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Approximation and existence of fixed points via interpolative enriched contractions

Shivam Rawat[a,*], Ayush Bartwal[b], R.C. Dimri[a]

[a]*Department of Mathematics, H.N.B. Garhwal University, Uttarakhand 246174, India*
[b]*Department of Mathematics, Himwant Kavi Chandra Kunwar Bartwal Govt. P.G. College, Nagnath Pokhari, Uttarakhand 246473, India*

Abstract. In this paper, we define interpolative enriched contractions of Kannan type, Hardy-Rogers type and Matkowski type, by enriching existing interpolative contractions, in the setting of convex metric space. For these newly introduced contractions, we prove existence of fixed points and approximation results using Krasnoselskij iteration. Examples are also given to indicate the relevance of our results in comparison to some of the existing ones in the literature.

1. Introduction

Banach developed the Banach contraction principle [6], a fundamental consequence of fixed point theory on metric spaces, in 1922. This approach has been developed and extended in numerous ways as a result of its applications in diverse domains of nonlinear analysis and applied mathematical analysis. As the map following the Banach contraction principle is a continuous map, it was logical to ask whether a discontinuous map with similar contractive criteria has a fixed point or not. Kannan [17] gave an affirmative answer to this question by defining a contractive condition for a discontinuous map T as follows:

$$d(Tx, Ty) \leq c[d(x, Tx) + d(y, Ty)],$$

for all $x, y \in X$ and $c \in [0, \frac{1}{2})$. Moreover, he proved the existence and uniqueness of fixed points in the setting of complete metric spaces.

Takahashi [37] defined a convex structure in a metric space and referred to it as a convex metric space. He also investigated numerous features of this space to verify the presence of a fixed point for nonexpansive mappings in the context of convex metric space. Introducing convexity provides a basic tool for the building of different fixed point iterative algorithms ([2],[7]-[9],[32]-[36],[38]). Recently, Berinde [11] incorporated enrichment to contractive type mappings in the setting of Banach spaces in order to generalize the literature. A self-mapping T on X is called an enriched contraction or (b, θ)-enriched contraction if there exist two constants, $b \in [0, \infty)$ and $\theta \in [0, b + 1)$ such that for all $x, y \in X$

$$\|b(x - y) + Tx - Ty\| \leq \theta\|x - y\|.$$

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* Corresponding author: Shivam Rawat
Email addresses: rawat.shivam09@gmail.com (Shivam Rawat), ayushbartwal@gmail.com (Ayush Bartwal), dimrirc@gmail.com (R.C. Dimri)

The fact that enriched contraction is an extension of the Banach contraction class has supported the idea that in the context of a Banach space, Hilbert space and a convex metric space, a fixed point x^* will exist and Krasnoselskij iteration can approximate that point [10, 12, 13].

In 2018, Karapinar [18] used the technique of interpolation to revisit the Kannan type contraction. However, prior to Karapinar, interpolative theory was a vital technique in functional analysis. In 1982, Krein et al. [27] defined the notion of interpolative triples in the following way. Two Banach spaces A and B are known as a Banach couple denoted as (A, B), when they are algebraically and topologically embedded in different topological linear space. If the embedding $A \cap B \subset E \subset A + B$ holds, the Banach space E is said to be an intermediate for the spaces of the Banach pair (A, B). Let (A, B) and (C, D) be two Banach couples. If the limitations of T to the spaces A and B are, respectively bounded operators from A to C and B to D, then the linear mapping T operating from the space $A + B$ to $C + D$ is referred to as a bounded operator from (A, B) to (C, D). The linear space of all bounded operators from (A, B) to (C, D) is denoted by $L(AB, CD)$. This is a Banach space in the norm $\|T\|_{L(AB,CD)} = \max\{\|T\|_{A\to B}, \|T\|_{C\to D}\}$. Let (A, B) and (C, D) be two Banach couples, and E (respectively F) be an intermediate for the spaces of the Banach couple (A, B) (respectively (C, D)). The triple (A, B, E) is called an interpolation triple, relative to (C, D, F), if every bounded operator from (A, B) to (C, D) maps E to F. A triple (A, B, E) is said to be an interpolation triple of type α $(0 \le \alpha \le 1)$, relative to (C, D, F), if it is an interpolation triple and the following inequality holds: $\|T\|_{E\to F} \le c\|T\|^{\alpha}.\|T\|^{1-\alpha}$. Karapinar defined the interpolative Kannan type contraction as follows:

$$d(Tx, Ty) \le \lambda([d(x, Tx)]^{\alpha}.[d(y, Ty)]^{1-\alpha}),$$

for all $x, y \in X\backslash Fix(T)$, where $Fix(T) = \{x \in X, Tx = x\}$ and $\lambda \in [0, 1)$. Some recent works in the direction of interpolative Ćirić-Reich-Rus type contraction [4, 5, 31], Meir-Keeler type contraction [20, 25] may be reffered to. Many authors followed this new dimension in the study of fixed point theory [1, 3, 14–16, 19, 22–24, 26, 30, 39].

Motivated by the recent results, in this paper, we define interpolative enriched contractions of Kannan type, Hardy-Rogers type and Matkowski type to prove existence of fixed points and approximation results using Krasnoselskij iteration in a convex metric space. Examples are also given in order to indicate the relevance of our newly obtained results in comparison to some of the existing ones in the literature.

2. Preliminaries

Definition 2.1. *[18] Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be an interpolative Kannan contraction mapping if there exist some $c \in [0, 1)$ and $\alpha \in (0, 1)$ such that*

$$d(Tx, Ty) \le c[d(x, Tx)]^{\alpha}.[d(y, Ty)]^{1-\alpha}$$

for all $x, y \in X$ with $x \neq Tx$.

The main result in [18] is the following.

Theorem 2.2. *[18] Let (X, d) be a complete metric space and T be an interpolative Kannan type contraction. Then T has a unique fixed point in X.*

Karapınar, Agarwal and Aydi [21] gave a counter-example to Theorem 2.2, showing that the fixed point may be not unique. The revised version of Theorem 2.2 is the following.

Theorem 2.3. *[21] Let (X, d) be a complete metric space. Let $T : X \to X$ be a given mapping such that*

$$d(Tx, Ty) \le c[d(x, Tx)]^{\alpha}.[d(y, Ty)]^{1-\alpha},$$

for all $x, y \in X\backslash Fix(T)$. Then T has a fixed point in X.

Definition 2.4. *[22] Let (X, d) be a metric space. We say that the self-mapping $T : X \to X$ is an interpolative Hardy-Rogers type contraction if there exist $c \in [0, 1)$ and $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta + \gamma < 1$, such that*

$$d(Tx, Ty) \leq c[d(x, y)]^{\beta}.[d(x, Tx)]^{\alpha}.[d(y, Ty)]^{\gamma}.[\frac{1}{2}(d(x, Ty) + d(y, Tx))]^{1-\alpha-\beta-\gamma},$$

for all $x, y \in X \backslash Fix(T)$.

Theorem 2.5. *[22] Let (X, d) be a complete metric space and T be an interpolative Hardy-Rogers type contraction. Then, T has a fixed point in X.*

Definition 2.6. *[28] Let Φ be a set of functions, $\phi : [0, \infty) \to [0, \infty)$ such that*
(i) ϕ is non decreasing,
(ii) $\lim_{n\to\infty} \phi^n(t) = 0$ for each $t > 0$.

Lemma 2.7. *[29] Let $\phi \in \Phi$, then $\phi(t) < t$ for all $t > 0$ and $\phi(0) = 0$.*

Definition 2.8. *[23] Let (X, d) be a metric space. We say that the self mapping $T : X \to X$ is an interpolative Matkowski type contraction, if there exist exist $\alpha, \beta, \gamma \in (0, 1)$ and $\phi \in \Phi$ such that*

$$d(Tx, Ty) \leq \phi([d(x, y)]^{\beta}.[d(x, Tx)]^{\alpha}.[d(y, Ty)]^{\gamma}.[\frac{1}{2}(d(x, Ty) + d(y, Tx))]^{1-\alpha-\beta-\gamma}),$$

for all $x, y \in X \backslash Fix(T)$.

Theorem 2.9. *[23] Let (X, d) be a complete metric space and T be an interpolative Matkowski type contraction. Then T has a fixed point in X.*

Now, we give some basic preliminaries regarding convex metric spaces.

Definition 2.10. *[37] Let X be a metric space. A continuous function $W : X \times X \times [0, 1] \to X$ is said to be a convex structure on X, if for all $x, y \in X$ and $\lambda \in [0, 1]$ the following inequality holds*

$$d(u, W(x, y; \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y), \text{ for any } u \in X. \tag{1}$$

A metric space X endowed with a convex structure W is called a Takahashi convex metric space or simply a convex metric space and is usually denoted by (X, d, W).

Example 2.11. *[37] Let I be the unit interval $[0, 1]$ and X be the family of closed intervals $[a_i, b_i]$, such that $0 \leq a_i \leq b_i \leq 1$. For $I_i = [a_i, b_i], I_j = [a_j, b_j]$ and $\lambda(0 \leq \lambda \leq 1)$, we define a mapping W by $W(I_i, I_j; \lambda) = [\lambda a_i + (1 - \lambda)a_j, \lambda b_i + (1 - \lambda)b_j]$ and define a metric d on X by the Hausdorff distance, i.e.*

$$d(I_i, I_j) = \sup_{a\in I}\{|\inf_{b\in I_i}\{|a - b|\} - \inf_{c\in I_j}\{|a - c|\}|\}.$$

(X, d, W) is a convex metric space.

The next lemmas present some fundamental properties of a convex metric space.

Lemma 2.12. *[37] Let (X, d, W) be a convex metric space. For all $x, y \in X$ and any $\lambda \in [0, 1]$, we have*

$$d(x, y) = d(x, W(x, y; \lambda)) + d(W(x, y; \lambda), y).$$

Lemma 2.13. *[37] Let (X, d, W) be a convex metric space. For all $x, y \in X$ and any $\lambda \in [0, 1]$, we have*

$$d(x, W(x, y; \lambda)) = (1 - \lambda)d(x, y) \text{ and } d(W(x, y; \lambda), y) = \lambda d(x, y).$$

Lemma 2.14. *[2] Let (X, d, W) be a convex metric space. For each $x, y \in X$ and $\lambda, \lambda_1, \lambda_2 \in [0, 1]$, we have the following:*

1. $W(x, x; \lambda) = x; W(x, y; 0) = y$ *and* $W(x, y; 1) = x$.
2. $|\lambda_1 - \lambda_2| d(x, y) \leq d(W(x, y; \lambda_1), W(x, y; \lambda_2))$.

Lemma 2.15. *[2] Let (X, d, W) be a convex metric space and $T : X \to X$ be a self mapping. Define the mapping $T_\lambda : X \to X$ by*

$$T_\lambda x = W(x, Tx; \lambda),\ x \in X. \tag{2}$$

Then, for any $\lambda \in [0, 1)$, we have $Fix(T) = Fix(T_\lambda)$.

In 2021, Berinde and Păcurar [13] introduced the notion of enriched contraction in convex metric spaces and proved some fixed point results.

Definition 2.16. *[13] Let (X, d, W) be a convex metric space. A mapping $T : X \to X$ is said to be an enriched contraction if there exist $c \in [0, 1)$ and $\lambda \in [0, 1)$ such that*

$$d(W(x, Tx; \lambda), W(y, Ty; \lambda)) \leq c d(x, y),$$

for all $x, y \in X$.

Theorem 2.17. *[13] Let (X, d, W) be a complete convex metric space and let $T : X \to X$ be an enriched contraction. Then,*

1. $Fix(T) = \{p\}$ *for some* $p \in X$.
2. *The sequence $\{x_n\}_{n=0}^{\infty}$ obtained from the iterative process*

$$x_{n+1} = W(x_n, Tx_n; \lambda) \to p, \text{ for any } x_0 \in X \text{ and } n \geq 0. \tag{3}$$

3. *The following estimate holds*

$$d(x_{n+i-1}, p) \leq \frac{c^i}{1-c}.d(x_n, x_{n-1}),\ n = 1, 2, ...;\ i = 1, 2, ... \tag{4}$$

3. Main results

We enrich the interpolative Kannan type contraction as follows:

Definition 3.1. *Let (X,d,W) be a convex metric space. A self-mapping $T : X \to X$ is an interpolative enriched Kannan type contraction if there exist $\lambda \in [0, 1), c \in [0, 1)$ and $\alpha \in (0, 1)$, such that*

$$d(W(x, Tx; \lambda), W(y, Ty; \lambda)) \leq c[d(x, W(x, Tx; \lambda))]^\alpha.[d(y, W(y, Ty; \lambda))]^{1-\alpha} \tag{5}$$

for all $x, y \in X \backslash Fix(T)$. We also call T an interpolative (λ, k)-enriched Kannan type contraction.

It is easily seen that an interpolative $(0, k)$-enriched Kannan type contraction is a usual interpolative Kannan type contraction.

Theorem 3.2. *Let (X, d, W) be a complete convex metric space. $T : X \to X$ be a continuous interpolative enriched Kannan type contraction. Then,*

1. $Fix(T) \neq \phi$.
2. *The following estimate holds*

$$d(x_{n+i-1}, p) \leq \frac{c^i}{1-c}.d(x_n, x_{n-1}),\ n = 1, 2, ...;\ i = 1, 2, ... \tag{6}$$

Proof. Using the interpolative enriched Kannan type contraction condition (5), the mapping $T_\lambda : X \to X$ defined by (2) satisfies

$$d(T_\lambda x, T_\lambda y) \leq c[d(x, T_\lambda x)]^\alpha.[d(y, T_\lambda y)]^{1-\alpha} \tag{7}$$

that is, T_λ is an interpolative Kannan type contraction. The Picard iteration associated with T_λ is actually the Krasnoselskij iterative process $\{x_n\}_{n=0}^\infty$ associated to T, which is defined by $x_{n+1} = W(x_n, Tx_n; \lambda)$, i.e.,

$$x_{n+1} = T_\lambda x_n,\ n \geq 0. \tag{8}$$

Without loss of generality, we assume that $x_n \neq x_{n+1}$ for each nonnegative integer n. Indeed, if there exists a nonnegative integer n_0, such that $x_{n_0} = x_{n_0+1} = T_\lambda x_{n_0}$. Then x_{n_0} is a fixed point. Thus, we have

$$d(x_n, T_\lambda x_n) = d(x_n, x_{n+1}) > 0, \text{ for each nonnegative integer } n.$$

Taking $x = x_n$ and $y = x_{n-1}$ in (7), we obtain

$$d(x_{n+1}, x_n) \leq c[d(x_n, x_{n+1})]^\alpha.[d(x_{n-1}, x_n)]^{1-\alpha},$$

which further gives

$$[d(x_n, x_{n+1})]^{1-\alpha} \leq c[d(x_{n-1}, x_n)]^{1-\alpha}.$$

We deduce that

$$d(x_n, x_{n+1}) \leq cd(x_{n-1}, x_n) \leq c^n d(x_0, x_1). \tag{9}$$

Since, $c < 1$ and letting $n \to \infty$ in the above inequality, we get $d(x_n, x_{n+1}) \to 0$. Now, we will prove that $\{x_n\}$ is a Cauchy sequence. Using the triangular inequality, we obtain

$$\begin{aligned} d(x_n, x_{n+r}) &\leq d(x_n, x_{n+1}) + ... + d(x_{n+r-1}, x_{n+r}) \\ &\leq c^n d(x_0, x_1) + ... + c^{n+r-1} d(x_0, x_1) \\ &\leq \frac{c^n}{1-c}(1-c^r) d(x_0, x_1). \end{aligned} \tag{10}$$

Letting $n \to \infty$ in the above inequality, we get that $\{x_n\}$ is a Cauchy sequence. Since, (X, d, W) is a complete convex metric space, there exists a $x^* \in X$ such that $\lim_{n\to\infty} d(x_n, x^*) = 0$.
On substituting $x = x_n$ and $y = x^*$ in (7), we obtain

$$d(T_\lambda x_n, T_\lambda x^*) \leq c[d(x_n, T_\lambda x_n)]^\alpha.[d(x^*, T_\lambda x^*)^{1-\alpha}].$$

Letting $n \to \infty$, we obtain $T_\lambda x^* = x^*$, i.e., $x^* \in Fix(T_\lambda)$.
To prove the last statement of our theorem, we first deduce the following result from (9),

$$d(x_{n+r}, x_n) \leq \frac{c}{1-c} d(x_n, x_{n-1}). \tag{11}$$

Now, letting $r \to \infty$ in (10) and (11), we get

$$d(x_n, x^*) \leq \frac{c^n}{1-c} d(x_0, x_1), n \geq 1, \tag{12}$$

and

$$d(x_n, x^*) \leq \frac{c}{1-c} d(x_n, x_{n-1}), n \geq 1, \tag{13}$$

respectively. From (12) and (13), we get the unifying error estimate (6). □

Example 3.3. *Let $X = [0,5]$ and define the distance function $d : X \times X \to [0,\infty)$ as $d(x,y) = |x-y|$, for any $x, y \in X$. The mapping $W : X \times X \times [0,1] \to X$ is defined as $W(x,y;\lambda) = \lambda x + (1-\lambda)y$, for all $x, y \in X$ and let $T : X \to X$ be given by*

$$Tx = \frac{5-x}{9}. \tag{14}$$

For any $x, y, z \in X$, we have

$$\begin{aligned} d(z, W(x,y,\lambda)) &= |\lambda(z-x) + (1-\lambda)(z-y)| \\ &\leq \lambda|z-x| + (1-\lambda)|z-y| \\ &= \lambda d(z,x) + (1-\lambda)d(z,y). \end{aligned}$$

Hence, (X,d,W) is a convex metric space. Also, for $\lambda = \frac{1}{10}$

$$W(x,Tx;\lambda) = T_\lambda x = \frac{1}{2}. \tag{15}$$

So, we have $d(T_\lambda x, T_\lambda y) = 0$, where $x, y \in [0,5]\backslash Fix(T)$. Therefore, for any $\alpha \in (0,1)$, T is an interpolative enriched Kannan type contraction. Thus, by Theorem 3.2, T has a fixed point. In this case, $Fix(T) = \{\frac{1}{2}\}$.

Corollary 3.4. *Let (X,d) be a complete metric space and T be an interpolative Kannan type contraction. Then T has a fixed point in X.*

Proof. Taking $\lambda = 0$ in the above theorem, the result holds. □

Now, we obtain fixed point results by enriching the interpolative Hardy-Rogers type contraction as given in the following definition.

Definition 3.5. *Let (X,d,W) be a convex metric space. A self-mapping $T : X \to X$ is an interpolative enriched Hardy-Rogers type contraction if there exist $\lambda \in [0,1), c \in [0,1)$ and $\alpha, \beta, \gamma \in (0,1)$ with $\alpha + \beta + \gamma < 1$, such that*

$$\begin{aligned} d(W(x,Tx;\lambda), W(y,Ty;\lambda)) \leq c[d(x,y)]^\beta.[d(x,W(x,Tx;\lambda))]^\alpha.[d(y,W(y,Ty;\lambda))]^\gamma \\ .[\frac{1}{2}(d(x,W(y,Ty;\lambda)) + d(y,W(x,Tx;\lambda)))]^{1-\alpha-\beta-\gamma} \end{aligned} \tag{16}$$

for all $x, y \in X\backslash Fix(T)$. We also call T an interpolative (λ, k)-enriched Hardy-Rogers type contraction.

It is easily seen that an interpolative $(0,k)$-enriched Hardy-Rogers type contraction is a usual interpolative Hardy-Rogers type contraction.

Theorem 3.6. *Let (X,d,W) be a complete convex metric space. $T : X \to X$ be a continuous interpolative enriched Hardy-Rogers type contraction. Then,*

1. *$Fix(T) \neq \phi$.*
2. *The following estimate holds*

$$d(x_{n+i-1}, p) \leq \frac{c^i}{1-c}.d(x_n, x_{n-1}),\ n = 1,2,...;\ i = 1,2,.... \tag{17}$$

Proof. Using the interpolative enriched Hardy-Rogers type contraction condition (16), we have that the mapping $T_\lambda : X \to X$ defined by (2) satisfies

$$d(T_\lambda x, T_\lambda y) \leq c[d(x,y)]^\beta.[d(x,T_\lambda x)]^\alpha.[d(y,T_\lambda y)]^\gamma.[\frac{1}{2}(d(x,T_\lambda y) + d(y,T_\lambda x))]^{1-\alpha-\beta-\gamma}, \tag{18}$$

that is, T_λ is an interpolative Hardy-Rogers type contraction. The Picard iteration associated with T_λ is actually the Krasnoselskij iterative process $\{x_n\}_{n=0}^{\infty}$ associated to T, which is defined by $x_{n+1} = W(x_n, Tx_n; \lambda)$, i.e.,

$$x_{n+1} = T_\lambda x_n,\ n \geq 0. \tag{19}$$

Without loss of generality, we assume that $x_n \neq x_{n+1}$ for each nonnegative integer n. Indeed, if there exist a nonnegative integer n_0, such that $x_{n_0} = x_{n_0+1} = T_\lambda x_{n_0}$. Then x_{n_0} is a fixed point. Thus, we have

$$d(x_n, T_\lambda x_n) = d(x_n, x_{n+1}) > 0,\ \text{for each nonnegative integer } n.$$

Taking $x = x_n$ and $y = x_{n-1}$ in (18), we obtain

$$\begin{aligned} d(x_{n+1}, x_n) &\leq c[d(x_n, x_{n-1})]^{\beta}.[d(x_n, T_\lambda x_n)]^{\alpha}.[d(x_{n-1}, T_\lambda x_{n-1})]^{\gamma}.[\frac{1}{2}(d(x_n, T_\lambda x_{n-1}) + d(x_{n-1}, T_\lambda x_n))]^{1-\alpha-\beta-\gamma} \\ &\leq c[d(x_n, x_{n-1})]^{\beta}.[d(x_n, x_{n+1})]^{\alpha}.[d(x_{n-1}, x_n)]^{\gamma}.[\frac{1}{2}(d(x_n, x_n) + d(x_{n-1}, x_{n+1}))]^{1-\alpha-\beta-\gamma} \\ &\leq c[d(x_n, x_{n-1})]^{\beta}.[d(x_n, x_{n+1})]^{\alpha}.[d(x_{n-1}, x_n)]^{\gamma}.[\frac{1}{2}(d(x_{n-1}, x_n) + d(x_n, x_{n+1}))]^{1-\alpha-\beta-\gamma}. \end{aligned} \tag{20}$$

Suppose that $d(x_{n-1}, x_n) < d(x_n, x_{n+1})$ for some $n \geq 1$. Thus,

$$\frac{1}{2}(d(x_{n-1}, x_n) + d(x_n, x_{n+1})) \leq d(x_n, x_{n+1}).$$

Consequently, the inequality (20) yields that

$$[d(x_n, x_{n+1})]^{\beta+\gamma} \leq c[d(x_{n-1}, x_n)]^{\beta+\gamma},$$

which implies that

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n),$$

which is a contradiction. Thus, we have $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$ for all $n \geq 1$. Hence, $\{d(x_{n-1}, x_n)\}$ is a non-increasing sequence with positive terms. Set $l = \lim_{n\to\infty} d(x_{n-1}, x_n)$. We have

$$\frac{1}{2}(d(x_{n-1}, x_n) + d(x_n, x_{n+1})) \leq d(x_n, x_{n+1}),\ \text{for all } n \geq 1.$$

By a simple elimination, the inequality (20) implies that

$$[d(x_n, x_{n+1})]^{1-\alpha} \leq c[d(x_{n-1}, x_n)]^{1-\alpha},\ \text{for all } n \geq 1.$$

We deduce that

$$d(x_n, x_{n+1}) \leq cd(x_{n-1}, x_n) \leq c^n d(x_0, x_1). \tag{21}$$

Since, $c < 1$ and letting $n \to \infty$ in the above inequality, we get $l = 0$. Now, we will prove that $\{x_n\}$ is a Cauchy sequence. Using the triangular inequality, we obtain

$$\begin{aligned} d(x_n, x_{n+r}) &\leq d(x_n, x_{n+1}) + ... + d(x_{n+r-1}, x_{n+r}) \\ &\leq c^n d(x_0, x_1) + ... + c^{n+r-1} d(x_0, x_1) \\ &\leq \frac{c^n}{1-c}(1-c^r) d(x_0, x_1). \end{aligned} \tag{22}$$

Letting $n \to \infty$ in the above inequality, we get that $\{x_n\}$ is a Cauchy sequence. Since, (X, d, W) is a complete convex metric space, there exists a $x^* \in X$ such that $\lim_{n\to\infty} d(x_n, x^*) = 0$.
On substituting $x = x_n$ and $y = x^*$ in (18), we obtain

$$d(T_\lambda x_n, T_\lambda x^*) \leq c[d(x_n, x^*)]^\beta.[d(x_n, T_\lambda x_n)]^\alpha.[d(x^*, T_\lambda x^*)]^\gamma.[\frac{1}{2}(d(x_n, T_\lambda x^*) + d(x^*, T_\lambda x_n))]^{1-\alpha-\beta-\gamma}.$$

Letting $n \to \infty$, we obtain $d(x^*, T_\lambda x^*) = 0$, i.e., $x^* \in Fix(T_\lambda)$.
To prove the last statement of our theorem, we first deduce the following result from (21),

$$d(x_{n+r}, x_n) \leq \frac{c}{1-c} d(x_n, x_{n-1}). \tag{23}$$

Now, letting $r \to \infty$ in (22) and (23), we get

$$d(x_n, x^*) \leq \frac{c^n}{1-c} d(x_0, x_1), n \geq 1, \tag{24}$$

and

$$d(x_n, x^*) \leq \frac{c}{1-c} d(x_n, x_{n-1}), n \geq 1, \tag{25}$$

respectively. From (24) and (25) we get the unifying error estimate (17). □

Example 3.7. *Let $X = [-1, 2]$ and define the distance function $d : X \times X \to [0, \infty)$ as $d(x, y) = |x - y|$, for any $x, y \in X$. The mapping $W : X \times X \times [0, 1] \to X$ is defined as $W(x, y; \lambda) = \lambda x + (1 - \lambda)y$, for all $x, y \in X$ and let $T : X \to X$ be given by*

$$Tx = \begin{cases} \frac{35-2x}{38} & \text{if } x \in [-1, 0), \\ \frac{35}{38} & \text{if } x \in [0, 2]. \end{cases} \tag{26}$$

For any $x, y, z \in X$, we have

$$\begin{aligned} d(z, W(x, y, \lambda)) &= |\lambda(z - x) + (1 - \lambda)(z - y)| \\ &\leq \lambda|z - x| + (1 - \lambda)|z - y| \\ &= \lambda d(z, x) + (1 - \lambda)d(z, y). \end{aligned}$$

Hence, (X, d, W) is a convex metric space. Also, for $\lambda = \frac{1}{20}$

$$W(x, Tx; \lambda) = T_\lambda x = \begin{cases} \frac{7}{8} & \text{if } x \in [-1, 0), \\ \frac{x}{20} + \frac{7}{8} & \text{if } x \in [0, 2]. \end{cases} \tag{27}$$

Choose $c = \frac{1}{2}$, $\alpha = \frac{1}{3}, \beta = \frac{1}{2}$ and $\gamma = \frac{1}{7}$. Then, we have to check that (16) *holds.*

Case-I: For $x, y \in [-1, 0)$, we obtain $d(T_\lambda x, T_\lambda y) = 0$. Thus, (16) *holds.*

Case-II: For $x \in [-1, 0)$ and $y \in [0, 2]$, the left hand side of equation (16) *becomes*

$$d(T_\lambda x, T_\lambda y) = d\left(\frac{7}{8}, \frac{y}{20} + \frac{7}{8}\right) = \left|\frac{y}{20}\right| \leq \frac{1}{10},$$

where the maximum value on the left is attained for $y = 2$. Thus, for $y = 2$, the right hand side of the equation is

$$\begin{aligned} c.[d(x, y)]^{\frac{1}{2}}.[d(x, T_\lambda x)]^{\frac{1}{3}}.[d(y, T_\lambda y)]^{\frac{1}{7}}.[\frac{1}{2}(d(x, T_\lambda y) + d(y, T_\lambda x))]^{\frac{1}{42}} &> \frac{1}{2}.(2)^{\frac{1}{2}}.\left(\frac{7}{8}\right)^{\frac{1}{3}}.\left(\frac{82}{80}\right)^{\frac{1}{7}}.\left[\frac{1}{2}\left(\frac{78}{80} + \frac{9}{8}\right)\right]^{\frac{1}{42}} \\ &= (0.5).(2)^{\frac{1}{2}}.(0.875)^{\frac{1}{3}}.(1.025)^{\frac{1}{7}}.(1.05)^{\frac{1}{42}} \\ &= 0.679502 > \frac{1}{10}. \end{aligned}$$

Therefore, (16) *holds.*

Case-III: For $x, y \in [0,2]$, the left hand side of equation (16) *becomes*

$$d(T_\lambda x, T_\lambda y) = d\left(\frac{x}{20} + \frac{7}{8}, \frac{y}{20} + \frac{7}{8}\right) = \left|\frac{x-y}{20}\right| \leq \frac{1}{10},$$

where the maximum value on the left is attained when $x = 0$ and $y = 2$ or when $x = 2$ and $y = 0$. Thus, for $x = 0$ and $y = 2$, the right hand side of the equation is

$$\begin{aligned} c.[d(x,y)]^{\frac{1}{2}}.[d(x,T_\lambda x)]^{\frac{1}{3}}.[d(y,T_\lambda y)]^{\frac{1}{7}}.[\frac{1}{2}(d(x,T_\lambda y)+d(y,T_\lambda x))]^{\frac{1}{42}} &> \frac{1}{2}.(2)^{\frac{1}{2}}.\left(\frac{7}{8}\right)^{\frac{1}{3}}.\left(\frac{82}{80}\right)^{\frac{1}{7}}.\left[\frac{1}{2}\left(\frac{78}{80}+\frac{9}{8}\right)\right]^{\frac{1}{42}} \\ &= (0.5).(2)^{\frac{1}{2}}.(0.875)^{\frac{1}{3}}.(1.025)^{\frac{1}{7}}.(1.05)^{\frac{1}{42}} \\ &= 0.679502 > \frac{1}{10}. \end{aligned}$$

Similarly, when $x = 2$ and $y = 0$, the above equation holds. Thus, in both the possibilities, (16) *holds.*

From all the above three cases, we obtain that T is an interpolative enriched Hardy-Rogers type contraction. Thus, by Theorem 3.6, T has a fixed point which is $\frac{35}{38}$.

Corollary 3.8. *Let (X,d) be a complete metric space and T be an interpolative Hardy-Rogers type contraction. Then T has a fixed point in X.*

Proof. Taking $\lambda = 0$ in the above theorem, the result holds. □

Now, we obtain fixed point results by enriching the interpolative Matkowski type contraction as follows.

Definition 3.9. *Let (X,d,W) be a convex metric space. A self mapping $T : X \to X$ is an interpolative enriched Matkowski type contraction if there exist $\lambda \in [0,1)$, $\phi \in \Phi$ and $\alpha, \beta, \gamma \in (0,1)$ such that*

$$\begin{aligned} d(W(x,Tx;\lambda), W(y,Ty;\lambda)) \leq \phi([d(x,y)]^\beta.[d(x,W(x,Tx;\lambda))]^\alpha.[d(y,W(y,Ty;\lambda))]^\gamma \\ [\frac{1}{2}(d(x,W(y,Ty;\lambda)) + d(y,W(x,Tx;\lambda)))]^{1-\alpha-\beta-\gamma}) \end{aligned} \tag{28}$$

It is easily seen that an interpolative $(0,k)$-enriched Matkowski type contraction is a usual interpolative Matkowski type contraction.

Theorem 3.10. *Let (X,d,W) be a complete convex metric space. $T : X \to X$ be a continuous interpolative enriched Matkowski type contraction. Then T has a fixed point in X.*

Proof. Using the interpolative enriched Matkowski type contraction condition (28), we have that the mapping $T_\lambda : X \to X$ defined by (2) satisfies

$$d(T_\lambda x, T_\lambda y) \leq \phi([d(x,y)]^\beta.[d(x,T_\lambda x)]^\alpha.[d(y,T_\lambda y)]^\gamma.[\frac{1}{2}(d(x,T_\lambda y) + d(y,T_\lambda x))]^{1-\alpha-\beta-\gamma}), \tag{29}$$

that is, T_λ is an interpolative Matkowski type contraction. The Picard iteration associated with T_λ is actually the Krasnoselskij iterative process $\{x_n\}_{n=0}^\infty$ associated to T, which is defined by $x_{n+1} = W(x_n, Tx_n; \lambda)$, i.e.

$$x_{n+1} = T_\lambda x_n, n \geq 0. \tag{30}$$

Without loss of generality, we assume that $x_n \neq x_{n+1}$ for each nonnegative integer n. Indeed, if there exist a nonnegative integer n_0, such that $x_{n_0} = x_{n_0+1} = T_\lambda x_0$. Then x_{n_0} is a fixed point. Thus, we have

$$d(x_n, T_\lambda x_n) = d(x_n, x_{n+1}) > 0, \text{ for each nonnegative integer n.}$$

Taking $x = x_n$ and $y = x_{n-1}$ in (29), we obtain

$$\begin{aligned} d(x_{n+1}, x_n) \leq& \phi([d(x_n, x_{n-1})]^{\beta}.[d(x_n, T_\lambda x_n)]^{\alpha}.[d(x_{n-1}, T_\lambda x_{n-1})]^{\gamma}.[\frac{1}{2}(d(x_n, T_\lambda x_{n-1}) + d(x_{n-1}, T_\lambda x_n))]^{1-\alpha-\beta-\gamma}) \\ =& \phi([d(x_n, x_{n-1})]^{\beta}.[d(x_n, x_{n+1})]^{\alpha}.[d(x_{n-1}, x_n)]^{\gamma}.[\frac{1}{2}(d(x_n, x_n) + d(x_{n-1}, x_{n+1}))]^{1-\alpha-\beta-\gamma}) \\ \leq& \phi([d(x_n, x_{n-1})]^{\beta}.[d(x_n, x_{n+1})]^{\alpha}.[d(x_{n-1}, x_n)]^{\gamma}.[\frac{1}{2}(d(x_{n-1}, x_n) + d(x_n, x_{n+1}))]^{1-\alpha-\beta-\gamma}). \end{aligned} \tag{31}$$

Suppose that

$$d(x_{n-1}, x_n) < d(x_n, x_{n+1}), \text{ for some } n \geq 1.$$

Then

$$\frac{1}{2}(d(x_{n-1}, x_n) + d(x_n, x_{n+1})) \leq d(x_n, x_{n+1}).$$

Consequently, the inequality (31) yields that

$$0 < d(x_n, x_{n+1}) \leq \phi([d(x_n, x_{n-1})]^{\beta+\gamma}.[d(x_n, x_{n+1})]^{1-\beta-\gamma}) \leq [d(x_n, x_{n-1})]^{\beta+\gamma}.[d(x_n, x_{n+1})]^{1-\beta-\gamma}, \tag{32}$$

which implies that

$$d(x_n, x_{n+1}) \leq d(x_n, x_{n-1}),$$

which is a contradiction. Thus we have $d(x_n, x_{n+1}) \leq d(x_n, x_{n-1})$, for all $n \geq 1$. Hence, $\{d(x_n, x_{n-1})\}$ is a non-increasing sequence of positive terms. Using (32), we have

$$d(x_n, x_{n+1}) \leq \phi([d(x_n, x_{n-1})]^{\beta+\gamma}[d(x_n, x_{n+1})]^{1-\beta-\gamma}) \leq \phi(d(x_{n-1}, x_n)).$$

On repeating the same argument, we have

$$d(x_n, x_{n+1}) \leq \phi(d(x_{n-1}, x_n)) \leq \cdots \leq \phi^n(d(x_0, x_1)). \tag{33}$$

As $\phi \in \Phi$, we have $\lim_{n\to\infty} \phi^n(d(x_0, x_1)) = 0$. Therefore, (33) implies that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Now, we shall show that sequence $\{x_n\}$ is Cauchy. For any $\epsilon > 0$, there exists some $n_0 \in \mathbb{N}$, such that for $n \geq n_0$

$$\phi^n(d(x_0, x_1)) < \epsilon - \phi(\epsilon).$$

From (33), we have

$$0 < d(x_n, x_{n+1}) < \epsilon - \phi(\epsilon). \tag{34}$$

We claim that

$$0 < d(x_n, x_m) < \epsilon, \text{ for all } m \geq n \geq n_0. \tag{35}$$

We prove the above inequality by mathematical induction. Since $\epsilon - \phi(\epsilon) < \epsilon$, from (34), we conclude that (35) holds for $m = n + 1$. Now, suppose that (35) holds for $m = k$. For $m = k + 1$, we have

$$\begin{aligned} d(x_n, x_{k+1}) \leq& d(x_n, x_{n+1}) + d(x_{n+1}, x_{k+1}) \\ \leq& \epsilon - \phi(\epsilon) + d(T_\lambda x_n, T_\lambda x_k) \\ \leq& \epsilon - \phi(\epsilon) + \phi([d(x_n, x_k)]^{\beta}.[d(x_n, T_\lambda x_n)]^{\alpha}.[d(x_k, T_\lambda x_k)]^{\gamma}.[\frac{1}{2}(d(x_n, T_\lambda x_k) + d(x_k, T_\lambda x_n))]^{1-\alpha-\beta-\gamma}). \end{aligned}$$

Since $k \geq n \geq n_0$, we have

$$[d(x_n, x_k)]^{\beta}.[d(x_n, T_\lambda x_n)]^{\alpha}.[d(x_k, T_\lambda x_k)]^{\gamma}.[\frac{1}{2}(d(x_n, T_\lambda x_k) + d(x_k, T_\lambda x_n))]^{1-\alpha-\beta-\gamma} \leq \epsilon.$$

Thus, we have

$$d(x_n, x_{k+1}) \leq \epsilon - \phi(\epsilon) + \phi(\epsilon) = \epsilon.$$

So, (35) holds for $m = k + 1$, which implies that the sequence $\{x_n\}$ is Cauchy. Therefore, there exists $x^* \in X$, such that $\lim_{n\to\infty} d(x_n, x^*) = 0$. Suppose that $x^* \neq T_\lambda x^*$. Since $x_n \neq T_\lambda x_n$ for each n, by (29), we have

$$d(x_{n+1}, T_\lambda x^*) = d(T_\lambda x_n, T_\lambda x^*) \leq \phi([d(x_n, x^*)]^{\beta}.[d(x_n, T_\lambda x_n)]^{\alpha}.[d(x^*, T_\lambda x^*)]^{\gamma}.[\frac{1}{2}(d(x_n, T_\lambda x^*)+d(x^*, T_\lambda x_{n+1}))]^{1-\alpha-\beta-\gamma}). \tag{36}$$

It is obvious that there exists $N \in \mathbb{N}$, such that for each $n \geq N$,

$$[d(x_n, x^*)]^{\beta}.[d(x_n, T_\lambda x_n)]^{\alpha}.[d(x^*, T_\lambda x^*)]^{\gamma}.[\frac{1}{2}(d(x_n, T_\lambda x^*) + d(x^*, T_\lambda x_{n+1}))]^{1-\alpha-\beta-\gamma} < d(x^*, T_\lambda x^*).$$

Since ϕ is non decreasing, by insertion of inequality (36), we have

$$d(x_{n+1}, T_\lambda x^*) \leq \phi(d(x^*, T_\lambda x^*)), \text{ for all } n \geq N. \tag{37}$$

Letting $n \to \infty$, we obtain that

$$0 < d(x^*, T_\lambda x^*) \leq \phi(d(x^*, T_\lambda x^*)),$$

which is a contradiction. Thus $x^* = T_\lambda x^*$. So from Lemma 2.15, T has a fixed point. □

The following example justifies the previous theorem.

Example 3.11. *Let $X = [-1, 1]$ and define the distance function $d : X \times X \to [0, \infty)$ as $d(x, y) = |x - y|$, for any $x, y \in X$. The mapping $W : X \times X \times [0, 1] \to X$ is defined as $W(x, y; \lambda) = \lambda x + (1 - \lambda)y$, for all $x, y \in X$ and let $T : X \to X$ be given by*

$$Tx = \begin{cases} -x & \text{if } x \in [-1, 0), \\ x & \text{if } x \in [0, 1]. \end{cases} \tag{38}$$

For any $x, y, z \in X$, we have

$$\begin{aligned} d(z, W(x, y, \lambda)) &= |\lambda(z - x) + (1 - \lambda)(z - y)| \\ &\leq \lambda|z - x| + (1 - \lambda)|z - y| \\ &= \lambda d(z, x) + (1 - \lambda)d(z, y) \end{aligned}$$

Hence, (X, d, W) is a convex metric space. Also, for $\lambda = \frac{1}{2}$

$$W(x, Tx; \lambda) = T_\lambda x = \begin{cases} 0 & \text{if } x \in [-1, 0), \\ x & \text{if } x \in [0, 1]. \end{cases} \tag{39}$$

So, we have $d(T_\lambda x, T_\lambda y) = 0$, where $x, y \in [-1, 1]\backslash Fix(T)$. Therefore, for any $\phi \in \Phi$ and any $\alpha, \beta, \gamma \in (0, 1)$, T is an interpolative enriched Matkowski type contraction. Thus, by Theorem 3.10, T has a fixed point. In this case, $Fix(T) = [0, 1]$, i.e. T has infinitely many fixed point.

Corollary 3.12. *Let (X, d) be a complete metric space and T be an interpolative Matkowski type contraction. Then T has a fixed point in X.*

Proof. Taking $\lambda = 0$ in the above theorem, the result holds. □

4. Conclusion

In this paper, we introduced larger classes of contractive mappings, namely, interpolative enriched Kannan type contraction, interpolative enriched Hardy-Rogers type contraction and interpolative enriched Matkowski type contraction, by enriching usual interpolative Kannan type contraction, interpolative Hardy-Rogers type contraction and interpolative Matkowski type contraction, respectively, in the Takahashi convex metric space setting. For the given contractions, we proved existence of fixed points and approximation results using Krasnoselskij iteration. Our results generalized the results presented in [18], [22], [23] and some other results in the existing literature. Examples are also given for validating the usefulness of our results. The results presented in this paper will enlighten the way for researchers to enrich the results given in [5], [19]-[21] and [24]-[26].

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