Filomat 37:16 (2023), 5485–5507
https://doi.org/10.2298/FIL2316485I

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the p-Laplacian type equation with logarithmic nonlinearity: Existence, decay and blow up

Nazlı Irkıl[a]

[a] *Dicle University, Department of Mathematics, Diyarbakır, Turkey*

Abstract. This work is deal with a problem of wave equation with p-Laplacian, strong damping and logarithmic source terms under initial-boundary conditions. The global existence of weak solution was proved for related to the equation. Global existence results of solutions are obtained using the potential well method, Galerkin method and compactness approach corresponding to the logarithmic source term. Besides, we established the energy functional decaying polynomially to zero as the time goes to infinity due to Nakao's inequality and some precise priori estimates on logarithmic nonlinearity. For suitable conditions we proved the finite time blow up results of solutions. The proof is based on the concavity method, perturbation energy method and differential–integral inequality technique. Additionally, under suitable assumptions on initial data, the infinite time blow up result is investigated with negative initial energy.

1. Introduction

We consider the following a class of hyperbolic p-Laplacian type equation

$$\begin{cases} u_{tt} - div\left(|\nabla u|^{p-2}\nabla u\right) - \Delta u_t = |u|^{q-2}u\ln|u|, & x\in\Omega,\ t>0, \\ u(x,t)=0, & x\in\partial\Omega,\ t\geq 0, \\ u(x,0)=u_0(x),\ \ u_t(x,0)=u_1(x), & x\in\Omega, \end{cases} \tag{1}$$

where $u_0 \in W_0^{1,p}(\Omega)\setminus\{0\}$ and $u_1 \in H_0^1(\Omega)$ are given initial data. Let $\Omega\subset R^n$ $(n\geq 1)$ be a bounded domain with smooth boundary $\partial\Omega$.

The problem (1) with polynomial source term (in the case absence of the logarithmic source term) arise in physics. For example, the equation (2)

$$u_{tt} - div\left(|\nabla u|^{p-2}\nabla u\right) - \Delta u_t = 0, \tag{2}$$

represents the motion of fixed membrane with strong viscosity. The global existence of weak solution and stability of smooth solutions for $n=1$ case was investigated by Greenberg et. al [15]. Later, qualitative theory

2020 *Mathematics Subject Classification*. 35B44; 35A01; 35B40; 35G60; 35L05.
Keywords. Blow up; Boundary value problem; Global existence; Partial differential equation; Polynomial decay; p-Laplacian equation; Logarithmic nonlinearity.
Received: 10 July 2022; Accepted: 14 July 2023
Communicated by Maria Alessandra Ragusa
Email address: nazliirkil@gmail.com (Nazlı Irkıl)

of solutions to the equation (1) has been analyzed by many mathematicians through various approaches (see [2, 5, 6, 11, 21, 24, 27, 31, 32, 34, 37]).

Hyperbolic wave equations with strong damping terms and logarithmic source term occured naturally in different areas of physics (see [14, 26]). During the past decades, with much literature related to strong damping and logarithmic source term also investiagates constantly in partial differantial equation, see e.g [3, 8, 10, 12, 16, 23, 29, 38].

In particular, problem (1) for $p = q$ and without strong damping term was studied by Ye in [36]. He studied global existence of solution by applying Galerkin method and the logarithmic Sobolev inequality. Based on concavity method, the global nonexistence was established with positive initial energy. In [18], Irkıl and Pişkin considered equation

$$u_{tt} - div\left(|\nabla u|^{p-2}\nabla u\right) - \Delta u_t + |u_t|^{k-2}u_t = |u|^{p-2}u\ln|u|, \tag{3}$$

where $p > k > 2$. The local existence of weak solution has been obtained by using Banach fixed theorem. In the same paper, the blow up result in finite time of the solution has been considered for $E(0) < 0$. Yang and Han [35] studied the problem (3) where $p > 2$ and $k = 2$. They obtained blow up results at different initial energy case. Later on, for the case $k = 2$ and with $|u|^{p-2}u$ term the problem was studied by Pişkin et al. [28]. They established global existence for weak solutions, decay and growth results.

On the other hand, in [17], for the following logarithmic p-Laplacian parabolic type equation

$$u_t - div\left(|\nabla u|^{p-2}\nabla u\right) - \Delta u_t = |u|^{q-2}u\ln|u| \tag{4}$$

was studied, where $p\left(1+\frac{2}{n}\right) > q > p > 2$. Authors studied results decay and blow-up of solutions. In [9], p and q exponents which are more general than the conditions in [17]. Also, in [7], Dai, Mu and Xu generalized those results by discussing the asymptotic behavior of the weak solution for problem (4).

We denote that p-Laplacian operator $div\left(|\nabla u|^{p-2}\nabla u\right)$ and logarithmic source term $|u|^{p-2}u\ln|u|$ of the equations have the same power exponent p in order to utilize Sobolev embedding theorems or the logarithmic Sobolev inequality. However, the appearance of p-Laplacian operator and logarithmic nonlinearity $|u|^{q-2}u\ln|u|$ $(2 < p < q < p\left(1+\frac{2}{n}\right))$ of the wave equation cause some difficulties. For this reason less results are, at the present time, known for the p-Laplacian wave equation with logarithmic source term source $|u|^{q-2}u\ln|u|$ and many problems remain unsolved. We aim to find some new modified methods to overcome this difficulty when global existence, uniqueness, energy decay estimates and finite time blow-up of solutions for problem were studied. To the best of our knowledge, there are no qualitative theory results on problem (1). We hope that our results fill in the gaps in previous studies on this type of models.

The rest of this paper is organized as follows: some lemmas which will be used the proof of our results were given in Section 2. Section 3 is related with potential well theory of the problem (1). In section 4, we established global existence of weak solutions for the problem. Later, the polynomial decay results were obtained in section 5. Finally, the blow up results were studied for $E(0) < d$ (d is defined in (12)) and $E(0) < 0$ case with different method, in section 6.

2. Preliminaries

In order to state the main results to problem (1) more clearly, we start to our work by introducing some notations and lemma which will be used in this paper. Throughout this paper, we denote $u(t) = u$ and

$$\|u\|_m = \|u\|_{L^m(\Omega)}, \quad \|u\|_{1,m} = \|u\|_{W_0^{1,m}(\Omega)} = \left(\|u\|_m^m + \|\nabla u\|_m^m\right)^{\frac{1}{m}},$$

for $1 < m < \infty$. We consider $W_0^{-1,m'}(\Omega)$ to define the dual space of $W_0^{1,m'}(\Omega)$ where m' is Hölder conjugate exponent for $m > 1$ (see [1, 30], for details).

Lemma 2.1. *[1]. For $u \in H_0^1(\Omega)$ and $p > 2$, we get*

$$\|u\|_{q+\alpha} \leq C \|\nabla u\|, \tag{5}$$

where C was taken for the best embedding fixed, and

$$\alpha = \begin{cases} \frac{p}{2}\left(1+\frac{2}{n}\right) - \frac{q}{2} > 0, & \text{if } n = 1,2 \\ \frac{1}{2}\min\left\{\frac{2n}{n-2}, p\left(1+\frac{2}{n}\right)\right\} - \frac{q}{2} > 0 & \text{if } n \geq 3, \end{cases} \tag{6}$$

as well as q satisfies $1 \leq q \leq \frac{2n}{n-2}$ if $n \geq 3$; $1 \leq q < \infty$ if $n = 1,2$.

Definition 2.2. *(Weak solution) A function $u(t)$ is called a weak solution to problem (1) on $\Omega \times [0,T)$, if*

$$u \in L^\infty\left(0,T; W_0^{1,p}(\Omega)\right)$$

and

$$u_t \in L^\infty\left(0,T; H_0^1(\Omega)\right)$$

satisfy initial conditions $u(x,0) = u_0(x)$, $\quad u_t(x,0) = u_1(x)$ and

$$\begin{cases} \int_\Omega u_{tt}(x,t)\,\phi(x)\,dx + \int_\Omega \nabla u_t(x,t)\,\nabla\phi(x)\,dx \\ \quad + \int_\Omega |\nabla u(x,t)|^{p-2}\,\nabla u(x,t)\,\nabla\phi(x)\,dx \\ \quad = \int_\Omega \ln|u(x,t)|\,u^{q-2}(x,t)\,\phi(x)\,dx, \end{cases}$$

where $\forall w \in H_0^1(\Omega)$ and $t \in [0,T)$.

Definition 2.3. *(Existence of solution) Suppose $(u_0,u_1) \in W_0^{1,p}(\Omega) \times H_0^1(\Omega)$ and $2 < p < q < p\left(1+\frac{2}{n}\right)$ for every $T > 0$. Then for problem (1) can be obtained weak solution such that*

$$u \in C\left([0,T); W_0^{1,p}(\Omega)(\Omega)\right),\ u_t \in C\left([0,T); H_0^1(\Omega)\right).$$

3. Potential Well

We recall the total energy function $E(u(t))$ for $t \geq 0$ as

$$E(u) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{q}\int_\Omega |u|^q \ln|u|\,dx + \frac{1}{q^2}\|u\|_q^q. \tag{7}$$

Let us define some useful funcionals as follows

$$J(u) = \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{q}\int_\Omega |u|^q \ln|u|\,dx + \frac{1}{q^2}\|u\|_q^q, \tag{8}$$

and

$$I(u) = \|\nabla u\|_p^p - \int_\Omega |u|^q \ln|u|\,dx. \tag{9}$$

Gagliardo–Nirenberg interpolation inequality is a result in the theory of Sobolev spaces that relates the L^p of different weak derivatives of a function through an interpolation inequality. Inequalities of this type play a crucial role in improving regularity and integrability assertions for solutions of nonlinear partial differential equations and in clarifying how solutions $u(x,t)$ of evolutionary equations [4, 13, 25]. Morever, it is clear that $J(u)$ and $I(u)$ are continuous by the Gagliardo-Nirenberg multiplicative embedding inequality. Then, by (8) and (9), it tells us that

$$J(u) = \frac{1}{q} I(u) + \left(\frac{1}{p} - \frac{1}{q}\right) \|\nabla u\|_p^p + \frac{1}{q^2} \|u\|_q^q, \tag{10}$$

and

$$E(t) = \frac{1}{2} \|u_t\|_2^2 + J(u). \tag{11}$$

We can define the mountain-pass level

$$d = \inf_{u \in \aleph} J(u), \tag{12}$$

where $\aleph$ is the Nehari manifold, which is defined as follows

$$\aleph = \left\{ u \in W_0^{1,p}(\Omega) \setminus \{0\} : I(u) = 0 \right\}.$$

We put the potential well depth of the problem (1) such that

$$0 < d = \inf_u \left\{ \sup_{\lambda \geq 0} J(\lambda u) : u \in W_0^{1,p}(\Omega), \|u\|_p^p \neq 0 \right\}. \tag{13}$$

Now, we introduce the potential well W and its corresponding set V

$$W = \left\{ u \in W_0^{1,p}(\Omega) : I(u) > 0,\ J(u) < d \right\} \cup \{0\},$$

$$V = \left\{ u \in W_0^{1,p}(\Omega) : I(u) < 0,\ J(u) < d \right\}.$$

Lemma 3.1. *Let $(u_0, u_1) \in W_0^{1,p}(\Omega) \times H_0^1(\Omega)$ holds. $E(t)$ be a nonincreasing function, for $t \geq 0$*

$$E'(t) = -\|\nabla u_t\|_2^2 \leq 0. \tag{14}$$

Proof. Multiplying the equation (1) by u_t and integrating on Ω, we have

$$\int_\Omega u_{tt} u_t dx - \int_\Omega div\left(|\nabla u|^{p-2} \nabla u\right) u_t dx + \int_\Omega \nabla u_t \nabla u_t dx = \int_\Omega u^{q-2} u \ln|u|\, u_t dx,$$

$$\frac{d}{dt}\left(\frac{1}{2} \|u_t\|_2^2 + \frac{1}{p} \|\nabla u\|_p^p - \frac{1}{q} \int_\Omega |u|^q \ln|u|\, dx + \frac{1}{q^2} \|u\|_q^q \right) = -\|\nabla u_t\|_2^2,$$

$$E'(t) = -\|\nabla u_t\|_2^2.$$

□

Lemma 3.2. *Suppose that $\lambda > 0$, $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ and $\|u\|_q^q \neq 0$. Then we get*
i) $\lim_{\lambda \to 0} J(\lambda u) = 0,\ \lim_{\lambda \to \infty} J(\lambda u) = -\infty;$
ii) there exists a unique λ^* such that

$$\frac{d}{d\lambda} J(\lambda u) \mid_{\lambda = \lambda^*} = 0;$$

iii) $J(\lambda u)$ is strictly decreasig on $\lambda^* < \lambda < \infty$, strictly increasing on $0 \le \lambda \le \lambda^*$, and takes maximum at $\lambda = \lambda^*$;
iv) For any $\lambda \ge 0$, we get

$$I(\lambda u) = \lambda \frac{d}{d\lambda} J(\lambda u) \begin{cases} > 0, & 0 < \lambda < \lambda^*, \\ = 0, & \lambda = \lambda^*, \\ < 0, & \lambda^* < \lambda < \infty. \end{cases} \tag{15}$$

Proof. i) $J(\lambda u)$ was obtained as

$$\begin{aligned} J(\lambda u) &= \frac{1}{p} \|\lambda \nabla u\|_p^p + \frac{1}{q^2} \|\lambda u\|_q^q - \frac{1}{q} \int_\Omega (\lambda u)^q \ln |\lambda u| \, dx \\ &= \frac{\lambda^p}{p} \|\nabla u\|_p^p + \frac{\lambda^q}{q^2} \|u\|_q^q - \frac{\lambda^q}{q} \ln |\lambda| \|u\|_q^q - \frac{\lambda^q}{q} \int_\Omega \ln |u| |u|^q \, dx, \end{aligned}$$

by using definition of $J(u)$. Clearly, we obtain $\lim_{\lambda \to 0} g(\lambda) = 0$, $\lim_{\lambda \to \infty} g(\lambda) = -\infty$.Here, $\|u\|_q^q \neq 0$ is taken.

ii) Now, differentiating $J(\lambda u)$ with respect to λ, we obtain

$$\begin{aligned} \frac{d}{d\lambda} J(\lambda u) &= \lambda^{p-1} \|\nabla u\|_p^p - \lambda^{q-1} \ln |\lambda| \|u\|_q^q - \lambda^{q-1} \int_\Omega |u|^q \ln |u| \, dx \\ &= \lambda \left(\lambda^{p-2} \|\nabla u\|_p^p - \lambda^{q-2} \ln |\lambda| \|u\|_q^q - \lambda^{q-2} \int_\Omega |u|^q \ln |u| \, dx \right) \\ &= \lambda \varphi(\lambda), \end{aligned} \tag{16}$$

where

$$\varphi(\lambda) = \lambda^{p-2} \|\nabla u\|_p^p - \lambda^{q-2} \ln |\lambda| \|u\|_q^q - \lambda^{q-2} \int_\Omega |u|^q \ln |u| \, dx.$$

We observe from $2 < p < q$ that

$$\begin{aligned} \varphi(\lambda) &= \lambda^{p-2} \|\nabla u\|_p^p - \lambda^{q-2} \ln |\lambda| \|u\|_q^q - \lambda^{q-2} \int_\Omega |u|^q \ln |u| \, dx \\ &= \lambda^{q-2} \left(\lambda^{p-q} \|\nabla u\|_p^p - \ln |\lambda| \|u\|_q^q - \int_\Omega |u|^q \ln |u| \, dx \right) \\ &= \lambda^{q-2} \left(x \lambda^{p-q} - y \ln |\lambda| - z \right), \end{aligned}$$

where $x = \|\nabla u\|_p^p \ge 0$, $y = \|u\|_q^q \ge 0$ and $z = \int_\Omega |u|^q \ln |u| \, dx$. Also we obtain

$$\begin{aligned} \varphi'(\lambda) &= (q-2) \lambda^{q-3} \left(x \lambda^{p-q} - y \ln |\lambda| - z \right) + \lambda^{q-3} \left(x (p-q) \lambda^{p-q} - y \right) \\ &= \lambda^{q-3} \left[(p-2) x \lambda^{p-q} - y \left((q-2) \ln |\lambda| + 1 \right) - (q-2) z \right]. \end{aligned}$$

Let

$$g(\lambda) = (p-2) x \lambda^{p-q} - y \left((q-2) \ln |\lambda| + 1 \right) - (q-2) z,$$

which together with $2 < p < q$ satisfies that

$$\lim_{\lambda\to 0} g(\lambda) = \infty, \ \lim_{\lambda\to\infty} g(\lambda) = -\infty,$$

and

$$g'(\lambda) = \frac{(p-q)(p-1)\lambda^{p-q} - (q-1)z}{\lambda} < 0.$$

Now, we deduce that there exists a unique λ_0 such that $g(\lambda)\,|_{\lambda=\lambda_0} = 0$, which satisfies

$$\begin{cases} \varphi'(\lambda) > 0, \text{ for } 0 < \lambda < \lambda_0, \\ \varphi'(\lambda) = 0, \text{ for } \quad \lambda = \lambda_0, \\ \varphi'(\lambda) < 0, \text{ for } \quad \lambda > \lambda_0. \end{cases}$$

Therefore, we conclude that there exists a unique $\lambda_1 > \lambda_0$ such that $\varphi(\lambda)\,|_{\lambda=\lambda_1} = 0$ and $\varphi(\lambda)$ is monotone decreasing $\lambda > \lambda_1$. Hence, there exists $\lambda^* > \lambda_1$ such that $\left(\|\nabla u\|^2 + \varphi(\lambda)\right) = 0$,which means $\frac{d}{d\lambda} J(\lambda u)\,|_{\lambda=\lambda^*}$.

iii) From (ii), we can see clearly

$$\frac{d}{d\lambda} J(\lambda u) > 0 \text{ for } 0 \le \lambda \le \lambda^*,$$

$$\frac{d}{d\lambda} J(\lambda u) < 0 \text{ for } \lambda^* < \lambda < \infty,$$

which gives (iii).

iv) Thus, by definition of $I(u)$ we have the desired results such that

$$\begin{aligned} I(\lambda u) &= \lambda^p \|\nabla u\|_p^p - \lambda^q \ln|\lambda| \|u\|_q^q - \lambda^q \int_\Omega |u|^q \ln|u|\, dx \\ &= \lambda \frac{d}{d\lambda} J(\lambda u). \end{aligned} \tag{17}$$

We obtain (15) from the proof of the (ii) and (17). □

Lemma 3.3. *i) d is positive and there exists a positive function $u \in \aleph$ such that $J(u) = d$.*

ii) The depth of potential well d is defined as

$$d = \left(\frac{q-p}{pq}\right)\left(\frac{e\alpha}{C}\right)^{\frac{p}{q+\alpha-p}}.$$

Proof. i) By (10), our aim is to show that there is a positive function $u \in \aleph$ such that $J(u) = d$. Let $\{u_m\}_{m=1}^\infty \subset \aleph$ be a minimum sequence of $J(u)$, i.e.

$$\lim_{m\to\infty} J(u_m) = d.$$

We can see clearly that $\{|u_m|\}_{m=1}^\infty \subset \aleph$ is a minimum sequence of $J(u)$. Morever, we can assume that $u_m > 0$ a.e. for all $m \in \mathbb{N}$.

Otherwise, we have already observed that, $J(u)$ is coercive on $\aleph$ which satisfies that $\{u_m\}_{m=1}^\infty \subset \aleph$ is bounded in $u \in W_0^{1,p}(\Omega)$. Let $\alpha > 0$ is a sufficiently small such that $q + \alpha < \frac{np}{n-p}$, so the embedding $W_0^{1,p} \hookrightarrow L^{q+\alpha}$ is compact, and there is a function u and subsequence $\{u_m\}_{m=1}^\infty$, still denoted by $\{u_m\}_{m=1}^\infty$, such that

$$\begin{aligned} u_m &\to u, \text{ weakly in } W_0^{1,p}(\Omega), \\ u_m &\to u, \text{ strongly in } L^{q+\alpha}(\Omega), \\ u_m &\to u, \text{ a.e. in } \Omega. \end{aligned}$$

Thus, we get $u \geq 0$ a.e. in Ω. By Lebesgue dominated convergence theorem, we see that

$$\int_\Omega |u|^q \ln|u|\, dx = \lim_{m\to\infty} \int_\Omega |u_m|^q \ln|u_m|\, dx, \tag{18}$$

$$\int_\Omega |u|^q\, dx = \lim_{m\to\infty} \int_\Omega |u_m|^q\, dx. \tag{19}$$

The weak lower semicontinuity of $\|.\|_{W_0^{1,p}}$ implies

$$\|\nabla u\|_p \leq \lim_{m\to\infty} \inf \|\nabla u_m\|_p\,. \tag{20}$$

Combining definition of the $J(u)$ and $I(u)$, (18) - (20), we conclude that

$$J(u) \leq \lim_{m\to\infty} \inf J(u_m) = d, \tag{21}$$

$$I(u) \leq \lim_{m\to\infty} \inf I(u_m) = 0. \tag{22}$$

Thanks to $u_m \in \aleph$ one has $u_m \in W_0^{1,p}(\Omega)$ and $I(u_m) = 0$. Therefore, by using the fact

$$\ln x \leq \frac{1}{e\alpha} x^\alpha \text{ for } x \geq 1, \tag{23}$$

and the Sobolev embedding inequality, we have

$$\begin{aligned}
\|\nabla u_m\|_p^p &= \int_\Omega |u_m|^q \ln|u_m|\, dx \\
&= \int_{\{x\in\Omega:|u_m(x)|\geq 1\}} |u_m|^q \ln|u_m|\, dx + \int_{\{x\in\Omega:|u_m(x)|<1\}} |u_m|^q \ln|u_m|\, dx \\
&\leq \int_{\{x\in\Omega:|u_m(x)|\geq 1\}} |u_m|^q \ln|u_m|\, dx \\
&\leq \frac{1}{e\alpha} \int_{\{x\in\Omega:|u_m(x)|\geq 1\}} |u_m|^{q+\alpha}\, dx \\
&\leq C\, \|\nabla u_m\|_{p+\alpha}^{p+\alpha},
\end{aligned}$$

for some positive constant C, which implies

$$\int_\Omega |u_m|^q \ln|u_m|\, dx = \|\nabla u_m\|_p^p \geq C. \tag{24}$$

From (24) and (18), we reproduce

$$\int_\Omega |u|^q \ln|u|\, dx \geq C.$$

Therefore, we obtain $u \in W_0^{1,p}(\Omega)$. By (22), we easily have $I(u) \leq 0$. Now, we show that $I(u) = 0$. Indeed, if it false, we get $I(u) < 0$, then by Lemma 5, there exists a λ^* such that $0 < \lambda^* < 1$ and $I(\lambda^* u) = 0$. Thus, we conclude that

$$\begin{aligned}
d &\leq J(\lambda^* u) \\
&= \frac{1}{q} I(\lambda^* u) + \left(\frac{1}{p} - \frac{1}{q}\right) \|\nabla(\lambda^* u)\|_p^p + \frac{1}{q^2} \|\lambda^* u\|_q^q \\
&= \left(\frac{1}{p} - \frac{1}{q}\right) \|\nabla(\lambda^* u)\|_p^p + \frac{1}{q^2} \|\lambda^* u\|_q^q \\
&\leq (\lambda^*)^p \left(\left(\frac{1}{p} - \frac{1}{q}\right) \|\nabla u\|_p^p + \frac{1}{q^2} \|u\|_q^q\right) \\
&\leq (\lambda^*)^p \lim_{m\to\infty} \inf \left(\left(\frac{1}{p} - \frac{1}{q}\right) \|\nabla u_m\|_p^p + \frac{1}{q^2} \|u_m\|_q^q\right) \\
&\leq (\lambda^*)^p \lim_{m\to\infty} \inf J(u_m) \\
&= (\lambda^*)^p d \\
&< d.
\end{aligned}$$

This is impossible, so we derive $I(u) = 0$ and $u_m \in \aleph$. From (21) and (12), we obtain $J(u) = d$, and the proof of (i) is complete.

ii) By $I(u) = 0$ and the definition of $I(u)$, we obtain

$$\|\nabla u\|_p^p = \int_\Omega |u|^q \ln|u|\, dx. \tag{25}$$

Then, by using tha fact (23) and Sobolev embedding theorem, (25) becomes

$$\begin{aligned}
\|\nabla u\|_p^p &< \frac{1}{e\alpha} \|u\|_{q+\alpha}^{q+\alpha} \\
&\leq \frac{C}{e\alpha} \|\nabla u\|_p^{q+\alpha},
\end{aligned}$$

where $C > 0$, which means that

$$\left(\frac{e\alpha}{C}\right)^{\frac{1}{q+\alpha-p}} \leq \|\nabla u\|_p. \tag{26}$$

From the (i) we know that, $u \in \aleph$. By $I(u) = 0$, (10) and (26), we note that

$$\begin{aligned}
J(u) &= \frac{1}{q} I(u) + \left(\frac{1}{p} - \frac{1}{q}\right) \|\nabla u\|_p^p + \frac{1}{q^2} \|u\|_q^q \\
&\geq \left(\frac{1}{p} - \frac{1}{q}\right) \|\nabla u\|_p^p \\
&\geq \left(\frac{q-p}{pq}\right) \left(\frac{e\alpha}{C}\right)^{\frac{p}{q+\alpha-p}},
\end{aligned}$$

where $q > p$, which implies that

$$d = \left(\frac{q-p}{pq}\right) \left(\frac{e\alpha}{C}\right)^{\frac{p}{q+\alpha-p}}.$$

This completes the proof. □

4. Global Existence

In this part, we prove the global existence of solution for the problem (1).

Lemma 4.1. *Let* $(u_0, u_1) \in W_0^{1,p}(\Omega) \times H_0^1(\Omega)$ *and* $2 < p < q < p\left(1 + \frac{2}{n}\right)$ *and* u *be a weak solution to problem (1). If* $E(0) < d$ *and* $u_0 \in W$, *then* $u \in W$.

Proof. Let $u(t)$ be any weak solution to problem (1) with condition $E(0) < d$ and $u_0 \in W$. We define T is the maximum existence time of the $u(x,t)$. Because of Lemma 2 we obtain that $E(t) < E(0) < d$ which means $I(u(t)) > 0$ for $0 < t < T$. Arguing by contradiction, we assume that there is $t^* \in (0, T)$ such that $I(u(t^*)) \leq 0$. By the continuity of $I(u(t))$ about time, there exists a $t_1 \in (0, T)$ to provide $I(u(t_1)) = 0$. Then by using (13) and Lemma 6, we obtain

$$d > E(0) \geq E(u(t_1)) \geq J(u(t_1)) \geq d,$$

which is a contradiction. □

Theorem 4.2. *Let* $(u_0, u_1) \in W_0^{1,p}(\Omega) \times H_0^1(\Omega)$. *Suppose that* $E(0) < d$ *and* $\|u\|_q \neq 0$, *then problem (1) admits a global weak solution* $u(t) \in L^\infty\left(0, \infty; W_0^{1,p}(\Omega)\right)$ *with* $u_t(t) \in L^\infty\left(0, \infty; H_0^1(\Omega)\right)$.

Proof. Let $h_j(x)$ be a system of base function $W_0^{1,p}(\Omega)$. We establish the approximate solution $u_m(x,t)$ of problem (1).

$$V_m = \text{span}\{w_1, w_2, ..., w_m\}$$

Let the projections of the initial data on the finite time be given by

$$\begin{aligned} u_{m_0}(x) &= \sum_{j=1}^{m} f_{mj} w_j(x) \to u_0(x) \text{ in } W_0^{1,p}(\Omega), \\ u_{m_1}(x) &= \sum_{j=1}^{m} g_{mj} w_j(x) \to u_1(x) \text{ in } H_0^1(\Omega), \end{aligned}$$

for $j = 1, 2, ..., m$.

Now our aim is looking for approximate solution such that

$$u_m(x,t) = \sum_{j=1}^{m} h_{jm}(t) w_j(x),$$

for the approximate problem

$$(u_{mtt}, w_s) + \left(|\nabla u^m|^{p-2} \nabla u^m, \nabla w_s\right) + (\nabla u_{mt}, w_s) = \left(|u_m|^{q-1} \ln|u_m|, w_s\right),\ s = 1, 2, ..., m. \tag{27}$$

Multiplying equation (27), summing for s and integrating over $(0, t)$ we obtain

$$\frac{1}{2}\|u_{mt}\|_2^2 + J(u_m) + \int_0^t \|\nabla u_{m\tau}\|_2^2 \, d\tau = E_m(0). \tag{28}$$

By virtue problem of (27) initial data, while $m \to \infty$ we obtain $E^m(0) \to E(0)$. By choosing of large m we get

$$\frac{1}{2}\|u_{mt}\|_2^2 + J(u_m) + \int_0^t \|\nabla u_{m\tau}\|_2^2 \, d\tau < d. \tag{29}$$

Then from (10) we can denote that

$$J(u_m) = \frac{1}{q}I(u_m) + \left(\frac{1}{p} - \frac{1}{q}\right)\|\nabla u_m\|_p^p + \frac{1}{q^2}\|u_m\|_q^q. \tag{30}$$

By $u_0 \in W$,

$$\frac{1}{2}\left\|u_t^m(0)\right\|_2^2 + J(u^m(0)) = E(0), \tag{31}$$

and initial data, for taking large m and $0 \le t < \infty$, we get $u_m(0) \in W$. Byusing (29) and similar argument to Lemma 7 and by choosing large m and $0 \le t < \infty$, we obtain $u_m(t) \in W$. Thanks of (29), (30) and (31), it yields that

$$\begin{aligned} d &> \frac{1}{2}\|u_{mt}\|_2^2 + \frac{1}{q}I(u_m) + \left(\frac{1}{p} - \frac{1}{q}\right)\|\nabla u_m\|_p^p \\ &\quad + \frac{1}{q^2}\|u_m\|_q^q + \int_0^t \|\nabla u_{m\tau}\|_2^2 d\tau \\ &> \frac{1}{2}\|u_{mt}\|_2^2 + \frac{q-p}{pq}\|\nabla u_m\|_p^p + \int_0^t \|\nabla u_{m\tau}\|_2^2 d\tau, \end{aligned} \tag{32}$$

where $0 \le t < \infty$ and $2 < p < q < p\left(1 + \frac{2}{n}\right)$. For a sufficiently large m and $0 \le t < \infty$, (32) gives

$$\left\|u_t^m\right\|_2^2 < 2d,$$

$$\|\nabla u_m\|_p^p < \frac{pq}{q-p}d,$$

$$\int_0^t \|\nabla u_{m\tau}\|_2^2 d\tau < d.$$

By using the definition of $I(u)$ and Sobolev embedding inequality and taking care the inequality $x^\alpha \ln x \le (e\alpha)^{-1}$ for all $x \in [1, \infty)$, we get

$$\begin{aligned} \|\nabla u_m\|_p^p &= \int_\Omega |u_m|^q \ln|u_m|\, dx \\ &\le \frac{1}{e\alpha}\|u^m\|_{q+\alpha}^{q+\alpha} \\ &\le \frac{C^{q+\alpha}}{e\alpha}\|\nabla u^m\|^{q+\alpha}, \end{aligned}$$

which implies

$$\begin{aligned} \|\nabla u^m\|_2^2 &\ge \left(\frac{e\alpha}{C^{q+\alpha}}\|\nabla u_m\|_p^p\right)^{\frac{2}{q+\alpha}} \\ &\ge \left(\frac{e\alpha}{C^{q+\alpha}}\frac{pq}{q-p}d\right)^{\frac{2}{q+\alpha}}, \end{aligned}$$

where $\alpha > 0$ which was defined in (6).

Hence, we obtain

$$\begin{cases} u^m, \text{ is uniformly bounded in } L^\infty\left(0,\infty; W_0^{1,p}(\Omega)\right), \\ u_t^m, \text{ is uniformly bounded in } L^\infty\left(0,\infty; H_0^1(\Omega)\right). \end{cases}$$

Then integrating (27) over to t , for $0 \le t < \infty$, it can be written as

$$\int_\Omega u_t w_s dx = \int_\Omega u_1 w_s dx + \int_0^t \int_\Omega \ln |u^m| \, |u^m|^{q-1} w_s dx ds - \int_0^t \int_\Omega u_t^m w_s dx ds$$
$$- \int_0^t \int_\Omega |\nabla u^m|^{p-2}(s) \nabla u^m(s) \nabla w_s dx ds. \tag{33}$$

Therefore, after passing through the limit in (33), and we get a weak solution u to problem (1) with the above regularity. On the other hand, initial data conditions in (27) we may conclude $(u(x,0)) = (u_0)$ in $W_0^{1,p}$ and $(u_t(x,0)) = (u_1)$ in $L^2(\Omega)$. □

5. Decay results of solution

In this part the decay of solution for the problem (1) was studied .

Theorem 5.1. *Let $u_0(t) \in W$, $u_1(t) \in H_0^1(\Omega)$ and $E(0) < d$ and $2 < p < q < p\left(1 + \frac{2}{n}\right)$ hold. There is a positive fixed S_0 such that $E(t)$ satisfies the following polynomial decay estimate for $\forall t \in [0,\infty)$*

$$E(t) \le \frac{S_0}{1+t}.$$

Proof. It follows from Lemma 7 that $u \in W$ on $[0,T]$. By using the definition of the d, (14), (10) and (11) we obtain the following inequality

$$\begin{aligned} d &> E(0) \ge E(t) + \int_0^t \|\nabla u_t\|_2^2 d\tau \\ &= \frac{1}{2}\|u_t\|_2^2 + J(u) + \int_0^t \|\nabla u_t\|_2^2 d\tau \\ &= \frac{1}{2}\|u_t\|_2^2 + \frac{1}{q} I(u) + \left(\frac{1}{p} - \frac{1}{q}\right)\|\nabla u\|_p^p + \frac{1}{q^2}\|u\|_q^q + \int_0^t \|\nabla u_t\|_2^2 d\tau \\ &\ge \frac{1}{2}\|u_t\|_2^2 + \left(\frac{1}{p} - \frac{1}{q}\right)\|\nabla u\|_p^p + \frac{1}{q^2}\|u\|_q^q + \int_0^t \|\nabla u_t\|_2^2 d\tau, \end{aligned} \tag{34}$$

which means that

$$\frac{1}{2}\|u_t\|_2^2 \le d \text{ or } \frac{pqd}{q-p} \le \|\nabla u\|_p^p, \tag{35}$$

and

$$\int_0^t \|u_t\|_2^2 d\tau \le \frac{1}{\mu}\int_0^t \|\nabla u_t\|_2^2 d\tau \le \frac{d}{\mu}, \tag{36}$$

where μ is the optimal constant.

From $I(u) \geq 0$ and Lemma 5, we claim that there is a constant $\lambda^* \geq 1$ such that $I(\lambda^* u) = 0$. Therefore, from (10) and (12), we conclude

$$\begin{aligned} d &\leq J(\lambda^* u) = \frac{1}{q} I(\lambda^* u) + \left(\frac{1}{p} - \frac{1}{q}\right) \|\nabla (\lambda^* u)\|_p^p + \frac{1}{q^2} \|(\lambda^* u)\|_q^q \\ &= \frac{q-p}{pq} (\lambda^*)^p \|\nabla u\|_p^p + \frac{(\lambda^*)^q}{q^2} \|(u)\|_q^q \\ &= (\lambda^*)^q \left(\frac{q-p}{pq} (\lambda^*)^{p-q} \|\nabla u\|_p^p + \frac{1}{q^2} \|(u)\|_q^q\right) \\ &\leq (\lambda^*)^q \left(\frac{q-p}{pq} \|\nabla u\|_p^p + \frac{1}{q^2} \|(u)\|_q^q\right) \\ &< (\lambda^*)^q E(0), \end{aligned} \tag{37}$$

which satisfies that

$$\lambda^* > \left(\frac{d}{E(0)}\right)^{\frac{1}{q}} > 1. \tag{38}$$

On the other hand, by using $I(\lambda^* u) = 0$ equality and definition of $I(u)$, we get

$$\begin{aligned} 0 &= I(\lambda^* u) = (\lambda^*)^p \|\nabla u\|_p^p - \int_\Omega |\lambda^* u|^q \ln |\lambda^* u| \, dx \\ &= (\lambda^*)^p \|\nabla u\|_p^p - (\lambda^*)^q \int_\Omega |u|^q \ln |u| \, dx - (\lambda^*)^q \ln |\lambda^*| \, \|(u)\|_q^q \\ &= (\lambda^*)^p \|\nabla u\|_p^p + (\lambda^*)^q I(u) - (\lambda^*)^q \|\nabla u\|_p^p - (\lambda^*)^q \ln |\lambda^*| \, \|(u)\|_q^q \\ &= (\lambda^*)^q I(u) - \left[(\lambda^*)^q - (\lambda^*)^p\right] \|\nabla u\|_p^p - (\lambda^*)^q \ln |\lambda^*| \, \|(u)\|_q^q . \end{aligned} \tag{39}$$

By combining (38) with (39), we arrive at

$$\begin{aligned} I(u) &\geq \left[\frac{(\lambda^*)^q - (\lambda^*)^p}{(\lambda^*)^q}\right] \|\nabla u\|_p^p + \ln |\lambda^*| \, \|(u)\|_q^q \\ &\geq \left[1 - (\lambda^*)^{p-q}\right] \|\nabla u\|_p^p \\ &= \beta \|\nabla u\|_p^p , \end{aligned} \tag{40}$$

where $\beta = 1 - (\lambda^*)^{p-q} \in (0, 1)$.

Next, we multiple the equation of (1) by u and integrate over $\Omega \times (0, t)$.Then, , we obtain

$$\int_0^t \int_\Omega u_{tt} u dx d\tau + \int_0^t \int_\Omega |\nabla u|^{p-1} \nabla u dx d\tau + \int_0^t \int_\Omega \nabla u_t \nabla u dx d\tau = \int_0^t \int_\Omega |u|^{q-1} u \ln |u| \, dx d\tau,$$

$$\int_0^t \|\nabla u\|_p^p \, d\tau - \int_0^t \int_\Omega |u|^{q-1} u \ln |u| \, dx d\tau = -\int_0^t \int_\Omega u_{tt} u dx dt - \int_0^t \int_\Omega \nabla u_t \nabla u dx d\tau. \tag{41}$$

From the definition of $I(u)$, (41) and using of Young and Hölder inequality, (41) implies that

$$\begin{aligned}\int_0^t I(u)\,d\tau &= -\int_0^t\int_\Omega u_{tt}u\,dx\,dt - \int_0^t\int_\Omega \nabla u_t \nabla u\,dx\,d\tau \\ &= -\int_0^t\int_\Omega \frac{d}{dt}(u_t,u)\,dx\,dt + \int_0^t \|u_t\|_2^2\,d\tau - \frac{1}{2}\int_0^t \frac{d}{dt}\|\nabla u\|_2^2\,d\tau \\ &= \int_0^t \|u_t\|_2^2\,d\tau - (u_t(t),u(t)) + (u_1,u_0) - \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|\nabla u_0\|_2^2 \\ &\le \int_0^t \|u_t\|_2^2\,d\tau + \frac{1}{2}\|u\|_2^2 + \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|u_0\|_2^2 \\ &\quad + \frac{1}{2}\|u_1\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|\nabla u_0\|_2^2. \end{aligned} \tag{42}$$

Inserting (35) and (36) into (42), for $0 < t < \infty$ we get

$$\int_0^t I(u)\,d\tau \le C. \tag{43}$$

Morever, the combination of (40) and (43), it follows that

$$\int_0^t \|\nabla u\|_p^p \le \frac{1}{1-(\lambda^*)^{p-q}}\int_0^t I(u)\,d\tau \le C, \tag{44}$$

and

$$\int_0^t \|(u)\|_q^q \le \frac{1}{\ln|\lambda^*|}\int_0^t I(u)\,d\tau \le C. \tag{45}$$

By using Lemma 4, we consider that

$$\begin{aligned}[(1+t)E(t)]' &= (1+t)E'(t) + E(t) \\ &\le E(t). \end{aligned} \tag{46}$$

Integrating the (46) over $(0,t)$ and using (11) and (10), it implies that

$$\begin{aligned}(1+t)E(t) &\le E(0) + \int_0^t E(\tau)\,d\tau \\ &= E(0) + \frac{1}{2}\int_0^t \|u_t\|_2^2\,d\tau + \frac{1}{q}\int_0^t I(u)\,d\tau \\ &\quad + \left(\frac{1}{p}-\frac{1}{q}\right)\int_0^t \|\nabla u\|_p^p\,d\tau + \frac{1}{q^2}\int_0^t \|u\|_q^q\,d\tau. \end{aligned} \tag{47}$$

Consequently, inserting (36), (43),(44) and (45) into (47), we prove that there is a positive fixed S_0 such that

$$(1+t)\,E(t) \le S_0.$$

This inequality finished the proof of the Theorem 9. □

6. Blow up results

In this section, we consider the finite time blow up results of solutions for problem (1). The interested reader can look to proof of Theorem 1 of paper [17] to obtain local existence of problem (1) by using similar method.

6.1. Blow up results for $E(0) < d$

Next, we give the following lemma which will have an essential role in our proof of Theorem 11.

Lemma 6.1. *[20, 22]. Let $B(t)$ be a positive C^2 function, which satisfies, for $t > 0$, inequality*

$$B(t)\,B''(t) - (1+\theta)\,[B'(t)]^2 \ge 0, \tag{48}$$

with some $\theta > 0$. If $B(0) > 0$ and $B'(0) > 0$, then there exist a time $T^* \le \frac{B(0)}{\beta B'(0)}$ such that

$$\lim_{t\to T^{*-}} B(t) = \infty. \tag{49}$$

Theorem 6.2. *Assume that $u_0(t) \in V$, $u_1(t) \in H_0^1(\Omega)$. Suppose that $2 < p < q < p\left(1+\frac{2}{n}\right)$ and $E(0) < d$ hold, then the solution u of problem (1) blow up in finite time; that is the maximum existence time T^* of $u(t)$ is finite and*

$$\lim_{t\to T^{*-}} \left(\|u\|_2^2 + \int_0^t \|\nabla u\|_2^2\, d\tau \right) = +\infty. \tag{50}$$

Consequently, the upper bound for blow up time T^* is given by

$$T^* \le \frac{2bT_0^2 + 2\,\|u_0\|_2^2}{(p-2)\,bT_0 + (p-2)\int_\Omega u_0 u_1 dx - 2\,\|\nabla u_0\|_2^2}, \tag{51}$$

where b and T_0 will be chosen in (61) and (62).

Proof. By contradiction, we assume that u is global, then $T^* = +\infty$. For any $T > 0$, we assume that $\Phi : [0,T] \to R^+$ defined by

$$B(t) = \|u\|^2 + \int_0^t \|\nabla u\|_2^2\, d\tau + (T-t)\,\|\nabla u_0\|_2^2 + b\,(T_0+t)^2, \tag{52}$$

where b and T_0 are positive constants which will be specified later.

Firstly, we compute the first order differential and second order differential of $\Phi(t)$, respectively, as follows:

$$\begin{aligned} B'(t) &= 2\int_\Omega u_t u dx + \|\nabla u\|_2^2 - \|\nabla u_0\|_2^2 + 2b\,(T_0+t) \\ &= 2\int_\Omega u_t u dx + 2\int_0^t \int_\Omega \nabla u \nabla u_t dx d\tau + 2b\,(T_0+t), \end{aligned} \tag{53}$$

and

$$\begin{aligned} B''(t) &= 2\int_\Omega |u_t|^2\,dx + 2\int_\Omega u_{tt}u\,dx + 2\int_\Omega \nabla u\nabla u_t dx + 2b \\ &= 2\int_\Omega |u_t|^2\,dx + 2\int_\Omega u_{tt}u\,dx - 2\int_\Omega u\Delta u_t dx + 2b \\ &= 2\int_\Omega |u_t|^2\,dx + 2\int_\Omega u\left[u_{tt} - \Delta u_t\right]dx + 2b \\ &= 2\int_\Omega |u_t|^2\,dx + 2\int_\Omega u\left[div\left(|\nabla u|^{p-2}\nabla u\right) + |u|^{q-2}u\ln|u|\right]dx + 2b \\ &= 2\int_\Omega |u_t|^2\,dx - 2\left[\int_\Omega |\nabla u|^p\,dx - \int_\Omega |u|^q\ln|u|\,dx\right] + 2b \\ &= 2\int_\Omega |u_t|^2\,dx - 2\left[\|\nabla u\|_p^p - \int_\Omega |u|^q\ln|u|\,dx\right] + 2b \\ &= 2\|u_t\|^2 - 2I(u) + 2b. \end{aligned} \tag{54}$$

Through a direct calculation, we have

$$\begin{aligned} &B(t)B''(t) - \frac{q+2}{4}\left[B'(t)\right]^2 \\ = \;& 2B(t)\left(\|u_t\|_2^2 - \|\nabla u\|_p^p + \int_\Omega |u|^q\ln|u|\,dx + b\right) \\ &+ (q+2)\left[G(t) - \left(B(t) - (T-t)\|\nabla u_0\|_2^2\right)\left(\|u_t\|_2^2 + \int_0^t \|\nabla u_t\|_2^2\,d\tau + b\right)\right], \end{aligned} \tag{55}$$

where

$$\begin{aligned} G(t) &= \left(\|u\|^2 + \int_0^t \|\nabla u\|_2^2\,d\tau + b(T_0+t)^2\right)\left(\|u_t\|^2 + \int_0^t \|\nabla u_t\|_2^2\,d\tau + b\right) \\ &\quad -\left(\int_\Omega u_t u\,dx + \int_0^t\int_\Omega \nabla u\nabla u_t dx d\tau + 2b(T_0+t)\right)^2. \end{aligned} \tag{56}$$

Using Schwarz inequality and Young inequality, it is not difficult to verify that $B(t) \geq 0$ for any $t \in [0,T]$. As a consequence, from (55) we arrive that

$$B(t)B''(t) - \frac{q+2}{4}\left[B'(t)\right]^2 \geq B(t)\,\xi(t), \tag{57}$$

where $\xi(t):[0,T]\to R$ is defined by

$$\begin{aligned}\xi(t) &= -q\|u_t\|^2-2\|\nabla u\|_p^p+2\int_\Omega |u|^q\ln|u|\,dx\\ &\quad-(q+2)\int_0^t\|\nabla u_t\|_2^2\,d\tau-qb. \end{aligned}\tag{58}$$

Furthermore, by the definition of $E(t)$ and Lemma 6, it follows that

$$\begin{aligned}\xi(t) &= -2qE(t)+\frac{2(q-p)}{p}\|\nabla u\|_p^p+\frac{2}{q}\|u\|_q^q-(q+2)\int_0^t\|\nabla u_t\|_2^2\,d\tau-qb\\ &\geq -2qd+\frac{2(q-p)}{p}\|\nabla u\|_p^p+\frac{2}{q}\|u\|_q^q-(q+2)\int_0^t\|\nabla u_t\|_2^2\,d\tau-qb. \end{aligned}\tag{59}$$

From $u_0(x)\in V$, $u_1(x)\in H_0^1(\Omega)$ and Lemma 6, we obtain $u(x)\in V$, $u_t(x)\in H_0^1(\Omega)$ for all $t\geq 0$, which implies that $I(u)<0$. Hence there exists a $\lambda_*\in(0,1)$ such that $I(\lambda_* u)=0$. Thus by the definition of d and (10), we get that

$$\begin{aligned}\left(\frac{q-p}{pq}\right)\|\nabla u\|_p^p+\frac{1}{q^2}\|u\|_q^q &\geq \frac{(q-p)\lambda_*^p}{pq}\|\nabla u\|_p^p+\frac{\lambda_*^q}{q^2}\|u\|_q^q\\ &\geq J(\lambda_* u)\\ &\geq d. \end{aligned}\tag{60}$$

Choosing b small enough shuch that

$$0<b\leq\frac{-2qd+\frac{2(q-p)}{p}\|\nabla u\|_p^p+\frac{2}{q}\|u\|_q^q+(q+2)\int_0^t\|\nabla u_t\|^2\,d\tau}{q}.\tag{61}$$

The combination of (59)-(61) implies that $\xi(t)\geq 0$. Hense, by the above discussion, we have

$$B(t)B''(t)-\frac{q+2}{4}[B'(t)]^2\geq 0.$$

From the definition of $B(t)$, it is easy to know that $B(0)=\|u_0\|_2^2+T\|\nabla u_0\|_2^2+bT_0^2>0$. We choose T_0 large enough shuch that

$$T_0>\frac{(q-p)\left(\|u_0\|_2^2+\|u_1\|_2^2\right)+4\|\nabla u_0\|_2^2}{2(q-p)b},\tag{62}$$

which fulfills the requirement of

$$B'(0)=2\int_\Omega u_0u_1dx+2bT_0\geq 2bT_0-\|u_0\|_2^2-\|u_1\|_2^2-\frac{4\|\nabla u_0\|_2^2}{q-p}>0.\tag{63}$$

Then, according to Lemma 10, we obtain that $B(t)$ goes to ∞ as t tends to some T^* satisfying

$$\begin{aligned} T^* &\leq \frac{4B(0)}{(q-p)B'(0)} \\ &\leq \frac{4B(0)}{(q-2)B'(0)} \\ &= \frac{2bT_0^2 + 2\|u_0\|_2^2 + 2T\|\nabla u_0\|_2^2}{(q-2)bT_0 + (q-2)\int\limits_\Omega u_0 u_1 dx}, \end{aligned}$$

which means that

$$T^* \leq \frac{4\left(bT_0^2 + \|u_0\|_2^2\right)}{(q-2)bT_0 + (q-2)\int\limits_\Omega u_0 u_1 dx - 2\|\nabla u_0\|_2^2}. \tag{64}$$

Finally, for fixed T_0, choose T as

$$T > \frac{2bT_0^2 + 2\|u_0\|^2}{2(q-2)bT_0 - 4\|\nabla u_0\|^2 - (q-2)\left(\|u_0\|^2 + \|u_1\|^2\right)}. \tag{65}$$

The combination of (64) and (65), we see that $T > T^*$. This contradicts to our assumption, which finished the proof. □

6.1.1. Blow up results for $E(0) < 0$

In this subsection we establish the blow up of the solution with $E(0) < 0$ by using the method of [33] with a modification in the energy functional due to the different nature of the problems.

Lemma 6.3. *[19]. There is $C > 0$ which is dependent on Ω only such that*

$$\left(\int\limits_\Omega u^q \ln|u|\, dx\right)^{\frac{s}{q}} \leq C\left[\int\limits_\Omega u^q \ln|u|\, dx + \|\nabla u\|_p^p\right], \tag{66}$$

where $\int\limits_\Omega u^q \ln|u|\, dx \geq 0$ for any $u \in L^{q+1}(\Omega)$ and $2 < p \leq s \leq q \leq p\left(1 + \frac{2}{n}\right)$.

Lemma 6.4. *There is $C > 0$ which is dependent on Ω only such that*

$$\|u\|_q^q \leq C\left[\int\limits_\Omega u^q \ln|u|\, dx + \|\nabla u\|_p^p\right], \tag{67}$$

where $\int\limits_\Omega u^q \ln|u|\, dx \geq 0$ for any $u \in L^q(\Omega)$.

Proof. We introduce

$$\Omega^+ = \{x \in \Omega : \ln|u| > 1\} \text{ and } \Omega^- = \{x \in \Omega : \ln|u| \leq 1\}. \tag{68}$$

Therefore, by using embedding and (68) we arrive at

$$\begin{aligned}
\|u\|_q^q &= \int_{\Omega^+} u^q dx + \int_{\Omega^-} u^q dx \\
&\le \int_{\Omega^+} u^q \ln|u|\, dx + \int_{\Omega^-} u^q \ln|u|\, dx \\
&\le \int_{\Omega^+} u^q \ln|u|\, dx + \int_{\Omega^-} \left|\frac{u}{e}\right|^q e^q dx \\
&\le \int_{\Omega^+} u^q \ln|u|\, dx + e^q \int_{\Omega^-} \left|\frac{u}{e}\right|^p dx \\
&\le \int_{\Omega^+} u^q \ln|u|\, dx + e^{q-p} \int_{\Omega} |u|^p\, dx \\
&\le C\left[\int_{\Omega} u^q \ln|u|\, dx + \|\nabla u\|_p^p\right].
\end{aligned}$$

Thus, the result was obtained. □

Corollary 6.5. *Let the assumptions of the Lemma 13 hold. Using the fact that* $\|u\|_p^p \le C\|u\|_q^q \le C\left(\|u\|_q^q\right)^{\frac{p}{q}}$ *. Then we obtain the following*

$$\|u\|_p^p \le C\left[\left(\int_{\Omega} u^q \ln|u|\, dx\right)^{\frac{p}{q}} + \|\nabla u\|_p^{\frac{p^2}{q}}\right].$$

Lemma 6.6. *There is* $C > 0$ *which is dependent on* Ω *only such that*

$$\|u\|_q^s \le C\left[\|u\|_q^q + \|\nabla u\|_p^p\right],$$

for any $u \in L^q(\Omega)$ *and* $2 < p \le s \le q \le p\left(1 + \frac{2}{n}\right)$.

Theorem 6.7. *Let conditions in (6) hold. and.Morever, for negative initial energy* ($E(0) < 0$)*, the solution of problem (1) blows up in finite time for* $\xi, \alpha > 0$.

$$T^* \le \frac{1-\alpha}{\xi \frac{\alpha}{1-\alpha} L^{\frac{\alpha}{1-\alpha}}(0)}.$$

Proof. For obtaining the proof of the Theorem 16, we start with defining auxiliary function

$$H(t) = -E(t). \tag{69}$$

If we use the definition of $H(t)$ and (14), it is clearly that

$$H'(t) = -E'(t) = \|u_t\|_2^2 \ge 0. \tag{70}$$

Consequently by virtue of (7), (69) and (70) we have

$$\frac{1}{q}\int_{\Omega} u^q \ln|u|\, dx \ge H(t) \ge H(0) > 0. \tag{71}$$

We set

$$L(t) = H^{1-\alpha}(t) + \varepsilon \int_{\Omega} u u_t dx, \tag{72}$$

for ε small to be chosen later and

$$\frac{2\left(q^2 - 2p\right)}{q^3} < \alpha < \frac{q-2}{2q}. \tag{73}$$

Now, differentiating $L(t)$ with respect to t and we obtain from (1) and (7)

$$\begin{aligned} L'(t) &= (1-\alpha) H^{-\alpha}(t) H'(t) + \varepsilon \int_{\Omega} |u_t|^2 dx + \int_{\Omega} u u_{tt} dx \\ &= (1-\alpha) H^{-\alpha}(t) \|\nabla u_t\|_2^2 + \varepsilon \|u_t\|_2^2 \\ &\quad + \varepsilon \int_{\Omega} u \left(div \left(|\nabla u|^{p-2} \nabla u \right) + \Delta u_t + |u|^{q-2} u \ln |u| \right) dx \\ &= (1-\alpha) H^{-\alpha}(t) \|\nabla u_t\|_2^2 + \varepsilon \|u_t\|_2^2 - \varepsilon \|\nabla u\|_p^p \\ &\quad - \varepsilon \int_{\Omega} \nabla u \nabla u_t dx + \varepsilon \int_{\Omega} u^q \ln |u| dx. \end{aligned} \tag{74}$$

Adding and subtracting $\varepsilon q H(t)$ in (74), we obtain$\frac{1}{2} \|u_t\|^2 + \frac{1}{p} \|\nabla u\|_p^p - \frac{1}{q} \int_{\Omega} |u|^q \ln |u| dx + \frac{1}{q^2} \|u\|_q^q$

$$\begin{aligned} L'(t) &= (1-\alpha) H^{-\alpha}(t) \|\nabla u_t\|_2^2 + \varepsilon \left(\frac{q+2}{2} \right) \|u_t\|_2^2 \\ &\quad - \varepsilon \left(1 - \frac{q}{p} \right) \|\nabla u\|_p^p - \varepsilon \int_{\Omega} \nabla u \nabla u_t dx \\ &\quad + \varepsilon \frac{1}{q} \|u\|_q^q + \varepsilon q H(t). \end{aligned}$$

Exploiting Hölder's and Young's inequalities, for any $\mu > 0$, (74) takes form

$$\begin{aligned} L'(t) &\geq (1-\alpha) H^{-\alpha}(t) \|\nabla u_t\|_2^2 + \varepsilon \left(\frac{q+2}{2} \right) \|u_t\|_2^2 \\ &\quad - \varepsilon \left(1 - \frac{q}{p} \right) \|\nabla u\|_p^p + \varepsilon \frac{1}{q} \|u\|_q^q \\ &\quad + \varepsilon q H(t) \\ &\quad + \varepsilon \mu \|\nabla u_t\|_2^2 + \varepsilon \frac{1}{4\mu} \|\nabla u\|_2^2. \end{aligned} \tag{75}$$

According to Sobolev emebedding theorems, we get

$$\|u\|_2^p < \|u\|_p^p < \|\nabla u\|_2^p = \left(\|\nabla u\|_2^2 \right)^{\frac{p}{2}}. \tag{76}$$

Hence, by using the inequality (76) and by choosing μ so that $\mu = M_1 H^{-\alpha}(t)$, for M_1 to be specified later, (75) yields that

$$\begin{aligned} L'(t) &\geq [(1-\alpha)+\varepsilon M_1] H^{-\alpha}(t) \|\nabla u_t\|_2^2 + \varepsilon\left(\frac{q+2}{2}\right)\|u_t\|_2^2 \\ &-\varepsilon\left(1-\frac{q}{p}\right)\|\nabla u\|_p^p + \varepsilon\frac{1}{q}\|u\|_q^q + \varepsilon\frac{H^\alpha(t)}{4M_1}\left(\|u\|_p^p\right)^{\frac{2}{p}} \\ &+\varepsilon q H(t). \end{aligned} \tag{77}$$

By using of the Corallary 12 and Young's inequality, we obtain

$$\begin{aligned} H^\alpha(t)\left(\|u\|_p^p\right)^{\frac{2}{p}} &\leq \left(\frac{1}{q}\int_\Omega u^q \ln|u|\,dx\right)^\alpha \left(\|u\|_p^p\right)^{\frac{2}{p}} \\ &\leq C\left(\int_\Omega u^p \ln|u|\,dx\right)^\alpha \left[\left(\int_\Omega u^q \ln|u|\,dx\right)^{\frac{2}{q}} + \|\nabla u\|_p^{\frac{2p}{q}}\right] \\ &\leq C\left[\left(\int_\Omega u^p \ln|u|\,dx\right)^{\alpha+\frac{2}{q}} + \left(\int_\Omega u^p \ln|u|\,dx\right)^\alpha \|\nabla u\|_p^{\frac{2p}{q}}\right] \\ &\leq C\left[\left(\int_\Omega u^p \ln|u|\,dx\right)^{\frac{q\alpha+2}{q}} + \left(\int_\Omega u^p \ln|u|\,dx\right)^\alpha \|\nabla u\|_q^{\frac{2p}{q}}\right] \\ &\leq C\left[\left(\int_\Omega u^p \ln|u|\,dx\right)^{\frac{q\alpha+2}{q}} + \left(\int_\Omega u^p \ln|u|\,dx\right)^{\frac{\alpha q^2}{q^2-2p}} + \|\nabla u\|_q^q\right]. \end{aligned} \tag{78}$$

We also exploit

$$2 < q\alpha + 2 \leq q \text{ and } 2 < \frac{\alpha q^3}{q^2-2p} \leq q,$$

to obtain

$$H^\alpha(t)\left(\|u\|_p^p\right)^{\frac{2}{p}} \leq C\left[\int_\Omega u^q \ln|u|\,dx + \|\nabla u\|_q^q\right]. \tag{79}$$

Inserting (79) into (77) and using embedding inequality $\frac{\|u\|_q^q}{C_1} < \|\nabla u\|_q^q$, we deduce

$$\begin{aligned} L'(t) &\geq [(1-\alpha)+\varepsilon M_1] H^{-\alpha}(t) \|\nabla u_t\|_2^2 + \varepsilon\left(\frac{q+2}{2}\right)\|u_t\|_2^2 \\ &-\varepsilon\left(1-\frac{q}{p}\right)\|\nabla u\|_p^p + \varepsilon\left(\frac{1}{q}+\frac{C}{4C_1M_1}\right)\|u\|_q^q \\ &+\frac{\varepsilon C}{4M_1}\int_\Omega u^q \ln|u|\,dx + \varepsilon q H(t). \end{aligned} \tag{80}$$

At this point, we choose $0 < \alpha < 1$ small that

$(1-\alpha) + \varepsilon M_1 > 0,$

and

$$H(0) + \varepsilon \int_{\Omega} u_0 u_1 dx > 0. \tag{81}$$

Therefore, from (80) we have

$$L'(t) \geq \omega \left[H(t) + \|u_t\|_2^2 + \|\nabla u\|_p^p + \|u\|_q^q + \int_{\Omega} u^q \ln|u|\, dx \right], \tag{82}$$

and

$$L(0) < L(t) \ \text{ for } t \geq 0.$$

Otherwise, thanks to inequality $(a+b)^k \leq 2^{k-1}\left(a^k + b^k\right)$,(72) can be written as

$$\begin{aligned} L(t)^{\frac{1}{1-\alpha}} &= \left[H^{1-\alpha}(t) + \varepsilon \int_{\Omega} u u_t dx \right]^{\frac{1}{1-\alpha}} \\ &\leq C \left[H(t) + \varepsilon \left(\int_{\Omega} u u_t dx \right)^{\frac{1}{1-\alpha}} \right]. \end{aligned} \tag{83}$$

From Hölder's inequality and the embedding $L^q(\Omega) \hookrightarrow L^2(\Omega)$,we get

$$\begin{aligned} \left| \int_{\Omega} u u_t dx \right| &\leq \|u_t\|_2 \|u\|_2 \\ &\leq \|u_t\|_2 \|u\|_q . \end{aligned}$$

So, there exists C which is a positive fixed and it satisfies

$$\left| \int_{\Omega} u u_t dx \right|^{1/(1-\alpha)} \leq \|u_t\|_2^{1/(1-\alpha)} \|u\|_q^{1/(1-\alpha)} . \tag{84}$$

By thanks to Young's inequality which was used for the right-hand side of the(84), we have

$$\left| \int_{\Omega} u u_t dx \right|^{1/(1-\alpha)} \leq C \left[\|u_t\|_2^{\theta/(1-\alpha)} + \|u\|_q^{\mu/(1-\alpha)} \right],$$

for $\frac{1}{\mu} + \frac{1}{\theta} = 1$. We take $\theta = 2(1-\alpha)$, thus $\mu = 2(1-\alpha)/(1-2\alpha)$, to obtain

$$\left| \int_{\Omega} u u_t dx \right|^{1/(1-\alpha)} \leq C \left[\|u_t\|_2^2 + \|u\|_q^{2/(1-2\alpha)} \right].$$

Now, if we use Poincare's inequality, above inequality can be obtained as

$$\left| \int_{\Omega} u u_t dx \right|^{1/(1-\alpha)} \leq C \left[\|u_t\|_2^2 + \|u\|_q^{2/(1-2\alpha)} \right].$$

Hence, Lemma 15 gives

$$\left|\int_{\Omega} uu_t dx\right|^{1/(1-\alpha)} \leq C\left[\|u_t\|_2^2 + \|u\|_q^q + \|\nabla u\|_p^p\right]. \tag{85}$$

Inserting (85) into (83) yields that

$$L(t)^{\frac{1}{1-\alpha}} \leq C\left[H(t) + \|u_t\|_2^2 + \|u\|_q^q + \|\nabla u\|_p^p\right]. \tag{86}$$

By combining (86) and (82) we reach

$$L'(t) \geq \xi L^{\frac{1}{1-\alpha}}(t), \tag{87}$$

where ξ is a positive constant. Integration of (87) over $(0,t)$ we yield

$$L^{\frac{\alpha}{1-\alpha}}(t) \geq \frac{1}{L^{-\frac{\alpha}{1-\alpha}}(0) - \frac{\xi \alpha t}{1-\alpha}}.$$

Therefore $L(t)$ blows up in afinite time and T^* is

$$T^* \leq \frac{1-\alpha}{\xi \frac{\alpha}{1-\alpha} L^{\frac{\alpha}{1-\alpha}}(0)}.$$

Consequently we completed our proof. □

References

[1] R. A. Adams, J. J. F. Fournier, *Sobolev Spaces*, Academic Press, New York, 2003.

[2] A. Benaissa, S. Mokeddem, *Decay estimates for the wave equation of p-Laplacian type with dissipation of m-Laplacian type*, Math. Methods Appl. Sci. **30(2)**(2007), 237-247.

[3] S. M. Boulaaras, A. Choucha, A. Zara, M. Abdalla, B. B. Cheri, *Global existence and decay estimates of energy of solutions for a new class of-Laplacian heat equations with logarithmic nonlinearity*, J. Funct. Spaces. **2021** (2021), 5558818.

[4] H. Brezis, P. Mironescu, *Gagliardo-Nirenberg, composition and products in fractional Sobolev spaces*, J. Evol. Equ. **1(4)**(2001), 387-404.

[5] C. Chen, H. Yao, L. Shao, *Global existence, uniqueness, and asymptotic behavior of solution for-Laplacian type wave equation*, J. Inequal. Appl. **2010** (2010), 1-15.

[6] A. Choucha, S. Boulaaras, D. Ouchenane, *Exponential decay and global existence of solutions of a singular nonlocal viscoelastic system with distributed delay and damping terms*, Filomat. **35(3)**(2021), 795-826.

[7] P. Dai, C. Mu, G. Xu, *Blow-up phenomena for a pseudo-parabolic equation with p-Laplacian and logarithmic nonlinearity terms*, J. Math. Anal. Appl. **481 (1)**(2020), 123439.

[8] H. Di, Y. Shang, Z. Song, *Initial boundary value problem for a class of strongly damped semilinear wave equations with logarithmic nonlinearity*, Nonlinear Anal. Real World Appl. **51** (2020), 102968.

[9] H. Ding, J. Zhou, *Global existence and blow-up for a mixed pseudo-parabolic p-Laplacian type equation with logarithmic nonlinearity*, J. Math. Anal. Appl. **478** (2019), 393-420.

[10] H. Ding, R. Wang, J. Zhou, *Infinite time blow-up of solutions to a class of wave equations with weak and strong damping terms and logarithmic nonlinearity*, Stud. Appl. Math. **147(3)**(2021), 914-934.

[11] Y. Ebihara,*On some nonlinear evolution equations with the strong dissipation*, J. Differential Equations. **30** (1978), 149-164.

[12] J. Ferreira, E. Pişkin, N. Irkıl, C. Raposo, *Blow up results for a viscoelastic Kirchhoff-type equation with logarithmic nonlinearity and strong damping*, Math. Morav. **25(2)**(2021), 125-141.

[13] E. Gagliardo, *Ulteriori proprieta di alcune classi di funzioni in piu variabili*, Ricerche Mat. **8** (1959), 24-51.

[14] P. Gorka, *Logarithmic Klein–Gordon equation*, Acta Phys. Polon. B. **40(1)**(2009), 59–66.

[15] J. Greenberg, R. MacCamy, V. Mizel, *On the existence, uniqueness and stability of solutions of the equation* $\sigma'(u_x)u_{xx} + \lambda u_{xtx} = \rho u_{tt}$, J. Math. Mech. **17** (1968), 707-728.

[16] T. G. Ha, S. H. Park, *Blow-up phenomena for a viscoelastic wave equation with strong damping and logarithmic nonlinearity*, Adv. Difference Equ. **2020** (2020), 235.

[17] Y. He, H. Gao, H. Wang, *Blow-up and decay for a class of pseudo-parabolic p-Laplacian equation with logarithmic nonlinearity*, Comput. Math. Appl. **75**(2018), 459-469.

[18] N. Irkıl, E. Pişkin, *Local existence and blow up for p-Laplacian equation with logarithmic nonlinearity*, Miskolc Math. Notes. **23 (1)** (2022), 231-251.

[19] M. Kafini, S. Messaoudi, *Local existence and blow up of slutions to a logarithmic nonlinear wave equation with delay*, Appl. Anal. **99(3)** (2020), 530-547.

[20] V. K. Kalantarov, O. A. Ladyzhenskaya, *The occurrence of collapse for quasilinear equations of parabolic and hyperbolic type*, J. Soviet Math. **10(1)**(1978), 53-70.

[21] T. Kobayashi, H. Pecher, Y. Shibata,*On a global in time existence theorem of smooth solutions to nonlinear wave equation with viscosity*, Math. Annalen. **296** (1993), 215-234.

[22] H. A. Levine, *Instability and nonexistence of global solutions of nonlinear wave equations of the form* $Pu_{tt} = Au + F(u)$,, Trans. Amer. Math. Soc. **192**(1974), 1-21.

[23] L. Ma, Z. B. Fang, *Energy decay estimates and infinite blow-up phenomena for a strongly damped semilinear wave equation with logarithmic nonlinear source*, Math. Methods Appl. Sci. **41(7)**(2018), 2639-2653.

[24] M. Nakao, *Energy decay for the quasilinear wave equation with viscosity*, Math Z. **219** (1995), 289-299.

[25] L. Nirenberg, *On elliptic partial differential equations*, Ann. Scuola Norm. Sup. Pisa. **(3)13** (1959),115-162.

[26] V. Pata, S. Zelik, *Smooth attractors for strongly damped wave equations*, Nonlinearity. **19(7)** (2006), 1495-1506.

[27] P. Pei, M. A. Rammaha, D. Toundykov, *Weak solutions and blow-up for wave equations of p-Laplacian type with supercritical sources*, J. Math. Phys.**56(8)** (2015), 081503.

[28] E. Pişkin, S. Boulaaras, N. Irkil, *Qualitative analysis of solutions for the p-Laplacian hyperbolic equation with logarithmic nonlinearity*, Math. Methods Appl. **44(6)** (2021), 4654-4672.

[29] E. Pişkin, S. Boulaaras, N. Irkil, *Local existence and blow up for a nonlinear viscoleastic Kirchhoff-type equation with logarithmic nonlinearity*, Kragujevac J. Math. **49(3)** (2025),335-351.

[30] E. Pişkin, B. Okutmuştur, *An Introduction to Sobolev Spaces*, Bentham Science, Bentham Science Publisher, 2021.

[31] M. A. Ragusa, A. Razani, F. Safari, *Existence of positive radial solutions for a problem involving weighted Heisenberg* $p(\cdot)$*-Laplacian operator*, AIMS Math. **8** (2022), 404-422.

[32] Y. M. Qin, X. Liu, S. X. Deng, *Decay rate of quasilinear wave equation with viscosity*, Acta Math. Appl. Sin. Engl. Ser. **28** (2012), 591-596.

[33] V. Georgiev, G. Todorova, *Existence of a solution of the wave equation with nonlinear damping and source terms*, J. Differential Equations. **109(2)**(1994), 295-308.

[34] Z. Yang, *Initial-boundary value problem and Cauchy problem for a quasilinear evolution equation*, Acta Math. Sci. **19(5)**(1999), 487-496.

[35] H. Yang, Y. Han, *Blow-up for a damped p-Laplacian type wave equation with logarithmic nonlinearity*, J. Differential Equations. **306** (2022), 569-589.

[36] Y. Ye, Q. Zhu, *Existence and nonexistence of global solutions for logarithmic hyperbolic equation*, ERA. **30(3)** (2022), 1035-1051.

[37] Y. Zhijian, *Existence and asymptotic behaviour of solutions for a class of quasi-linear evolution equations with non-linear damping and source terms*, Math. Methods Appl. Sci. **25(10)** (2002), 795-814.

[38] G.Zu, B. Guo, *Bounds for lifespan of solutions to strongly damped semilinear wave equations with logarithmic sources and arbitrary initial energy*, Evol. Equ. Control Theory. **10 (2)**(2021), 259-270.