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Existence and L^∞-estimates for non-uniformly elliptic equations with non-polynomial growths

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Abstract. In the current paper, we investigate the existence and regularity of weak solutions to a class of non-uniformly elliptic equations with degenerate coercivity and non-polynomial growth. The model case is given as follows:

$$\operatorname{div}\Big(\frac{\exp(1+|Du|)}{(1+|u|)^2}Du\Big) + \frac{M(|Du|)}{(1+|u|)^2}.u = f \quad \text{in} \quad \omega.$$

An L^∞- estimate of solutions is also obtained for an L^1-datum f.

1. Introduction

Let ω be a bounded open set in $\mathbb{R}^d$ that satisfies the segment property, ($d \geq 2$). The goal of the current research is to prove the existence and $L^\infty-$ estimates of weak solutions to the nonlinear and non-degenerate equations with non-polynomial growth equations:

$$\begin{cases} -\operatorname{div}(\Gamma(x,u,Du)) + \mathrm{B}(x,u,Du) = f & \text{on} \quad \omega, \\ u = 0 & \text{on} \quad \partial\omega. \end{cases} \tag{1}$$

Here, $\Gamma : \omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is a Carathéodory function that satisfies the assumptions below:
for a.e. $x \in \omega$ and for all $s \in \mathbb{R}$, $\xi, \xi^* \in \mathbb{R}^d$, $\xi \neq \xi^*$, there exist two N-functions M and P (See Definition below) such that:

$$\Gamma(x,s,\xi).\xi \geq g(|s|)M(|\xi|), \tag{2}$$

where $g : \mathbb{R}^+ \to \mathbb{R}^+_*$ is a continuous decreasing function with $g(0) = 1$, and set the primitive $G(s) = \displaystyle\int_0^s \frac{1}{g(t)}dt$.

$$|\Gamma(x,s,\xi)| \leq \nu(a_0(x) + \overline{M}^{-1}P(k_1|s|)) + \overline{M}^{-1}M(k_2|\xi|)), \tag{3}$$

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where $\nu > 0, k_1 > 0, k_2 > 0, a_0(.) \in E_{\overline{M}}(\Omega)$.

$$(\Gamma(x,s,\xi) - \Gamma(x,s,\xi^*))(\xi - \xi^*) > 0, \tag{4}$$

and B : $\omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is a Carathéodory function such that

$$|B(x,s,\xi)| \leq h(s)M(|\xi|), \tag{5}$$

where $h : \mathbb{R} \to \mathbb{R}^+$ is a continuous function,

$$B(x,s,\xi).s \geq 0, \tag{6}$$

We suppose that $t \mapsto \frac{h(t)}{g(|t|)}$ belongs to $L^1(\mathbb{R})$ and defining $\psi(r) = \displaystyle\int_0^r \frac{h(t)}{g(|t|)}dt$ for all $r \in \mathbb{R}$, this implies that,

$$\psi \text{ is bounded.} \tag{7}$$

and

$$f \in L^1(\omega) \text{ and } \int_r^{+\infty} \Big(\frac{t}{M(t)}\Big)^p dt < \infty, \text{ with } \quad p = \frac{1}{d-1} \text{ and } r > 0. \tag{8}$$

In the case of uniform ellipticity, i.e., $g(s) = const$, the existence of bounded solutions of equation (1) has been the subject of several papers in functional frameworks of classical Sobolev spaces, as well as in general functional frameworks, see for example [1–3, 6, 13–15, 21], and their references. However, due to assumption 2, the operator degenerates as soon as the solution u is unbounded. Indeed, for large values of the solution u, a slow but steady diffusion effect can occur. The function $\Gamma(x,s,\xi)$ strongly degenerates when $|s|$ grows to infinity because when $|s|$ is large, $g(|s|)$ vanishes. This lack of coercivity prohibits us from using classical approaches.
For the results dealing with the non-coercivity case, we give the following overview of the pioneering work of Boccardo L. et al. in [9, 17], who studied (1) with $\Gamma(x,u,Du) \geq \frac{g}{(1+|u|)^\theta}Du$, $B = 0$ and $f \in L^m(\omega)$ with $m \geq 1$ and $\theta \in]0;1]$. After that, Croce G. in [29] introduced the term $B(x,u,\nabla u) = |u|^{p-1}u$, which has a regulating effect on the solution u. In the case of weighted Sobolev spaces $W_0^{1,p}(\omega,\nu)$, Ammar K. in [10] established the existence of a renormalized solution in the L^1−frame under the condition $\Gamma(x,s,\xi)\xi \geq g\nu(x)|\xi|^p$ and B satisfies the sign condition. Aharouch L. et al. [7] investigated problem (1) in the presence of an obstacle, where the right-hand side $f \in L^1(\omega)$ and the lower-order term B satisfies $|B(x,s,\xi)| \leq \gamma(s) + g(s)\displaystyle\sum_{i=1}^N \nu_i(x)\,|\xi_i|^p$.
Later, in [11], the authors showed the same results using the following conditions:

$$\Gamma(x,s,\xi)\cdot\xi \geq b(s)^{p-1}\sum_{i=1}^N \nu_i(x)\,|\xi_i|^p\,, \quad \int_{-\infty}^{+\infty} b(s)ds < +\infty \text{ and } B = 0.$$

For further results, we suggest that the reader consult [3, 4, 8, 9, 13, 16, 25] and the references therein.

This research intends to generalize the previous results (See also [1, 2, 18–20, 26, 27]) in the framework of Orlicz spaces. Moreover, to prove the existence and L^∞−estimates of the solutions of (1) by assuming the coercivity condition (2). Therefore, we use rearrangement techniques to surmount this task, approximate problems, and choose suitable test functions.

The paper's layout is as follows. Section 2 gives some preliminaries and technical lemmas in Orlicz Spaces. In Section 3, we prove the existence of weak solutions. In the appendix, we establish an L^∞−estimate of the solution to (1).

2. Auxiliary Outcomes and Mathematical Context

This section shows the notation, goes over some basic definitions, and collects the propositions and facts we need to show our main result.

Definition 2.1. *[2] Let* $M : \mathbb{R}^+ \to \mathbb{R}^+$ *be an N-function, that is, M is continuous, convex, with* $M(s) > 0$ *for* $s > 0$, $\dfrac{M(s)}{s} \to 0$ *as* $s \to 0$, *and* $\dfrac{M(s)}{s} \to +\infty$ *as* $s \to +\infty$. *The N-function* $\overline{M}$ *conjugate to M is defined by* $\overline{M}(s) = \sup\limits_{t>0}(st - M(t))$.

We will extend these N-functions into even functions on all $\mathbb{R}$.

Let P and Q be two N-functions. $P \ll Q$ means that P grows essentially less rapidly than Q, that is, for each $\varepsilon > 0$, $\lim\limits_{s\to+\infty} \dfrac{P(s)}{Q(\varepsilon s)} = 0$.

Definition 2.2. *[4] We define the Orlicz class* $K_M(\omega)$ *(resp. the Orlicz space* $L_M(\omega)$ *as the set of (equivalence classes of) real valued measurable functions u on* ω *such that*

$$\int_\Omega M(v(x))dx < +\infty \quad (resp. \quad \int_\omega M(\frac{v(x)}{\alpha})dx < +\infty \quad for\ some \quad \alpha > 0).$$

The set $L_M(\omega)$ *is Banach space under the norm*

$$\|v\|_{M,\omega} = inf\Big\{\alpha > 0 : \int_\Omega M(\frac{v(x)}{\alpha})dx \leq 1\Big\},$$

and $K_M(\omega)$ *is a convex subset of* $L_M(\omega)$.

- The closure in $L_M(\omega)$ of the set of bounded measurable functions with compact support in $\overline{\omega}$ is denoted by $E_M(\omega)$.
- The dual $E_M(\omega)$ can be identified with $L_{\overline{M}}(\omega)$ by means of the pairing $\int_\omega uvdx$ and the dual norm of $L_{\overline{M}}(\omega)$ is equivalent to $\|v\|_{\overline{M},\omega}$.
- The Orlicz-Sobolev space, $W^1L_M(\omega)$ (resp. $W^1E_M(\omega)$) is the space of all functions v such that v and its distributional derivatives up to order 1 lie in $L_M(\omega)$ (resp. $E_M(\omega)$). It is a Banach space under the norm
$$\|v\|_{1,M} = \sum_{|\alpha|\leq 1} \|D^\alpha v\|_{M,\omega}.$$
- Thus, $W^1L_M(\omega)$ and $W^1E_M(\omega)$ can be identified with subspaces of product of $N+1$ copies of $L_M(\omega)$. Denoting this product by ΠL_M. We will use the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$.

Definition 2.3. *[6] We define the space* $W_0^1E_M(\omega)$ *as the (norm) closure of the Schwartz space* $\mathcal{D}(\omega)$ *in* $W^1E_M(\omega)$ *and the space* $W_0^1L_M(\omega)$ *as the* $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ *closure of* $\mathcal{D}(\omega)$ *in* $W^1L_M(\omega)$.

We denote by $W^{-1}L_{\overline{M}}(\omega)$ (resp. $W^{-1}E_{\overline{M}}(\omega)$) the space of distributions on ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{M}}(\omega)$ (resp. $E_{\overline{M}}(\omega)$). It is also a Banach space under the usual quotient norm. For more details, we refer the reader to [6].

2.1. *Rearrangement*

Denote by $|\omega|$ the Lebesgue measure of ω. Assume that v is a measurable function from ω into $\mathbb{R}$. The distribution function μ_v of v is defined as follows:

$$\mu_v(t) = |\{x \in \omega; |v(x)| > t\}|\,, t \geq 0.$$

The decreasing rearrangement v_* of v defined on $]0, |\omega|[$ by

$$v_*(s) = \inf\{t \geq 0; \mu_v(t) \leq s\}$$

$$v_*(0) = ess\,sup|v|. \tag{9}$$

Furthermore, for all $t \geq 0$, we have

$$v_*(\mu_v(t)) \leq t. \tag{10}$$

Finally, let $\Theta(t) = te^{\sigma t^2}$, $\sigma > 0$. It's obvious that when $\sigma = (\frac{\lambda_1}{\lambda_2})^2$, $\lambda_1 > 0$, $\lambda_2 > 0$, one has

$$\Theta'(t) - \frac{\lambda_1}{\lambda_2}|\Theta(t)| \geq \frac{1}{2} \quad \text{for all } t \in \mathbb{R}. \tag{11}$$

3. Main results

In what follows, we will assume that M and P are two N-functions such that $H(s) = \dfrac{M(s)}{s}$ is a convex function.

Definition 3.1. *A measurable function $u \in W_0^1L_M(\omega)$ is called a weak solution to problem (1), if $\Gamma(x, u, Du) \in (L_{\overline{M}}(\omega))^d$ and*

$$\int_\omega \Gamma(x, u, Du)\nabla\varphi dx + \int_\omega B(x, u, \nabla u)\varphi dx = \int_\Omega f\varphi dx, \ \forall\varphi \in \mathcal{D}(\omega). \tag{12}$$

Theorem 3.2. *Assume that (3)-(5) hold. Given $f \in L^1(\omega)$ with the condition (8), then there exists a bounded weak solution $u \in W_0^1L_M(\omega) \cap L^\infty(\omega)$ to problem (1).*

Step 1: Approximate problems

For every $n > 0$, we define the following approximations:
$\Gamma_n(x, s, \xi) = \Gamma(x, T_n(s), \xi)$, $B_n(x, s, \xi) = \frac{B(x,s,\xi)}{1+\frac{1}{n}|B(x,s,\xi)|}$, a.e. $x \in \omega$, for all $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^d$, where $T_n(s) = \max(-n, \min(n, s))$.
Denoting by $(f_n)_n$ the sequence of smooth functions such that $f_n \to f$ strongly in $L^1(\omega)$, and

$$\|f_n\|_{L^1(\omega)} \leq \|f\|_{L^1(\omega)}. \tag{13}$$

and consider the approximated equations

$$\int_\omega \Gamma_n(x, T_n(u_n), Du_n)\nabla\varphi dx + \int_\omega B_n(x, u_n, Du_n)\varphi dx = \int_\omega f_n\varphi dx, \ \forall\varphi \in W_0^1L_M(\omega) \tag{14}$$

Now, since $g(.)$ is decreasing and by (2), we have

$$\Gamma_n(x, T_n(s), \xi).\xi \geq g(|T_n(s)|)M(|\xi|) \geq g(n)M(|\xi|).$$

We have also $|B_n(x,s,\xi)| \leq |B(x,s,\xi)|$, $|B_n(x,s,\xi)| \leq n$ and $B_n(x,s,\xi)s \geq 0$.

As a consequence of the [23], since $\Gamma_n(x,s,\xi) + B_n(x,s,\xi)$ verify the assumption (A_4) of Proposition 5, there exists $u_n \in W_0^1L_M(\omega)$ solution of the problem (14).

Step 2: A priori Estimates

According to (13) and (43) (see appendix), there exists a constant still denoted c_0 such that

$$\|u_n\|_{L^\infty(\omega)} \leq c_0. \tag{15}$$

Let $n > c_0$, then $T_n(u_n) = u_n$. Choosing $\varphi = \Theta(u_n)$ as a test function of (14), by (2) and (5), we have

$$\int_\omega g(|u_n|)M(|\nabla u_n|)\Theta'(u_n)dx \leq \int_\omega h(u_n)M(|\nabla u_n|)|\Theta(u_n)|dx + \int_\omega |f_n||\Theta(u_n)|dx,$$

using (13) and Dominated Convergence Theorem, we have

$$\int_\omega \Big(g(|u_n|)\Theta'(u_n) - h(u_n)|\Theta(u_n)|\Big)M(|\nabla u_n|)dx \leq \Theta(c_0)\|f_n\|_{L^1(\omega)},$$

using (11) with $\sigma = \left(\frac{h(u_n)}{g(|u_n|)}\right)^2$, we obtain

$$\int_\omega M(|\nabla u_n|)dx \leq \frac{2\Theta(c_0)}{g(c_0)}\|f\|_{L^1(\omega)}. \tag{16}$$

As a result, one has $\{u_n\}_n$ is bounded in $W_0^1L_M(\omega)$. If required, we go to a subsequence and suppose that

$$u_n \overset{weakly}{\rightharpoonup} u \text{ in } W_0^1L_M(\omega) \text{ for } \sigma(\Pi L_M, \Pi E_{\overline{M}})\text{, strongly in } E_M(\omega)\text{, and a.e. in } \omega. \tag{17}$$

and usnig the compact embedding of $W_0^1L_M(\omega)$ in $E_M(\omega)$, we have also

$$u_n \longrightarrow u \text{ strongly in } E_M(\omega) \text{ and a.e. in } \omega. \tag{18}$$

We will demonstrate that $\{\Gamma(x, T_n(u_n), Du_n)\}_n$ is bounded in $\left(L_{\overline{M}}(\omega)\right)^d$.
For this, we take $\nu \in (E_M(\omega))^d$, and by (4) we get,

$$(\Gamma(x, T_n(u_n), Du_n) - \Gamma(x, T_n(u_n), \frac{\nu}{k_2}))(Du_n - \frac{\nu}{k_2}) > 0,$$

then

$$\int_\omega \Gamma(x, T_n(u_n), Du_n)dx \leq I + J,$$

where

$$I = k_2\int_\omega \Gamma(x, T_n(u_n), Du_n)\nabla u_n dx,$$

and

$$J = \int_\omega \Gamma(x, T_n(u_n), \frac{\nu}{k_2})\nu dx - k_2\int_\omega \Gamma(x, T_n(u_n), \frac{\nu}{k_2})Du_n dx.$$

From (2), (14) and using the same previous techniques to establish that

$$\int_\omega \Gamma(x, T_n(u_n), Du_n)Du_n dx \leq 2|\Theta(c_0)|\|f\|_{L^1(\omega)}, \tag{19}$$

and then,

$$I \leq C_I, \tag{20}$$

where C_I is a positive constant independent of n.
By (3), the convexity of $\overline{M}$, and the fact that $P << M$, we have

$$\int_\omega \overline{M}\Big(\frac{\mathrm{A}(x, T_n(u_n), \frac{\nu}{k_2})}{3\nu}\Big)dx \leq \frac{1}{3}\int_\omega \Big(\overline{M}(a_0(x)) + M(k_1|T_n(u_n)|) + M(|\nu|)\Big)dx + C,$$

thus $\{\Gamma(x, T_n(u_n), \frac{\nu}{k_2})\}_n$ is bounded in $(L_{\overline{M}}(\omega))^d$.
Returning to J, we have

$$J \leq 2\|\Gamma(x, T_n(u_n), \frac{\nu}{k_2})\|_{\overline{M},\omega}\|\nu\|_{M,\omega} + 2k_2\|\Gamma(x, T_n(u_n), \frac{\nu}{k_2})\|_{\overline{M},\omega}\|Du_n\|_{M,\omega},$$

and by (16), we obtain

$$J \leq C_J,$$

where C_J is a positive constant independent of n.
So,

$$\int_\omega \Gamma(x, T_n(u_n), Du_n)wdx \leq C, \tag{21}$$

with C is a positive constant that is independent of n.
Finally, according to the Banach-Steinhaus Theorem, $\{\Gamma(x, T_n(u_n), Du_n)\}_n$ remains bounded in $(L_{\overline{M}}(\omega))^d$. Hence

$$\Gamma_n(x, T_n(u_n), Du_n) \overset{weakly}{\rightharpoonup} \xi, \quad \text{in} \quad (L_{\overline{M}}(\omega))^d, \tag{22}$$

for $\sigma(\Pi L_{\overline{M}}, \Pi E_M)$.

Step 3 : Almost everywhere convergence of Du_n
Let $v_j \in \mathcal{D}(\omega) \overset{modular}{\rightarrow} u$, in $W_0^1L_M(\Omega)$ (cf. [24]). Let $W_n^j = u_n - v_j$ and $W^j = u - v_j$.
Plug the test function $\Theta(W_n^j)$ in (14), we get,

$$\int_\omega \Gamma(x, u_n, Du_n).D\Theta(W_n^j)dx + \int_\omega B_n(x, u_n, Du_n)\Theta(W_n^j)dx = \int_\omega f_n\Theta(W_n^j)dx. \tag{23}$$

For $i \geq 1$, we denote by $\varepsilon_i(n, j)$ the various sequences of real numbers which satisfy

$$\lim_{j\to+\infty}\lim_{n\to+\infty}\varepsilon_i(n, j) = 0.$$

The first term in (23) is written as follows

$$\begin{aligned}\int_\omega \Gamma(x, u_n, Du_n).D\Theta(W_n^j)dx \;&= \int_\omega [\Gamma(x, u_n, Du_n) - \Gamma(x, u_n, Dv_j\chi_j^s)]\\ &\times[Du_n - Dv_j\chi_j^s)]\Theta'(W_n^j)dx\\ &+ \int_\omega \Gamma(x, u_n, Dv_j\chi_j^s).[Du_n - Dv_j\chi_j^s)]\Theta'(W_n^j)dx\\ &- \int_{\omega\backslash\omega_j^s} \Gamma(x, u_n, Du_n).Dv_j\Theta(W_n^j)'dx,\end{aligned}$$

where χ_j^s denotes the characteristic function of the subset $\omega_j^s = \{x \in \omega : |Dv_j| \leq s\}$.
Starting with the third term, since $Dv_j\chi_{\omega\backslash\omega_j^s} \in (E_M(\omega))^d$, (17) and (22), we have

$$\int_{\omega\backslash\omega_j^s} \Gamma(x, u_n, Du_n).Dv_j\Theta'(W_n^j)dx \to \int_{\omega\backslash\omega_j^s} \xi.Dv_j\Theta'(W^j)dx \quad \text{as } n \to \infty,$$

using the modular convergence of $\{v_j\}$, we get

$$\int_{\omega\backslash\omega_j^s} \xi.Dv_j\Theta'(W^j)dx \to \int_{\omega\backslash\omega_j^s} \xi.Dud\,x \text{ as } j \to \infty,$$

it will allow us to write

$$\int_{\omega\backslash\omega_j^s} \Gamma(x, u_n, Du_n).Dv_j\Theta'(W_n^j)dx = \int_{\omega\backslash\omega_j^s} \xi.Dud\,x + \varepsilon_1(n, j). \tag{24}$$

For the second term of (23), remark that

$$\int_\omega \Gamma(x, u_n, \Gamma v_j\chi_j^s).[Du_n - Dv_j\chi_j^s)]\Theta'(W_n^j)dx$$

$$\to \int_\omega \Gamma(x, u, Dv_j\chi_j^s).[Du - Dv_j\chi_j^s)]\Theta'(W^j)dx$$

as $n \to +\infty$, since $\Gamma(x, u_n, Dv_j\chi_j^s)\Theta'(W_n^j) \to \Gamma(x, u, Dv_j\chi_j^s)\Theta'(W^j)$ strongly in $(E_M(\omega))^d$ as $n \to \infty$ (becauses of lemma 1, page 405 in [12], and (18)), while $Du_n \to Du$ weakly in $(L_M(\omega))^d$.
We have also, $Dv_j\chi_j^s \to Du\chi^s$ strongly in $(E_M(\omega))^d$ as $j \to +\infty$, then it is easy to see that

$$\int_\omega \Gamma(x, u, Dv_j\chi_j^s).[Du - Dv_j\chi_j^s)]\Theta'(W^j)dx \to 0 \quad \text{as } j \to \infty,$$

and

$$\int_\omega \Gamma(x, u_n, Dv_j\chi_j^s).[Du_n - Dv_j\chi_j^s)]\Theta'(W_n^j)dx = \varepsilon_2(n, j), \tag{25}$$

where $\omega^s = \{x \in \omega : |Du| \leq s\}$.
Now combing (23), (24) and (25), we obtain

$$\int_\omega \Gamma(x, u_n, Du_n).D\Theta(W_n^j)dx = \varepsilon_3(n, j) - \int_{\omega\backslash\omega^s} \xi_k.Dudx$$

$$+ \int_\omega [\Gamma(x, u_n, Du_n) - \Gamma(x, u_n, Dv_j\chi_j^s)].[Du_n - Dv_j\chi_j^s)]\Theta'(W_n^j)dx. \tag{26}$$

Returning to the second term on the left-hand side of (23). We have

$$\begin{aligned} \left|\int_\omega B_n(x, u_n, Du_n)\Theta(W_n^j)dx\right| &\leq \int_\omega h(u_n)M(|Du_n|)|\Theta(W_n^j)|dx \\ &\leq \int_\omega \frac{h(u_n)}{g(|u_n|)}\Gamma(x, u_n, Du_n).Du_n|\Theta(W_n^j)|dx \\ &\leq \frac{\|h(u_n)\|_{L^\infty}}{g(c_0)} \int_\omega \Gamma(x, u_n, Du_n).Du_n|\Theta(W_n^j)|dx \\ &\leq g_0 \int_\omega [\Gamma(x, u_n, Du_n) - \Gamma(x, u_n, Dv_j\chi_j^s)] \\ &\times[Du_n - Dv_j\chi_j^s]|\Theta(W_n^j)|dx \\ &+g_0 \int_\omega \Gamma(x, u_n, Du_n).Dv_j\chi_j^s|\Theta(W_n^j)|dx \\ &+g_0 \int_\omega \Gamma(x, u_n, Dv_j\chi_j^s)].[Du_n - Dv_j\chi_j^s]|\Theta(W_n^j)|dx, \end{aligned} \tag{27}$$

where $g_0 = \dfrac{\|h(u_n)\|_{L^\infty}}{g(c_0)}$.
In a similar way as above, we have

$$g_0 \int_\omega \Gamma(x,u_n,Du_n).Dv_j\chi_j^s|\Theta(W_n^j)|dx = \varepsilon_4(n,j),$$

$$g_0 \int_\omega \Gamma(x,u_n,Dv_j\chi_j^s)].[Du_n - Dv_j\chi_j^s]|\Theta(W_n^j)|dx = \varepsilon_5(n,j).$$

Hence

$$\begin{aligned}\Big|\int_\omega B_n(x,u_n,Du_n\Theta(W_n^j)dx\Big| \leq g_0 \int_\omega \quad & [\Gamma(x,u_n,Du_n) - \Gamma(x,u_n,Dv_j\chi_j^s)] \\ & \times[Du_n - Dv_j\chi_j^s]|\Theta(W_n^j)|dx + \varepsilon_6(n,j).\end{aligned} \tag{28}$$

Regarding the term on the right side of (23), since $\Theta(W_n^j) \overset{weakly*}{\rightarrow} \Theta(W^j)$, in $L^\infty(\omega)$ for $\sigma(L^\infty, L^1)$ as $n \to \infty$, one has

$$\int_\omega f_n\Theta(W_n^j)dx \to \int_\omega f\Theta(W^j)dx,$$

we have also $v_j \overset{weakly*}{\rightarrow} u$, in $L^\infty(\omega)$ for $\sigma(L^\infty, L^1)$ as $j \to \infty$, we get

$$\int_\omega f_n\Theta(W_n^j)dx = \varepsilon_7(n,j). \tag{29}$$

Finally, by (23), (26), (28) and (29), we obtain

$$\int_\omega [\Theta'(W_n^j) - g_0|\Theta(W_n^j)|][a(x,u_n,\nabla u_n) - a(x,u_n,\nabla v_j\chi_j^s)].[\nabla u_n - \nabla v_j\chi_j^s]dx$$

$$\leq \int_{\omega\setminus\omega^s} \xi.\nabla u dx + \epsilon_8(n,j),$$

and then

$$\int_\omega [\Gamma(x,u_n,Du_n) - \Gamma(x,u_n,Dv_j\chi_j^s)].[Du_n - Dv_j\chi_j^s]dx \leq 2\int_{\omega\setminus\omega^s} \xi.Dudx + 2\varepsilon_8(n,j).$$

On the other hand

$$\begin{aligned}\int_\omega [\Gamma(x,u_n,Du_n)- \quad & \Gamma(x,u_n,Du\chi^s)][Du_n - Du)\chi^s]dx \\ & = \int_\omega [\Gamma(x,u_n,Du_n) - \Gamma(x,u_n,Dv_j\chi_j^s)][Du_n - Dv_j\chi_j^s]dx \\ & + \int_\omega \Gamma(x,u_n,Du_n).[Dv_j\chi_j^s - Du\chi^s]dx \\ & - \int_\omega \Gamma(x,u_n,Du\chi^s).[Du_n - Du\chi^s]dx \\ & + \int_\omega \Gamma(x,u_n,Dv_j\chi_j^s).[Du_n - Dv_j\chi_j^s]dx.\end{aligned} \tag{30}$$

We will pass to the limit in n and j in the last three terms on the right side of the above equality. Tools similar to those in (24) give

$$\int_\omega \Gamma(x,u_n,Du_n).[Dv_j\chi_j^s - Du\chi^s]dx = \varepsilon_9(n,j),$$

$$\int_{\omega} \Gamma(x, u_n, Du\chi^s).[Du_n - Du\chi^s]dx = \varepsilon_{10}(n, j),$$

and

$$\int_{\omega} \Gamma(x, u_n, Dv_j\chi_j^s).[Du_n - Dv_j\chi_j^s]dx = \varepsilon_{11}(n, j), \tag{31}$$

which imply that

$$\int_{\omega} [\Gamma(x, u_n, Du_n) - \Gamma(x, u_n, Du\chi^s)].[Du_n - Du\chi^s]dx \leq 2\int_{\omega\backslash\omega^s} \xi Dudx + 2\varepsilon_{12}(n, j).$$

For $r \leq s$, one has

$$\begin{aligned} 0 &\leq \int_{\omega_r} [\Gamma(x, u_n, Du_n) - \Gamma(x, u_n, Du)].[Du_n - Du]dx \\ &= \int_{\omega_s} [\Gamma(x, u_n, Du_n) - \Gamma(x, u_n, Du\chi^s)].[Du_n - Du\chi^s]dx \\ &\leq \int_{\omega} [\Gamma(x, u_n, Du_n) - \Gamma(x, u_n, Du\chi^s)].[Du_n - Du\chi^s]dx \\ &\leq 2\int_{\omega\backslash\omega_s} \xi.Dud\,x + \varepsilon_{13}(n, j). \end{aligned}$$

Using the fact that $\xi.Du \in L^1(\omega)$ and $|\omega\backslash\omega_s| \to 0$ as $s \to +\infty$, we get

$$\int_{\omega_r} [\Gamma(x, u_n, Du_n) - \Gamma(x, u_n, Du)].[Du_n - Du]dx \to 0 \quad \text{as} \quad n \to +\infty.$$

As a result, we conclude that there exists a subsequence still denoted by u_n such that

$$[\Gamma(x, u_n, Du_n) - \Gamma(x, u_n, Du)].[Du_n - Du] \to 0 \quad \text{a.e. in} \quad \omega_r.$$

On the other hand, for every $x \in \omega^r\backslash Z$ with $|Z| = 0$, one has by (3) and (2),

$$\begin{aligned} [\Gamma(x, u_n, Du_n) - \Gamma(x, u_n, Du)].[Du_n - Du] \quad &\geq g(c_0)M(|Du_n|) \\ &-\Gamma(x, u_n, Du_n).Du - \Gamma(x, u_n, Du).Du_n \\ &\geq g(c_0)M(|Du_n|) \\ &-C\Big(1 + |Du_n| + \overline{M}^{-1}M(|Du_n|)\Big), \end{aligned} \tag{32}$$

where C is a constant not depend on n.
Following all the previous results, $\{Du_n\}$ is bounded in $\mathbb{R}^N$, and for a subsequence of u_n, there exists $\xi \in \mathbb{R}^d$ such that

$$\nabla u_n \to \xi \quad \text{in} \quad \mathbb{R}^d,$$

and

$$[\Gamma(x, u, \xi) - \Gamma(x, u, \nabla u)].[\xi - Du] = 0.$$

Thus $\xi = Du$ and $Du_n \to Du \quad$ a.e. in $\quad \omega^r$.
Since r is arbitrary, we construct a subsequence such that

$$Du_n \to Du \quad \text{a.e. in} \quad \omega. \tag{33}$$

From (22), (17), and (33), it follows that

$$\Gamma(x, T_n(u_n), Du_n) \overset{weakly}{\rightharpoonup} \Gamma(x, u, Du) \in (L_M(\omega))^d, \quad \text{for} \quad \sigma\,(\Pi L_{\bar{M}}, \Pi E_M) \tag{34}$$

Step 4: $Du_n \overset{modular}{\longrightarrow} Du$

Let $n > c_0$, by (4), we obtain

$$\begin{aligned}\int_\omega \Gamma(x,u_n,Du_n).Du_n dx \quad &\leq \int_\omega \Gamma(x,u_n,Du_n).Dv_j\chi_j^s dx \\ &+ \int_\omega \Gamma(x,u_n,Dv_j\chi_j^s).[Du_n - Dv_j\chi_j^s]dx \\ &+2\int_{\omega\backslash\omega^s} \Gamma(x,u,Du).Dud\,x + 2\varepsilon_8(n,j).\end{aligned} \tag{35}$$

We return to (31) to have

$$\begin{aligned}\int_\omega \Gamma(x,u_n,Du_n).Du_n dx \quad &\leq \int_\omega \Gamma(x,u_n,Du_n).Dv_j\chi_j^s dx \\ &+2\int_{\omega\backslash\omega^s} \Gamma(x,u,Du).Dud\,x + \varepsilon_{15}(n,j),\end{aligned} \tag{36}$$

letting $n \to \infty$ and $j \to \infty$, we get

$$\begin{aligned}\limsup_{n\to\infty}\int_\omega \Gamma(x,u_n,Du_n).Du_n dx \quad &\leq \int_\omega \Gamma(x,u,Du).Du\chi^s d\,x \\ &+2\int_{\omega\backslash\omega^s} \Gamma(x,u,Du).Dudx.\end{aligned} \tag{37}$$

Passing to the limit as $s \to \infty$, we get

$$\limsup_{n\to\infty}\int_\omega \Gamma(x,u_n,Du_n).Du_n dx \leq \int_\omega \Gamma(x,u,Du).Dudx,$$

and by Fatou's Lemma, we deduce that

$$\lim_{n\to\infty}\int_\omega \Gamma(x,u_n,Du_n).Du_n dx = \int_\omega \Gamma(x,u,Du).Dudx.$$

Using Lemma 4 page 164 in [22], we get

$$\Gamma(x,u_n,Du_n).Du_n \to \Gamma(x,u,Du).Du \quad \text{strongly in} \quad L^1(\omega). \tag{38}$$

On the other hand, since $g(c_0) \leq g(|u_n|)$ and using Young inequality, one has

$$\begin{aligned}M(\frac{|Du_n - Du|}{2}) \quad &\leq \tfrac{g(|u_n|)}{2g(c_0)}M(|Du_n|) + \tfrac{g(|u|)}{2g(c_0)}M(|Du|) \\ &\leq \frac{1}{2g(c_0)}\Gamma(x,u_n,Du_n).Du_n + \frac{1}{2g(c_0)}\Gamma(x,u,Du).Du.\end{aligned} \tag{39}$$

As a result of (33) and Lebesgue Theorem, we reach to our result. Hence, according to (33) and Dominated Convergence Theorem, we deduce our result.

Step 5: Equi-integrability of $\{B(x,u_n,Du_n)\}_n$

We aim to establish that

$$B(x,u_n,Du_n) \to B(x,u,Du) \text{ strongly in } L^1(\omega). \tag{40}$$

Indeed, by (17) and (33), one gets $B(x,u_n,Du_n) \to B(x,u,Du)$ a.e. in ω. However, because u_n is bounded and h is continuous, choosing $h_0 = \|h(u_n)\|_{L^\infty(\omega)}$, and by (5), we have

$$|B(x,u_n,Du_n)| \leq h(|u_n|)M(|Du_n|) \leq h_0 M(|Du_n|).$$

Now, let $E \subset \omega$, then

$$\int_E |B(x,u_n,Du_n)|dx \leq \frac{h_0}{g(c_0)}\int_E \Gamma(x,u_n,Du_n).Du_n dx$$

As we have (38), we can apply the equi-integrability of $\{\Gamma(x,u_n,Du_n)\}_n$ and finish our proof with Vitali's theorem.

Remark 3.3. *we can find the same result if we replaced* (8) *by*

$$f \in L^r(\omega) \text{ sucth that } r = \frac{pd}{p+1} \text{ and } p > \frac{1}{d-1}.$$

Step 6: Passing to the limit.
Taking $v \in \mathcal{D}(\omega)$ as a test function in (14) yields

$$\int_\omega \Gamma_n(x,u_n,Du_n)\cdot Dvdx + \int_\omega B_n(x,u_n,Du_n)\,vdx = \int_\omega f_n vdx. \tag{41}$$

By (34), (40) and (42) respectively, we get

$$\int_\omega \Gamma_n(x,u_n,Du_n)\cdot Dvdx \to \int_\omega \Gamma(x,u,Du)\cdot Dvdx.$$

$$\int_\omega B_n(x,u_n,Du_n)\,vdx \to \int_\omega B(x,u,Du)\,vdx$$

and

$$\int_\omega f_n vdx \to \int_\omega fvdx$$

This complets the proof.

Appendix

Theorem 3.4. *Assume that* (3) - (7) *hold. Given* $f \in L^1(\omega)$ *with the condition* (8), *then any weak solution* u *to problem* (1) *(in the sense of Definition 3.1) satisfied*

$$\|u\|_{L^\infty(\omega)} \leq c_0, \tag{42}$$

where c_0 *is a constant depending only on* d.

Proof of Theorem 3.4

We define a decreasing and convex function $K(.)$ as $K(s) = \frac{1}{H^{-1}(s)}$ where $H^{-1}(s) = \sup\{r \geq 0, H(r) \leq s\}$. Using Jensen's inequality, the definition of H and the fact that $g(.)$ is decreasing function such that $g(0) = 1$, we have

$$\begin{aligned} K\Big(\int_{\{t\leq|u|\leq t+h\}} \frac{g(|u|)M(|\nabla u|)}{\int_{\{t\leq|u|\leq t+h\}}|\nabla u|ds}ds\Big) &= K\Big(\int_{\{t\leq|u|\leq t+h\}} \frac{g(|u|)H(|\nabla u|)|\nabla u|}{\int_{\{t\leq|u|\leq t+h\}}|\nabla u|ds}ds\Big) \\ &\leq \int_{\{t\leq|u|\leq t+h\}} \frac{K\big(g(|u|)H(|\nabla u|)\big)|\nabla u|}{\int_{\{t\leq|u|\leq t+h\}}|\nabla u|ds}ds \\ &\leq \frac{g(|t|)(\mu(t)-\mu(t+h))}{\int_{\{t\leq|u|\leq t+h\}}|\nabla u|ds}. \end{aligned}$$

Letting $h \to 0$, we get

$$K\Big(-\frac{d}{dt}\int_{\{|u|>t\}} \frac{g(|u|)M(|\nabla u|)}{-\frac{d}{dt}\displaystyle\int_{\{|u|>t\}} |\nabla u| ds} ds\Big) \leq \frac{-g(|t|)\mu'(t)}{-\frac{d}{dt}\displaystyle\int_{\{|u|>t\}} |\nabla u| ds}.$$

From Lemma (See [24], Lemma 2, page 72), we have

$$-\frac{d}{dt}\int_{\{|u|>t\}} |\nabla u| dx \geq dC_d^{\frac{1}{d}}\mu(t)^{1-\frac{1}{d}},$$

where C_d is the measure of the unit ball of $\mathbb{R}^d$. By the same arguments in Lemma 3.3 in [5] we have

$$\begin{aligned}\frac{1}{g(|t|)} &\leq \frac{-\mu'(t)}{dC_d^{\frac{1}{d}}\mu(t)^{1-\frac{1}{d}}}H^{-1}\Big(\frac{-\frac{d}{dt}\displaystyle\int_{\{|u|>t\}} g(|u|)M(|\nabla u|)ds}{dC_d^{\frac{1}{d}}\mu(t)^{1-\frac{1}{d}}}\Big)\\ &\leq \frac{-\mu'(t)}{dC_d^{\frac{1}{d}}\mu(t)^{1-\frac{1}{d}}}H^{-1}\left(\frac{c_1\displaystyle\int_{|u|>t} |f| ds}{dC_d^{\frac{1}{d}}\mu(t)^{1-\frac{1}{d}}}\right).\end{aligned}$$

By integrating between 0 and r, we obtain

$$G(r) \leq \frac{1}{dC_d^{1/d}}\int_0^r \frac{-\mu'(t)}{\mu(t)^{1-\frac{1}{d}}}H^{-1}\left(\frac{c_1\|f\|_{L^1(\omega)}}{dC_d^{1/d}\mu(t)^{1-\frac{1}{d}}}\right)dt,$$

a change of variables gives

$$G(r) \leq \frac{1}{dC_d^{1/d}}\int_{\mu(r)}^{|\omega|} H^{-1}\left(\frac{c_1\|f\|_{L^1(\omega)}}{dC_d^{1/d}s^{1-\frac{1}{d}}}\right)\frac{ds}{s^{1-\frac{1}{d}}},$$

as above, taking $r = u^*(t)$ gives

$$G\left(u^*(t)\right) \leq \frac{1}{dC_d^{1/d}}\int_t^{|\omega|} H^{-1}\left(\frac{c_1\|f\|_{L^1(\omega)}}{dC_d^{1/d}s^{1-\frac{1}{d}}}\right)\frac{ds}{s^{1-\frac{1}{d}}}.$$

Then, we have

$$G\left(\|u\|_\infty\right) \leq \frac{1}{dC_d^{1/d}}\int_0^{|\omega|} H^{-1}\left(c_1\frac{\|f\|_{L^1(\omega)}}{dC_d^{1/d}s^{1-\frac{1}{d}}}\right)\frac{ds}{s^{1-\frac{1}{d}}},$$

a change of variables gives

$$G\left(\|u\|_\infty\right) \leq \frac{\left(c_1\|f\|_{L^1(\omega)}\right)^p}{d^pC_d^{\frac{p+1}{d}}}\int_{c_0}^{+\infty} pt^{-p-1}H^{-1}(t)dt,$$

where $c_0 = \frac{c_1\|f\|_{L^1(\omega)}}{dC_d^{1/d}|\omega|^{1-\frac{1}{d}}}$. And using integration by parts we get

$$G\left(\|u\|_{L^\infty(\omega)}\right) \leq \frac{\left(c_1\|f\|_{L^1(\omega)}\right)^p}{d^pC_d^{\frac{p+1}{d}}}\left(\frac{H^{-1}\left(c_0\right)}{c_0^p} + \int_{H^{-1}(c_0)}^{+\infty}\left(\frac{r}{M(r)}\right)^p dr\right).$$

Thus

$$\|u\|_{L^{\infty}(\omega)} \leq G^{-1}\left(\frac{\left(c_1\|f\|_{L^1(\omega)}\right)^p}{d^p C_d^{\frac{p+1}{d}}}\left(\frac{H^{-1}(c_0)}{c_0^p}+\int_{H^{-1}(c_0)}^{+\infty}\left(\frac{r}{M(r)}\right)^p dr\right)\right). \tag{43}$$

Then, we get the L^{∞}-estimates of u.

Example 3.5. *Taking* $M(t) = t^2\exp(t)$, *and* $g(u) = \frac{1}{(1+|u|)^2}$.

$$\Gamma(x,u,Du) = \frac{\exp(1+|Du|)}{(1+|u|)^2}Du; \quad B(x,u,Du) = g(u).M(|Du|).$$

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