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On the categories of probabilistic approach groups: actions

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Abstract. Starting with the category of probabilistic approach groups, we show that the category of approach groups can be embedded into the category of probabilistic approach groups as a bicoreflective subcategory; further, considering a category of probabilistic topological convergence groups, we show that the category of probabilistic topological convergence groups is *isomorphic* to the category of probabilistic approach groups under so-called triangle function $\tau\colon \Delta^+ \times \Delta^+ \longrightarrow \Delta^+$, where Δ^+ is the set of all *distance distribution functions* that plays a central role for probabilistic metric spaces. Moreover, if we allow this triangle function τ to be *sup-continuous*, then we can show that the category of probabilistic metric groups can be embedded into the category of probabilistic approach groups as a coreflective subcategory. Furthermore, we demonstrate that every T_1 probabilistic topological convergence group satisfying so-called (PM) axiom is probabilistic metrizable. Finally, among others, introducing a category of probabilistic approach transformation groups, we show that the category of probabilistic topological convergence transformation groups is isomorphic to the category of probabilistic approach transformation groups; this solves an open problem that proposed in one of our earlier papers. Moreover, we prove that the category of probabilistic metric transformation groups is isomorphic to the category of probabilistic metric probabilistic convergence transformation groups.

1. Introduction

There are various types of generalization of classical metric spaces; herein this text, we are concerned about two types of such generalizations: one, *approach spaces*, which is based on *point-to-set distances*, instead of *point-to-point distances*. Given a metric space (S, d) or more generally an extended pseudometric space, one can define an induced map $d\colon S \times P(S) \longrightarrow [0, \infty]$ by $d(x, A) = \inf\{d(x, a)\colon a \in A\}$. With the point of view of this example, a pair (S, d), where $d\colon S \times P(S) \longrightarrow [0, \infty]$ is a distance function, called an *approach space* if the following axioms are satisfied:

(AP1) $d(p, \{p\}) = 0$, for all $p \in S$;
(AP2) $d(p, \emptyset) = \infty$, for all $p \in S$;
(AP3) $d(p, A \cup B) = d(p, A) \wedge d(p, B)$, for all $p \in S$ and $A, B \in P(S)$;

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(AP4) $d(p, A) \leq d(p, A^{(\epsilon)}) + \epsilon$, for all $p \in S$, $A \subseteq S$ and $\epsilon \in [0, \infty]$, whence $A^{(\epsilon)} = \{p \in S \colon d(p, A) \leq \epsilon\}$.

A mapping $f \colon (S, d) \longrightarrow (S', d')$ between approach spaces is called a *contraction* mapping if $d'(f(p), f(A)) \leq d(p, A)$, for all $p \in S$ and $A \in P(S)$. The category of all approach spaces and contraction mappings is denoted by **Ap**, and this category is attributed to Lowen. [25]; this category is further studied by many authors from various perspectives, cf. [10, 12, 13, 18, 20, 21, 24, 26]. The other one is, the generalization in which the distances between points are specified by *probability distributions rather than numbers*. The general notion for this type of generalization originated from the work of K. Menger, [28], and has since been developed by a number of authors. A very useful book where a comprehensive treatment of the subject is given by B. Schweizer and S. Sklar, [32] (see also, [11, 15, 16, 19, 31, 33]).

In 2017, Jäger [20], argued that it is reasonable to assign a point $p \in S$ and a subset $A \subseteq S$ a distance distribution function $\delta(p, A)$, whose value at x, $\delta(p, A)(x)$ is then interpreted as the *probability that the distance between p and A is less than x*. With this point of view, he generalized the concept of approach spaces by introducing a category of probabilistic approach spaces, **ProbAp**, where the codomain $([0, \infty], \geq, +)$ so-called Lawvere quantale, [14, 27], is replaced by $(\Delta^+, \leq, \tau)$, where Δ^+ is the set of distance distribution functions. In doing so, it is proved in [20] that the category of approach spaces is isomorphic to a simultaneously bireflective and bicoreflective subcategory, and that the category of probabilistic quasi-metric spaces is isomorphic to a bicoreflective subcategory of the category of probabilistic approach spaces. In this respect, it is paramount important to note that the most significant result that he obtained is the *isomorphism* between the category **ProbAp** and **probTConv**, the category of probabilistic topological convergence spaces.

We introduced in [2], the concept of probabilistic convergence groups and studied its two important aspects, one is *uniformization* and the other is, *metrization*, of probabilistic convergence groups. We also introduced a category of probabilistic convergence transformation groups, $\mathbf{ProbConvTrGrp}_\tau$ under the triangle function τ in [3]. In this present paper, starting with the notions of probabilistic approach group, and probabilistic topological convergence group, we show, among others, that the category **ApGrp** can be embedded into $\mathbf{ProbApGrp}_{\tau_*}$ under the triangle function $\tau_* \colon \Delta^+ \times \Delta^+ \longrightarrow \Delta^+$ where $*$ is a continuous t-norm. Also, we show that the category of probabilistic topological convergence groups, **PobTopConvGrp** is isomorphic to the category of probabilistic approach groups, **ProbApGrp**. Needless to mention that this is possible due to the isomorphism between **ProbAp** and **ProbTopConv** initiated in [20]. It is worth mentioning that in [5], considering arbitrary quantale, we introduced and studied quantale-valued approach spaces, where we were unable to produce an isomorphism between the category of quantale-valued approach groups and the category of quantale-valued topological convergence groups, proving just only one part of it. In studying all these structures we are able to produce natural examples such as probabilistic metric spaces and probabilistic metric groups; it goes without saying that the probabilistic metric spaces is of paramount importance in random functional analysis, especially, due to its extensive applications in random differential as well as random integral equations, by saying this, we mean the study of the category of probabilistic metric spaces and its subcategories deserve specials attention.

Finally, introducing a category of probabilistic approach transformation groups, $\mathbf{ProbApTrGrp}_\tau$, a subcategory of $\mathbf{ProbAp}_\tau$, and extending the existing category of probabilistic convergence transformation groups into the category of probabilistic topological transformation groups, $\mathbf{ProbTopConvTrGrp}_\tau$, we show that these two categories are isomorphic, that is, $\mathbf{ProbApTrGrp}_\tau$ is isomorphic to $\mathbf{ProbTopConvTrGrp}_\tau$. In [3], we could only prove that every probabilistic metric transformation group is probabilistic convergence group, we extend these results to prove that the category of probabilistic metric transformation groups is in fact isomorphic to the category of probabilistic metric probabilistic convergence transformation groups.

2. Preliminaries

We recall some notions from [32] that are used in the sequel.

A function $\varphi : [0, \infty] \longrightarrow [0, 1]$, which is non-decreasing, left-continuous on $(0, \infty)$ and satisfies $\varphi(0) = 0$ and $\varphi(\infty) = 1$ is called a *distance distribution function*(ddf). The set of all ddf is denoted by Δ^+. For example, for each $0 \leq a < \infty$ the functions

$$\epsilon_a(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq a \\ 1 & \text{if } a < x \leq \infty \end{cases}$$

$$\epsilon_\infty(x) = \begin{cases} 0 & \text{if } 0 \leq x < \infty \\ 1 & \text{if } x = \infty \end{cases}$$

are in Δ^+. The set Δ^+ is ordered point-wise, i.e., for $\varphi, \psi \in \Delta^+$ we define $\varphi \leq \psi$ if for all $x \geq 0$ we have $\varphi(x) \leq \psi(x)$. The smallest element of Δ^+ is then ϵ_∞ and the largest element is ϵ_0. Observe that $(\Delta^+, \leq)$ is a complete lattice with ϵ_0 as the top element and ϵ_∞ as the bottom element. It follows from [17] that φ is called *well-below* ψ written as $\varphi \triangleleft \psi$ if $\forall D \subseteq \Delta^+$ such that $\psi \leq \vee D$, there exists $\varrho \in D$ such that $\varphi \leq \varrho$.

On Δ^+, we consider the modified *Lévy metric*, [32] which is defined below.
Let $\varphi, \psi \in \Delta^+$ and $\epsilon > 0$. Consider the following properties:

$$A(\varphi, \psi, \epsilon) \Leftrightarrow \varphi(x - \epsilon) - \epsilon \leq \psi(x), \text{ if } x \in [0, \tfrac{1}{\epsilon});$$
$$\text{and } B(\varphi, \psi, \epsilon) \Leftrightarrow \varphi(x + \epsilon) + \epsilon \geq \psi(x), \text{ if } x \in [0, \tfrac{1}{\epsilon}).$$

Then the *modified Lévy metric* D_L on Δ^+ is given by

$$D_L(\varphi, \psi) = \bigwedge\{\epsilon > 0 \colon A(\varphi, \psi, \epsilon) \text{ and } B(\varphi, \psi, \epsilon) \text{ holds }\}.$$

A binary operation $\tau \colon \Delta^+ \times \Delta^+ \longrightarrow \Delta^+$, which is commutative, associative, non-decreasing in each place and satisfies the boundary condition $\tau(\varphi, \epsilon_0) = \varphi$ for all $\varphi \in \Delta^+$, is called a *triangle function*. The largest triangle function is the point-wise minimum $\mu(\varphi, \psi) = \varphi \wedge \psi$. A triangle function τ is called *sup-continuous* if $\tau\left(\bigvee_j \varphi_j, \psi\right) = \bigvee_{j \in J} \tau\left(\varphi_j, \psi\right)$.

A triangle function τ is *continuous* if it is continuous with respect to the topology and product topology induced by the Lévy metric. In view of [32] (see also, [23]) for a continuous t-norm, the mapping τ_* is defined by $\tau_*(\varphi, \psi)(x) = \bigvee_{u+v=x} \varphi(u) * \psi(v)$ for any $\varphi, \psi \in \Delta^+$. Among the important examples of continuous t-norms, we recall the minimum t-norm, $\alpha * \beta = \alpha \wedge \beta$, the product $\alpha * \beta = \alpha\beta$ and Lukasiewich t-norm $\alpha * \beta = (\alpha + \beta - 1) \vee 0$.

For a set S, we denote by $P(S)$ the power set. The set of all filters on a set S is denoted by $\mathbb{F}(S)$ while the set of all ultrafilters on S is denoted by $\mathbb{U}(S)$. We order this set by set inclusion. If $\mathbb{F} \in \mathbb{F}(S)$ and $\mathbb{G} \in \mathbb{F}(T)$, then the filters on $S \times T$ is generated by the sets of the form $\{F \times G \colon F \in \mathbb{F}, G \in \mathbb{G}\}$ is denoted by $\mathbb{F} \times \mathbb{G}$. If $(S, \cdot)$ is a group and $\mathbb{F}, \mathbb{G} \in \mathbb{F}(S)$, we define $\mathbb{F} \odot \mathbb{G}$ as the filter generated by the sets $F \cdot G = \{pq \colon p \in F, q \in G\}$, where $F \in \mathbb{F}$ and $G \in \mathbb{G}$. The filter $\mathbb{F}^{-1}$ is generated by the sets $F^{-1} = \{p^{-1} \colon p \in F\}$, where $F \in \mathbb{F}$.

Definition 2.1. ([32]) *A probabilistic metric space under a triangle function* τ is a pair (S, F), where $\mathsf{F} \colon S \times S \longrightarrow \Delta^+$ such that for all $p, q, r \in S$ the following properties hold:

(PM1) $\mathsf{F}(p, p) = \epsilon_0$;

(PM2) $\mathsf{F}(p, q) = \mathsf{F}(q, p)$;

(PM3) $\tau(\mathsf{F}(p, q), F(q, r)) \leq \mathsf{F}(p, r)$.

The function $F(p, q)$ is usually denoted by F_{pq} or $\mathsf{F}_{p,q}$, and $\mathsf{F}(p, q)(x)$, its value at x, is interpreted as the probability that the distance between p and q is less than x.

A mapping $f \colon (S, \mathsf{F}) \longrightarrow (S', \mathsf{F}')$ is called *non-expansive* if $\mathsf{F}_{p,q} \leq \mathsf{F}'_{f(p), f(q)}$ for all $p, q \in S$.

The category of all probabilistic metric spaces and non-expansive maps is denoted by $\mathbf{ProbMet}_\tau$.

On the space $S \times S$, we consider the product probabilistic metric $\mathsf{F} \otimes_\tau \mathsf{F}\,((p_1, p_2), (q_1, q_2)) = \tau\left(\mathsf{F}_{p_1, q_1}, \mathsf{F}_{p_2, q_2}\right)$ in the sense of Tardiff, [33].

Definition 2.2. ([2]) A triple $(S, \cdot, \mathsf{F})$ is called a *probabilistic metric group* under a triangle function τ provided $(S, \cdot)$ is a group and (S, F) is a probabilistic metric space under the triangle function τ with F is an *invariant* probabilistic metric, that is, $\mathsf{F}(p, q) = \mathsf{F}(pr, qr) = \mathsf{F}(rp, rq)$ for all $p, q, r \in S$.

The category of probabilistic metric groups consists of all probabilistic metric groups as objects and all mappings which a contractive group homomorphisms as morphisms, this category is denoted by $\mathbf{ProbMetGrp}_\tau$.

For the notions of category theory, we refer to Adámek et al., [1]. However, for the convenience of the reader we quote those notions that are used frequently in the sequel.

A *functor* $\mathcal{F}: \mathfrak{C} \longrightarrow \mathfrak{D}$ is a morphism between categories, consists of mappings between objects of $\mathfrak{C}$ and objects of $\mathfrak{D}$ (sometimes we write as $|\mathfrak{C}|$ to denote the objects of $\mathfrak{C}$) and the mapping between morphisms of $\mathfrak{C}$ and morphisms of $\mathfrak{D}$ such that (i) if $f: S \longrightarrow T$, then $\mathcal{F}(f): \mathfrak{C}(S) \longrightarrow \mathfrak{D}(T)$; (ii) $\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$, whenever $f \circ g$ is defined; (iii) $\mathcal{F}(id_S) = id_{\mathcal{F}(S)}$. The functor $\mathcal{F}$ is called an *embedding* if it is injective on objects. If $\mathfrak{E}$ is a category, then by a *concrete category over* $\mathfrak{E}$, we understand a pair $(\mathfrak{G}, \mathcal{F})$, where $\mathfrak{C}$ is a category and $\mathcal{F}: \mathfrak{G} \longrightarrow \mathfrak{E}$ is a faithful functor.

A *construct* is a concrete category over **Set**, the category of sets and mappings, and we consider the objects of a construct as structured set (S, ξ), and morphisms are suitable mappings between the underlying sets. A construct is called *topological* if it allows *initial constructions*, that is, for any source $\left(f_j: S \longrightarrow (S_j, \varsigma_j)\right)_{j\in J}$, there is a unique structure ς on S such that a mapping $g: (T, \beta) \longrightarrow (S, \varsigma)$ is a morphism if and only if for each $j \in J$ the composition $f_j \circ g: (T, \beta) \longrightarrow (S_j, \varsigma_j)$ is a morphism, where (T, β) is a structured set.

A functor $\mathcal{F}: \mathfrak{C} \longrightarrow \mathfrak{D}$ between categories $\mathfrak{C}$ and $\mathfrak{D}$ is called an *isomorphism* if there is a functor $\mathcal{H}: \mathfrak{D} \longrightarrow \mathfrak{C}$ such that $\mathcal{H} \circ \mathcal{F} = id_{\mathfrak{C}}$ and $\mathcal{F} \circ \mathcal{H} = id_{\mathfrak{D}}$. Two categories $\mathfrak{C}$ and $\mathfrak{D}$ are said to be *isomorphic* if there is an isomorphism.

Let $\mathfrak{C}$ be a subcategory of a category $\mathfrak{A}$. Then $\mathfrak{C}$ is said to be *reflective* in $\mathfrak{A}$ (or $\mathfrak{C}$ is a reflective subcategory of $\mathfrak{A}$) if for each $\mathbb{X} \in |\mathfrak{A}|$ there exists a $\mathfrak{C}$-object $\mathbb{X}_{\mathfrak{C}}$ and an $\mathfrak{A}$-morphism $r_{\mathbb{X}}: \mathbb{X} \longrightarrow \mathbb{X}_{\mathfrak{C}}$ such that for each $\mathfrak{C}$-object $\mathbb{C}$ and each $\mathfrak{A}$-morphism $f: \mathbb{X} \longrightarrow \mathbb{C}$ there is a unique morphism $f': \mathbb{X}_{\mathfrak{C}} \longrightarrow \mathbb{C}$ such that $f' \circ r_{\mathbb{X}} = f$. The notion of *coreflective subcategory* is defined dually. The category of groups and group homomorphisms is denoted by **Grp**.

3. Category of approach groups, ApGrp

Definition 3.1. ([25]) A pair (S, d), where $d: S \times P(S) \longrightarrow [0, \infty]$, a distance function, is called an *approach space* if the following axioms are satisfied:

(AP1) $d(p, \{p\}) = 0$, for all $p \in S$;

(AP2) $d(p, \emptyset) = \infty$, for all $p \in S$;

(AP3) $d(p, A \cup B) = d(p, A) \wedge d(p, B)$, for all $p \in S$ and $A, B \in P(S)$;

(AP4) $d(p, A) \leq d(p, A^{(\epsilon)}) + \epsilon$, for all $p \in S$, $A \subseteq S$ and $\epsilon \in [0, \infty]$,

where $A^{(\epsilon)} = \{p \in S: d(p, A) \leq \epsilon\}$. A mapping $f: (S, d) \longrightarrow (S', d')$ between approach spaces is called a *contraction* if $d'(f(p), f(A)) \leq d(p, A)$, for all $p \in S$ and $A \in P(S)$. The category of all approach spaces and contraction mappings is denoted by **Ap**.

Example 3.2. ([25]) Let (S, T) be a topological space and define the function $d_{\mathsf{T}}: S \times P(S) \longrightarrow [0, \infty]$ by

$$d_{\mathsf{T}}(p, A) = \begin{cases} 0 & \text{if } p \in cl_{\mathsf{T}}(A) \\ \infty & \text{if } p \notin cl_{\mathsf{T}}(A) \end{cases}$$

Then (S, d_{T}) is an approach space.

Definition 3.3. ([26]) A triple $(S, \cdot, d)$ is called an *approach group* if $(S, \cdot)$ is a group and (S, d) is an approach space such that the following conditions are fulfilled:

(APGM) $d(pq, AB) \leq d(p, A) + d(q, B)$ for all $p, q \in S$ and $A, B \subseteq S$;

(APGI) $d(p^{-1}, A^{-1}) \leq d(p, A)$, for all $p \in S$ and $A \subseteq S$.

The category of all approach groups and contractive group homomorphisms is denoted by **ApGrp**.

Example 3.4. Let $(S, \cdot, \mathsf{T})$ be a topological group and A be a subgroup of S. Then $(S, \cdot, d_{\mathsf{T}}) \in |\mathbf{ApGrp}|$, where the function $d_{\mathsf{T}}: S \times P(S) \longrightarrow [0, \infty]$ by

$$d_{\mathsf{T}}(p, A) = \begin{cases} 0 & \text{if } p \in cl_{\mathsf{T}}(A) \\ \infty & \text{if } p \notin cl_{\mathsf{T}}(A) \end{cases}$$

where $cl_{\mathrm{T}}(A)$ is the closure of A.

Remark 3.5. In view of the Theorem 2.2.6[25], one can show that the category of topological groups, **TopGrp** can be embedded as concretely bireflective subcategory of **APGrp** (see also, [26]). Note that every invariant metric group is an approach group. There are various stimulating examples on approach groups and semigroups that can be seen in [26]. Since approach spaces and approach systems are equivalent concept, we can quote here another example on approach system group based on τ_m-Menger space (S, F, τ_m), τ_m being is a Lukasiewich t-norm, where for any $s \in S$ and $\epsilon > 0$ if one defines $d_s^{\epsilon}: S \longrightarrow [0, \infty]$, $t \longmapsto 1 - \mathsf{F}(s,t)(\epsilon)$. Then one can obtain an approach system $(\mathfrak{R}(s))_{s\in S} = (\{d_s^{\epsilon}: \epsilon \in (0,\infty)\})_{s\in S}$ compatible with additive group structure. The interesting point here is to observe that this approach system in question has a close connection with so-called *strong topology* widely used in application relating to probabilistic metric spaces that deals with problems in functional analysis, cf. [7, 29, 32]. This demands special attention to look into the connection between this strong topology and the Tardiff neighborhood systems that we considered in [2], and elsewhere in recent years.

4. Category of probabilistic approach groups, $\mathbf{ProbApGrp}_\tau$

Definition 4.1. ([20]) A pair (S, δ) where $S \in |\mathbf{Set}|$ and $\delta: S \times P(S) \longrightarrow \Delta^+$, $((p, A) \longmapsto \delta(p, A): [0, \infty] \longrightarrow [0, 1], x \mapsto \delta(p, A)(x) \in [0, 1])$ is called a *probabilistic approach space* (under triangle function τ) if for all $p \in S$, $A, B \in S$ the following are fulfilled:

(PA1) $\delta(p, \{p\}) = \epsilon_0$;

(PA2) $\delta(p, \emptyset) = \epsilon_\infty$;

(PA3) $\delta(p, A) \vee \delta(p, B) = \delta(p, A \cup B)$;

(PA4) $\tau\left(\delta(p, \overline{A}^{\varphi}), \varphi\right) \leq \delta(p, A)$, for all $\varphi \in \Delta^+$, where $\overline{A}^{\varphi} = \{p \in S: \delta(p, A) \geq \varphi\}$.

A mapping $f: (S, \delta) \longrightarrow (S', \delta')$ between probabilistic approach spaces (S, δ), (S', δ') is called a *contraction* if $\delta(p, A) \leq \delta'(f(p), f(A))$, for all $p \in S$ and $A \subseteq S$.

The category of all probabilistic approach spaces under triangle function τ and all contracting mappings is denoted by $\mathbf{ProbAp}_\tau$.

The value $\delta(p, A)(x)$ can be interpreted as the *probability that the distance between p and A is less than x*.

Definition 4.2. Let $(S, \cdot) \in |\mathbf{Grp}|$ and $(S, \delta) \in |\mathbf{ProbAp}_\tau|$. Then the triple $(S, \cdot, \delta)$ is called a *probabilistic approach group* under triangle function τ if the following conditions are fulfilled:

(PAGM) $\tau(\delta(p, A), \delta(q, B)) \leq \delta(pq, AB)$, for all $p, q \in S$ and $A, B \subseteq S$;

(PAGI) $\delta(p, A) \leq \delta(p^{-1}, A^{-1})$.

The category of all probabilistic approach groups and all contractive group homomorphisms, is denoted by $\mathbf{ProbAPGrp}_\tau$.

Example 4.3. Every probabilistic approach system group in the sense of [5] is a probabilistic approach group under sup-continuous triangle function τ. The transition goes as follows. If $(\mathfrak{R}(s))_{s\in S}$ is a probabilistic approach system on S, then the probabilistic approach structure for any $A \subseteq S$ and $s \in S$ is given by: $\delta^{\mathfrak{R}}(s, A) = \bigwedge_{\nu\in\mathfrak{R}(s)} \bigvee_{a\in A} \nu(a)$, cf. [21].

Example 4.4. Every probabilistic metric group is a probabilistic approach group under the triangle function τ. In fact, if $(S, \cdot, \mathsf{F})$ is a probabilistic metric group, then in view of the Lemma 3.2[5], $\left(S, \cdot, \delta^{\mathsf{F}}\right)$ is a probabilistic approach group, where for any $A \subseteq S$ and $s \in S$, $\delta^{\mathsf{F}}(s, A) = \bigvee_{a\in A} d(s, a)$.

There are some other interesting examples of probabilistic approach groups; one of those examples is, probabilistic gauge group. Every probabilistic gauge group is a probabilistic approach group. We refer to [5] for the notion of probabilistic gauge group, and some other categorical connections. In view of the Theorem 3.3[5], it follows that given a probabilistic gauge group $(S, \cdot, \mathbf{G}))$, we can obtain $\left(S, \cdot, \delta^{\mathbf{G}}\right)$ probabilistic approach group, where the probabilistic approach structure is given for any $A \subseteq S$ and $s \in S$ by $\delta^{\mathbf{G}}(s, A) = \bigwedge_{d\in\mathbf{G}} \bigvee_{a\in A} d(s, a)$.

Let (S, d) be an approach space, then in view of [20] $\delta^d(p, A) = \epsilon_{d(p,A)}$. Upon using this definition, we obtain the following.

Lemma 4.5. *The category **ApGrp** is embedded into the category **ProbApGrp**$_{\tau_*}$.*

Proof. Define the functor $\mathcal{F}$ as follows:

$$\mathcal{F} : \begin{cases} \textbf{ApGrp} & \longrightarrow & \textbf{ProbApGrp}_{\tau_*} \\ (S,\cdot,d) & \longmapsto & \left(S,\cdot,\delta^d\right) \\ f & \longmapsto & f \end{cases}$$

Let $(S,\cdot,d) \in |\textbf{ApGrp}|$. Upon using a property of triangular norm, cf. [23], pp.4, we have $\tau_*\left(\delta^d(p,A),\delta^d(q,B)\right)(z) = \tau_*\left(\epsilon_{d(p,A)},\epsilon_{d(q,B)}\right)(z) = \bigvee_{x+y=z}\epsilon_{d(p,A)}(x)*\epsilon_{d(q,B)}(y) = \epsilon_{d(p,A)+d(q,B)}(x+y) = \epsilon_{d(p,A)+d(q,B)}(z) \le \epsilon_{d(pq,AB)}(z) = \delta^d(pq,AB)(z)$, i.e., $\tau_*\left(\delta^d(p,A),\delta^d(q,B)\right) \le \delta^d(pq,AB)$. In view of Proposition 6.2 [20], one obtains: $f\colon (X,\cdot,d) \longrightarrow (S',\cdot,d')$ is contractive group homomorphism if and only if $f\colon (S,\cdot,\delta^d) \longrightarrow (S,\cdot,\delta^{d'})$ is contractive group homomorphism. It also follows from the same proposition that this functor $\mathcal{F}$ is injective on objects and hence it is an embedding. Thus, following Proposition 6.1 in conjunction with the Proposition 6.2 we conclude that **ApGrp** is embedded into **ProbApGrp**$_{\tau_*}$. □

If (S,δ) is a probabilistic approach space, and $\alpha > 0$, then for $p \in S$ and $A \subseteq S$, it is defined in [20] that $d^\delta_\alpha(p,A) = \bigwedge\{x\colon \delta(p,A)(x) \ge \alpha\}$. Then one obtains the following

Lemma 4.6. *The category **ApGrp** is embedded into **ProbApGrp**$_{\tau_\wedge}$.*

Proof. Define the functor $\mathcal{G}$ as follows:

$$\mathcal{G} : \begin{cases} \textbf{ProbApGrp}_{\tau_\wedge} & \longrightarrow & \textbf{ApGrp} \\ (S,\cdot,\delta) & \longmapsto & \left(S,\cdot,d^\delta_\alpha\right) \\ f & \longmapsto & f \end{cases}$$

Let $p,q \in S$ and $A,B \subseteq S$. Then upon using (PAGM), we have for any $\alpha > 0$:

$$d^\delta_\alpha(p,A) + d^\delta_\alpha(q,B) = \bigwedge\{x : \delta(p,A)(x) \ge \alpha\} + \bigwedge\{y : \delta(q,B)(y) \ge \alpha\}$$

$$= \bigwedge\{x+y : \delta(p,q)(x) \ge \alpha, \delta(q,B)(y) \ge \alpha\}$$

$$\ge \bigwedge\{x+y : \delta(p,A)(x) \wedge \delta(q,B)(y) \ge \alpha\}$$

$$= \bigwedge\{x+y : \alpha \le \bigvee_{u+v=x+y} \delta(p,A)(u) \wedge \delta(q,B)(v)\} = \bigwedge\{x+y : \tau_\wedge\left(\delta(p,A),\delta(q,B)\right)(x+y) \ge \alpha\}$$

$$\ge \bigwedge\{w : \delta(pq,AB)(w) \ge \alpha\} = d^\delta_\alpha(pq,AB)$$

In view of Proposition 6.4 [20], one obtains: $f\colon (S,\cdot,d) \longrightarrow (S',\cdot,d)$ is continuous group homomorphism, then $f\colon (S,\cdot,d^\delta_\alpha) \longrightarrow (S',\cdot,d'^\delta_\alpha)$ is contractive group homomorphism. This together with the fact that this functor is injective on objects yields that it is an embedding functor. In view of the Theorem 6.11, one can show that **ApGrp** can be embedded into **ProbApGrp**$_{\tau_\wedge}$. □

5. Category of probabilistic topological convergence groups, ProbTopConvGrp$_\tau$

Definition 5.1. ([19, 20]) Let $S \in |\textbf{Set}|$. A family of mappings $\left(c_\varphi\colon \mathbb{F}(S) \longrightarrow P(S)\right)_{\varphi\in\Delta^+}$ which satisfies the following axioms:

(PC1) $p \in c_\varphi([p])$, $p \in S$, $\varphi \in \Delta^+$;
(PC2) if $\mathbb{F} \le \mathbb{G}$, then $c_\varphi(\mathbb{F}) \subseteq c_\varphi(\mathbb{G})$, $\forall \mathbb{F},\mathbb{G} \in \mathbb{F}(S)$ and $\forall \varphi \in \Delta^+$;
(PC3) $\varphi \le \psi$, $c_\psi(\mathbb{F}) \subseteq c_\varphi(\mathbb{F})$, $\forall \mathbb{F} \in \mathbb{F}(S)$, $\forall \varphi,\psi \in \Delta^+$;

(PC4) $p \in c_{\epsilon_\infty}(\mathbb{F})$, $\forall p \in S$, $\forall \mathbb{F} \in \mathbb{F}(S)$,
is called a *probabilistic convergence structure* on S. The pair $\left(S, \overline{c} = (c_\varphi)_{\varphi \in \Delta^+}\right)$ is called a *probabilistic convergence space*.

A mapping $f\colon (S, \overline{c}) \longrightarrow \left(S', \overline{c'}\right)$ between probabilistic convergence spaces, is called *continuous* if $\forall p \in S$, $\forall \mathbb{F} \in \mathbb{F}(S)$, $p \in c_\varphi(\mathbb{F}) \Rightarrow f(p) \in c'_\varphi(f(\mathbb{F}))$.

PCONV denotes the category of probabilistic convergence spaces and continuous mappings.

Definition 5.2. A probabilistic convergence space $(S, \overline{c})$ is called *probabilistic pretopological* if:
(PPT) $\bigcap_{j \in J} c_\varphi(\mathbb{F}_j) \subseteq c_\varphi(\bigwedge_{j \in J} \mathbb{F}_j)$ whenever $\varphi \in \Delta^+$ and $\mathbb{F}_j \in \mathbb{F}(S)$.
It is called left-continuous provided it satisfies:
(PLC) $p \in c_{\vee A}(\mathbb{F})$ whenever $p \in c_\varphi(\mathbb{F})$ $\forall \varphi \in A \subseteq \Delta^+$.
A left-continuous and pretopological probabilistic convergence space is called *probabilistic topological convergence space* if it satisfies *Kowlasky diagonal axom (τ-PK)*:
$\forall \mathbb{G}, \mathbb{F}_q \in \mathbb{F}(S), q \in S$, one obtains $p \in c_{\tau(\varphi,\psi)}\left(\kappa(\mathbb{G}, (\mathbb{F}_q)_{q \in S})\right)$ whenever $p \in c_\varphi(\mathbb{G})$ and $q \in c_\varphi(\mathbb{F}_q)$, for all $q \in S$.

The category of probabilistic topological convergence spaces under triangle function τ and continuous mappings between them is denoted by $\mathbf{ProbTopConv}_\tau$. This category $\mathbf{ProbTopConv}_\tau$ is a full subcategory **ProbConv**, of probabilistic convergence spaces.

Definition 5.3. A triple $\left(S, \cdot, \overline{c} = (c_\varphi)_{\varphi \in \Delta^+}\right)$ is called a *probabilistic topological convergence group* under a triangle function τ if
(PTCG1) $(S, \cdot) \in |\mathbf{Grp}|$;
(PTCG2) $(S, \overline{c}) \in |\mathbf{ProbTopConv}_\tau|$;
(PTCGM) $pq \in c_{\tau(\varphi,\psi)}(\mathbb{F} \odot \mathbb{G})$, $\forall \mathbb{F}, \mathbb{G} \in \mathbb{U}(S)$, that is, for all ultrafilters $\mathbb{F}$ and $\mathbb{G}$ on S, whenever $p \in c_\varphi(\mathbb{F})$, and $q \in c_\psi(\mathbb{G})$;
(PTCGI) $p^{-1} \in c_\varphi(\mathbb{F}^{-1})$ whenever $p \in c_\varphi(\mathbb{F})$, $\forall \mathbb{G} \in \mathbb{U}(S)$, that is, for all ultrafilters $\mathbb{G}$, on S, and $p \in S$.
The category of probabilistic topological convergence groups and continuous group homomorphisms is denoted by $\mathbf{PobTopConvGrp}_\tau$.

Example 5.4. ([2]) Let $(S, \cdot, F) \in |\mathbf{ProbMetGrp}_\tau|$. Then $\left(S, \cdot, \overline{c} = (c_\varphi)_{\varphi \in \Delta^+}\right) \in |\mathbf{ProbTopConvGrp}_\tau|$.

Let $\left(S, \overline{c} = (c_\varphi)_{\varphi \in \Delta^+}\right) \in |\mathbf{ProbTopConv}_\tau|$. We define approach distance function $\delta^c(p, A) = \bigvee_{\mathcal{U} \in \mathbb{U}(S), A \in \mathcal{U}} \bigvee_{\varphi: p \in c_\varphi(\mathcal{U})} \varphi$.
Note that the proof of the following result is given for an arbitrary quantal in [5], we deduce it for the convenience of the reader, while for further details one can consult the quoted paper.

Lemma 5.5. *Let* $\left(S, \cdot, \overline{c} = (c_\varphi)_{\varphi \in \Delta^+}\right) \in |\mathbf{ProbTopConvGrp}_\tau|$. *Then* $(S, \cdot, \delta^c) \in |\mathbf{ProbApGrp}_\tau|$.

Proof. Let $\left(S, \cdot, \overline{c} = (c_\varphi)_{\varphi \in \Delta^+}\right) \in |\mathbf{ProbTopConvGrp}_\tau|$. Assume that $p, q \in S$, $A, B \in P(S)$. We need to show (PAGM), that is, $\tau\left(\delta^c(p, A), \delta^c(q, B)\right) \leq \delta^c(pq, AB)$. Let us assume that $\varepsilon \triangleleft \tau\left(\delta^c(p, A), \delta^c(q, B)\right)$. Then there are $\mathcal{U}, \mathcal{V} \in \mathbb{U}(S)$ and $\varphi, \psi \in \Delta^+$ such that $A \in \mathcal{U}, B \in \mathcal{V}$ and that $p \in c_\varphi(\mathcal{U})$ and $q \in c_\psi(\mathcal{V})$ with $\varepsilon \leq \tau(\varphi, \psi)$. Then $AB \in \mathcal{U} \odot \mathcal{V}$ and $pq \in c_{\tau(\varphi,\psi)}(\mathcal{U} \odot \mathcal{V})$. For $\mathcal{M} \in \mathbb{U}(S)$, $\mathcal{M} \geq \mathcal{U} \odot \mathcal{V}$, we have $AB \in \mathcal{M}$ with $pq \in c_{\tau(\varphi,\psi)}(\mathcal{M})$. Hence $\delta^c(pq, AB) = \bigvee_{AB \in \mathcal{M}, \mathcal{M} \in \mathbb{U}(S)} \bigvee_{pq \in c_\gamma(\mathcal{M})} \gamma \geq \tau(\varphi, \psi) \geq \varepsilon$, and applying complete distributivity of Δ^+, we have $\tau\left(\delta^c(p, A), \delta^c(q, B)\right) \leq \delta^c(pq, AB)$. Similarly, (PAGI) can be obtained. □

Lemma 5.6. *If* $f\colon (S, \cdot, \overline{c}) \longrightarrow (S, \cdot, \overline{c'})$ *is a continuous group homomorphism between probabilistic topological convergence groups under triangle function* τ, *then* $f\colon (S, \cdot, \delta^c) \longrightarrow (S', \cdot, \delta^{c'})$ *is a contractive group homomorphism between probabilistic approach groups under triangle function* τ.

Hence there is a functor

$$\mathcal{H} : \begin{cases} \mathbf{ProbTopConvGrp}_\tau & \longrightarrow & \mathbf{ProbApGrp}_\tau \\ \left(S, \cdot, \overline{c} = (c_\varphi)_{\varphi \in \Delta+}\right) & \longmapsto & (S, \cdot, \delta^c) \\ f & \longmapsto & f \end{cases}$$

Let $(S, \delta) \in |\mathbf{ProbAp}_\tau|$. Define for $\varphi \in \Delta^+$, $p \in S$ and $\mathcal{U} \in \mathbb{U}(S)$, $p \in c^\delta_\varphi(\mathcal{U}) \Leftrightarrow \bigwedge_{A \in \mathcal{U}} \delta(p, A) \geq \varphi$.

Remark 5.7. While we prepared the article [5], the following result was an open question as mentioned in the conclusion section of [5]. Following the paper of Jäger [20] , we learned that the category of probabilistic approach spaces is isomorphic to the category of probabilistic topological convergence spaces, upon using the isomorphism theorem we are in a position to answer the aforementioned open question.

Lemma 5.8. *Let $(S,\cdot,\delta) \in |\boldsymbol{ProbApGrp}_\tau|$. Then $\left(S,\cdot,\overline{c}^\delta = (c^\delta_\varphi)_{\varphi\in\Delta^+}\right) \in |\boldsymbol{ProbTopConvGrp}_\tau|$*

Proof. We only prove (PTCGM). Let $p,q \in S$, $\mathcal{U},\mathcal{V} \in \mathbb{U}(S)$ be ultrafilters on S. Further, assume that for any $\varphi,\psi \in \Delta^+$, $p \in c^\delta_\varphi(\mathcal{U})$ and $q \in c^\delta_\psi(\mathcal{V})$. Then in view of the Lemma 5.1(4)[20], we get that $\bigwedge_{U\in\mathcal{U}} \delta(p,U) \geq \varphi$, and $\bigwedge_{V\in\mathcal{V}} \delta(q,V) \geq \psi$. Upon using the properties of τ, and (PAGM), we have $\bigwedge_{UV\in\mathcal{U}\odot\mathcal{V}} \delta(pq,UV) \geq \tau\left(\bigwedge_{U\in\mathcal{U}} \delta(p,U), \bigwedge_{V\in\mathcal{V}} \delta(q,V)\right) \geq \tau(\varphi,\psi)$ implying that $\bigwedge_{UV\in\mathcal{U}\odot\mathcal{V}} \delta(pq,UV) \geq \tau(\varphi,\psi)$ which gives that $pq \in c^\delta_{\tau(\varphi,\psi)}(\mathcal{U}\odot\mathcal{V})$. The missing part follows at ease. □

Lemma 5.9. *If $f\colon (S,\cdot,\delta) \longrightarrow (S,\cdot,\delta))$ is a contractive group homomorphism between probabilistic approach groups, then $f\colon \left(S,\cdot,\overline{c^\delta}\right) \longrightarrow \left(S',\cdot,\overline{c^{\delta'}}\right)$ is a continuous group homomorphism.*

Proof. Since group homomorphism remains as it is, the result follows from Lemma 5.3[20] in conjunction with Lemma 5.1[20]. □

Hence there is a functor

$$\mathcal{I}: \begin{cases} \mathbf{ProbApGrp}_\tau & \longrightarrow & \mathbf{ProbTopConvGrp}_\tau \\ (S,\cdot,\delta) & \longmapsto & \left(S,\cdot,\overline{c^\delta} = (c^\delta_\varphi)_{\varphi\in\Delta+}\right) \\ f & \longmapsto & f \end{cases}$$

Theorem 5.10. *If $(S,\cdot,\delta) \in |\boldsymbol{ProbApGrp}_\tau|$, then $\left(S,\cdot,\overline{c^\delta}\right) \in |\boldsymbol{ProbTopConvGrp}_\tau|$. Conversely, if $(S,\cdot,\overline{c}) \in |\boldsymbol{ProbTopConvGrp}_\tau|$, then $(S,\cdot,\delta^c) \in |\boldsymbol{ProApGrp}_\tau|$. Furthermore, there are functors*

$$\begin{cases} \boldsymbol{ProbApGrp}_\tau & \xrightarrow{\mathcal{I}} & \boldsymbol{ProbTopConvGrp}_\tau \\ \boldsymbol{ProbApGrp}_\tau & \xleftarrow{\mathcal{H}} & \boldsymbol{ProbTopConvGrp}_\tau \end{cases}$$

such that $\mathcal{I}\circ\mathcal{H} = id_{\mathbf{ProbTopConvGrp}_\tau}$ and $\mathcal{H}\circ\mathcal{I} = id_{\mathbf{ProbApGrp}_\tau}$. That is, the categories $\boldsymbol{ProbApGrp}_\tau$ and $\boldsymbol{ProbTopConvGrp}_\tau$ are isomorphic.

Proof. Consider the following arrows

$$\mathbf{ProbTopConvGrp}_\tau \xrightarrow{\mathcal{H}} \mathbf{ProbApGrp}_\tau \xrightarrow{\mathcal{I}} \mathbf{ProbTopConvGrp}_\tau.$$

Then $\mathcal{I}\circ\mathcal{H} = id_{\mathbf{ProbTopConvGrp}}$, i.e., if $(S,\cdot,\overline{c}) \in |\mathbf{ProbTopConvGrp}_\tau|$, then by applying proposition 5.8[20] coupled with the Lemma 5.5, we get

$$\mathcal{I}(\mathcal{H}(S,\cdot,\overline{c})) = \mathcal{I}(S,\cdot,\delta^c) = \left(S,\cdot,\overline{c^{\delta^c}}\right) = (S,\cdot,\overline{c}) = id_{\mathbf{ProbTopConv}_\tau}(S,\cdot,\overline{c}).$$

On the other hand, we consider the arrows below

$$\mathbf{ProbApGrp}_\tau \xrightarrow{\mathcal{I}} \mathbf{ProbTopConvGrp}_\tau \xrightarrow{\mathcal{H}} \mathbf{ProbApGrp}_\tau.$$

Then $\mathcal{H}\circ\mathcal{I} = id_{\mathbf{ProbApGrp}}$, i.e., if $(S,\cdot,\delta) \in |\mathbf{ProbApGrp}_\tau|$, then by applying Proposition 5.8[20] in conjunction with the Lemma 5.6, we get

$$\mathcal{I}(\mathcal{H}(S,\cdot,\delta)) = \mathcal{I}\left(S,\cdot,\overline{c^\delta}\right) = \left(S,\cdot,\delta^{c^\delta}\right) = id_{\mathbf{ProbApGrp}_\tau}(S,\cdot,\delta).$$

Hence the result follows from the preceding lemmas. □

6. Category of probabilistic metric groups, $\mathbf{ProbMetGrp}_\tau$

Upon using the Definitions 2.2, we have the following.

Proposition 6.1. *Let $(S,\cdot,\mathsf{F}) \in |\boldsymbol{ProbMetGrp}_\tau|$, and let τ be sup-continuous triangle function. Then $\left(S,\cdot,\delta^{\mathsf{F}}\right) \in |\boldsymbol{ProbApGrp}_\tau|$, where $\delta^{\mathsf{F}} \colon S \times P(S) \longrightarrow \Delta^+$ is defined by $\delta^{\mathsf{F}}(p,A) = \bigvee_{q\in A} \mathsf{F}_{pq}$.*

Proof. Due to the Proposition 7.1[20], we only need to proof the conditions (PAGM) and (PAGI). $\tau\left(\delta^{\mathsf{F}}(p,A),\delta^{\mathsf{F}}(q,B)\right) = \tau\left(\bigvee_{a\in A} \mathsf{F}(p,a), \bigvee_{b\in B} \mathsf{F}(q,b)\right) = \bigvee_{a\in A,b\in B} \tau\left(\mathsf{F}(p,a),\mathsf{F}(q,b)\right)$
$\leq \bigvee_{a\in A,b\in B} \tau\left(\mathsf{F}(pq,ar),\mathsf{F}(ar,ab)\right) \leq \bigvee_{a\in A,b\in B} \tau(pq,ab) \leq \bigvee_{c\in AB} \tau(pq,c) = \delta^{\mathsf{F}}(pq,AB)$. Similarly, one can prove that $\delta^{\mathsf{F}}(p,A) \leq \delta^{\mathsf{F}}(p^{-1},A^{-1})$. □

Lemma 6.2. *Let $f \colon (S,\cdot,\mathsf{F}) \longrightarrow (S',\cdot,\mathsf{F}')$ be a non-expansive group homomorphism between probabilistic metric groups. Then $f \colon \left(S,\cdot,\delta^{\mathsf{F}}\right) \longrightarrow \left(S',\cdot,\delta^{\mathsf{F}'}\right)$ is a contractive group homomorphism between probabilistic approach spaces.*

Proof. This follows at once from the Proposition 7.2[20]. □

Lemma 6.3. *Let τ be a sup-continuous triangle function, then there is an embedding functor:*

$$\mathcal{J} : \begin{cases} \boldsymbol{ProbMetGrp}_\tau & \longrightarrow & \boldsymbol{ProbApGrp}_\tau \\ (S,\cdot,\mathsf{F}) & \longmapsto & \left(S,\cdot,\delta^{\mathsf{F}}\right) \\ f & \longmapsto & f \end{cases}$$

In fact, $\mathbf{ProbMetGrp}_\tau$ is isomorphic to a coreflective subcategory of the category $\mathbf{ProbApGrp}_\tau$.

Proof. This follows from proposition 6.1 and Lemma 6.3 in conjunction with the Corollary 7.3[20]. □

Let $(S,\cdot,\delta) \in |\mathbf{ProbApGrp}_\tau|$. Define $\mathsf{F}^\delta_{pq} = \delta(p,\{q\})$.

Lemma 6.4. *Let $(S,\cdot,\delta) \in |\boldsymbol{ProbApGrp}_\tau|$. Then $\left(S,\cdot,\mathsf{F}^\delta\right) \in |\boldsymbol{ProbMetGrp}_\tau|$.*

Proof. Let $(S,\cdot,\delta) \in |\mathbf{ProbApGrp}_\tau|$. In view of the Proposition 7.5[20], we just check (PMGM). In fact, $\tau\left(\mathsf{F}^\delta(p_1,p_2),\mathsf{F}^\delta(q_1,q_2)\right) = \tau\left(\delta(p_1,\{p_2\}),\delta(q_1,\{q_2\})\right) \leq \delta(p_1q_1,\{p_2q_2\}) = \mathsf{F}^\delta(p_1q_1,p_2q_2)$ □

Lemma 6.5. *If $f \colon (S,\cdot,\delta) \longrightarrow (S',\cdot,\delta')$ is a contractive group homomorphism between probabilistic approach groups, then $f \colon (S,\cdot,\mathsf{F}^\delta) \longrightarrow (S',\cdot,\mathsf{F}^{\delta'})$ is a non-expansive group homomorphism between probabilistic metric groups.*

Proof. This is immediate from the Proposition 7.6[20] since group homomorphism remains unchanged. □

Hence there is a functor

$$\mathcal{K} : \begin{cases} \mathbf{ProbApGrp}_\tau & \longrightarrow & \mathbf{ProbMetGrp}_\tau \\ (S,\cdot,\delta) & \longmapsto & \left(S,\cdot,\mathsf{F}^\delta\right) \\ f & \longmapsto & f \end{cases}$$

Theorem 6.6. *If the triangle function τ is* sup-continuous*, then the category of probabilistic metric groups $\boldsymbol{ProbMetGrp}_\tau$ can be embedded into the category $\boldsymbol{ProbApGrp}_\tau$ as a coreflective subcategory.*

Proof. In view of the preceding results, we only need to show two items:

$$\begin{array}{ccc} \mathbf{ProbMetGrp}_\tau & \xrightarrow{\mathcal{J}} & \mathbf{ProApGrp}_\tau \\ & id_{\mathbf{ProbMetGrp}_\tau} \searrow & \mathcal{K}\downarrow \\ & & \mathbf{ProbMetGrp}_\tau \end{array}$$

whence $\mathcal{K} \circ \mathcal{J} = id_{\mathbf{ProbMetGrp}_\tau}$ and $\mathcal{J} \circ \mathcal{K} \leq id_{\mathbf{ProbApGrp}_\tau}$, where

$$\mathbf{ProbApGrp}_\tau \xrightarrow{\mathcal{K}} \mathbf{ProbMetGrp}_\tau \xrightarrow{\mathcal{J}} \mathbf{ProbApGrp}_\tau.$$

In fact, in view of the Proposition 7.7[20], if $(S, \cdot, \mathsf{F}) \in |\mathbf{ProbMetGrp}_\tau|$, then

$$\mathcal{K} \circ \mathcal{J}(S, \cdot, \mathsf{F}) = \mathcal{K}(\mathcal{J}(S, \cdot, \mathsf{F})) = \mathcal{K}\left(S, \cdot, \mathsf{F}^{\delta^{\mathsf{F}}}\right) = (S, \cdot, \mathsf{F}) = id_{\mathbf{ProbMetGrp}_\tau}(S, \cdot, \mathsf{F}).$$

In a similar fashion, one can show the other part upon using Proposition 7.8[20]. Hence the result follows. □

We recall the following notions from [19] (see also, [2], pp.998). A probabilistic convergence space $\left(S, \overline{c} = (c_\varphi)_{\varphi \in \Delta^+}\right)$ is called τ-transitive [19], if for all $p, q, r \in S$ and all $\varphi, \psi \in \Delta^+$, $p \in c_\varphi([q])$ and $q \in c_\psi([r])$ implies $p \in c_{\tau(\varphi,\psi)}([r])$. It is called *symmetric* [19], if $p \in c_\varphi([q])$ implies $q \in c_\varphi([p])$ for all $p, q \in S$ and $\varphi \in \Delta^+$.

Lemma 6.7. ([2]) *Every probabilistic convergence group under triangle function τ is symmetric and τ-transitive.*

We also recall from [19] that a probabilistic convergence space $(S, \overline{c})$ is called T_1 if $\bigvee_{p \in c_\varphi([q])} \varphi = \epsilon_0$ implies $p = q$. Furthermore, we need from [2], pp. 998 (see also, [19]) the following axiom stating that a probabilistic convergence space $(S, \overline{c})$ satisfies *(PM) axiom* if :

$$\forall p \in S, \forall \varphi \in \Delta^+, \forall \mathbb{U} \in \mathbb{U}(S), p \in c_\varphi(\mathbb{U}) \Leftrightarrow \forall U \in \mathbb{U}, \forall \epsilon > 0, \exists q \in U \text{ s.t. } \bigvee_{\psi : p \in c_\psi([q])} \psi(x + \epsilon) + \epsilon \geq \varphi(x),$$
$$\forall x \in [0, \tfrac{1}{\epsilon}).$$

Definition 6.8. A probabilistic topological convergence space $(S, \overline{c})$ is called *probabilistic metrizable* if there exists a probabilistic metric F on S such that $\overline{c^{\mathsf{F}}} = \overline{c}$.

Now combining the Definition 5.1 and Definition 5.2, one obtains the following metrization theorem, for the proof we refer to the Theorem 8.2[2].

Theorem 6.9. *Every T_1 probabilistic topological convergence group under sup-continuous and continuous triangle function τ satisfying* (PM) axiom *is probabilistic metrizable.*

Let us denote T_1-$\mathbf{PMTopConvGrp}_\tau$, the category of all T_1 probabilistic topological convergence groups under sup-continuous and continuous triangle function τ as objects, and all continuous group-homomorphisms as morphisms. Let $\mathbf{PMConvGrp}_\tau$ denote the category of all probabilistic convergence groups under sup-continuous and continuous triangle function τ and continuous group-homomorphisms such that the underlying probabilistic convergence space is pretopological, T_1 and satisfying (PM) axiom.

Remark 6.10. In view of the preceding statements we can clearly see that the category T_1-$\mathbf{PMTopConvGrp}_\tau$ is a full subcategory of the category $\mathbf{PMConvGrp}_\tau$ developed in [2], and furthermore, it is shown in [4] that the category $\mathbf{PMConvGrp}_\tau$ is isomorphic to the category $\mathbf{PMetGrp}_\tau$. At this stage, it is evident that neither T_1-$\mathbf{PMTopConvGrp}_\tau$ nor $\mathbf{ProbMetGrp}_\tau$ is isomorphic to the category $\mathbf{ProbApGrp}_\tau$. It may be noted here that there is another way of defining T_1 in a topological category in [9, 10]. It is shown in [10], that T_1 in [10] and T_1 in [25] are not the same in the category **AP**. It would be interesting to characterize T_1 separation axiom in [9, 10], in topological categories that we studied herein this text.

7. Action of probabilistic approach groups on probabilistic approach spaces

In 2016, Colebunders et. al. first introduced the concept of action of convergence approach spaces [13]; following this idea, in 2020, we introduced the concept of convergence approach transformation groups [6]. In this section, firstly, we formulate a category of probabilistic approach transformation groups, and a category of probabilistic topological convergence approach transformation groups in an attempt to show isomorphism between these two; secondly, we show that he category of probabilistic metric group under sup-continuous and continuous triangle function is isomorphic to the category of probabilistic metric probabilistic convergence transformation group. Finally, we derive the relationship between some other categories.

Definition 7.1. Let $(T,\cdot,\delta)\in|\mathbf{ProbApGrp}_\tau|$, $(S,\delta)\in|\mathbf{ProbAp}_\tau|$, and $\xi\colon T\times S\longrightarrow S,(t,s)\longmapsto ts$. Then the triple$((T,\cdot,\delta),(S,\delta),\xi)$ is called a *probabilistic approach transformation group under triangle function* τ if the following conditions are fulfilled:
(PATG1) $\xi(t\cdot u,s)=\xi(t,\xi(u,s))$, for all $t,u\in T$ and $s\in S$;
(PATG2) $\xi(e,s)=s$, for all $s\in S$;
(PATG3) For all $A\subseteq T$, $B\subseteq S$, and for all $t\in T$, $s\in S$, $\tau(\delta,(t,A),\delta(s,B))\leq\delta(ts,AB)$,
where ξ is called an *action* of T on S.

Denote by $\mathbf{ProbApTrGrp}_\tau$, the category of probabilistic approach transformation groups having objects consisting of all triples $((T,\cdot,\delta),(S,\delta),\xi)$ or (T,δ,ξ) (satisfying (PATG1)-(PATG3)), and morphisms are all pairs of mappings $(f,h)\colon(T,S,\xi)\longrightarrow(T',S',\xi')$ such that
(PATG4) $f\colon T\longrightarrow T'$ a morphism in $\mathbf{ProbApGrp}_\tau$;
(PATG5) $h\colon S\longrightarrow S'$ a morphism in $\mathbf{ProbAp}_\tau$;
(PATG6) $\xi'\circ(f\times h)=h\circ\xi$.

Example 7.2. Any probabilistic approach group $(T,\cdot,\delta)$ can be made into a probabilistic approach transformation group $((T,\cdot,\delta),(T,\delta),\xi)$ on itself, where condition (PATG3) stands as follows:
$\tau(\delta(t,A),\delta(s,B))\leq\delta(ts,AB)$, for all $A,B\subseteq T$ and $t,s\in T$.

Definition 7.3. ([3]) Let $\left(S,\bar{c}=(c_\varphi)_{\varphi\in\Delta^+}\right)\in|\mathbf{ProbTopConv}_\tau|$ and $\left(T,\cdot,\bar{c}=(c_\varphi)_{\varphi\in\Delta^+}\right)\in|\mathbf{ProbTopConvGrp}_\tau|$. Then the triple $\left(\left(T,\cdot,\bar{c}=(c_\varphi)_{\varphi\in\Delta^+}\right),\left(S,\bar{c}=(c_\varphi)_{\varphi\in\Delta^+}\right),\xi\right)$ or in short, (T,S,ξ), where $\xi\colon T\times S\longrightarrow S$, $(t,s)\longmapsto ts$, is called a *probabilistic topological convergence transformation group, under the triangle function* τ if the following properties are fulfilled:
(PTTG1) $\xi(t\cdot v,s)=\xi(t,\xi(v,s))$ for all $t,v\in T$ and $s\in S$.;
(PTTG2) $\xi(e,s)=s$ for all $s\in S$;
(PTTG3) For all $\mathcal{U}\in\mathbb{U}(T)$, $\mathcal{V}\in\mathbb{U}(S)$, ultrafilters on T and S, $t\in T$ and $s\in S$, $ts\in c_{\tau(\varphi,\psi)}(\mathcal{U}\odot\mathcal{V})$ provided $t\in c_\varphi(\mathcal{U})$ and $s\in c_\psi(\mathcal{V})$.

We denote $\mathbf{ProbTopConvTrGrp}_\tau$, as the category of probabilistic topological convergence transformation groups having objects consisting of all triples $((T,\cdot,\bar{c}),(S,\bar{c}),\xi)$ (or in short (T,S,ξ)), where $(T,\cdot,\bar{c})\in|\mathbf{ProbTopConvGrp}_\tau|$, $(S,\bar{c})\in|\mathbf{ProbTopConv}|$ with ξ a continuous action of probabilistic topological convergence group on probabilistic topological convergence space, and morphisms are all pairs of mappings $(f,h)\colon(T,S,\xi)\longrightarrow(T',S',\xi')$ such that
(PATTG4) $f\colon T\longrightarrow T'$ a morphism in $\mathbf{ProbtopConvGrp}_\tau$;
(PATTG5) $h\colon S\longrightarrow S'$ a morphism in $\mathbf{ProbTopConv}_\tau$;
(PATTG6) $\xi'\circ(f\times h)=h\circ\xi$.

Lemma 7.4. *Let* $((T,\cdot,\delta),(S,\delta),\xi)\in|\boldsymbol{ProbApTrGrp}_\tau|$. *Then* $\left(\left(T,\cdot,\overline{c^\delta}\right),(S,\overline{c^\delta}),\xi\right)\in|\boldsymbol{ProbTopConvTrGrp}_\tau|$, *where* $\tau(\varphi,\varphi)=\varphi$, *for all* $\varphi\in\Delta^+$, *the largest triangle function* τ, *and* $\xi(t,s)=ts$.

Proof. Assume that $((T,\cdot,\delta),(S,\delta),\xi)\in|\mathbf{ProbApTrGrp}_\tau|$. Since every probabilistic approach group is a probabilistic topological convergence group, we only need to check (PTTG3). For, let $\mathcal{U}\in\mathbb{U}(T)$, $\mathcal{V}\in\mathbb{U}(S)$, $t\in T$ and $s\in S$. Furthermore, let $t\in c^\delta_\varphi(\mathcal{U})$ and $s\in c^\delta_\psi(\mathcal{V})$, for any $\varphi,\psi\in\Delta^+$. Then using the Definition 5.1(PGA1)[3], and the given assumption, in view of [20], we have $\bigwedge_{U\in\mathcal{U}}\delta(t,U)\geq\varphi$ and $\bigwedge_{V\in\mathcal{V}}\delta(s,V)\geq\psi$. Then following the same route as in the proof of Lemma 5.8, we arrive at $ts\in c_{\tau(\varphi,\psi)}(\mathcal{U}\odot\mathcal{V})$. □

Then in view of the Lemma 5.8, Lemma 7.4 and the Lemma 5.3[20], there is a functor

$$\mathcal{L}:\begin{cases}\mathbf{ProbApTrGrp}_\tau & \longrightarrow & \mathbf{ProbTopConvTrGrp}_\tau\\ ((T,\cdot,\delta),(S,\delta)),\xi) & \longmapsto & \left(T,\cdot,\overline{c^\delta},(S,\overline{c^\delta}),\xi\right)\\ (f,g) & \longmapsto & (f,g)\end{cases}$$

Lemma 7.5. *Let* $\left(\left(T,\cdot,\bar{c}=c_\varphi)_{\varphi\in\Delta^+}\right),(S,\bar{c}=(c_\varphi)_{\varphi\in\Delta^+}),\xi\right)\in|\boldsymbol{ProbTopConvTrGrp}_\tau|$, *under the largest triangle function* τ. *Then* $((T,\cdot,\delta^c),(S,\delta^c),\xi)\in|\boldsymbol{ProbApTrGrp}_\tau|$.

Proof. It follows from the Lemma 5.5 that every probabilistic topological convergence group under τ is a probabilistic approach group, we only need check the item (PATGM).
Let us assume that $\left(\left(T,\cdot,\overline{c}=c_{\varphi})_{\varphi\in\Delta^+}\right),(S,\overline{c}=(c_{\varphi})_{\varphi\in\Delta^+}),\xi\right)\in|\mathbf{ProbTopConvTGrp}_\tau|$, under the largest triangle function τ. Let $A\subseteq T$, $B\subseteq S$, $t\in T$ and $s\in S$. Then following the similar route as in the proof of the Lemma 5.5, we deduce that $\tau\left(\delta^c(t,A),\delta^c(s,B)\right)\leq\delta(ts,AB)$. □

In view of the Lemma 5.5, Lemma 5.9 and Lemma 5.7[20], we obtain the following functor

$$\mathcal{M}:\begin{cases}\mathbf{ProbTopConvTrGrp}_\tau & \longrightarrow & \mathbf{ProbApTrGrp}_\tau\\ ((T,\cdot,\overline{c}),(S,\overline{c}),\xi) & \longmapsto & ((T,\cdot,\delta^c),(S,\delta^c),\xi)\\ (f,g) & \longmapsto & (f,g)\end{cases}$$

Theorem 7.6. ***ProbTopConvTrGrp***$_\tau\cong$ ***ProbApTrGrp***$_\tau$.

Proof. This follows from the preceding results in conjunction with the Theorem 5.10. □

Definition 7.7. Let $(T,\cdot,d)\in|\mathbf{ApGrp}|$, $(S,d)\in|\mathbf{Ap}|$, and $\xi\colon T\times S\longrightarrow S$, $(t,s)\longmapsto ts$. Then the triple $((T,\cdot,d),(S,d),\xi)$ is called an approach transformation group if the following conditions are fulfilled:
(APTG1) $\xi(t\cdot u,s)=\xi(t,\xi(u,s))$, for all $t,u\in T$ and $s\in S$;
(APTG2) $\xi(e,s)=s$, for all $s\in S$;
(APTG3) For all $A\subseteq T$, $B\subseteq S$, and for all $t\in T$, $s\in S$, $d(ts,AB)\leq d(t,A)+d(s,B)$,
where ξ is called an *action* of T on S.

Denote by **ApTrGrp**, the category of approach transformation groups having objects consisting of all triples $((T,\cdot,\delta),(S,\delta),\xi)$ (or in short, (T,S,ξ), where $(T,\cdot,d)\in|\mathbf{ApGrp}|$, $(S,d)\in|\mathbf{Ap}|$ with $\xi\colon T\times S\longrightarrow,(t,s)\longmapsto ts$, satisfying the conditions (APTG1)-(APTG3), and morphisms are all pairs of mappings $(f,h)\colon(T,S,\xi)\longrightarrow(T',S',\xi')$ such that
(APTG4) $f\colon T\longrightarrow T'$ a morphism in **ApGrp**;
(APTG5) $h\colon S\longrightarrow S'$ a morphism in **Ap**;
(APTG6) $\xi'\circ(f\times h)=h\circ\xi$.

Theorem 7.8. *Let* $((T,\cdot,d),(S,d),\xi)\in|\textit{\textbf{ApTrGrp}}|$. *Then* $\left(\left(T,\cdot,\delta^d\right),(S,\delta^d),\xi\right)\in|\textit{\textbf{ProbApTrGrp}}_{\tau_*}|$.

Proof. It follows from Section 4 that every approach group is a probabilistic approach group under triangle function τ_*. we need to check the condition (APTG3). For, let $A\subseteq T$, $B\subseteq S$, and $t\in T$, $s\in S$. But then upon using the same technique used in the proof of the Lemma 4.5, we deduce that $\tau_*\left(\delta^d(t,A),\delta^d(s,B)\right)\leq\delta^d(ts,AB)$, where $\delta^d(t,A)=\epsilon_{d(t,A)}$ and $\delta^d(s,B)=\epsilon_{d(s,B)}$. □

Hence there is a functor

$$\mathcal{N}:\begin{cases}\mathbf{APTrGrp}_\tau & \longrightarrow & \mathbf{ProbApTrGrp}_\tau\\ ((T,\cdot,d),(S,d),\xi) & \longmapsto & \left(\left(T,\cdot,\delta^d\right),(S,\delta^d),\xi\right)\\ (f,g) & \longmapsto & (f,g)\end{cases}$$

Definition 7.9. ([3]) Let $\left(T,\cdot,F^T\right)\in|\mathbf{ProbMetGrp}_\tau|$, $(S,F^S)\in|\mathbf{ProbMet}_\tau|$ and $\xi\colon T\times S\longrightarrow S$, $(t,s)\longmapsto ts$ be a non-expansive mapping such that $\xi(t\cdot v,s)=\xi(t,\xi(v,s))$ for all $t,v\in T$ and $s\in S$ and $\xi(e,s)=s$ for all $s\in S$, where $T\times S$ is equipped with min-product probabilistic metric structure $F^T\otimes_\wedge F^S$. Then the triple $\left(\left(T,\cdot,F^T\right),(S,F^S),\xi\right)$ is called a *probabilistic metric transformation group* on the probabilistic metric space (S,F^S) with respect to ξ.

Theorem 7.10. *Let* $((T,\cdot,F),(S,F),\xi)\in|\textit{\textbf{ProbMetTrGrp}}_\tau|$. *Then* $\left(\left(T,\cdot,\delta^F\right),(S,\delta^F),\xi\right)\in|\textit{\textbf{ProbApTrGrp}}_\tau|$.

Proof. In view of the Proposition 6.1, we only need to check (PATG3) but that too follows from the same proof of the Proposition 6.1 upon using the definition $\delta^F(p,A)=\bigvee_{q\in A}F_{pq}$. □

Corollary 7.11. ([3]) *Let* $\left(\left(T,\cdot,F^T\right),(S,F^S),\xi\right)$ *be a probabilistic metric transformation group under a continuous triangle function* τ*, on the probabilistic metric group* (S,F^S) *with respect to* $\xi\colon T\times S\longrightarrow S,(t,s)\longmapsto ts$*. Then* $\left(\left(T,\cdot,\overline{c^{F^T}}\right),(S,\overline{c^{F^S}},\xi\right)$ *is a probabilistic topological convergence transformation group with respect* ξ*, if* $T\times S$ *is equipped with min-product* $F^T\otimes_\wedge F^S$ *.*

Proof. This follows from the Theorem 5.18[3] in conjunction with the Lemma 6.2 and Corollary 5.3 [19]. □

Lemma 7.12. *Let* $((T,\cdot,\delta),(S,\delta),\xi)\in|$***ProbApTrGrp***$_\tau|$*. Then* $\left(\left(T,\cdot,\mathsf{F}^{T^\delta}\right),(S,\mathsf{F}^{S^\delta})\right)\in|$***ProbMetTrGrp***$_\tau|$ *with min-product structure.*

Proof. This follows at once upon using the Lemma 6.4. In fact, for any $(t,s),(t',s')\in T\times S$, upon using (PAGM), we have

$$\mathsf{F}^{T^\delta}\otimes_\wedge\mathsf{F}^{S^\delta}((t,s),(t',s'))=\tau\left(\mathsf{F}^{T^\delta}(t,t'),\mathsf{F}^{S^\delta}(s,s')\right)=\tau\left(\delta(t,\{t'\}),\delta(s,\{s'\})\right)\leq\delta(ts,\{t's'\})=\mathsf{F}^\delta(ts,t's').$$

□

Theorem 7.13. *If the triangle function* τ *is* sup-continuous*, then the category of probabilistic metric transformation groups* ***ProbMetTrGrp***$_\tau$ *can be embedded into the category* ***ProbApTrGrp***$_\tau$ *as a coreflective subcategory.*

Proof. In view of the Lemma 6.3 (see also, Theorem 4.2[5]), The category **ProbMetGrp**$_\tau$ is isomorphic to a coreflective subcategory of the category **ProbApGrp**$_\tau$. For the embedding part, we refer to the Theorem 3.8[22]. However, one can check that if $((T,\cdot,\mathsf{F}),(S,\mathsf{F}),\xi)\neq((T,\cdot,\mathsf{F}'),(S,\mathsf{F}'),\xi)$, then $\left(\left(T,\cdot,\delta^{\mathsf{F}}\right),(S,\delta^{\mathsf{F}}),\xi\right)\neq\left(\left(T,\cdot,\delta^{\mathsf{F}'}\right),(S,\delta^{\mathsf{F}'}),\xi\right)$ by using Theorem 3.8[22]. □

Theorem 7.14. *The category* ***ProbMetTrGrp***$_\tau$ *under sup-continuous and continuous triangle function* τ *is isomorphic to the category* ***ProbMetConvTrGrp***$_\tau$

Proof. It follows from the Theorem 4.2[4] that **ProbMetGrp**$_\tau\cong$ **ProbMetConvGrp**$_\tau$ under sup-continuous triangle function τ. In view of the Theorem 5.18[3], we know that every probabilistic metric transformation group is a probabilistic convergence transformation group. We only provide the proof for the missing part, that is, show that the mapping $\xi\colon T\times S\longrightarrow S,(t,s)\longmapsto ts$ is non-expansive. For, let $(t,s),(t',s')\in T\times S$. Then applying Lemma 6.7 and the continuity ξ, we have

$$\begin{aligned}\mathsf{F}^T\otimes_\wedge\mathsf{F}^S((t,s),(t',s'))&=\tau\left(\mathsf{F}^T(t,t'),\mathsf{F}^S(s,s')\right)=\tau\left(\bigvee_{\varphi:t\in c^T_\varphi([t'])}\varphi,\bigvee_{\varphi:s\in c^S_\varphi([s'])}\varphi\right)\\&=\bigvee_{\varphi:(t,s)\in c^{\mathsf{F}^T}_\varphi\times c^{\mathsf{F}^S}_\varphi([t'],[s'])}\varphi\leq\bigvee_{\varphi:ts\in c^S_\varphi([t's'])}\varphi=\mathsf{F}^S(ts,t's'),\text{ i.e., }\mathsf{F}^T\otimes\mathsf{F}^S((t,s),(t',s'))\leq\mathsf{F}^S(ts,t's').\end{aligned}$$

□

8. Conclusion

In this paper, starting with the category of approach group, **ApGrp**, we showed that this category can be embedded into the category of probabilistic approach groups; also, we showed that the category **ApGrp**, of probabilistic approach groups is isomorphic to **ProbApGrp**$_\tau$, the category of probabilistic topological convergence groups, **ProbTopConvGrp**$_\tau$. Furthermore, it is proved that the category of probabilistic metric groups, **ProbMetGrp**$_\tau$ can be embedded into the category **ProbApGrp**$_\tau$ of the category of probabilistic approach groups as a bicoreflective subcategory. Finally, we showed that the category of probabilistic approach transformation groups is isomorphic to the category of probabilistic topological convergence transformation groups. However, it would be interesting to show under what condition a probabilistic approach transformation group is a probabilistic metric transformation group, in this respect. It would be interesting to provide with the metrization theorem for probabilistic approach group by a direct approach, it would also be interesting to construct function space structure for probabilistic approach spaces, we take this issue including the one raised in Remark 3.5, and Remark 6.10 in a separate paper. Furthermore, we intend to look in a separate paper into the relationships among T_1 separation property in our sense, T_1 separation property defined in [8] and T_1 separation property used in [11] for the categories of probabilistic convergence spaces.

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References

[1] J. Adámek, H. Herrlich, G.E. Strecker, Abstract and Concrete Categories, J. Wiley & Sons, New York, 1990.
[2] T.M.G. Ahsanullah, G. Jäger, Probabilistic uniformization and probabilistic metrization of probabilistic convergence groups, Math. Slovaca 67(2017) 1447–1464.
[3] T.M.G. Ahsanullah, G. Jäger, Probabilistic convergence transformation groups, Math. Slovaca 68 (2018) 1447–1464.
[4] T.M.G. Ahsanullah, Tesnim Meryem Baran, Fawzi Al-Thukair, On the probabilistic convergence spaces: Monad and its Eilenberg-Moore category, New Math. & Nat. Comput. 18 (2022) 1–21.
[5] T.M.G. Ahsanullah, G. Jäger, Quantal-valued generalization of approach groups, New Math. & Nat. Comput. 15 (2019) 1–30.
[6] T.M.G. Ahsanullah, F. Al-Thukair, J. Al-Mufarrij, On the continuous action of enriched lattice-valued convergence groups: Some examples, Filomat 34 (2020) 3045–3064.
[7] M. Bachir, The space of probabilistic 1-Lipschitz maps, Aequat. Math. 93 (2019) 955–983.
[8] M. Baran, Separation properties, Indian J. Pure Appl. Math. 33 (1991) 333–341.
[9] M. Baran, Closure operators in convergence spaces, Acta Math. Hungarica 87 (2000) 33–45.
[10] M. Baran, M. Qasim, T_1 approach spaces, Fac. Sci. Univ. Ank. Series A1 Math. State. 6 (2019) 784–800.
[11] P. Brock, Probabilistic convergence spaces and generalized metric spaces, Int. J. Math & Math. Sci. 21 (1998) 439–452.
[12] P. Brock, D.C. Kent, Approach spaces, limit tower spaces, and probabilistic convergence spaces, Applied Categor. Struct. 5 (1997) 99–110.
[13] E. Colebunders, H. Boustique, P Mikusinski, G. Richardson, Convergence approach spaces: Actions, Appl. Categor. Struct. 24 (2016) 147–161.
[14] R.C. Flagg, Quantales and continuity spaces, Algebra Universalis 37 (1997) 257–276.
[15] L.C. Florescu, Probabilistic convergence structures, Aequat. Math. 38 (1989) 123–145.
[16] R. Fritsche, Topologies for probabilistic metric spaces, Fund. Math. 72 (1971) 7–16.
[17] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lowson, M. W. Mislove, D. S. Scott, Contuous Lattices and Domains, Cambridge University Press, 2003.
[18] D. Hofmann, G.J. Seal, W. Tholen (eds.), Monoidal Topology: A categorical Approach to Order, Metric and Topology. Cambridge University Press, London, 2014.
[19] G. Jäger, A convergence theory for probabilistic metric spaces, Quaest. Math. 38 (2015) 587–599.
[20] G. Jäger, Probabilistic approach spaces, Math. Bohemica 142 (2017) 277–298.
[21] G. Jäger, Quantal-valued generalizations of approach spaces: L-approach systems, Top. Proc. 51 (2016) 253–276.
[22] G. Jäger, W. Yao, Quantale-valued gauge spaces, Iranian J. Fuzzy Syst. 15 (2018) 103–122.
[23] E.P. Klement, R. Mesiar, E. Pap, Triangular Norms, Kluwer Academic Publishers, 2000.
[24] W. Li, D. Zhang, Scott approach distance on metric spaces, Appl. Categor. Struct. 26 (2015) 1067–1093.
[25] R. Lowen, Approach spaces : The Missing Link in the Topology-Uniformity-Metric Triade, Clarendon Press, Oxford, London, 1997.
[26] R. Lowen, B. Windels, Approach groups, Rocky Mountain. J. Math. 30(3) (2000) 1057–1073.
[27] F.W. Lawvere, Metric spaces, generalized logic, and closed categories, Theory and Applications of categories 1 (2002) 1–37.
[28] K. Menger, Statistical metrices, Proc. Natl. Acad. Sci. 28 (1942) 535–537.
[29] K. Nourouzi, A.R. Pourmoslemi, Probabilistic normed groups, Iranian J. Fuzzy Syst. 14 (2017) 90–113.
[30] G. Preuss, *Foundations of Topology: An Approach to Convenient Topology*. Kluwer Academic Publishers, Dordrecht, 2002
[31] G.D. Richardson, D.C. Kent, Probabilistic convergence spaces, J. Austral. Math. Soc. 61 (1996) 400–420.
[32] B. Schweizer, A. Sklar, Probabilistic Metric Spaces, North Holland, New York, 1983.
[33] R.M. Tardiff, Topologies for probabilistic metric spaces, Pacific. J. Math. 65 (1976) 233–251.