivial solution for the following boundary value problem

$$\begin{cases} -\mathrm{div}(\omega(x)|\nabla u(x)|^{N-2}\nabla u(x))+V(x)|u|^{N-2}u & = & f(x,u) & \text{in} & B\\ u & = & 0 & \text{on} & \partial B, \end{cases}$$

where B is the unit ball in $\mathbb{R}^N$, $N\geq 2$, the weight $\omega(x)=\Big(\log\frac{e}{|x|}\Big)^{N-1}$ is of logarithmic type, the function $f(x,u)$ is continuous in $B\times\mathbb{R}$ and behaves like $\exp(e^{\alpha t^{\frac{N}{N-1}}})$ as $t\to+\infty$, for some $\alpha>0$. The authors proved that there is a non-trivial solution to this problem using minimax techniques combined with Trudinger-Moser inequality.

In order to motivate our study, we begin by giving a brief survey on Trudinger-Moser inequalities. In the past few decades, Moser [30, 33] gives the famous result about the Trudinger-Moser inequality. So, many applications take place as in conformal deformation theory on manifolds, the study of the prescribed Gauss curvature and mean field equations. After that, a logarithmic Trudinger-Moser inequality was used in crucial way in [29] to study the Liouville equation of the form

$$\begin{cases} -\Delta u & = & \lambda\dfrac{e^u}{\int_\Omega e^u} & \text{in} & \Omega\\ u & = & 0 & \text{on} & \partial\Omega, \end{cases} \tag{5}$$

where Ω an open domain of $\mathbb{R}^N$, $N\geq 2$ and λ a positive parameter.
The equation (5) has a long history and has been derived in the study of multiple condensate solution in

the Chern-Simons-Higgs theory [31, 32]. It also appeared in the study of Euler Flow [7, 8, 14, 25].

Later, The Trudinger-Moser inequality was improved to weighted inequalities [1, 9, 10, 13]. The influence of weight in the Sobolev norm was studied as the compact embedding [19, 26].
When the weight is of logarithmic type, Calanchi and Ruf [11] extend the Trudinger-Moser inequality and give some applications when $N = 2$ and for prescribed nonlinearities. After that, Calanchi et al. [12] consider a more general nonlinearities and prove the existence of radial solutions.

We mention that Figueiredo and Severo [21] studied the following problem

$$\begin{cases} -m\Big(\int_B |\nabla u|^2\,dx\Big)\triangle u &=& f(x,u) & \text{in} & \Omega \\ u &>& 0 & \text{in} & \Omega \\ u &=& 0 & \text{on} & \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^2$, the nonlinearity f behaves like $\exp(\alpha t^2)$ as $t \to +\infty$, for some $\alpha > 0$. $m : (0, +\infty) \to (0, +\infty)$ is a continuous function satisfying some conditions. The authors proved that this problem has a positive ground state solution. The existence result was proved by combining minimax techniques and Trudinger-Moser inequality.

Recently, Sitong Chen, Xianhua Tang and Jiuyang Wei [16], studied the last problem. They have developed some new approaches to estimate precisely the minimax level of the energy functional and prove the existence of Nehari-type ground-state solutions and nontrivial solutions for the above problem.
It should be noted that recently , the following nonhomogeneous Kirchhoff-Schrödinger equation

$$\begin{cases} -M(\int_{\mathbb{R}^2} |\nabla u|^2 + \xi(|x|)u^2\,dx)(-\triangle u + \xi(|x|)u) &=& Q(x)\overline{h}(u) + \varepsilon h(x), \\ u(x) \to 0 \text{ as } |x| \to +\infty, \end{cases}$$

has been studied in [2], where ε is a positive parameter, $M : \mathbb{R}^+ \to \mathbb{R}^+$, $\xi, Q : (0. +\infty) \to \mathbb{R}$, are continuous functions that satisfy some mild conditions. The nonlinearity $\overline{h} : \mathbb{R} \to \mathbb{R}$ is continuous and behaves like $\exp(\alpha t^2)$ as $t \to +\infty$, for some $\alpha > 0$. The authors proved the existence of at least two weak solutions for this equation by combining the Mountain Pass Theorem and Ekeland's Variational Principle.

Inspired by the last work cited above, we investigate our problem in adapted weighted Sobolev space setting, and use Trudinger-Moser inequality to study and prove the existence of solutions to (1).

Let $\Omega \subset \mathbb{R}^2$, be a bounded domain and $\sigma \in L^1(\Omega)$ be a nonnegative function. We define the following weighted Sobolev space as

$$H_0^1(\Omega, \sigma) = \text{closure}\ \{u \in C_0^\infty(\Omega)\ \mid\ \int_B \sigma(x)|\nabla u|^2\,dx < \infty\}.$$

We will limit our attention to radial functions and then consider the subspace,

$$H^1_{0,rad}(B, \sigma) = \text{closure}\ \{u \in C^\infty_{0,rad}(B)\ \mid\ \int_B \sigma(x)\ |\nabla u|^2\,dx < \infty\} \tag{6}$$

equipped with the norm

$$\|u\|_{H^1_{0,rad}} = \Big(\int_B \sigma(x)\ |\nabla u|^2\,dx\Big)^{\frac{1}{2}}.$$

The choice of the weight and the space $H^1_{0,rad}(B, \sigma)$ are motivated by the following exponential inequalities.

Theorem 1.1. *[9] Let σ given by (2), then*

$$\int_B e^{e^{u^2}}\,dx < +\infty, \quad \forall\ u \in H^1_{0,rad}(B,\sigma), \tag{7}$$

and

$$\sup_{\substack{u\in H^1_{0,rad}(B,\sigma)\\ \|u\|_{H^1_{0,rad}}\leq 1}} \int_B e^{\beta e^{2\pi u^2}}\,dx \leq +\infty \quad \Leftrightarrow \quad \beta \leq 2. \tag{8}$$

Due to (7) and (8), it will be said that f is sub-critical growth at $+\infty$ if

$$\lim_{s\to+\infty}\frac{|f(x,s)|}{\exp(2e^{\alpha s^2})} = 0, \quad \text{for all } \ \alpha > 0 \tag{9}$$

and f f is critical growth at $+\infty$, if there exists some $\alpha_0 > 0$ such that

$$\lim_{s\to+\infty}\frac{|f(x,s)|}{\exp(2e^{\alpha s^2})} = 0,\ \ \forall\,\alpha > \alpha_0 \quad \text{and} \quad \lim_{s\to+\infty}\frac{|f(x,s)|}{\exp(2e^{\alpha s^2})} = +\infty,\ \ \forall\,\alpha < \alpha_0. \tag{10}$$

Now we define the Kirchhoff function g and give the conditions on it. The function g is continuous on $\mathbb{R}^+$ and verifies the conditions:

(G_1) There exists $g_0 > 0$ sucht that $g(t) \geq g_0$ for all $t \geq 0$ and

$$G(t+s) \geq G(t) + G(s)\ \ \forall\ s,t \geq 0;$$

where

$$G(t) = \int_0^t g(s)ds,$$

(G_2) $t \mapsto \dfrac{g(t)}{t}$ is nonincreasing for $t > 0$.

The assmption (G_2) implies that $\dfrac{g(t)}{t} \leq g(1)$ for all $t \geq 1$. As a consequence of (G_2), a simple calculation shows that

$$t \mapsto \frac{1}{2}G(t) - \frac{1}{4}g(t)t \ \text{ is nondecreasing for } \ t \geq 0.$$

Consequently, one has

$$\frac{1}{2}G(t) - \frac{1}{4}g(t)t \geq 0,\ \ \forall t \geq 0. \tag{11}$$

A typical example of a function g fulfilling the conditions (G_1), (G_2) and (G_3) is given by

$$g(t) = g_0 + at \ \ , g_0, a > 0.$$

Another example is given by $g(t) = 1 + \ln(1+t)$.

The potential V is continuous on $\overline{B}$ and verifies

(V_1) $V(x) \geq V_0 > 0$ in B for some $V_0 > 0$.

The condition (V_1) implies that the function $\dfrac{1}{V}$ belongs to $L^1(B)$.

In this paper, we consider the problem (1) with subcritical and critical growth nonlinearities $f(x,t)$. Furthermore, we suppose that $f(x,t)$ satisfies the following hypothesis:

(H_1) The non-linearity $f : \overline{B} \times \mathbb{R} \to \mathbb{R}$ is positive, continuous, radial in x, and $f(x,t) = 0$ for $t \leq 0\cdot$

(H_2) There exist $t_0 > 0$ and $M_0 > 0$ such that for all $t > t_0$ and for all $x \in B$ we have

$$0 < F(x,t) \leq M_0 f(x,t),$$

where

$$F(x,t) = \int_0^t f(x,s)ds.$$

(H_3) For each $x \in B$, $\dfrac{f(x,t)}{t^3}$ is increasing for $t > 0$.

(H_4) In the critical case, there exists a constant γ_0 with $\gamma_0 > \dfrac{2g(\frac{2\pi}{\alpha_0})}{\alpha_0 e^2\Big(1+e^{-m}\Big)}$

$$\lim_{t\to\infty} \frac{f(x,t)t}{\exp(2e^{\alpha_0 t^2})} \geq \gamma_0 \quad \text{uniformly in } x,$$

where $m = \max\limits_{x\in\bar{B}} V(x)$.

The condition (H_2) implies that for any $\varepsilon > 0$, there exists a real $t_\varepsilon > 0$ such that

$$F(x,t) \leq \varepsilon t f(x,t), \ \ \forall |t| > t_\varepsilon, \ \text{ uniformly in } x \in \overline{B}. \tag{12}$$

Also, we have that the condition (H_3) leads to

$$\lim_{t\to 0} \frac{f(x,t)}{t^\theta} = 0 \ \text{ for all } \ \ 0 \leq \theta < 3 \ \text{ uniformly in } x \in \overline{B}. \tag{13}$$

The asymptotic condition (H_4) will be crucial to identify the min-max level of the energy associated to problem (1).

We give an example of f. Let $f(t) = F'(t)$, with $F(t) = \dfrac{t^4}{4} + t^4 \exp(2e^{\alpha_0 t^2})$. A simple calculation shows that f verifies the conditions (H_1), (H_2), (H_3) and (H_4).

To study the solvability of problem (1), consider the space

$$\mathcal{E} = \{u \in H^1_{0,rad}(B) \ | \ \int_B V(x)u^2\,dx < +\infty\},$$

endowed with the norm

$$||u|| = \left(\int_B \sigma(x)\,|\nabla u|^2\,dx + \int_B V(x)u^2\,dx\right)^{\frac{1}{2}}. \tag{14}$$

We note that this norm is induced from the product scalar

$$\langle u,v\rangle = \int_B (\nabla u.\nabla v \log\frac{e}{|x|} + V(x)uv)\,dx.$$

It will be said that u is a solution to problem (1), if u is a weak solution in the following sense.

Definition 1.2. *A function u is called a solution to (1) if $u \in \mathcal{E}$ and*

$$g(\|u\|^2)\Big[\int_B \Big(\sigma(x)\ \nabla u\ \nabla \varphi\Big)\,dx + \int_B V(x)u\varphi\,dx\Big] = \int_B f(x,u)\,\varphi\,dx, \quad \text{for all} \quad \varphi \in \mathcal{E} \tag{15}$$

The energy functional, also known as the Euler-Lagrange functional associated to (1) is defined by $\mathcal{J} : \mathcal{E} \to \mathbb{R}$

$$\mathcal{J}(u) = \frac{1}{2}G(\|u\|^2) - \int_B F(x,u)\,dx, \tag{16}$$

where

$$F(x,t) = \int_0^t f(x,s)ds.$$

It is quite clear that finding weak solutions to problem (1) is equivalent to finding non-zero critical points of the functional $\mathcal{J}$ over $\mathcal{E}$.

The major difficulty in this problem lie in the concurrence between the growths of g and f. To avoid this difficulty, many authors usually assume that g is increasing or bounded.(see[3, 4, 15, 21]).

Our results are as follows :
In the subcritical exponential growth case, we have:

Theorem 1.3. *Let $f(x,t)$ be a function that has a subcritical growth at $+\infty$ and satisfies (H_1), (H_2) and (H_3). In addition, suppose that (V_1), (G_1) and (G_2) hold, then problem (1) has a non trivial radial solution.*

In the context of the critical double exponential growth, the study of problem (1) becomes more difficult than in the subcritical case. Our Euler-Lagrange function is losing compactness at a certain level. To overcame this lack of compactness we choose test functions, which are extremal for the Trudinger-Moser inequality (8). In this case we have:

Theorem 1.4. *Assume that $f(x,t)$ has a critical growth at $+\infty$ for some α_0 and satisfies the conditions (H_1), (H_2), (H_3) and (H_4). If in addition (V_1), (G_1) and (G_2) are satisfied, then the problem (1) has a nontrivial solution.*

To the best of our knowledge, the present papers results have not been covered yet in the literature.
This paper is organized as follows. In Section 2, we present some necessary preliminary knowledge about functional space. In Section 3, we give some useful lemmas for the compactness analysis. In Section 4, we prove that the energy $\mathcal{J}$ satisfies the two geometric properties. Section 5 is devoted to estimate the min-max level of the energy. Finally, we conclude with the proofs of the main results in Section 6.
Through this paper, the constant C may change from one line to another and we sometimes index the constants in order to show how they change.

2. Weighted Lebesgue and Sobolev Spaces setting

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain in $\mathbb{R}^N$ and let $w \in L^1(\Omega)$ be a nonnegative function. To deal with weighted operator, we need to introduce some functional spaces $L^p(\Omega,w)$, $W^{m,p}(\Omega,w)$, $W_0^{m,p}(\Omega,w)$. Let $S(\Omega)$ be the set of all measurable real-valued functions defined on Ω and two measurable functions are considered as the same element if they are equal almost everywhere.

Following Drabek et al. and Kufner in [19, 26], the weighted Lebesgue space $L^p(\Omega,w)$ is defined as follows:

$$L^p(\Omega,w) = \{u : \Omega \to \mathbb{R} \text{ measurable}; \quad \int_\Omega w(x)|u|^p\,dx < \infty\},$$

for any real number $1 \le p < \infty$.
This is a normed vector space equipped with the norm

$$\|u\|_{p,w} = \Big(\int_\Omega w(x)|u|^p\, dx\Big)^{\frac{1}{p}}.$$

For $w(x) = 1$, we find the standard Lebesgue space $L^p(\Omega)$ and its norm

$$\|u\|_p = \Big(\int_\Omega |u|^p\, dx\Big)^{\frac{1}{p}}.$$

In [19], the corresponding weighted Sobolev space was defined as

$$W^{1,p}(\Omega, w) = \{u \in L^p(\Omega);\ \ \nabla u \in L^p(\Omega, w)\}$$

and equipped with the norm defined on $W^{1,p}(\Omega)$ by

$$\|u\|_{W^{1,p}(\Omega,w)} = \Big(\|u\|_p^p + \|\nabla u\|_{p,w}^p\Big)^{\frac{1}{p}}. \tag{17}$$

The spaces $L^p(\Omega, w)$ and $W^{1,p}(\Omega, w)$ are separable, reflexive Banach spaces provided that $w(x)^{\frac{-1}{p-1}} \in L^1_{loc}(\Omega)$. If we suppose also that $w(x) \in L^1_{loc}(\Omega)$, then $C_0^\infty(\Omega)$ is a subset of $W^{1,p}(\Omega, w)$, then we can introduce the space $W_0^{1,p}(\Omega, w)$, as the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega, w)$.
For the space $W_0^{1,p}(\Omega, w)$, we have the following norm,

$$\|u\|_{W_0^{1,p}(w,\Omega)} = \Big(\int_\Omega w(x)|\nabla u|^p\, dx\Big)^{\frac{1}{p}} \tag{18}$$

which is equivalent to the one given by (17).
Also, we will use the spaces $H_0^1(\Omega, w)$, which is the closure of $C_0^\infty(\Omega)$ in $W^{1,2}(\Omega, w)$, equipped with the norm

$$\|u\|_{H_0^1(\Omega,w)} = \Big(\int_\Omega w(x)|\nabla u|^2\, dx\Big)^{\frac{1}{2}},$$

witch is equivalent to the norm given by (17) when $p = 2$. We have then

$$\|u\|_{H_0^1(\Omega,\sigma)} = \|\nabla u\|_{2,w}.$$

Let the subspace

$$H^1_{0,rad}(B, w) = \text{closure } \{u \in C^\infty_{0,rad}(B) \ \mid\ \int_B \sigma(x)|\nabla u|^2\, dx < \infty\},$$

with $\sigma(x) = \log \frac{e}{|x|}$. Then the space $\mathcal{E} = \{u \in H^1_{0,rad}(B) \ \mid\ \int_B V(x)u^2 dx < +\infty\}$ is a Banach and reflexive space provided (V_1) is satisfied. The space $\mathcal{E}$ is endowed with the norm

$$\|u\| = \left(\int_B \sigma(x)|\nabla u|^2 dx + \int_B V(x)u^2 dx\right)^{\frac{1}{2}}$$

which is equivalent to the following norm (see lemma 3.1)

$$\|u\|_{H^1_{0,rad}(B,\sigma)} = \Big(\int_B \sigma(x)|\nabla u|^2\, dx\Big)^{\frac{1}{2}}.$$

We also have the continuous embedding

$$\mathcal{E} \hookrightarrow L^q(B) \ \text{ for all } \ q \ge 1.$$

Moreover, $\mathcal{E}$ is compactly embedded in $L^q(B)$ for all $q \ge 1$ (see lemma 3.1).

3. Preliminaries for the compactness analysis

In this section, we will derive several technical lemmas for our use later. First we begin by the radial lemma.

Lemma 3.1. *[22] Assume that V is continuous and verifies (V_1).*

(i) Let u in $C^1_0(B)$ a radially symmetric function then,

$$|u(x)| \leq \frac{1}{\sqrt{2\pi}} \log^{\frac{1}{2}}(\log(\frac{e}{|x|}))(\int_B |\nabla u|^2 w(x)dx)^{\frac{1}{2}} \leq \frac{1}{\sqrt{2\pi}} \log^{\frac{1}{2}}(\log(\frac{e}{|x|}))\|u\|.$$

(ii) There exists a positive contant C such that for all $u \in \mathcal{E}$

$$\int_B V(x)u^2dx \leq C\|u\|^2$$

and then the norms $\|.\|$ and $\|.\|_{H^1_{0,rad}(B,w)} = \Big(\int_\Omega \sigma|\nabla .|^2\, dx\Big)^{\frac{1}{2}}$ are equivalents.

(iii) The following embedding is continuous

$$\mathcal{E} \hookrightarrow L^q(B) \text{ for all } q \geq 1.$$

(iv) $\mathcal{E}$ is compactly embedded in $L^q(B)$ for all $q \geq 1$.

Proof

(i) The function u is radially symmetric; then $u(x) = v(|x|)$ and by using the Hölder's inequality,

$$\begin{aligned}
|u(x)| = |v(|x|) - v(1)| &= \Big|\int_1^{|x|} v'(t)dt\Big| \\
&= \int_{|x|}^1 |v'(t)|t^2|\log\frac{e}{t}|^{\frac{1}{2}}|t^{-2}|\log\frac{e}{t}|^{\frac{-1}{2}}dt \\
&\leq \Big[\int_{|x|}^1 |v'(t)|^2 t|\log\frac{e}{t}|dt\Big]^{\frac{1}{2}}\Big[\int_{|x|}^1 \frac{1}{t|\log\frac{e}{t}|}dt\Big]^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{2\pi}}\Big[2\pi\int_{|x|}^1 |v'(t)|^2 t|\log\frac{e}{t}|dt\Big]^{\frac{1}{2}} \log^{\frac{1}{2}}(\log(\frac{e}{|x|})) \\
&\leq \frac{1}{\sqrt{2\pi}} \log^{\frac{1}{2}}(\log(\frac{e}{|x|}))(\int_B |\nabla u|^2 w(x)dx)^{\frac{1}{2}} \\
&\leq \frac{1}{\sqrt{2\pi}} \log^{\frac{1}{2}}(\log(\frac{e}{|x|}))\|u\|.
\end{aligned}$$

(ii) From (i) we have for all $u \in \mathcal{E}$,

$$\int_B V(x)u^2dx \leq m\int_B u^2dx \leq \frac{m}{2\pi}\|u\|^2\int_B \log(\log(\frac{e}{|x|}))dx \leq \frac{m}{2\pi}\|u\|^2\int_B \log(\frac{e}{|x|})dx \leq C\|u\|^2,$$

where $m = \max\limits_{x\in\overline{B}} V(x)$. Then (ii) is proved.

(iii) From (i) and (ii), we have that the following embedding are continuous

$$\mathcal{E} \hookrightarrow H^1_{0,rad}(B) \hookrightarrow L^q(B) \;\; \forall q \geq 2.$$

We also have by the Hölder inequality and (V_1),

$$\int_B |u|dx \leq \Big(\int_B \frac{1}{V}dx\Big)^{\frac{1}{2}}\Big(\int_B Vu^2dx\Big)^{\frac{1}{2}} \leq \Big(\int_B \frac{1}{V}dx\Big)^{\frac{1}{2}}\|u\|\cdot$$

For any $1 < \beta < 2$, there holds

$$\int_B |u|^\beta dx \leq \int_B (|u| + |u|^2) dx \leq \Big(\int_B \frac{1}{V} dx \Big)^{\frac{1}{2}} \|u\| + \frac{1}{V_0} \|u\|^2.$$

Thus, we get the continous embedding $\mathcal{E} \hookrightarrow L^q(B)$ for all $q \geq 1$.
(iv) The above embedding is also compact. Indeed, let $u_k \subset \mathcal{E}$ be a sequence such that $\|u_k\| \leq C$ for all k. Then $\|u_k\|_{H^1_{0,rad}} \leq C$, for all k. On the other hand, we have the following compact embedding [19] $H^1_{0,rad}(B) \hookrightarrow L^q(B)$ for all q such that $1 \leq q < 2s$, with $s > 1$, then up to a subsequence, there exists some $u \in H^1_{0,rad}$, such that u_k convergent to u strongly in $L^q(B)$ for all q such that $1 \leq q < 2s$. Without loss of generality, we may assume that

$$\begin{cases} u_k & \rightharpoonup u \quad \text{weakly in } \mathcal{E} \\ u_k & \to u \quad \text{strongly in } L^1(B) \\ u_k(x) & \to u(x) \quad \text{almost everywhere in } B. \end{cases} \tag{19}$$

For $q > 1$, it follows from (19) and the continuous embedding $\mathcal{E} \hookrightarrow L^p(B)$ $(p \geq 1)$ that

$$\begin{aligned} \int_B |u_k - u|^q dx &= \int_B |u_k - u|^{\frac{1}{2}} |u_k - u|^{q-\frac{1}{2}} dx \\ &\leq \Big(\int_B |u_k - u| dx \Big)^{\frac{1}{2}} \Big(\int_B |u_k - u|^{2q-1} dx \Big)^{\frac{1}{2}} \\ &\leq C \Big(\int_B |u_k - u| dx \Big)^{\frac{1}{2}} \to 0. \end{aligned}$$

This concludes the lemma. □

In the next, we give the following useful lemma.

Lemma 3.2. *[20] Let $\Omega \subset \mathbb{R}^{\mathbb{N}}$ be a bounded domain and $f : \overline{\Omega} \times \mathbb{R}$ a continuous function. Let $(u_n)_n$ be a sequence in $L^1(\Omega)$ converging to u in $L^1(\Omega)$. Assume that $f(x, u_n)$ and $f(x, u)$ are also in $L^1(\Omega)$. If*

$$\int_\Omega |f(x, u_n) u_n| dx \leq C,$$

where C is a positive constant, then

$$f(x, u_n) \to f(x, u) \ \ in \ \ L^1(\Omega).$$

In order to prove a compactness condition for the energy $\mathcal{J}$, we need a Lions type result [28] about an improved TM-inequality when we deal with weakly convergent sequences and double exponential case.

Lemma 3.3. *[12] Let $(u_k)_k$ with $\|u_k\| = 1$ be a sequence in $\mathcal{E}$ converging weakly to a non zero function u. Then*

$$\sup_k \int_B \exp\Big(e^{2\pi p u_k^2}\Big) dx < +\infty,$$

for all $1 < p < P$, where

$$P := \begin{cases} (1 - \|u\|^2)^{-1} & \text{if } \|u\| < 1 \\ +\infty & \text{if } \|u\| = 1. \end{cases}$$

Proof By Young inequality, we have

$$\exp(2e^{a+b}) \leq \frac{1}{q} \exp(2e^{qa}) + \frac{1}{q'} \exp(2e^{q'b}), \ \ \forall a, b \in \mathbb{R}, \ q > 1,$$

where $\frac{1}{q}+\frac{1}{q'}=1$. Also, we estimate u_k^2,

$$u_k^2=(u_k-u+u)^2\le(1+\varepsilon)(u_k-u)^2+(1+\frac{1}{\varepsilon})u^2,\ \ \forall\ \ \varepsilon>0.$$

Therefore, for any $p>1$, using the above inequalities, we get

$$\int_B\exp\left(2e^{2\pi pu_k^2}\right)dx\le\frac{1}{q}\int_B\exp(2e^{2pq\pi(1+\varepsilon)(u_k-u)^2})dx+\frac{1}{q'}\int_B\exp(2e^{2pq'\pi(1+\frac{1}{\varepsilon})u^2})dx.$$

From (7) the last integral is finite and to complete the proof, we should prove that for every p such that $1<p<P$, we have

$$\sup_k\int_B\exp(2e^{2pq\pi(1+\varepsilon)(u_k-u)^2})dx<+\infty\ \text{, for some}\ \ \varepsilon>0\ \text{ and }\ q>1.$$

We may assume that $\|u\|<1$, the proof in the case $\|u\|=1$ is similar. If $\|u\|<1$, then for

$$p<\frac{1}{1-\|u\|^2},$$

there exists $\nu>0$ such that

$$p(1-\|u\|^2)(1+\nu)<1.$$

On the other hand, we have

$$\lim_{k\to+\infty}\|u_k-u\|^2=1+\|u\|^2-\lim_{k\to+\infty}\langle u_k,u\rangle=1-\|u\|^2.$$

Therefore, for every $\varepsilon>0$, there exists $k_\varepsilon\ge1$ such that

$$\|u_k-u\|^2\le(1+\varepsilon)(1-\|u\|^2)\ \ \forall\ \ k\ge k_\varepsilon.$$

Then, for $q=1+\varepsilon$ with ε such that $\varepsilon=\sqrt[3]{1+\nu}-1$, and for every $k\ge k_\varepsilon$ we get

$$pq(1+\varepsilon)\|u_k-u\|^2\le p(1+\varepsilon)^3(1-\|u\|^2)=p(1+\nu)(1-\|u\|^2)<1.$$

Using (8), we get

$$\begin{aligned}\int_B\exp(2e^{2\pi pq(1+\varepsilon)\|u_k-u\|^2})dx &\le \int_B\exp(2e^{2\pi(1+\varepsilon)pq(\frac{u_k-u}{\|u_k-u\|})^2\|u_k-u\|^2})dx\\ &\le \int_B\exp(2e^{2\pi(\frac{(u_k-u)}{\|u_k-u\|})^2})dx\\ &\le \sup_{\|v\|\le1,v\in E}\int_B e^{2\pi v^2}dx<+\infty\end{aligned}$$

and Lemma 3.3 is proved.

4. The mountain pass geometry of the energy

Since the nonlinearity $f(x,t)$ is critical or subcritical at $+\infty$, there exist positive constants $a,\ C>0$ and there exists $t_2>1$ such that

$$|f(x,t)|\le C\exp(e^{a\ t^2}),\qquad\forall|t|>t_2.\tag{20}$$

So the functional $\mathcal{J}$ defined by (16), is well defined and of class C^1.

In order to prove the existence of nontrivial solution to problem (1), we will prove the existence of nonzero critical point of the functional $\mathcal{J}$ by using the theorem introduced by Ambrosetti and Rabinowitz in [5] (Mountain Pass Theorem).

Definition 4.1. *Let (u_n) be a sequence in a Banach space E and $J \in C^1(E, \mathbb{R})$ and let $c \in \mathbb{R}$. We say that the sequence (u_n) is a Palais-Smale sequence at level c (or $(PS)_c$ sequence) for the functional J if*

$$J(u_n) \to c \ \ in \ \ \mathbb{R}, \ \ as \ \ n \to +\infty$$

and

$$J'(u_n) \to 0 \ \ in \ \ E', \ \ as \ \ n \to +\infty.$$

We say that the functional J satisfies the Palais-Smale condition $(PS)_c$ at the level c if every $(PS)_c$ sequence (u_n) is relatively compact in E.

Theorem 4.2. *[5] Let E be a Banach space and $J : E \to \mathbb{R}$ a C^1 functional satisfying $J(0) = 0$. Suppose that*

(i) There exist $\rho, \ \beta > 0$ such that $\forall u \in \partial B(0, \rho), J(u) \geq \beta$;
(ii) There exists $x_1 \in E$ such that $\|x_1\| > \rho$ and $J(x_1) < 0$;
(iii) J satisfies the Palais-Smale condition (PS), that is any Palais Smale sequence (u_n) in E is relatively compact.

Then, J has a critical point u and the critical value $c = J(u)$ verifies

$$c := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t))$$

where $\Gamma := \{\gamma \in C([0, 1], E)$ such that $\gamma(0) = 0$ and $\gamma(1) = x_1\}$ and $c \geq \beta$.

Before starting the proof of the geometric properties for the functional $\mathcal{J}$, it follows from the continuous embedding $\mathcal{E} \hookrightarrow L^q(B)$ for all $q \geq 1$, that there exists a constant $C > 0$ such that $\|u\|_{2q} \leq c\|u\|$, for all $u \in \mathcal{E}$.

In the next Lemmas, we prove that the functional $\mathcal{J}$ has the mountain pass geometry of Theorem 4.2.

Lemma 4.3. *Suppose that f has critical growth at $+\infty$. In addition if (H_1), (H_3), (V_1) and (G_1) hold, then, there exist $\rho, \ \beta > 0$ such that $\mathcal{J}(u) \geq \beta$ for all $u \in \mathcal{E}$ with $\|u\| = \rho$.*

Proof. It follows from (13) that there exists $\delta_0 > 0$

$$F(x, t) \leq \epsilon|t|^2, \qquad \text{for } \ |t| < \delta_0. \tag{21}$$

From (H_3), (20) and for all $q > 2$, there exist a positive constant $C > 0$ such that

$$F(x, t) \leq C|t|^q exp(e^{a \ t^2}), \ \ \forall \ |t| > \delta_1. \tag{22}$$

So, using the continuity of F, we get

$$F(x, t) \leq \epsilon|t|^2 + C|t|^q exp(2e^{a \ t^2}), \qquad \text{for all } \ t \in \mathbb{R}. \tag{23}$$

Since

$$\mathcal{J}(u) = \frac{1}{2}G(\|u\|^2) - \int_B F(x, u)dx,$$

we get from (G_1)

$$\mathcal{J}(u) \geq \frac{g_0}{2}\|u\|^2 - \varepsilon \int_B u^2 dx - C \int_B |u|^q \exp(e^{a \ u^2}) \, dx.$$

From the Hölder inequality, we obtain

$$\mathcal{J}(u) \geq \frac{g_0}{2}\|u\|^2 - \varepsilon \int_B u^2 dx - C\Big(\int_B \exp(2e^{a \ u^2})dx\Big)^{\frac{1}{2}} \|u\|_{2q}^q. \tag{24}$$

From Theorem 1.1, if we choose $u \in \mathcal{E}$ such that

$$a\|u\|^2 \leq 2\pi, \tag{25}$$

we get

$$\int_B \exp(2e^{a\ u^2})dx = \int_B \exp(2e^{a\ \|u\|^2(\frac{|u|}{\|u\|})^2})dx < C, \quad \text{with } C \text{ not depending on } u.$$

On the other hand $\|u\|_{2q} \leq C_1\|u\|$, so for fixed ϵ such that $\frac{g_0}{2C_1} > \epsilon$,

$$\mathcal{J}(u) \geq \frac{g_0}{2}\|u\|^2 - \varepsilon C_1\|u\|^2 - C\|u\|^q = \|u\|^2\Big(\frac{g_0}{2} - \varepsilon C_1 - C\|u\|^{q-2}\Big),$$

for all $u \in \mathcal{E}$ satisfying (25). Since $2 < q$, we can choose $\rho = \|u\| \leq \sqrt{\frac{2\pi}{a}}$ and for ϵ such that $\frac{g_0}{2C_1} > \epsilon$, there exists $\beta = \rho^2\Big((\frac{g_0}{2} - \varepsilon)C_1 - C\rho^{q-2}\Big) > 0$ with $\mathcal{J}(u) \geq \beta > 0$. □

By the following lemma, we prove the second geometric property for the functional $\mathcal{J}$.

Lemma 4.4. *Suppose that (H_1) , (H_2), (V_1) and (G_2) hold. Then there exists $e \in \mathcal{E}$ with $\mathcal{J}(e) < 0$ and $\|e\| > \rho$.*

Proof. From the condition (G_2), for all $t \geq 1$, we have that

$$G(t) \leq \frac{g(1)}{2}t^2. \tag{26}$$

It follows from the condition (H_2) that

$$f(x,t) = \frac{\partial}{\partial t}F(x,t) \geq \frac{1}{M}F(x,t),$$

for all $t \geq t_0$. So

$$F(x,t) \geq C\, e^{\frac{t}{M}}, \ \ \forall\ \ t \geq t_0. \tag{27}$$

In particular, for $p > 4$ there exists C_1 and C_2 such that

$$F(x,t) \geq C_1|t|^p - C_2, \ \ \forall t \in \mathbb{R}, \ \ x \in B.$$

Next, one arbitrarily picks $\bar{u} \in \mathcal{E}$ such that $\|\bar{u}\| = 1$. Thus from (26) and (27), for all $t \geq 1$

$$\mathcal{J}(t\bar{u}) \leq \frac{g(1)}{4}t^4 - C_1\|\bar{u}\|_p^p\ t^p - \pi C_2.$$

Therefore,

$$\lim_{t\to+\infty} \mathcal{J}(t\bar{u}) = -\infty.$$

We take $e = \bar{t}\bar{u}$, for some $\bar{t} > 0$ large enough. So, Lemma 4.4 follows. □

5. The minimax estimate of the energy

According to Lemmas 4.3 and 4.4, let

$$d_* := \inf_{\gamma\in\Lambda}\max_{t\in[0,1]}\mathcal{J}(\gamma(t)) > 0 \tag{28}$$

and

$$\Lambda := \{\gamma \in C([0,1],\mathcal{E}) \text{ such that } \quad \gamma(0)=0\,,\ \gamma(1)=e \quad \text{and} \quad \mathcal{J}(\gamma(1))<0\}.$$

We are going to estimate the minimax value of the functional $\mathcal{J}$. The idea is to construct a sequence of functions $(v_n) \in \mathcal{E}$, and estimate $\max\{\mathcal{J}(tv_n) : t \geq 0\}$. For this goal, let consider the following Moser function

$$w_n(x) = \frac{1}{\sqrt{2\pi}}\begin{cases} \dfrac{\log(\log(\frac{e}{|x|}))}{\log^{\frac{1}{2}}(1+n)} & \text{if } e^{-n} \leq |x| \leq 1\\ \log^{\frac{1}{2}}(1+n) & \text{if } 0 \leq |x| \leq e^{-n}. \end{cases}$$

Let $v_n(x) = \dfrac{w_n(x)}{\|w_n\|}$. Then $v_n \in \mathcal{E}$ and $\|v_n\| = 1$.

5.1. *Key lemmas*

We need two lemmas that we shall use later. We begin by the first lemma.

Lemma 5.1. *[22]Assume that for the continuous potential $V(x)$, the condition (V_1) is satisfied. Then*

$$1 + \frac{V_0}{\log(1+n)} + o_n(1) \leq \|w_n\|^2 \leq 1 + \frac{m}{\log(1+n)} + o_n(1)$$
$$\frac{1}{\|w_n\|^2} \geq 1 - \frac{m}{\log(1+n)} + o_n(1),$$

where $m = \max_{x\in\overline{B}} V(x)$ and $o_n(1) \to 0$ as $n \to +\infty$.

Proof. We have

$$\frac{1}{2\pi}\int_B \log(\frac{e}{|x|})|\nabla w_n|^2 dx = \frac{1}{\log(1+n)}\int_{e^{-n}}^1 \frac{1}{r}\frac{1}{\log(\frac{e}{r})}dr = 1.$$

Also,

$$\begin{aligned}\int_{e^{-n}\leq|x|\leq1} \log^2(\log(\frac{e}{|x|}))dx &\leq \int_{e^{-n}\leq|x|\leq1} \log^2(\frac{e}{|x|})dx\\ &= 2\pi\int_{e^{-n}}^1 r\log^2(\frac{e}{r})dr\\ &= 2\pi\Big(\tfrac{5}{4} - \tfrac{1}{2}e^{-2n}(n^2+n-\tfrac{1}{2})\Big).\end{aligned}$$

Hence,

$$\begin{aligned}\int_B |w_n(x)|^2dx &\leq \frac{1}{2\pi\log(1+n)}\int_{e^{-n}\leq|x|\leq1}\log^2(\frac{e}{|x|})dx + \frac{1}{2\pi}\int_{0\leq|x|\leq e^{-n}}\log(1+n)dx\\ &= \frac{1}{\log(1+n)}\int_{e^{-n}}^1 r\log^2(\frac{e}{r})dr + \frac{1}{2}e^{-2n}\log(1+n)\\ &= \frac{1}{\log(1+n)}\Big(\frac{5}{4} - \frac{1}{2}e^{-2n}(n^2+n-\frac{1}{2})\Big) + \frac{1}{2}e^{-2n}\log(1+n)\\ &\leq \frac{5}{4}\frac{1}{\log(1+n)} + \frac{1}{2}e^{-2n}\log(1+n) = \frac{1}{\log(1+n)} + o_n(1)\end{aligned}$$

and thus

$$\begin{aligned}\|w_n(x)\|^2 &= \int_B \log(\frac{e}{|x|})\, |\nabla w_n|^2 dx + \int_B V(x)\, |w_n(x)|^2 dx\\ &\leq 1 + \frac{m}{2} e^{-2n} \log(1+n) + \frac{5}{4}\frac{m}{\log(1+n)}\\ &\leq 1 + \frac{m}{\log(1+n)} + o_n(1)\cdot\end{aligned}$$

In the same way, using the condition (V_1) we obatin,

$$1 + \frac{V_0}{\log(1+n)} + o_n(1) \leq \|w_n\|^2.$$

□

Now, we present the second elementary lemma.

Lemma 5.2. *[22]*

$$\lim_{n\to+\infty} \int_{e^{-n}\leq|x|\leq 1} \exp(2e^{2\pi v_n^2})\, dx = \lim_{n\to+\infty} 2\pi \int_0^n \exp(2e^{\frac{\log^2(1+s)}{\log(1+n)\|w_n\|^2}} - 2s)\, ds \geq 2\pi e^2$$

Proof. We have,

$$\int_{e^{-n}\leq|x|\leq 1} \exp(2e^{2\pi v_n^2})\, dx = 2\pi \int_{e^{-n}}^1 r \exp(2e^{\frac{\log^2(\log(\frac{e}{r}))}{\|w_n\|^2 \log(1+n)}})dr\cdot$$

Let us pose $b(m,n) = b := 1 - \dfrac{m}{\log(1+n)} + o_n(1)$. We bring change of variable $|x| = e^{-s}$, $t = 1+s$, $k = n+1$, and using the result of lemma 5.1 , we get

$$\begin{aligned}I = \int_{e^{-n}\leq|x|\leq 1} \exp(2e^{2\pi v_n^2})\, dx = &\geq 2\pi \int_0^n \exp(2e^{b\frac{\log^2(1+s)}{\log(1+n)}} - 2s)\, ds\\ &= 2\pi \int_1^k \exp(2t^{(\frac{b\log t}{\log k})} - 2(t-1)\, dt\ ,\\ &= 2\pi e^2 \int_1^k \exp(2t^{(\frac{b\log t}{\log k})} - 2t)\, dt\cdot\end{aligned}$$

We claim that

$$\lim_{k\to+\infty} \int_1^k \exp(2t^{(\frac{b\log t}{\log k})} - 2t)dt = 1, \tag{29}$$

and so lemma4.4 follows.
For any $k > 4$, let

$$\psi_k(t) := 2t^{(\frac{b\log t}{\log k})} - 2t, \quad \text{with } t \geq 1.$$

and divide the interval $[1,k]$ as

$$[1,k] = [1, \sqrt{k}] \cup [\sqrt{k}, k - \sqrt{k}] \cup [k - \sqrt{k}, k].$$

First, we consider the interval $[1, \sqrt{k}]$. Since $b \leq 1$, and $\lim\limits_{k\to+\infty} b = 1$, then

$$\chi_{[1,\sqrt{k}]}(t)e^{\psi_k(t)} \leq e^{2t^{\frac{b}{2}} - 2t} \leq e^{2t^{\frac{1}{2}} - 2t} \in L^1([1,+\infty))$$

and

$$\chi_{[1,\sqrt{k}]}(t)e^{\psi_k(t)} \to e^{2-2t} \ \text{ for a.e } t \geq 1, \ \text{ as } k \to +\infty.$$

So, by the Lebesgue's convergence dominated theorem, we get

$$\lim_{k\to+\infty}\int_1^{\sqrt{k}} e^{2t^{(\frac{b\log t}{\log k})}-2t}dt = \lim_{k\to+\infty}\int_1^{k}\chi_{[1,\sqrt{k}]}(t)e^{\psi_k(t)}dt = \frac{1}{2}\cdot \tag{30}$$

Now, we study the limit of the integral on $[\sqrt{k}, k-\sqrt{k}]$. So, we calculate

$$\psi_k(\sqrt{k}) = -2\sqrt{k}\Big(1 - k^{\frac{b-2}{4}}\Big)$$

and then

$$\psi_k(\sqrt{k}) \le -\sqrt{k}, \ \text{ for all } \ k \ge 4\cdot \tag{31}$$

On the other hand, since $b \le 1$, we get,

$$\begin{array}{rcl}\psi_k(k-\sqrt{k}) & = & 2e^{\left(\frac{b}{\log k}\left(\log k+\log(1-\frac{1}{\sqrt{k}})\right)^2\right)} - 2(k-\sqrt{k})\\ & = & 2e^{\left(b\log k\left(1+\frac{\log(1-\frac{1}{\sqrt{k}})}{\log k}\right)^2\right)} - 2(k-\sqrt{k})\\ & = & 2e^{\left(b\log k\left\{1-2\frac{1}{\sqrt{k}\log k}+o(\frac{1}{\sqrt{k}\log k})\right\}-1\right)} + 2\sqrt{k}\\ & \le & 2k\Big[e^{\left(-2\frac{1}{\sqrt{k}}+o(\frac{1}{\sqrt{k}})\right)}-1] + 2\sqrt{k}.\end{array}$$

Therefore, for every $\epsilon \in (0,1)$ there exists $k_\epsilon \ge 1$ such that

$$\psi_k(k-\sqrt{k}) \le -2\sqrt{k}(-1+2(1-\epsilon)) \ \text{ for every } \ k \ge k_\epsilon\cdot \tag{32}$$

Let k fixed and large enough. A qualitative study carried out on ψ_k in $[1,+\infty)$, proves that there is a unique $t_k \in (1,k)$ such that $\psi_k'(t_k) = 0$ and so,

$$\int_{\sqrt{k}}^{k-\sqrt{k}} e^{\psi_k(t)}dt \le (k-2\sqrt{k})e^{\max(\psi_k(\sqrt{k},\psi_k(k-\sqrt{k}))}.$$

In addition, from (31) and (32) with $\epsilon \le \frac{3}{4}$, we obtain

$$\max[\psi_k(\sqrt{k}, \psi_k(k-\sqrt{k})] \le -\sqrt{k},$$

provided k sufficiently large. Hence, there exists $\bar{k} \ge 1$ such that

$$\int_{\sqrt{k}}^{k-\sqrt{k}} e^{\psi_k(t)}dt \le (k-2\sqrt{k})e^{-\sqrt{k}} \ \text{ for all } \ k \ge \bar{k}$$

and so

$$\lim_{k\to+\infty}\int_{\sqrt{k}}^{k-\sqrt{k}} e^{\left(2e^{t^{(\frac{b\log t}{\log k})}}-2t\right)}dt = 0. \tag{33}$$

Finally, we will study the limit of the integral on the interval $[k-\sqrt{k}, k]$.
We mention that for a fixed $k \ge 1$ large enough, ψ_k is a convex function on $[k-\sqrt{k}, +\infty)$, and $\psi_k(k) = 2k^b - 2k \le 0$, so, we can get this estimate

$$\psi_k(t) - \psi_k(k) \le \frac{k-t}{\sqrt{k}}\psi_k(k-\sqrt{k}), \ \ t \in [k-\sqrt{k}, k].$$

then,

$$\psi_k(t) \leq \frac{k-t}{\sqrt{k}}\psi_k(k-\sqrt{k}), \;\; t \in [k-\sqrt{k}, k].$$

Hence, in view of (31) and (32), if $\epsilon \in (0,1)$ and $k \geq k_\epsilon$ we have

$$\psi_k(t) \leq -2(2(1-\epsilon)-1)(t-k), \;\; t \in [k-\sqrt{k}, k], \tag{34}$$

furthermore, using the fact that ψ_k is convex on $[k-\sqrt{k}, +\infty)$ and $\psi_k'(k) = 4bk^{b-1} - 2$, we get

$$\psi_k(t) \geq \psi_k(k) + \varphi_k'(k)(t-k) = 2k^b - 2k + (4bk^{b-1}-2)(t-k), \;\; t \in [k-\sqrt{k}, k]\cdot \tag{35}$$

So,

$$\int_{k-\sqrt{k}}^{k} e^{\psi_k(t)}dt \geq \frac{e^{2k^b-2k}}{4k^{b-1}-2}\Big(1-e^{-\sqrt{k}}\Big). \tag{36}$$

Then by bringing together (34), (35) and (36), we deduce

$$\lim_{k\to+\infty}\frac{e^{2k^b-2k}}{4k^{b-1}-2}\Big(1-e^{-\sqrt{k}}\Big) \leq \lim_{k\to+\infty}\int_{k-\sqrt{k}}^{k} e^{\psi_k(t)}dt \leq \lim_{k\to+\infty}\frac{1}{2((1-\epsilon)2-1)}\Big(1-e^{-\sqrt{k}}\Big)\cdot$$

Since $\epsilon \in (0, \frac{1}{2})$ is arbitrary fixed, we can conclude that

$$\lim_{k\to+\infty}\int_{\sqrt{k}}^{k-\sqrt{k}} e^{\left(2e^{t^{\left(\frac{b\log t}{\log k}\right)}}-2t\right)}dt = \frac{1}{2}\cdot$$

So our claim (29) is proved, and the lemma follows.

5.2. Estimate of the energy $\mathcal{J}$

We are now going to the desired estimate.

Lemma 5.3. *Assume that (G_1), (G_2), (V_1), (H_1), (H_2) and (H_4) hold. Then the level given by (28) verifies*

$$d_* < \frac{1}{2}G(\frac{2\pi}{\alpha_0})\cdot$$

Proof. We have $v_n \geq 0$ and $\|v_n\| = 1$. Then from Lemma 4.4 $\mathcal{J}(tv_n) \to -\infty$ as $t \to +\infty$. As a consequence,

$$d_* \leq \max_{t\geq 0}\mathcal{J}(tv_n).$$

We argue by contradiction and we suppose that for all $n \geq 1$,

$$\max_{t\geq 0}\mathcal{J}(tv_n) \geq \frac{1}{2}G(\frac{2\pi}{\alpha_0}).$$

Since $\mathcal{J}$ possesses the mountain pass geometry, for any $n \geq 1$, there exists $t_n > 0$ such that

$$\max_{t\geq 0}\mathcal{J}(tv_n) = \mathcal{J}(t_n v_n) \geq \frac{1}{2}G(\frac{2\pi}{\alpha_0})$$

Using the fact that $F(x,t) \geq 0$ for all $(x,t) \in B \times \mathbb{R}$ we get

$$G(t_n^2) \geq G(\frac{2\pi}{\alpha_0}).$$

On one hand, the condition (G_1) implies that $G : [0, +\infty) \to [0, +\infty)$ is an increasing bijection. So

$$t_n^2 \geq \frac{2\pi}{\alpha_0}\cdot \tag{37}$$

On the other hand,

$$\frac{d}{dt}\mathcal{J}(tv_n)\Big|_{t=t_n} = g(t_n^2)t_n - \int_B f(x, t_n v_n)v_n dx = 0,$$

that is

$$g(t_n^2)t_n^2 = \int_B f(x, t_n v_n)t_n v_n dx. \tag{38}$$

Now, we claim that the sequence (t_n) is bounded in $(0, +\infty)$. Indeed, it follows from (H_4) that for all $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that

$$f(x, t)t \geq (\gamma_0 - \varepsilon)\exp\left(2e^{\alpha_0 t^2}\right) \ \forall |t| \geq t_\varepsilon, \ \text{ uniformly in } \ x \in B. \tag{39}$$

By Lemma 5.1, if $|x| \leq e^{-n}$ we have

$$v_n^2 \geq \frac{1}{2\pi}\frac{\log(1+n)}{1 + \frac{m}{\log(1+n)} + o_n(1)} = \frac{1}{2\pi}\log(1+n) - \frac{m}{2\pi} + o_n(1). \tag{40}$$

Using the condition (G_2), (39) and (40), we get

$$\begin{aligned} g(1)t_n^4 \geq g(t_n^2)t_n^2 \ &\geq \ (\gamma_0 - \epsilon)\int_{0\leq|x|\leq e^{-n}} \exp\left(2e^{\alpha_0 t_n^2 v_n^2}\right)dx \\ &\geq \ (\gamma_0 - \epsilon)\int_{0\leq|x|\leq e^{-n}} \exp\left(2e^{\alpha_0 t_n^2(\frac{1}{2\pi}\log(1+n) - \frac{m}{2\pi} + o_n(1))}\right)dx \\ &= \ \pi(\gamma_0 - \epsilon)\exp\left(2e^{\alpha_0 t_n^2(\frac{1}{2\pi}\log(1+n) - \frac{m}{2\pi} + o_n(1))} - 2n\right)\cdot \end{aligned} \tag{41}$$

Then, for n large enough, we obtain

$$1 \geq \pi(\gamma_0 - \varepsilon)\exp\left(2e^{\alpha_0 t_n^2(\frac{1}{2\pi}\log(1+n) - \frac{m}{2\pi} + o_n(1))} - 2n - \log\left(g(1)t_n^4\right)\right).$$

As a direct result, (t_n) is a bounded sequence. We must note that, if

$$\lim_{n\to+\infty} t_n^2 > \frac{2\pi}{\alpha_0}, \tag{42}$$

then we get a contradiction with the boundedness of (t_n). Indeed if (42) occurs , then there exists some $\delta > 0$ such that for n large enough,

$$t_n^2 \geq \delta + \frac{2\pi}{\alpha_0}\cdot$$

Thus

$$\frac{\alpha_0}{2\pi}t_n^2 \geq \ \frac{\alpha_0}{\pi}\delta + 1$$

and then the right hand of (41) tends to infinity which contradicts the boundedness of (t_n). Consequently (42) cannot hold, and we get

$$\lim_{n\to+\infty} t_n^2 = \frac{2\pi}{\alpha_0}\cdot \tag{43}$$

We claim that (43) leads to a contradiction with (H_5). Indeed, let us introduce the sets:

$$A_n = \{x \in B \mid t_n v_n \geq t_\varepsilon\} \text{ and } C_n = B \setminus A_n,$$

where t_ε is given in (32). We have

$$\begin{aligned} g(t_n^2)t_n^2 &= \int_B f(x, t_n v_n) t_n v_n dx = \int_{A_n} f(x, t_n v_n) t_n v_n dx + \int_{C_n} f(x, t_n v_n) t_n v_n \\ &\geq (\gamma_0 - \varepsilon) \int_{A_n} (2e^{\alpha_0 t_n^2 v_n^2})\, dx + \int_{C_n} f(x, t_n v_n) t_n v_n dx \\ &= (\gamma_0 - \varepsilon) \int_B \exp(2e^{\alpha_0 t_n^2 v_n^2})\, dx - (\gamma_0 - \varepsilon) \int_{C_n} \exp(2e^{\alpha_0 t_n^2 v_n^2})\, dx + \int_{C_n} f(x, t_n v_n) t_n v_n dx. \end{aligned}$$

Since $v_n \to 0$ a.e in B and $\chi_{C_n} \to 1$ a.e in B, therefore using the dominated convergence theorem, we get

$$\lim_{n \to +\infty} g(t_n^2)t_n^2 = g(\frac{2\pi}{\alpha_0})\frac{2\pi}{\alpha_0} \geq (\gamma_0 - \varepsilon) \lim_{n \to +\infty} \int_B \exp(2e^{\alpha_0 t_n^2 v_n^2})\, dx - (\gamma_0 - \varepsilon)\pi e^2.$$

Using the fact that

$$t_n^2 \geq \frac{2\pi}{\alpha_0},$$

we get

$$\int_B \exp(2e^{\alpha_0 t_n^2 v_n^2})\, dx \geq \int_{0 \leq |x| \leq e^{-n}} \exp(2e^{2\pi v_n^2})\, dx + \int_{e^{-n} \leq |x| \leq 1} \exp(2e^{2\pi v_n^2})\, dx.$$

On the one hand, we have by (40)

$$\begin{aligned} \int_{0 \leq |x| \leq e^{-n}} \exp(2e^{2\pi v_n^2})\, dx &\geq \int_{0 \leq |x| \leq e^{-n}} \exp\left(2e^{\log(1+n) - m + o_n(1))}\right) dx \\ &= \pi \exp\left(2 + 2n - m + o_n(1)\right) e^{-2n} \\ &= \pi \exp\left(2 - m + o_n(1)\right) \to \pi e^{2-m} \text{as } n \to +\infty. \end{aligned}$$

On the other hand, using the definition of v_n and the result of Lemma 4.4, with the change of variable $|x| = e^{-t}$, we get

$$\begin{aligned} \int_{e^{-n} \leq |x| \leq 1} \exp(2e^{2\pi v_n^2})\, dx &= \int_{e^{-n} \leq |x| \leq 1} \exp\left(2e^{\frac{\log^2(\log(\frac{e}{|x|}))}{\|w_n\|^2 \log(1+n)}}\right) dx \\ &= 2\pi \int_{e^{-n}}^1 \exp\left(2e^{\frac{\log^2(\log(\frac{e}{r}))}{\|w_n\|^2 \log(1+n)}}\right) r dr \\ &= 2\pi \int_0^n \exp\left(2e^{\frac{\log^2(1+t)}{\log(1+n)\|w_n\|^2}} - 2t\right) dt \\ &\geq 2\pi e^2. \end{aligned}$$

Hence,

$$\lim_{n \to +\infty} g(t_n^2)t_n^2 = g(\frac{2\pi}{\alpha_0})\frac{2\pi}{\alpha_0} \geq (\gamma_0 - \varepsilon)\pi e^2\left(1 + e^{-m}\right).$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\frac{2g(\frac{2\pi}{\alpha_0})}{\alpha_0 e^2\left(1 + e^{-m}\right)} \geq \gamma_0.$$

This contradicts (H_4) and the lemma is proved.

6. Proof of main results

Now, we consider the Nehari manifold associated to the functional $\mathcal{J}$, namely,

$$\mathcal{N} = \{u \in \mathcal{E} : \langle \mathcal{J}'(u), u\rangle = 0, u \neq 0\},$$

and the number $c = \inf_{u\in\mathcal{N}} \mathcal{J}(u)$. We have the following lemmas.

Lemma 6.1. *[21] Assume that the conditions (H_1) and (H_3) hold, then for each $x \in B$,*

$$tf(x,t) - 4F(x,t) \text{ is increasing for } t > 0.$$

In particular, $tf(x,t) - 4F(x,t) \geq 0$ for all $(x,t) \in B \times [0,+\infty)$.

Proof : Assume that $0 < t < s$. For each $x \in B$, we have

$$\begin{array}{rcl} tf(x,t) - 4F(x,t) & = & \dfrac{f(x,t)}{t^3}t^4 - 4F(x,s) + 4\displaystyle\int_t^s f(x,v)dv \\ & < & \dfrac{f(x,t)}{s^3}t^4 - 4F(x,s) + \dfrac{f(x,s)}{s^3}(s^4 - t^4) \\ & = & sf(x,s) - 4F(x,s)\cdot \end{array}$$

Lemma 6.2. *Assume that (G_2), (V_1), (H_1) and (H_3) are satisfied. Then the level d_* given by (28) verifies*

$$d_* \leq c.$$

Proof : Let $\bar{u} \in \mathcal{N}$ and consider the function $\psi : (0,+\infty) \to \mathbb{R}$ defined by $\psi(t) = \mathcal{J}(t\bar{u})$. The function ψ is differentiable and we have

$$\psi'(t) = \langle \mathcal{J}'(t\bar{u}), \bar{u}\rangle = g(t^2\|\bar{u}\|^2)t\|\bar{u}\|^2 - \int_B f(x,t\bar{u})\bar{u}\,dx, \text{for all } t > 0.$$

Since $\bar{u} \in \mathcal{N}$, we have $\langle \mathcal{J}'(\bar{u}), \bar{u}\rangle = 0$ and therefore $g(\|\bar{u}\|^2)\|\bar{u}\|^2 = \int_B f(x,\bar{u})\bar{u}\,dx$. Hence,

$$\psi'(t) = t^3\|\bar{u}\|^4\Big(\frac{g(t^2\|\bar{u}\|^2)}{t^2\|\bar{u}\|^2} - \frac{g(\|\bar{u}\|^2)}{\|\bar{u}\|^2}\Big) + t^3\int_B\Big(\frac{f(x,\bar{u})}{\bar{u}^3} - \frac{f(x,t\bar{u})}{(t\bar{u})^3}\Big)\,dx\cdot$$

We have that $\psi'(1) = 0$. We also have by the conditions (G_2) and (H_3) that $\psi'(t) > 0$ for all $0 < t < 1$, $\psi'(t) \leq 0$ for all $t > 1$. It follows that

$$\mathcal{J}(\bar{u}) = \max_{t\geq 0} \mathcal{J}(t\bar{u}).$$

We define the function $\lambda : [0,1] \to \mathcal{E}$ such that $\lambda(t) = t\bar{t}\bar{u}$, with $\mathcal{J}(\bar{t}\bar{u}) < 0$. We have $\lambda \in \Lambda$, and hence

$$d_* \leq \max_{t\in[0,1]} \mathcal{J}(\lambda(t)) \leq \max_{t\geq 0} \mathcal{J}(t\bar{u}) = \mathcal{J}(\bar{u}).$$

Since $\bar{u} \in \mathcal{N}$ is arbitrary then $d_* \leq c$.

The main difficulty in the approach to the critical problem of growth is the loss of compactness. Precisely the overall conditions of Palais-Smale are not verified except for a certain level of energy. In the following proposition, we identify the first non compactness level.

Proposition 6.3. *Let $\mathcal{J}$ be the energy associated to problem (1) defined by (16). Assume that the conditions (G_1), (G_2), (H_1), (H_2), (H_3) and (H_4) are satisfied. Then*

(i) In the subcritical case, the functional $\mathcal{J}$ satisfies the Palais-Smale condition $(PS)_d$ at all level $d \in \mathbb{R}$.

(ii) In the critical case, the functionnal $\mathcal{J}$ satisfies the Palais-Smale condition $(PS)_d$ only for level d such that $d < \dfrac{1}{2}G(\dfrac{2\pi}{\alpha_0})$.

Proof. We start with the second item.
(ii) Consider a $(PS)_d$ sequence (u_n) in $\mathcal{E}$, for some $d \in \mathbb{R}$, that is

$$\mathcal{J}(u_n) = \frac{1}{2}G(\|u_n\|^2) - \int_B F(x, u_n)\,dx \to d, \quad n \to +\infty \tag{44}$$

and

$$|\langle \mathcal{J}'(u_n), \varphi\rangle| = \Big|g(\|u_n\|^2)\Big[\int_B \sigma(x)\nabla u_n.\nabla\varphi\,dx + \int_B V(x)u\varphi\,dx\Big] - \int_B f(x, u_n)\varphi\,dx\Big| \le \epsilon_n\|\varphi\|, \tag{45}$$

for all $\varphi \in \mathcal{E}$, where $\epsilon_n \to 0$, when $n \to +\infty$.
From (44) for large enough n, there exists a constant $C > 0$ such that

$$\frac{1}{2}G(\|u_n\|^2) \le C + \int_B F(x, u_n)\,dx\cdot$$

From (12), for all $\epsilon > 0$, there exists $t_\epsilon > 0$ such that

$$F(x, t) \le \epsilon t f(x, t), \quad \text{for all } |t| > t_\epsilon \text{ and uniformly in } x \in B.$$

It follows that,

$$\frac{1}{2}G(\|u_n\|^2) \le C + \int_{|u_n|\le t_\epsilon} F(x, u_n)\,dx + \epsilon\int_B f(x, u_n)u_n\,dx\cdot$$

From (45) and (G_2) , we get

$$\frac{1}{4}g(\|u_n\|^2)\|u_n\|^2 \le \frac{1}{2}G(\|u_n\|^2) \le C_1 + \epsilon\epsilon_n\|u_n\| + \epsilon g(\|u_n\|^2)\|u_n\|^2,$$

for some constant $C_1 > 0$.
Using (11) and the condition (G_1), for all ϵ such that $0 < \epsilon < \frac{1}{4}$, we get

$$g_0(\frac{1}{4} - \epsilon)\|u_n\|^2 \le C_1 + \epsilon\epsilon_n\|u_n\|\cdot$$

We deduce that the sequence (u_n) is bounded in $\mathcal{E}$. Consequently, there exists $u \in \mathcal{E}$ such that, up to subsequence, $u_n \rightharpoonup u$ weakly in $\mathcal{E}$, $u_n \to u$ strongly in $L^q(B)$, for all $q \ge 1$ and $u_n(x) \to u(x)$ *a.e.* Furthermore, we have from (44) and (45), that

$$0 < \int_B |f(x, u_n)u_n| \le C, \tag{46}$$

and

$$0 < \int_B F(x, u_n) \le C. \tag{47}$$

By Lemma 3.2, we have

$$f(x, u_n) \to f(x, u) \text{ in } L^1(B) \text{ as } n \to +\infty. \tag{48}$$

It follows from (H_2) and the generalized Lebesgue dominated convergence theorem that

$$F(x, u_n) \to F(x, u) \text{ in } L^1(B) \text{ as } n \to +\infty. \tag{49}$$

So,

$$\lim_{n\to+\infty} G(\|u_n\|^2) = 2(d + \int_B F(x,u)\,dx). \tag{50}$$

Next, we are going to make some claims.

Claim 1. $u > 0$. Indeed, since (u_n) is bounded, up to a subsequence, $\|u_n\| \to \rho > 0$. In addition, $\mathcal{J}'(u_n) \to 0$ leads to

$$g(\rho^2)\Big[\int_B \sigma(x)\nabla u.\nabla\varphi + V(x)u\varphi\,dx\Big] = \int_B f(x,u)\varphi\,dx, \ \ \forall\varphi \in \mathcal{E}.$$

By taking $\varphi = u^-$, with $w^\pm = \max(\pm w, 0)$, we get $\|u^-\|^2 = 0$ and so $u = u^+ \geq 0$. Since the nonlinearity has critical growth at $+\infty$ and from Trudinger-Moser inequality (8), $f(.,u) \in L^p(B)$, for all $p \geq 1$. So, by elliptic regularity $u \in W^{2,p}(B,\sigma)$, for all $p \geq 1$. Therefore, by Sobolev imbedding $u \in C(\overline{B})$.
Let define $B_0 = \{x \in B : u(x) = 0\}$. The set $B_0 = \emptyset$. Indeed, suppose by contradiction that $B_0 \neq \emptyset$. Since $f(x,u) \geq 0$, by Harnack inequality see ([19], Theorem 1.9), we can deduce that B_0 is an open and closed set of B. In virtue of the connectedness of B, we reach a contradiction. Hence Claim 1 is proved.

Claim 2. $g(\|u\|^2)\|u\|^2 \geq \int_B f(x,u)u\,dx$. The claim holds in the case $\|u\| = 0$. So we can assume that $\|u\| \neq 0$.
Then, we proceed by contradiction and we suppose that $g(\|u\|^2)\|u\|^2 < \int_B f(x,u)u\,dx$. Hence, $\langle\mathcal{J}'(u),u\rangle < 0$. The function $\psi : t \to \psi(t) = \langle\mathcal{J}'(tu),u\rangle$ is positive for t small enough. Indeed, from (13) and the critical (resp subcritical) growth of the non linearity f, for every $\epsilon > 0$, for every $q > 2$, there exist positive constants C and c such that

$$|f(x,t)| \leq \epsilon t + Ct^q \exp(e^{c\ t^2}), \qquad \forall, (t,x) \in \mathbb{R}\times B.$$

Then using the condition (G_1), the last inequality and the Hölder inequality we obtain

$$\psi(t) = g(t^2\|u\|^2)t\|u\|^2 - \int_B f(x,tu)u\,dx \geq g_0 t\|u\|^2 - \epsilon t\int_B u^2\,dx - C\Big(\int_B \exp(2e^{c\ t^2u^2})\,dx\Big)^{\frac{1}{2}}\Big(\int_B u^{2q}\,dx\Big)^{\frac{1}{2}}.$$

In view of (8) the integral $\int_B \exp(2e^{c\ t^2u^2})\,dx \leq \int_B \exp(2e^{c\ t^2\frac{u^2}{\|u\|^2}\|u\|^2})\,dx \leq C$, provided $t \leq \frac{1}{\|u\|}\sqrt{\frac{2\pi}{c}}$. Using the radial Lemma 3.1 we get $\|u\|^2_{2q} \leq C'\|u\|^q$. Then,

$$\psi(t) \geq g_0 t\|u\|^2 - C_1\epsilon t\|u\|^2 - C_2\|u\|^q = \|u\|^2 t\Big[(g_0 - C_1\epsilon) - C_2 t^{q-1}\|u\|^{q-2}\Big].$$

We chose $\epsilon > 0$, such that $g_0 - C_1\epsilon > 0$ and since $q > 2$, for small t, we get $\psi : t \to \psi(t) = \langle\mathcal{J}'(tu),u\rangle > 0$. So there exists $\eta \in (0,1)$ such that $\psi(\eta u) = 0$. Therefore $\eta u \in \mathcal{N}$. Using the condition (G_2), the result of Lemma 5.2, the semicontinuity of norm and Fatou's Lemma we get

$$\begin{array}{rcl}
d \leq d_* \leq c \leq \mathcal{J}(\eta u) & = & \mathcal{J}(\eta u) - \frac{1}{4}\langle\mathcal{J}'(\eta u),\eta u\rangle \\
& = & \frac{1}{2}G(\|\eta u\|^2) - \frac{1}{4}g(\|\eta u\|^2)\|\eta u\|^2 + \frac{1}{4}\int_B \Big(f(x,\eta u)\eta u - 4F(x,\eta u)\Big)\,dx \\
& < & \frac{1}{2}G(\|u\|^2) - \frac{1}{4}g(\|u\|^2)\|u\|^2 + \frac{1}{4}\int_B \Big(f(x,u)u - 4F(x,u)\Big) \\
& \leq & \liminf_{n\to+\infty}\Big[\frac{1}{2}G(\|u_n\|^2) - \frac{1}{4}g(\|u_n\|^2)\|u_n\|^2\Big] \\
& + & \liminf_{n\to+\infty}\Big[\frac{1}{4}\int_B \Big(f(x,u_n)u_n - 4F(x,u_n)\Big)\,dx\Big] \\
& \leq & \lim_{n\to+\infty}\Big[\mathcal{J}(u_n) - \frac{1}{4}\langle\mathcal{J}'(u_n),u_n\rangle\Big] = d
\end{array}$$

which is absurd and the claim is well established.

On the other hand, by claim 2, (13) and Lemma 6.2 we obtain

$$\mathcal{J}(u) \geq \frac{1}{2}G(\|u\|^2) - \frac{1}{4}g(\|u\|^2)\|u\|^2 + \frac{1}{4}\int_B [f(x,u) - 4F(x,u)]\,dx \geq 0. \tag{51}$$

Now, using the semicontinuity of the norm and (50) we get,

$$\mathcal{J}(u) \leq \frac{1}{2}\liminf_{n\to\to\infty} G(\|u_n\|^2) - \int_B F(x,u)\,dx = d.$$

Therefore, $d \geq 0$.

We will finish the proof by considering three cases for the level d.

Case 1. $d = 0$. In this case

$$0 \leq \mathcal{J}(u) \leq \liminf_{n\to+\infty} \mathcal{J}(u_n) = 0.$$

So,

$$\mathcal{J}(u) = 0$$

and then

$$\lim_{n\to+\infty} \frac{1}{2}G(\|u_n\|^2) = \int_B F(x,u)\,dx = \frac{1}{2}G(\|u\|^2).$$

Consequently,

$$\|u_n\| \to \|u\|$$

and therefore $u_n \to u$ in $\mathcal{E}$.

Case 2. $d > 0$ and $u = 0$. We prove that this case cannot happen. Indeed, we argue by contradiction and suppose that $u = 0$. Therefore, $\int_B F(x,u_n)\,dx \quad \to 0$ and consequently we get

$$\frac{1}{2}G(\|u_n\|^2) \to d < \frac{1}{2}G(\frac{2\pi}{\alpha_0}).$$

So, there exist $n_0 \in \mathbb{N}$ and $\eta \in (0,1)$ such that $\alpha_0\|u_n\|^2 = (1-\eta)2\pi$, for all $n \geq n_0$. By (45), we also have

$$\left|g(\|u_n\|^2)\|u_n\|^2 - \int_B f(x,u_n)u_n\,dx\right| \leq C\epsilon_n.$$

First we claim that there exists $q > 1$ such that

$$\int_B |f(x,u_n)|^q\,dx \leq C. \tag{52}$$

So

$$g(\|u_n\|^2)\|u_n\|^2 \leq C\epsilon_n + \Big(\int_B |f(x,u_n)|^q\Big)^{\frac{1}{q}}\,dx\Big(\int_B |u_n|^{q'}\Big)^{\frac{1}{q'}}$$

where q' is the conjugate of q. Since (u_n) converge to $u = 0$ in $L^{q'}(B)$

$$\lim_{n\to+\infty} g(\|u_n\|^2)\|u_n\|^2 = 0.$$

From the condition (G_1), we obtain
$$\lim_{n\to+\infty} \|u_n\|^2 = 0.$$
Therefore, $\mathcal{J}(u_n) \to 0$ which is in contradiction with $d > 0$.
Now, we will prove the claim (52). Since f has critical growth, for every $\epsilon > 0$ and $q > 1$ there exists $t_\epsilon > 0$ and $C > 0$ such that for all $|t| \geq t_\epsilon$, we have
$$|f(x,t)|^q \leq C \exp(2e^{\alpha_0(\epsilon+1)t^2}). \tag{53}$$
Consequently,
$$\begin{aligned}\int_B |f(x,u_n)|^q\,dx &= \int_{\{|u_n|\leq t_\epsilon\}} |f(x,u_n)|^q\,dx + \int_{\{|u_n|>t_\epsilon\}} |f(x,u_n)|^q\,dx\\ &\leq \pi \max_{B\times[-t_\epsilon,t_\epsilon]} |f(x,t)|^q + C\int_B \exp\left(2e^{\alpha_0(\epsilon+1)|u_n|^2}\right) dx.\end{aligned}$$
Since, there exist $n_0 \in \mathbb{N}$ and $\eta \in (0,1)$ such that $\alpha_0\|u_n\|^2 = (1-\eta)2\pi$, for all $n \geq n_0$, then
$$\alpha_0(1+\epsilon)(\frac{|u_n|}{\|u_n\|})^2\|u_n\|^2 \leq (1+\epsilon)(1-\eta)2\pi\cdot$$
We choose $\epsilon > 0$ small enough to get
$$\alpha_0(1+\epsilon)\|u_n\|^2 \leq 2\pi,$$
therefore the second integral is uniformly bounded in view of (8).

Case 3. $d > 0$ and $u \neq 0$. In this case, we claim that $\mathcal{J}(u) = d$ and therefore we get
$$\lim_{n\to+\infty} G(\|u_n\|^2) = 2\Big(d + \int_B F(x,u)\,dx\Big) = G(\|u\|^2).$$
So,
$$\|u_n\| \to \|u\|.$$
Now, using the semicontinuity of the norm and (44) we get,
$$\mathcal{J}(u) \leq \frac{1}{2}\liminf_{n\to\infty} G(\|u_n\|^2) - \int_B F(x,u)\,dx = d.$$
Since (u_n) is bounded, up to a subsequence, $\|u_n\| \to \rho > 0$. Suppose that
$$\mathcal{J}(u) < d. \tag{54}$$
Then
$$\|u\|^2 < \rho^2. \tag{55}$$
In addition,
$$\frac{1}{2}G(\rho^2) = \frac{1}{2}\lim_{n\to+\infty} G(\|u_n\|^2) = \Big(d + \int_B F(x,u)\,dx\Big), \tag{56}$$
which means that
$$\rho^2 = G^{-1}(\Big(2(d + \int_B F(x,u)\,dx\Big)).$$
Set
$$v_n = \frac{u_n}{\|u_n\|}$$

and

$$v = \frac{u}{\rho}\cdot$$

We have $\|v_n\| = 1$, $v_n \rightharpoonup v$ in $\mathcal{E}$, $v \not\equiv 0$ and $\|v\| < 1$. So, by Lemma 3.3, we get

$$\sup_n \int_B \exp\left(2e^{2\pi p|v_n|^2}\right) dx < \infty$$

for $1 < p < (1 - \|v\|^2)^{-1}$.
On the other hand, by claim 1, (13) and Lemma 5.3, we obtain

$$\mathcal{J}(u) \geq \frac{1}{2}G(\|u\|^2) - \frac{1}{4}g(\|u\|^2)\|u\|^2 + \frac{1}{4}\int_B [f(x,u) - 4F(x,u)]\, dx \geq 0. \tag{57}$$

From (57), Lemma 5.2 and the following equality

$$2d - 2\mathcal{J}(u) = G(\rho^2) - G(\|u\|^2),$$

we get

$$G(\rho^2) \leq 2d + G(\|u\|^2) < G(\frac{2\pi}{\alpha_0}) + G(\|u\|^2).$$

Now, using the condition (G_1) one has

$$\rho^2 < G^{-1}\Big(G(\frac{2\pi}{\alpha_0}) + G(\|u\|^2)\Big) \leq \frac{2\pi}{\alpha_0} + \|u\|^2. \tag{58}$$

Since

$$\rho^2 = \frac{\rho^2 - \|u\|^2}{1 - \|v\|^2},$$

we deduce from (58) that

$$\rho^2 < \frac{\frac{2\pi}{\alpha_0}}{1 - \|v\|^2}\cdot$$

Then there exists $\delta \in (0, \frac{1}{2})$ such that $\rho^2 = (1 - 2\delta)\frac{\frac{2\pi}{\alpha_0}}{1 - \|v\|^2}\cdot$
On one hand, we have this estimate $\int_B |f(x,u_n)|^q\, dx < C$. Indeed, For $\epsilon > 0$,

$$\begin{aligned}
\int_B |f(x,u_n)|^q\, dx &= \int_{\{|u_n| \leq t_\epsilon} |f(x,u_n)|^q\, dx + \int_{\{|u_n| > t_\epsilon\}} |f(x,u_n)|^q\, dx \\
&\leq \pi \max_{B\times[-t_\epsilon, t_\epsilon]} |f(x,t)|^q + C\int_B \exp\left(2e^{\alpha_0(1+\epsilon)|u_n|^2}\right) dx \\
&\leq C_\epsilon + C\int_B \exp\left(2e^{\alpha_0(1+\epsilon)\|u_n\|^2|v_n|^2}\right) dx \leq C,
\end{aligned}$$

provided $\alpha_0(1+\epsilon)\|u_n\|^2 \leq 2\pi p$, for p such that $1 < p < (1 - \|v\|^2)^{-1}$.
On the other hand, since

$$\lim_{n\to+\infty} \|u_n\|^2 = \rho^2,$$

then for n large enough, we get

$$\alpha_0(1+\epsilon)\|u_n\|^2 \leq \alpha_0(1+\epsilon)\rho^2 \leq (1+\epsilon)(1-\delta)2\pi\frac{1}{1 - \|v\|^2}\cdot$$

We choose $\epsilon > 0$ small enough such that $(1+\epsilon)(1-\delta) < 1$ which means

$$\alpha_0(1+\epsilon)\|u_n\|^2 < 2\pi\frac{1}{1-\|v\|^2}.$$

So the sequence $(f(x,u_n))$ is bounded in $L^q(B)$, $q > 1$. Using the Hölder inequality, we deduce that

$$\begin{array}{rcl}\Big|\displaystyle\int_B f(x,u_n)(u_n-u)\,dx\Big| & \leq & \Big(\displaystyle\int_B |f(x,u_n)|^q\,dx\Big)^{\frac{1}{q}}(\displaystyle\int_B |u_n-u|^{q'})^{\frac{1}{q'}}\,dx\\ & \leq & C(\displaystyle\int_B |u_n-u|^{q'})^{\frac{1}{q'}}\,dx \to 0 \text{ as } n\to+\infty,\end{array}$$

where $\frac{1}{q}+\frac{1}{q'}=1$. Since $\langle \mathcal{J}'(u_n), u_n-u\rangle = o_n(1)$, it follows that

$$g(\|u_n\|^2)\Big[\int_B \sigma(x)\nabla u_n(\nabla u_n-\nabla u)+\int_B V(x)u_n(u_n-u)\,dx\Big]\to 0.$$

On the other side,

$$g(\|u_n\|^2)\Big[\int_B \sigma(x)\nabla u_n(\nabla u_n-\nabla u)+\int_B V(x)u_n(u_n-u)\,dx\Big] = g(\|u_n\|^2)\|u_n\|^2 - g(\|u_n\|^2)\langle u_n,u\rangle.$$

Passing to the limit in the last equality, we get

$$g(\rho^2)\rho^2 - g(\rho^2)\|u\|^2 = 0,$$

therefore $\|u\| = \rho$ and $\|u_n\| \to \|u\|$. This is in contradiction with (55). It follows that $G(\rho^2) = G(\|u\|)$ and consequently $\mathcal{J}(u) = d$. Also, from (45) and (48) we get

$$g(\|u\|^2)\Big[\int_B \sigma(x)\nabla u.\nabla\varphi + V(x)u\varphi\,dx\Big] = \int_B f(x,u)\varphi\,dx,\quad \forall\varphi\in\mathcal{E}.$$

So u is a solution of problem (1). We also have $u_n \to u$ strongly in $\mathcal{E}$.

(i) From the proof of (ii), up a subsequence (u_n), there exists $M > 0$, such that $\|u_n\| \leq M$. By the subcritical case of f at $+\infty$, for some $q > 1$, there exist $\alpha \leq \dfrac{2\pi}{M^2}$ and positive constants $C_{1,q}$ and $C_{2,q}$, such that

$$\begin{array}{rcl}\displaystyle\int_B |f(x,u_n)|^q\,dx & = & C_{1,q}+C_{2,q}\displaystyle\int_B \exp\left(2e^{\alpha|u_n|^2}\right)dx\\ & \leq & C_{1,q}+C_{2,q}\displaystyle\int_B \exp\left(2e^{\alpha\frac{|u_n|^2}{\|u_n\|^2}\|u_n\|^2}\right)dx\\ & \leq & C_{1,q}+C_{2,q}\displaystyle\int_B \exp\left(2e^{2\pi\frac{|u_n|^2}{\|u_n\|^2}}\right)dx < +\infty.\end{array}$$

We conclude as in (ii).

□

Proof of Theorem 1.3

Since $f(x,t)$ satisfies the condition (9) for all $\alpha_0 > 0$, then by Proposition 6.3, the functional $\mathcal{J}$ satisfies the (PS) condition (at each possible level d). So, by Lemma 4.3 and Lemma 4.4, we deduce that the functional $\mathcal{J}$ has a nonzero critical point u in $\mathcal{E}$. □

Proof of Theorem 1.4

In the critical case, again by proposition 6.3, Lemma 4.3 and Lemma 4.4 , the energy $\mathcal{J}$ satisfies the $(PS)_d$ condition for all $d < \dfrac{1}{2}G(\dfrac{2\pi}{\alpha_0})$. Therefore, $\mathcal{J}$ has a nonzero critical point u in $\mathcal{E}$.

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