



Domination and Korenblum constants for some function spaces

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Abstract. The paper shows the domination in M_λ which is associated with the analytic Morrey space $\mathcal{L}^{2,\lambda}$. We get the explicit expressions for upper bounds of Korenblum constants for the analytic Morrey space $\mathcal{L}^{2,\lambda}$, the weighted Dirichlet space \mathcal{D}_α^p and the general function space $F(p, q, s)$, etc. Furthermore, the domination in the Dirichlet space \mathcal{D}_α is investigated.

1. Preliminaries and basic facts

We denote by \mathbb{D} the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and \mathbb{T} the unit circle in the complex plane \mathbb{C} . Let $\mathcal{H}ol(\mathbb{D})$ be the space of all analytic functions on \mathbb{D} . For $0 < p < +\infty$, $-1 < \alpha$, the weighted Bergman space \mathcal{A}_α^p and the weighted Dirichlet space \mathcal{D}_α^p are consisted of functions $f \in \mathcal{H}ol(\mathbb{D})$ satisfying, respectively

$$\|f\|_{\mathcal{A}_\alpha^p} = \left(\int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\alpha dA(z) \right)^{\frac{1}{p}} < \infty,$$

and

$$\|f\|_{\mathcal{D}_\alpha^p}^p = |f(0)|^p + \|f'\|_{\mathcal{A}_\alpha^p}^p < \infty.$$

The normalized Lebesgue measure on \mathbb{D} will be denoted by $dA(z)$. Obviously,

$$dA(z) = \frac{1}{\pi} dx dy = \frac{r}{\pi} dr d\theta$$

for $z = x + iy = re^{i\theta}$. More introduction about the weighted Bergman spaces \mathcal{A}_α^p and the weighted Dirichlet spaces \mathcal{D}_α^p are mentioned in references [2][4][32][33]. If $p < \alpha + 1$, then $\mathcal{D}_\alpha^p = \mathcal{A}_{\alpha-p}^p$ [8]. If $p > \alpha + 2$, then $\mathcal{D}_\alpha^p \subset \mathcal{H}^\infty$. Trivially, $\mathcal{D}_1^2 = \mathcal{H}^2$, for $2 \leq p < \infty$, $\mathcal{H}^p \subset \mathcal{D}_{p-1}^p$ [21], and $\mathcal{D}_{p-1}^p \subset \mathcal{H}^p$ when $0 < p \leq 2$ [8][23]. In particular, \mathcal{D}_0^2 is the classical Dirichlet spaces \mathcal{D} . For $0 < p < \infty$, the space of Hardy \mathcal{H}^p consists of all analytic functions f defined on \mathbb{D} such that

$$\|f\|_{\mathcal{H}^p}^p = \left(\sup_{0 \leq r < 1} \int_{\mathbb{T}} |f(r\zeta)|^p \frac{|d\zeta|}{2\pi} \right)^{\frac{1}{p}} < \infty.$$

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For $p = \infty$, it is said that $h \in \mathcal{H}^\infty$ if h is a bounded analytic function on \mathbb{D} with

$$\|h\|_\infty = \sup\{|h(z)| : z \in \mathbb{D}\}.$$

For $p = 2$, it is well known that $f \in \mathcal{H}^2$ has the boundary value $f(\zeta) = \lim_{r \rightarrow 1^-} f(r\zeta)$ for all $\zeta \in \mathbb{T}$. For $\lambda \in \mathbb{R}$, define the Morrey space $\mathcal{L}^{2,\lambda}$ to be the sets of all $f \in \mathcal{H}^2$ such that

$$\sup_{I \subset \mathbb{T}} \frac{1}{|I|^\lambda} \int_I |f(\zeta) - f_I|^2 \frac{|d\zeta|}{2\pi} < \infty.$$

Clearly, for $\lambda \leq 0$, $\mathcal{L}^{2,\lambda} = \mathcal{H}^2$, $\mathcal{L}^{2,1} = BMOA$, and $\mathcal{L}^{2,\lambda}$ is Bloch type space if $\lambda > 1$. Hence, $0 < \lambda < 1$ is the most interesting range for Morrey space $\mathcal{L}^{2,\lambda}$ and this paper will study Morrey space $\mathcal{L}^{2,\lambda}$ for $0 < \lambda < 1$ and $BMOA$ when $\lambda = 1$. See [5][6][7][10][20][24] and the references therein for further details.

Let $f_I = \frac{1}{|I|} \int_I f(\zeta) \frac{|d\zeta|}{2\pi}$ and $S(I)$ be the Carleson square with

$$S(I) = \{z \in \mathbb{D} : |z| > 1 - |I|, \text{ as well as } z/|z| \in I\}.$$

Here $|I| = \int_I |d\zeta|/(2\pi)$ is the normalized length of arc I , which is an interval of \mathbb{T} . If $I = \mathbb{T}$, then $S(I) = \mathbb{D}$.

Wulan, Zhou [24] and Xiao [31] give some equivalent conditions for the norm of $\mathcal{L}^{2,\lambda}$. Thanks to their work, a analytic function $f \in \mathcal{L}^{2,\lambda}$ can be well defined with the following norms:

$$\begin{aligned} \|f\|_{\mathcal{L}^{2,\lambda}} &\approx |f(0)| + \sup_{I \subset \mathbb{T}} \left(\frac{1}{|I|^\lambda} \int_{S(I)} |f'(z)|^2 (1 - |z|^2) dA(z) \right)^{\frac{1}{2}} \\ &\approx |f(0)| + \sup_{a \in \mathbb{D}} \left((1 - |a|^2)^{1-\lambda} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\phi_a(z)|^2) dA(z) \right)^{\frac{1}{2}} \\ &\approx |f(0)| + \sup_{a \in \mathbb{D}} \left((1 - |a|^2)^{1-\lambda} \int_{\mathbb{D}} |f'(z)|^2 \log \frac{1}{|\phi_a(z)|} dA(z) \right)^{\frac{1}{2}} \end{aligned}$$

where we denote $\phi_a(z)$ as the conformal automorphisms with

$$\phi_a(z) = \frac{a - z}{1 - \bar{a}z}, \quad z \in \mathbb{D}.$$

For $0 < p, s < \infty, -2 < q < \infty, q + s > -1$, the general function space $F(p, q, s)$ is the class of each $f \in \mathcal{H}ol(\mathbb{D})$ satisfying

$$\|f\|_{F(p,q,s)} = \left(\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q (1 - |\phi_a(z)|^2)^s dA(z) \right)^{\frac{1}{p}} < \infty.$$

In addition, functions $f \in F_0(p, q, s)$ if $f \in \mathcal{H}ol(\mathbb{D})$ and satisfies

$$\lim_{|a| \rightarrow 1^-} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q (1 - |\phi_a(z)|^2)^s dA(z) = 0.$$

Zhao [34] systematically introduced the family of spaces $F(p, q, s)$, he showed that $F(p, q, s)$ is a Banach space with the norm $\|f\|_* = |f(0)| + \|f\|_{F(p,q,s)}$ and $F_0(p, q, s)$ is a closed subspace of $F(p, q, s)$ when $1 \leq p < \infty$. Also, $F(p, q, s)$ are well known spaces for certain parameters p, q, s . Namely,

$F(2, 0, s)$ is the space Q_s for $0 < s < \infty$;

$F(2, 1 - \lambda, \lambda)$ is the analytic Morrey space $\mathcal{L}^{2,\lambda}$, for $0 < \lambda \leq 1$;

$F(p, \alpha p + p - 2, q - (\alpha p + p - 2))$ is the Dirichlet-Morrey space $\mathcal{D}_\alpha^{p,q}$ when $0 < p < \infty, 0 \leq \alpha < \infty, \alpha p + p - 2 < q$;

$F(p, p - \lambda, \lambda)$ is the Bergman-Morrey space $\mathcal{A}^{p,\lambda}$ for $0 < p < \infty, 0 < \lambda < 2$, etc.

For $0 < p < \infty, -2 < q < \infty, 0 < s < \infty, q + s \leq -1$, $F(p, q, s)$ is trivial, that is, $F(p, q, s)$ only contains constants [34].

Similarly, for $0 < p < +\infty$, $-1 < \alpha$, $0 < \lambda \leq 1$, a analytic function f is said to belong to the spaces $M_{p,\alpha}$ or M_λ if

$$\|f\|_{M_{p,\alpha}} = \left(\int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\alpha dA(z) \right)^{\frac{1}{p}} < \infty$$

or

$$\|f\|_{M_\lambda} = \sup_{a \in \mathbb{D}} \left((1 - |a|^2)^{1-\lambda} \int_{\mathbb{D}} |f(z)|^2 (1 - |\phi_a(z)|^2) dA(z) \right)^{\frac{1}{2}} < \infty.$$

Clearly, $M_{p,\alpha}$ ($1 \leq p < +\infty, -1 < \alpha$) and M_λ ($0 < \lambda \leq 1$) are Banach spaces with the above norms and $M_{p,\alpha}$ is exactly the weighted Bergman space \mathcal{A}_α^p .

The Gamma function $\Gamma(z)$ and the Beta function $B(u, v)$ are essential tools in the process of our main results. Recall that, the Gamma function $\Gamma(z)$ is defined by the formula

$$\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} dt. \tag{1}$$

whenever the complex variable z has a positive real part, i.e., $\Re(z) > 0$. The Beta function $B(u, v)$ is defined by the formula

$$B(u, v) = \int_0^1 x^{u-1} (1-x)^{v-1} dx, \quad (\Re(u) > 0, \Re(v) > 0) \tag{2}$$

and satisfies the following property

$$B(u+1, v+1) = \frac{u}{u+v+1} B(u, v+1). \tag{3}$$

The formula relating between the beta function and the gamma function is the following:

$$B(u, v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}. \tag{4}$$

Furthermore, the gamma function has the following properties [22]:

- (i) $\Gamma(1) = 1, \Gamma(\frac{1}{2}) = \sqrt{\pi}$;
- (ii) $\Gamma(n+1) = n!, \quad n \in \mathbb{N}^+$.

Duren [3] characterized the theory of Hardy space \mathcal{H}^p . Indeed, he proved the following lemma :

Lemma 1.1. *Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in \mathbb{D} . Then*

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n} \tag{5}$$

for any $r, 0 \leq r < 1$.

By the definition of the upper bound, we can easily have the result.

Lemma 1.2.

$$\sup_{z \in \mathbb{D}} f(z) - \sup_{z \in \mathbb{D}} g(z) \leq \sup_{z \in \mathbb{D}} (f(z) - g(z)) \tag{6}$$

for any f, g are two real-valued functions in \mathbb{D} .

2. Introduction

Korenblum [16] introduced the idea of domination in the Bergman space \mathcal{A}^2 . Recent years, Wang [29] proved that

$$z^n + a < z(1 + \bar{a}z)$$

for $n \geq 2$ in \mathcal{A}^2 under certain conditions, and applies it to obtain a new upper bound for the Korenblum constant. The theory of domination has been a subject of high interest during the past decades. The domination has been studied quite extensively on \mathcal{A}^2 . Most of their efforts aimed at the Korenblum constants. There are more and more researchers focusing on the domination of spaces of analytic function to obtain a upper bound on Korenblum constant. Korenblum [16] conjectured that if $f, g \in \mathcal{A}^2$ with $|f(z)| \leq g(z)$ in the annulus $c < |z| < 1$, then there is an absolute constant c , $0 < c < 1$ such that $\|f\|_{\mathcal{A}^2} \leq \|g\|_{\mathcal{A}^2}$. Hayman [14] proved that Korenblum's conjecture for $c = 0.04$. Hinkkanen [11] improved Hayman's result that $c = 0.157 \dots$. Wang [25] showed that for any functions $f(z)$ and $g(z)$ (which depend on n and $a > 0$) defined in Theorem of [25], $c = 0.6947116 \dots$ is the best one. More about the theory of domination, see references [15][16][17][18][29] for the study of domination in Bergman space. Based on Wang's work [29], the paper studies the domination in the space \mathcal{D}_α . Later, Gao and Wang [9] prove that $z^2 + a < z(1 + \bar{a}z^2)$ in M_p (associated with the Q_p spaces) under the condition

$$|a| \leq \frac{3(p+6)}{p^2 + 14p + 60}.$$

Motivated by [9], this paper characterizes the domination relationships in the spaces M_λ and $M_{p,\alpha}$.

Božin and Karapetrović [1] proved that the maximum principle does not hold in Bergman space \mathcal{A}^p ($0 < p < 1$). However, Korenblum maximum principle holds in Bergman spaces \mathcal{A}^p if and only if $p \geq 1$. This is improved by Karapetrović [19], where it is shown that Korenblum maximum principle holds in weighted Bergman spaces \mathcal{A}_α^p if and only if $p \geq 1$. In addition, results related to Korenblum maximum principle in mixed norm spaces $H(p, q, \alpha)$ were also obtained in [19]. Hu and Lou [13] showed that the maximum principle in Fock space \mathcal{F}_α^p ($0 < p < 1$) is invalid. Subsequently, Wee and Le [30] gave the explicit expressions for the upper bounds of Korenblum constants for the weighted Fock spaces and the weighted Bergman spaces with exponential weights, and showed that there is a failure of the Korenblum Maximum Principle for weighted Bergman spaces \mathcal{A}_α^p ($0 < \alpha, 0 < p < 1$). Inspired by the above work [30], we study the Korenblum Maximum Principle on these analytic function spaces $\mathcal{L}^{2,\lambda}$, BMOA, M_λ , \mathcal{D}_α^p , $M_{p,\alpha}$, $F(p, q, s)$.

The paper is organized as follows:

In Section 3, we study the domination in the spaces M_λ and $M_{p,\alpha}$; In section 4, the Korenblum constants are investigated for the analytic Morrey space $\mathcal{L}^{2,\lambda}$, the weighted Dirichlet space \mathcal{D}_α^p , the general function space $F(p, q, s)$ and some function spaces associated with the above spaces; In section 5, the domination in the Dirichlet space \mathcal{D}_α is given.

3. Domination in certain spaces M_λ and $M_{p,\alpha}$

In this section, we investigate domination in certain spaces M_λ and $M_{p,\alpha}$, which are associated with $\mathcal{L}^{2,\lambda}$ and \mathcal{D}_α^p , respectively. And we get the following results:

(i) For the analytic function space M_λ :

- The domination between $f(z) = z^2 + a$ and $g(z) = z(1 + \bar{a}z^2)$ in M_λ , $0 < \lambda \leq 1$.
- The domination between $f(z) = z + a$ and $g(z) = z(1 + \bar{a}z)$ in M_λ , $0 < \lambda \leq 1$.
- The domination between $f(z) = z^m + a$ and $g(z) = z(1 + \bar{a}z^m)$ in M_λ , $m \geq 3$, $0 < \lambda \leq 1$.

(ii) For the analytic function space $M_{p,\alpha}$:

- The domination between $f(z) = z^2 + a$ and $g(z) = z(1 + \bar{a}z^2)$ in $M_{p,\alpha}$, $-1 < \alpha$, $0 < p < \infty$.
- The domination between $f(z) = z + a$ and $g(z) = z(1 + \bar{a}z)$ in $M_{p,\alpha}$, $-1 < \alpha$, $0 < p < \infty$.
- The domination between $f(z) = z_m + a$ and $g(z) = z(1 + \bar{a}z_m)$ in $M_{p,\alpha}$, $m \geq 3$, $-1 < \alpha$, $0 < p < \infty$.

Generally, let X is a Banach space, if $f, g \in X$, g is said to dominate f in X if $\|fh\|_X \leq \|gh\|_X$ for any arbitrary $h \in \mathcal{H}^\infty$, and we denote it by $f < g$.

3.1 Domination in M_λ

Theorem 3.1. Suppose $a \in \mathbb{C}$, $0 < \lambda \leq 1$, let

$$f(z) = z^2 + a, \quad g(z) = z(1 + \bar{a}z^2),$$

where

$$|a| \leq \sqrt{\frac{7}{43}}. \quad (7)$$

Then $f < g$ in M_λ .

Proof. Suppose $h(z)$ is analytic in the open disk \mathbb{D} and

$$H(z) = \frac{h(z)}{1 - \bar{b}z}$$

is also analytic in \mathbb{D} . Thus we can rewrite $H(z)$ as

$$H(z) = \sum_{n=0}^{\infty} a_n z^n,$$

where a_n is a function of a .

By the definition of domination and Lemma 1.2, we obtain

$$\begin{aligned} \|fh\|_{M_\lambda}^2 - \|gh\|_{M_\lambda}^2 &\leq \sup_{b \in \mathbb{D}} \left((1 - |b|^2)^{1-\lambda} \int_{\mathbb{D}} (|f(z)|^2 - |g(z)|^2) |h(z)|^2 (1 - |\phi_b(z)|^2) dA(z) \right) \\ &= \sup_{b \in \mathbb{D}} (1 - |b|^2)^{2-\lambda} \int_{\mathbb{D}} (|f(z)|^2 - |g(z)|^2) |H(z)|^2 (1 - |z|^2) dA(z) \\ &= \sup_{b \in \mathbb{D}} (1 - |b|^2)^{2-\lambda} (I_1(b) - I_2(b)), \end{aligned}$$

where

$$\begin{aligned} I_1(b) &= \int_{\mathbb{D}} |f(z)|^2 |H(z)|^2 (1 - |z|^2) dA(z). \\ I_2(b) &= \int_{\mathbb{D}} |g(z)|^2 |H(z)|^2 (1 - |z|^2) dA(z). \end{aligned}$$

It is easy to see that $I_2(b) < \infty$. Now, we proceed to show that $I_1(b) - I_2(b) \leq 0$ holds for any $b \in \mathbb{D}$. Notice that

$$\begin{aligned} f(z)H(z) &= a \sum_{n=0}^1 a_n z^n + \sum_{n=0}^{\infty} (a_n + aa_{n+2}) z^{n+2}, \\ g(z)H(z) &= \sum_{n=0}^1 a_n z^{n+1} + \sum_{n=0}^{\infty} (a_{n+2} + \bar{a}a_n) z^{n+3}. \end{aligned}$$

Employing Lemma 1.1, we get

$$\begin{aligned} I_1(b) &= 2 \int_0^1 \left(|a|^2 \sum_{n=0}^1 |a_n|^2 |z|^{2n} + \sum_{n=0}^{\infty} |a_n + aa_{n+2}|^2 |z|^{2(n+2)} \right) (1 - r^2) r dr \\ &= \int_0^1 \left(|a|^2 \sum_{n=0}^1 |a_n|^2 r^n + \sum_{n=0}^{\infty} |a_n + aa_{n+2}|^2 r^{n+2} \right) (1 - r) dr \\ &= |a|^2 \sum_{n=0}^1 |a_n|^2 B(n+1, 2) + \sum_{n=0}^{\infty} |a_n + aa_{n+2}|^2 B(n+3, 2). \end{aligned}$$

and

$$\begin{aligned} I_2(b) &= \sum_{n=0}^1 |a_n|^2 B(n+2, 2) + \sum_{n=0}^{\infty} |a_{n+2} + \bar{a}a_n|^2 B(n+4, 2) \\ &= \sum_{n=0}^1 \frac{n+1}{n+3} |a_n|^2 B(n+1, 2) + \sum_{n=0}^{\infty} \frac{n+3}{n+5} |a_{n+2} + \bar{a}a_n|^2 B(n+3, 2). \end{aligned}$$

Then

$$\begin{aligned} I_1(b) - I_2(b) &= \sum_{n=0}^1 |a_n|^2 \left(|a|^2 - \frac{n+1}{n+3} \right) B(n+1, 2) \\ &\quad + \sum_{n=0}^{\infty} \left(|a_n + aa_{n+2}|^2 - \frac{n+3}{n+5} \times |a_{n+2} + \bar{a}a_n|^2 \right) B(n+3, 2) \\ &= \sum_{n=0}^1 |a_n|^2 \left(|a|^2 - \frac{n+1}{n+3} \right) B(n+1, 2) \\ &\quad + \sum_{n=0}^{\infty} \left[|a_n|^2 + |a|^2 |a_{n+2}|^2 + 2\Re(a_n \bar{a}a_{n+2}) \right. \\ &\quad \left. - \frac{n+3}{n+5} \cdot (|a_{n+2}|^2 + |a|^2 |a_n|^2 + 2\Re(a_n \bar{a}a_{n+2})) \right] B(n+3, 2). \end{aligned}$$

Using the following inequality

$$2\Re(a_n \bar{a}a_{n+2}) \leq |a|^2 (|a_n|^2 + |a_{n+2}|^2), \quad (8)$$

we obtain

$$\begin{aligned} I_1(b) - I_2(b) &\leq \sum_{n=0}^1 \left(|a|^2 - \frac{n+1}{n+3} \right) |a_n|^2 B(n+1, 2) + \sum_{n=0}^{\infty} \left[|a_n|^2 + 2|a|^2 |a_{n+2}|^2 + |a|^2 |a_n|^2 \right. \\ &\quad \left. - \frac{n+3}{n+5} \cdot (|a_{n+2}|^2 + 2|a|^2 |a_n|^2 + |a|^2 |a_{n+2}|^2) \right] B(n+3, 2) \\ &= \sum_{n=0}^1 \left(|a|^2 - \frac{n+1}{n+3} \right) |a_n|^2 B(n+1, 2) + \sum_{n=0}^{\infty} \left(1 - \frac{n+1}{n+5} |a|^2 \right) |a_n|^2 B(n+3, 2) \\ &\quad + \sum_{n=0}^{\infty} \left(\frac{n+7}{n+5} |a|^2 - \frac{n+3}{n+5} \right) |a_{n+2}|^2 B(n+3, 2) \\ &= \sum_{n=0}^1 \left(|a|^2 - \frac{n+1}{n+3} \right) |a_n|^2 B(n+1, 2) + \sum_{n=0}^1 \left(1 - \frac{n+1}{n+5} |a|^2 \right) |a_n|^2 B(n+3, 2) \\ &\quad + \sum_{n=2}^{\infty} \left(1 - \frac{n+1}{n+5} |a|^2 \right) |a_n|^2 B(n+3, 2) \\ &\quad + \sum_{n=2}^{\infty} \left(\frac{n+5}{n+3} |a|^2 - \frac{n+1}{n+3} \right) |a_n|^2 B(n+1, 2) \\ &= \sum_{n=0}^1 \left(|a|^2 - \frac{n+1}{n+3} + \tau_1 \left(1 - \frac{n+1}{n+5} |a|^2 \right) \right) |a_n|^2 B(n+1, 2) \\ &\quad + \sum_{n=2}^{\infty} \left(\frac{n+5}{n+3} |a|^2 - \frac{n+1}{n+3} + \tau_1 \left(1 - \frac{n+1}{n+5} |a|^2 \right) \right) |a_n|^2 B(n+1, 2) \end{aligned}$$

where $\tau_1 = \frac{(n+1)(n+2)}{(n+3)(n+4)}$.

Then for

$$|a| \leq \left(\frac{\frac{n+1}{n+3} - \tau_1}{1 - \tau_1 \frac{n+1}{n+5}} \right)^{\frac{1}{2}}, \quad n = 0, 1,$$

and

$$|a| \leq \left(\frac{\frac{n+1}{n+3} - \tau_1}{\frac{n+5}{n+3} - \tau_1 \frac{n+1}{n+5}} \right)^{\frac{1}{2}}, \quad n = 2, 3, \dots.$$

we have

$$I_1(b) - I_2(b) \leq 0.$$

Thus

$$|a| \leq \min \left\{ \left\{ \left(\frac{\frac{n+1}{n+3} - \tau_1}{\frac{n+5}{n+3} - \tau_1 \frac{n+1}{n+5}} \right)^{\frac{1}{2}} : n = 2, 3, \dots \right\}, \left\{ \left(\frac{\frac{n+1}{n+3} - \tau_1}{1 - \tau_1 \frac{n+1}{n+5}} \right)^{\frac{1}{2}} : n = 0, 1 \right\} \right\}. \tag{9}$$

An easy calculation shows that the right side of the above formula takes the minimum value when n takes 2, then

$$|a| \leq \sqrt{\frac{7}{43}},$$

which completes the theorem. \square

Corollary 3.2. *It seems that , for*

$$f(z) = z + a, \quad g(z) = z(1 + \bar{a}z)$$

$f < g$ does not hold in M_λ unless $a = 0$.

Corollary 3.3. *Suppose $a \in \mathbb{C}$, $0 < \lambda \leq 1$, let*

$$f(z) = z^m + a, \quad g(z) = z(1 + \bar{a}z^m), \quad m \geq 3$$

where

$$|a| \leq \frac{1}{\sqrt{3}}. \tag{10}$$

Then $f < g$ in M_λ .

Proof. An application of Theorem 3.1 yields the following:

$$\begin{aligned} I_1(b) - I_2(b) &\leq \sum_{n=0}^{m-1} \left(|a|^2 - \frac{n+1}{n+3} + \tau_2 \left(1 - \frac{n+m-1}{n+m+3} |a|^2 \right) \right) |a_n|^2 B(n+1, 2) \\ &\quad + \sum_{n=m}^{\infty} \left(\frac{n+5}{n+3} |a|^2 - \frac{n+1}{n+3} + \tau_2 \left(1 - \frac{n+m-1}{n+m+3} |a|^2 \right) \right) |a_n|^2 B(n+1, 2) \end{aligned}$$

where

$$\tau_2 = \frac{(n+m)(n+m-1) \cdots (n+1)}{(n+m+2)(n+m+1) \cdots (n+3)} = \frac{(n+2)(n+1)}{(n+m+2)(n+m+1)}.$$

Also, for

$$|a| \leq \left(\frac{\frac{n+1}{n+3} - \tau_2}{1 - \tau_2 \frac{n+m-1}{n+m+3}} \right)^{\frac{1}{2}}, \quad n = 0, 1, \dots, m-1$$

and

$$|a| \leq \left(\frac{\frac{n+1}{n+3} - \tau_2}{\frac{n+5}{n+3} - \tau_2 \frac{n+m-1}{n+m+3}} \right)^{\frac{1}{2}}, \quad n = m, m+1, \dots,$$

we have

$$I_1(b) - I_2(b) \leq 0.$$

Let

$$L(x) = \frac{\frac{x+1}{x+3} - \tau_2}{1 - \tau_2 \frac{x+m-1}{x+m+3}}, \quad x \in [0, m-1].$$

Straightforward calculations show that

$$L(0) \leq L(1) \leq \dots \leq L(m-1),$$

so

$$\begin{aligned} |a|^2 &\leq L(0) \\ &= \frac{(m+3)(m^2+5m)}{3(m+1)(m+2)(m+3) - 6(m-1)} \\ &\leq \lim_{m \rightarrow \infty} \frac{(m+3)(m^2+5m)}{3(m+1)(m+2)(m+3) - 6(m-1)} \\ &= \frac{1}{3}. \end{aligned} \tag{11}$$

On the other hand, consider

$$N(x) = \frac{\frac{x+1}{x+3} - \tau_2}{\frac{x+5}{x+3} - \tau_2 \frac{x+m-1}{x+m+3}}, \quad x \in [m, +\infty).$$

Put $x = m, m+1, \dots$ into $N(x)$ for calculations and we get

$$N(m) \leq N(m+1) \leq \dots,$$

then

$$\begin{aligned} |a|^2 &\leq N(m) \\ &\leq \lim_{m \rightarrow \infty} \frac{6m^3 + 11m^2 - 5m - 12}{6m^3 + 47m^2 + 79m + 36} = 1. \end{aligned} \tag{12}$$

Hence, we obtain the desired result. \square

3.2 Domination in $M_{p,\alpha}$

Now, we study the domination in certain spaces $M_{p,\alpha}$ associated with weighted Dirichlet spaces \mathcal{D}_α^p .

Theorem 3.4. Suppose $0 < p < \infty$, $-1 < \alpha$, let

$$f(z) = z^2 + a, \quad g(z) = z(1 - \bar{a}z^2), \quad z \in \mathbb{D}$$

where

$$|a| \leq \begin{cases} \left(\frac{3(\alpha+6)}{2\alpha^2+25\alpha+102} \right)^{\frac{1}{2}}, & -1 < \alpha < 2, \\ \left(\frac{\alpha+4}{\alpha^2+8\alpha+20} \right)^{\frac{1}{2}}, & 2 \leq \alpha. \end{cases} \tag{13}$$

Then $f < g$ in $M_{p,\alpha}$.

Proof. Suppose $h(z)$ is analytic in the open disk \mathbb{D} hence we can write $h(z)$ as

$$h(z) = \sum_{n=0}^{\infty} a_n z^n.$$

where a_n is a function of a .

By the definition of domination, consider

$$\begin{aligned} \|fh\|_{M_{p,\alpha}}^p - \|gh\|_{M_{p,\alpha}}^p &= \int_{\mathbb{D}} (|f(z)|^p - |g(z)|^p) |h(z)|^p (1 - |z|^2)^\alpha dA(z) \\ &= I_1(b) - I_2(b). \end{aligned}$$

where

$$\begin{aligned} I_1(b) &= \int_{\mathbb{D}} |f(z)|^p |h(z)|^p (1 - |z|^2)^\alpha dA(z). \\ I_2(b) &= \int_{\mathbb{D}} |g(z)|^p |h(z)|^p (1 - |z|^2)^\alpha dA(z). \end{aligned}$$

Be the same as the proof of Theorem 3.1, we need only to consider the case $I_1(b) - I_2(b) \leq 0$ holds for any $b \in \mathbb{D}$.

With the proof technique of Theorem 3.1, one has

$$\begin{aligned} I_1(b) - I_2(b) &\leq \sum_{n=0}^1 \left(|a|^2 - \frac{n+1}{n+\alpha+2} + \tau_3 \left(1 - \frac{n-\alpha+2}{n+\alpha+4} |a|^2 \right) \right) |a_n|^2 B(n+1, \alpha+1) \\ &\quad + \sum_{n=2}^{\infty} \left(\frac{n+2\alpha+3}{n+\alpha+2} |a|^2 - \frac{n+1}{n+\alpha+2} + \tau_3 \left(1 - \frac{n-\alpha+2}{n+\alpha+4} |a|^2 \right) \right) \\ &\quad \cdot |a_n|^2 B(n+1, \alpha+1). \end{aligned}$$

where

$$\tau_3 = \frac{(n+1)(n+2)}{(n+\alpha+2)(n+\alpha+3)}.$$

For $n = 0, 1$, it follows that

$$\begin{aligned} |a| &\leq \min \left\{ \left(\frac{\alpha+4}{\alpha^2+8\alpha+20} \right)^{\frac{1}{2}}, \left(\frac{2(\alpha+5)}{\alpha^2+11\alpha+42} \right)^{\frac{1}{2}} \right\} \\ &= \left(\frac{\alpha+4}{\alpha^2+8\alpha+20} \right)^{\frac{1}{2}}. \end{aligned}$$

And for $n = 2, 3, \dots$

$$\begin{aligned} |a| &\leq \min \left\{ \left(\frac{\frac{n+1}{n+\alpha+2} - \tau_3}{\frac{n+2\alpha+3}{n+\alpha+2} - \tau_3 \frac{n-\alpha+2}{n+\alpha+4}} \right)^{\frac{1}{2}} \right\} \\ &= \left(\frac{3(\alpha+6)}{2\alpha^2+25\alpha+102} \right)^{\frac{1}{2}}. \end{aligned}$$

It is proper to verify that when

$$|a| \leq \begin{cases} \left(\frac{3(\alpha+6)}{2\alpha^2+25\alpha+102} \right)^{\frac{1}{2}}, & -1 < \alpha < 2, \\ \left(\frac{\alpha+4}{\alpha^2+8\alpha+20} \right)^{\frac{1}{2}}, & 2 \leq \alpha. \end{cases} \tag{14}$$

we obtain

$$I_1(b) - I_2(b) \leq 0.$$

This theorem is completed. \square

Corollary 3.5. *It seems that , for*

$$f(z) = z + a, \quad g(z) = z(1 + \bar{a}z)$$

$f < g$ does not hold in $M_{p,\alpha}$ unless $a = 0$.

Corollary 3.6. *Suppose $0 < p < \infty$, $-1 < \alpha$, let*

$$f(z) = z^m + a, \quad g(z) = z(1 - \bar{a}z^m), \quad m \geq 3$$

where

$$|a| \leq \min \left\{ \left(\frac{(\alpha + 3) \cdots (m + \alpha + 2) - m!(m + \alpha + 2)}{(\alpha + 2) \cdots (m + \alpha + 2) - m!(m - \alpha)} \right)^{\frac{1}{2}}, \left(\frac{\frac{m+1}{m+\alpha+2} - \tau_5}{\frac{m+2\alpha+3}{m+\alpha+2} - \tau_5 \frac{2m-\alpha}{2m+\alpha+2}} \right)^{\frac{1}{2}} \right\}, \tag{15}$$

with

$$\tau_5 = \frac{\Gamma(2m + 1)/\Gamma(m + 1)}{\Gamma(2m + \alpha + 2)/\Gamma(m + \alpha + 2)}.$$

Then $f < g$ in $M_{p,\alpha}$.

Proof. By the techniques of Theorems 3.1,3.4 and Corollary 3.3, we have

$$\begin{aligned} I_1(b) - I_2(b) &\leq \sum_{n=0}^{m-1} \left(|a|^2 - \frac{n+1}{n+\alpha+2} + \tau_4 \left(1 - \frac{n+m-\alpha}{n+m+\alpha+2} |a|^2 \right) \right) |a_n|^2 B(n+1, \alpha+1) \\ &\quad + \sum_{n=m}^{\infty} \left(\frac{n+2\alpha+3}{n+\alpha+2} |a|^2 - \frac{n+1}{n+\alpha+2} + \tau_4 \left(1 - \frac{n+m-\alpha}{n+m+\alpha+2} |a|^2 \right) \right) \\ &\quad \cdot |a_n|^2 B(n+1, \alpha+1). \end{aligned}$$

where

$$\tau_4 = \frac{\Gamma(n+m+1)/\Gamma(n+1)}{\Gamma(n+m+\alpha+2)/\Gamma(n+\alpha+2)}.$$

It is simple to show that for $n = 0$,

$$|a| \leq \left(\frac{(\alpha + 3) \cdots (m + \alpha + 2) - m!(m + \alpha + 2)}{(\alpha + 2) \cdots (m + \alpha + 2) - m!(m - \alpha)} \right)^{\frac{1}{2}}$$

and for $n = m$,

$$|a| \leq \left(\frac{\frac{m+1}{m+\alpha+2} - \tau_5}{\frac{m+2\alpha+3}{m+\alpha+2} - \tau_5 \frac{2m-\alpha}{2m+\alpha+2}} \right)^{\frac{1}{2}}, \quad \tau_5 = \frac{\Gamma(2m + 1)/\Gamma(m + 1)}{\Gamma(2m + \alpha + 2)/\Gamma(m + \alpha + 2)},$$

we get

$$I_1(b) - I_2(b) \leq 0.$$

Thus

$$|a| \leq \min \left\{ \left(\frac{(\alpha + 3) \cdots (m + \alpha + 2) - m!(m + \alpha + 2)}{(\alpha + 2) \cdots (m + \alpha + 2) - m!(m - \alpha)} \right)^{\frac{1}{2}}, \left(\frac{\frac{m+1}{m+\alpha+2} - \tau_5}{\frac{m+2\alpha+3}{m+\alpha+2} - \tau_5 \frac{2m-\alpha}{2m+\alpha+2}} \right)^{\frac{1}{2}} \right\}, \tag{16}$$

we get the desired result. \square

4. Korenblum constants for analytic spaces

In this section, we obtain the following results:

- (i) For the analytic Morrey space $\mathcal{L}^{2,\lambda}$:
 - The explicit expression for upper bounds of $\kappa_{\mathcal{L}^{2,\lambda}}$ in $\mathcal{L}^{2,\lambda}$, $0 < \lambda < 1$.
 - The explicit expression for upper bounds of κ_{BMOA} in $BMOA$.
 - The explicit expression for upper bounds of κ_{BMOA} in M_λ , $0 < \lambda \leq 1$.
- (ii) For the weighted Dirichlet space \mathcal{D}_α^p :
 - The explicit expression for upper bounds of $\kappa_{\mathcal{D}_\alpha^p}$ in \mathcal{D}_α^p , $0 < \alpha, p < \infty$.
 - The explicit expression for upper bounds of $\kappa_{M_{p,\alpha}}$ in $M_{p,\alpha}$, $-1 < \alpha, 0 < p < \infty$.
- (iii) For the general function space $F(p, q, s)$:
 - The explicit expression for upper bounds of $\kappa_{F(p,q,s)}$ in $F(p, q, s)$, $0 < p, s < \infty, -2 < q < \infty, -1 < q + s$.
- (iv) Open questions.

Throughout the paper, c is called a Korenblum constant and denote by κ as the largest value of c . To avoid ambiguity, for the remainder of this paper we denote the largest Korenblum constant, unless otherwise stated, by $\kappa_{\mathcal{L}^{2,\lambda}}$ for $\mathcal{L}^{2,\lambda}$, κ_{BMOA} for $BMOA$, etc.

4.1 For the analytic Morrey space $\mathcal{L}^{2,\lambda}$

Theorem 4.1. For any $0 < \lambda \leq 1$, we consider the spaces $\mathcal{L}^{2,\lambda}$.

(i) For $\lambda = 1$, suppose

$$\sup_{a \in \mathbb{D}} \left((1 - |a|^2) \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} |a|^{2n} \right)^{\frac{1}{2}} < c < 1. \tag{17}$$

(ii) For $0 < \lambda < 1$, suppose

$$\sup_{a \in \mathbb{D}} \left((1 - |a|^2)^{2-\lambda} \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} |a|^{2n} \right)^{\frac{1}{2}} < c < 1. \tag{18}$$

Then there exist analytic functions f and g defined on \mathbb{D} such that $|f(z)| < |g(z)|$ whenever $c < |z| < 1$, but $\|f\|_{BMOA} > \|g\|_{BMOA}$ (or $\|f\|_{\mathcal{L}^{2,\lambda}} > \|g\|_{\mathcal{L}^{2,\lambda}}$).

Therefore,

$$\kappa_{BMOA} \leq \sup_{a \in \mathbb{D}} \left((1 - |a|^2) \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} |a|^{2n} \right)^{\frac{1}{2}} \tag{19}$$

and

$$\kappa_{\mathcal{L}^{2,\lambda}} \leq \sup_{a \in \mathbb{D}} \left((1 - |a|^2)^{2-\lambda} \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} |a|^{2n} \right)^{\frac{1}{2}}. \tag{20}$$

Proof. Choose functions $f(z) = c, g(z) = z \in \mathcal{L}^{2,\lambda}$, one obtains

$$|f(z)| = |c| < |z| = |g(z)|$$

for any $|z| > c$.

We aim to proof

$$\|f\|_{\mathcal{L}^{2,\lambda}} > \|g\|_{\mathcal{L}^{2,\lambda}}.$$

or equivalently,

$$c > \sup_{a \in \mathbb{D}} \left((1 - |a|^2)^{1-\lambda} \int_{\mathbb{D}} (1 - |\phi_a(z)|^2) dA(z) \right)^{\frac{1}{2}}.$$

or,

$$c > \sup_{a \in \mathbb{D}} \left((1 - |a|^2)^{2-\lambda} \int_{\mathbb{D}} \frac{1 - |z|^2}{|1 - \bar{a}z|^2} dA(z) \right)^{\frac{1}{2}}.$$

Also, by [33, Lemma 3.10], we have

$$\begin{aligned} & \sup_{a \in \mathbb{D}} \left((1 - |a|^2)^{2-\lambda} \int_{\mathbb{D}} \frac{1 - |z|^2}{|1 - \bar{a}z|^2} dA(z) \right)^{\frac{1}{2}} \\ &= \sup_{a \in \mathbb{D}} \left((1 - |a|^2)^{2-\lambda} \sum_{n=0}^{\infty} \frac{n!}{(n+2)!} |a|^{2n} \right)^{\frac{1}{2}} \\ &= \sup_{a \in \mathbb{D}} \left((1 - |a|^2)^{2-\lambda} \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} |a|^{2n} \right)^{\frac{1}{2}}. \end{aligned}$$

Then,

$$\|f\|_{\mathcal{L}^{2,\lambda}} > \|g\|_{\mathcal{L}^{2,\lambda}} \iff c > \sup_{a \in \mathbb{D}} \left((1 - |a|^2)^{2-\lambda} \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} |a|^{2n} \right)^{\frac{1}{2}}. \tag{21}$$

Therefore, in order to get the Korenblum Maximum Principle, we must have

$$\kappa_{\mathcal{L}^{2,\lambda}} \leq \sup_{a \in \mathbb{D}} \left((1 - |a|^2)^{2-\lambda} \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} |a|^{2n} \right)^{\frac{1}{2}}. \tag{22}$$

□

Theorem 4.2. For any $0 < \lambda \leq 1$, we consider the space M_λ . Suppose

$$\sup_{a \in \mathbb{D}_\delta} (1 - |a|^2)^{2-\lambda} \sum_{n=0}^{\infty} (n+1) |a|^n B\left(\frac{n}{2} + 2, 2\right) < \sup_{a \in \mathbb{D}_\delta} (1 - |a|^2)^{2-\lambda} \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} |a|^{2n} c \tag{23}$$

with $c < 1$. Then there exist analytic functions f and g defined on \mathbb{D} such that $|f(z)| < |g(z)|$ whenever $c < |z| < 1$, but $\|f\|_{M_\lambda} > \|g\|_{M_\lambda}$.

Therefore,

$$\kappa_{M_\lambda} \leq \frac{\sup_{a \in \mathbb{D}_\delta} (1 - |a|^2)^{2-\lambda} \sum_{n=0}^{\infty} (n+1) |a|^n B\left(\frac{n}{2} + 2, 2\right)}{\sup_{a \in \mathbb{D}_\delta} (1 - |a|^2)^{2-\lambda} \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} |a|^{2n}}. \tag{24}$$

Proof. Choose functions $f(z) = c$, $g(z) = z \in M_\lambda$, one obtains

$$|f(z)| = |c| < |z| = |g(z)|$$

for any $|z| > c$.

Since

$$\begin{aligned} \|f\|_{M_\lambda}^2 &= c \cdot \sup_{a \in \mathbb{D}} (1 - |a|^2)^{1-\lambda} \int_{\mathbb{D}} 1 - |\phi_a(z)|^2 dA(z) \\ &= c \cdot \sup_{a \in \mathbb{D}} (1 - |a|^2)^{2-\lambda} \int_{\mathbb{D}} \frac{1 - |z|^2}{|1 - \bar{a}z|^2} dA(z) \\ &= c \cdot \sup_{a \in \mathbb{D}} (1 - |a|^2)^{2-\lambda} \sum_{n=0}^{\infty} \frac{\Gamma^2(n+1)}{n! \Gamma(n+3)} |a|^{2n} \\ &= c \cdot \sup_{a \in \mathbb{D}} (1 - |a|^2)^{2-\lambda} \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} |a|^{2n}. \end{aligned}$$

$$\begin{aligned}
 \|g\|_{M_\lambda}^2 &= \sup_{a \in \mathbb{D}} (1 - |a|^2)^{1-\lambda} \cdot \int_{\mathbb{D}} |z|^2 (1 - |\phi_a(z)|^2) dA(z) \\
 &= \sup_{a \in \mathbb{D}} (1 - |a|^2)^{2-\lambda} \cdot \int_{\mathbb{D}} |z|^2 (1 - |z|^2) \frac{1}{|1 - \bar{a}z|^2} dA(z) \\
 &= \sup_{a \in \mathbb{D}} (1 - |a|^2)^{2-\lambda} \cdot \sum_{n=0}^{\infty} \frac{\Gamma(n+2)}{n! \Gamma(2)} |a|^n \int_{\mathbb{D}} |z|^{n+2} (1 - |z|^2) dA(z) \\
 &= \sup_{a \in \mathbb{D}} (1 - |a|^2)^{2-\lambda} \cdot \sum_{n=0}^{\infty} (n+1) |a|^n \int_{\mathbb{D}} |z|^{n+2} (1 - |z|^2) dA(z) \\
 &= \sup_{a \in \mathbb{D}} (1 - |a|^2)^{2-\lambda} \cdot \sum_{n=0}^{\infty} (n+1) |a|^n \int_0^1 r^{\frac{n}{2}+1} (1-r) dr \\
 &= \sup_{a \in \mathbb{D}} (1 - |a|^2)^{2-\lambda} \cdot \sum_{n=0}^{\infty} (n+1) |a|^n B\left(\frac{n}{2} + 2, 2\right).
 \end{aligned}$$

Using Mathematica, we know that there exists $\delta > 0$ such that

$$(1 - |a|^2)^{2-\lambda} \cdot \sum_{n=0}^{\infty} (n+1) |a|^n B\left(\frac{n}{2} + 2, 2\right) < (1 - |a|^2)^{2-\lambda} \cdot \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} |a|^{2n}$$

where $a \in \mathbb{D}_\delta = \{z; |z| < \delta\}$. Then

$$\|f\|_{M_\lambda} > \|g\|_{M_\lambda}$$

if and only if (23) holds. Also, we get the Korenblum Maximum Principle (24). \square

4.2 For the weighted Dirichlet space \mathcal{D}_α^p

Theorem 4.3. For any $0 < \alpha, p < \infty$, let

$$c > \left(\frac{1}{\alpha + 1}\right)^{\frac{1}{p}}. \tag{25}$$

Then there exist analytic functions f and g on the open unit disk \mathbb{D} such that $|f(z)| < |g(z)|$ whenever $c < |z| < 1$, but $\|f\|_{\mathcal{D}_\alpha^p} > \|g\|_{\mathcal{D}_\alpha^p}$.
Therefore,

$$\kappa_{\mathcal{D}_\alpha^p} \leq \left(\frac{1}{\alpha + 1}\right)^{\frac{1}{p}}. \tag{26}$$

Proof. Similarly, we choose $f(z) = c, g(z) = z \in \mathcal{D}_\alpha^p$, and

$$|f(z)| = |c| < |z| = |g(z)|$$

for any $|z| > c$.

Next, we are going to proof

$$\|f\|_{\mathcal{D}_\alpha^p}^p > \|g\|_{\mathcal{D}_\alpha^p}^p.$$

that is,

$$c^p > \int_{\mathbb{D}} (1 - |z|^2)^\alpha dA(z) = \int_0^1 (1 - r)^\alpha dr = \frac{1}{\alpha + 1}.$$

Hence c satisfies (25).

Clearly, in order for the Korenblum Maximum Principle to satisfy, one must get (26). \square

Theorem 4.3 yields the corollary, Korenblum constant for Dirichlet space \mathcal{D} .

Corollary 4.4. *Let $c > 1$. There exist functions f and g in \mathcal{D} such that $|f(z)| < |g(z)|$ for all $|z| > c$, but $\|f\|_{\mathcal{D}} > \|g\|_{\mathcal{D}}$. Therefore,*

$$\kappa_{\mathcal{D}} \leq 1. \tag{27}$$

Wee and Le [30] studied the Korenblum Maximum Principle on the weighted Bergman space with exponential weight $e^{-\frac{p\alpha}{2}|z|^2}$, they get the explicit expression for the upper bounds of the largest Korenblum constant for the weighted Bergman spaces, $0 < \alpha, 1 \leq p < \infty$. and prove that a failure of the Korenblum Maximum Principle for the weighted Bergman spaces, $0 < \alpha, 0 < p < 1$. Now, we focus on the weighted Bergman spaces $M_{p,\alpha}$ with the weight $v_{\alpha}(z) = (1 - |z|^2)^{\alpha}$.

Theorem 4.5. *Let $0 < p < \infty, -1 < \alpha$. Let*

$$c > \left((\alpha + 1)B\left(\frac{p}{2} + 1, \alpha + 1\right) \right)^{\frac{1}{p}}. \tag{28}$$

There exist functions f and g on \mathbb{D} such that $|f(z)| < |g(z)|$ for all $|z| > c$, but $\|f\|_{M_{p,\alpha}} > \|g\|_{M_{p,\alpha}}$. Therefore,

$$\kappa_{M_{p,\alpha}} \leq \left((\alpha + 1)B\left(\frac{p}{2} + 1, \alpha + 1\right) \right)^{\frac{1}{p}}. \tag{29}$$

Proof. Choose

$$f(z) = c, \quad g(z) = z \in M_{p,\alpha}$$

such that $|f(z)| < |g(z)|$. Since

$$\begin{aligned} \|f\|_{M_{p,\alpha}}^p &= c^p \cdot \int_{\mathbb{D}} (1 - |z|^2)^{\alpha} dA(z) = \frac{c^p}{\alpha + 1}. \\ \|g\|_{M_{p,\alpha}}^p &= \int_{\mathbb{D}} |z|^p (1 - |z|^2)^{\alpha} dA(z) = B\left(\frac{p}{2} + 1, \alpha + 1\right). \end{aligned}$$

Then

$$\|f\|_{M_{p,\alpha}} > \|g\|_{M_{p,\alpha}} \Leftrightarrow c > \left((\alpha + 1)B\left(\frac{p}{2} + 1, \alpha + 1\right) \right)^{\frac{1}{p}}.$$

This completes the theorem. \square

4.3 For the general function space $F(p, q, s)$

Based on the analytic Morrey space $\mathcal{L}^{2,\lambda}$, we have the general case.

Theorem 4.6. *For any $0 < p, s < \infty, -2 < q < \infty, q + s > -1$, let*

$$c > \left(\sup_{a \in \mathbb{D}} h_a(q, s) \right)^{\frac{1}{p}}. \tag{30}$$

Then there exist analytic functions f and g on the open unit disk \mathbb{D} such that $|f(z)| < |g(z)|$ whenever $c < |z| < 1$, but $\|f\|_{F(p,q,s)} > \|g\|_{F(p,q,s)}$.

Therefore,

$$\kappa_{F(p,q,s)} \leq \left(\sup_{a \in \mathbb{D}} h_a(q, s) \right)^{\frac{1}{p}}. \tag{31}$$

where

$$\begin{aligned} h_a(q, s) &= \int_{\mathbb{D}} (1 - |z|^2)^q (1 - |\phi_a(z)|^2)^s dA(z) \\ &= (1 - |a|^2)^s \cdot \frac{\Gamma(q + s + 1)}{\Gamma^2(s)} \cdot \sum_{n=0}^{\infty} \frac{\Gamma^2(n + s)}{n! \Gamma(n + q + s + 2)} |a|^{2n}. \end{aligned} \tag{32}$$

Proof. Let $f(z) = c, g(z) = z \in F(p, q, s)$ with

$$|f(z)| = |c| < |z| = |g(z)|$$

for each $|z| > c$.

Since

$$\|f\|_{F(p,q,s)}^p = c^p,$$

and

$$\|g\|_{F(p,q,s)}^p = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} (1 - |z|^2)^q (1 - |\phi_a(z)|^2)^s dA(z) \tag{33}$$

$$= \sup_{a \in \mathbb{D}} (1 - |a|^2)^s \int_{\mathbb{D}} \frac{(1 - |z|^2)^{q+s}}{|1 - \bar{a}z|^{2s}} dA(z)$$

$$= \sup_{a \in \mathbb{D}} h_a(q, s). \tag{34}$$

where $h_a(q, s)$ is defined as (32).

So,

$$\|f\|_{F(p,q,s)} > \|g\|_{F(p,q,s)}$$

if and only if (30) holds, as desired. \square

Corollary 4.7. *Let $0 < s < \infty$. Suppose*

$$\left(\sup_{a \in \mathbb{D}} h_a(0, s) \right)^{\frac{1}{2}} < c < 1. \tag{35}$$

Then there exist functions f and g in Q_s such that $|f(z)| < |g(z)|$ for all $|z| > c$, but $\|f\|_{Q_s}^2 > \|g\|_{Q_s}^2$. Therefore,

$$\kappa_{Q_s} \leq \left(\sup_{a \in \mathbb{D}} h_a(0, s) \right)^{\frac{1}{2}} \tag{36}$$

$$= \left(\sup_{a \in \mathbb{D}} (1 - |a|^2)^s \cdot \frac{s}{\Gamma(s)} \cdot \sum_{n=0}^{\infty} \frac{\Gamma(n+s)}{n!(n+s)(n+s+1)} |a|^{2n} \right)^{\frac{1}{2}}.$$

4.4 Open questions

Theorems 4.1-4.6 lead us to have the following questions to consider.

Question 1. For $0 < \lambda \leq 1$, suppose

$$c = \sup_{a \in \mathbb{D}} \left((1 - |a|^2)^{2-\lambda} \cdot \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} |a|^{2n} \right)^{\frac{1}{2}}. \tag{37}$$

Do there exist analytic functions f and g defined on \mathbb{D} such that $|f(z)| < |g(z)|$ whenever $c < |z| < 1$, but $\|f\|_{\mathcal{L}^{2,\lambda}} > \|g\|_{\mathcal{L}^{2,\lambda}}$?

Question 2. For any $0 < \lambda \leq 1$, suppose

$$\sup_{a \in \mathbb{D}_\delta} (1 - |a|^2)^{2-\lambda} \sum_{n=0}^{\infty} (n+1) |a|^n B\left(\frac{n}{2} + 2, 2\right) = \sup_{a \in \mathbb{D}_\delta} (1 - |a|^2)^{2-\lambda} \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} |a|^{2n} c. \tag{38}$$

Do there exist analytic functions f and g defined on \mathbb{D} such that $|f(z)| < |g(z)|$ whenever $c < |z| < 1$, but $\|f\|_{M_\lambda} > \|g\|_{M_\lambda}$?

Question 3. For any $0 < \alpha, p < \infty$, let

$$c = \left(\frac{1}{\alpha + 1}\right)^{\frac{1}{p}}. \tag{39}$$

Do there exist analytic functions f and g in \mathcal{D}_α^p for which $|f(z)| < |g(z)|$ with $|z| > c$ and $\|f\|_{\mathcal{D}_\alpha^p} > \|g\|_{\mathcal{D}_\alpha^p}$?

Question 4. Let $c = 1$. Do there exist functions $f(z)$ and $g(z)$ in \mathcal{D} for which $|f(z)| < |g(z)|$ with $|z| > c$ and $\|f\|_{\mathcal{D}} > \|g\|_{\mathcal{D}}$?

Question 5. Let $-1 < \alpha, 0 < p < \infty$. Let

$$c = \left((\alpha + 1)B\left(\frac{p}{2} + 1, \alpha + 1\right)\right)^{\frac{1}{p}}. \tag{40}$$

Do there exist analytic functions f and g in $M_{p,\alpha}$ for which $|f(z)| < |g(z)|$ with $|z| > c$ and $\|f\|_{M_{p,\alpha}} > \|g\|_{M_{p,\alpha}}$?

Question 6. For any $0 < p, s < \infty, -2 < q < \infty, q + s > -1$, let

$$c = \left(\sup_{a \in \mathbb{D}} h_a(q, s)\right)^{\frac{1}{p}}. \tag{41}$$

Do there exist analytic functions f and g in $F(p, q, s)$ for which $|f(z)| < |g(z)|$ with $|z| > c$ and $\|f\|_{F(p,q,s)} > \|g\|_{F(p,q,s)}$?

5. Domination in Dirichlet space \mathcal{D}_α

For $\alpha \in \mathbb{R}$, the Dirichlet space \mathcal{D}_α is the set of functions $f(z) = \sum_{n=0}^\infty a_n z^n \in \mathcal{H}ol(\mathbb{D})$ for which

$$\|f\|_{\mathcal{D}_\alpha} = \left(\sum_{n=0}^\infty (n + 1)^{1-\alpha} |a_n|^2\right)^{\frac{1}{2}}. \tag{42}$$

Remark that \mathcal{D}_0 is the classical Dirichlet space, and \mathcal{D}_1 is the Hardy space \mathcal{H}^2 [2][3][32].

Motivated by the paper [29], in which the special case of domination on the space \mathcal{D}_α when $\alpha = 2$ was investigated, the following theorem generalizes the results to the general cases for $2 < \alpha < \infty$.

Theorem 5.1. Let $2 < \alpha < \infty, a \in \mathbb{C}$, and m be a positive integer. Suppose that one of the following conditions is satisfied :

(i) $m \geq 3$, and

$$|a| \leq \sqrt{\frac{m - 2}{2(m - 1)}} \cdot \frac{1}{(k + m + 1)^{\alpha-2}}, \quad k = 0, 1, \dots; \tag{43}$$

(ii) $m = 2$, and $|a| \leq |a^*|$, where $|a^*|$ is the solution of (49).

Then

$$\|(z^m + a)h(z)\|_{\mathcal{D}_\alpha} \leq \|z(1 + \bar{a}z^m)h(z)\|_{\mathcal{D}_\alpha}.$$

for any $h(z) = \sum_{n=0}^\infty c_n z^n \in \mathcal{H}^\infty$.

Proof. Define

$$f(z) = (z^m + a)h(z), \quad g(z) = z(1 + \bar{a}z^m)h(z)$$

where $h(z) = \sum_{k=0}^\infty c_k z^k$ is the power series expansion of $h(z)$ and $a \in \mathbb{C}, m \in \mathbb{N}$. Then

$$f(z) = a \sum_{k=0}^{m-1} c_k z^k + \sum_{k=0}^\infty (c_k + a c_{k+m}) z^{k+m},$$

$$g(z) = \sum_{k=0}^{m-1} c_k z^{k+1} + \sum_{k=0}^{\infty} (\bar{a}c_k + c_{k+m}) z^{k+m+1}.$$

Clearly, $\|g\|_{\mathcal{D}_\alpha} < \infty$.

Followed by (42), we have

$$\begin{aligned} \|f\|_{\mathcal{D}_\alpha}^2 - \|g\|_{\mathcal{D}_\alpha}^2 &= \sum_{k=0}^m |c_k|^2 \left(\frac{|a|^2}{(k+1)^{\alpha-1}} - \frac{1}{(k+2)^{\alpha-1}} \right) \\ &\quad + \sum_{k=0}^{\infty} \left(\frac{|c_k + ac_{k+m}|^2}{(k+m+1)^{\alpha-1}} - \frac{|\bar{a}c_k + c_{k+m}|^2}{(k+m+2)^{\alpha-1}} \right) \\ &= \sum_{k=0}^{\infty} |c_k|^2 |a|^2 \left(\frac{1}{(k+1)^{\alpha-1}} - \frac{1}{(k+m+2)^{\alpha-1}} \right) \\ &\quad - \sum_{k=0}^{\infty} |c_k|^2 \left(\frac{1}{(k+2)^{\alpha-1}} - \frac{1}{(k+m+1)^{\alpha-1}} \right) \\ &\quad + \sum_{k=0}^{\infty} 2\Re(\bar{a}c_k c_{k+m}) \left(\frac{1}{(k+m+1)^{\alpha-1}} - \frac{1}{(k+m+2)^{\alpha-1}} \right) \end{aligned} \tag{44}$$

$$\leq \sum_{k=0}^{\infty} (\alpha - 1) |c_k|^2 |a|^2 \left(\frac{1}{(k+1)^{\alpha-2}} \cdot \frac{m+1}{(k+1)(k+m+2)} \right) \tag{45}$$

$$\begin{aligned} &- \sum_{k=0}^{\infty} (\alpha - 1) |c_k|^2 \left(\frac{1}{(k+2)^{\alpha-2}} \cdot \frac{m-1}{(k+2)(k+m+1)} \right) \\ &\quad + \sum_{k=0}^{\infty} 2(\alpha - 1) \Re(\bar{a}c_k c_{k+m}) \left(\frac{1}{(k+m+1)^{\alpha-2}} \cdot \frac{1}{(k+m+1)(k+m+2)} \right), \end{aligned} \tag{46}$$

where a well-known inequality of [12, p39] is used in the inequality (45), which states that

$$py^{p-1}(x - y) \leq x^p - y^p \leq px^{p-1}(x - y) \tag{47}$$

whenever $x \geq 0, y \geq 0$ for $p \geq 1$.

Notice that

$$2\Re(\bar{a}c_k c_{k+m}) \leq |a|b \frac{k+2m}{k+m} |c_k|^2 + \frac{|a|}{b} \frac{k+m}{k+2m} |c_{k+m}|^2. \tag{48}$$

for all nonnegative integer k and all positive number b . So

$$\begin{aligned} &\sum_{k=0}^{\infty} 2\Re(\bar{a}c_k c_{k+m}) \left(\frac{1}{(k+m+1)^{\alpha-1}(k+m+2)} \right) \\ &\leq \sum_{k=0}^{\infty} |a|b \frac{k+2m}{k+m} \cdot \frac{|c_k|^2}{(k+m+1)^{\alpha-1}(k+m+2)} + \sum_{k=m}^{\infty} \frac{|a|}{b} \frac{k}{k+m} \cdot \frac{|c_k|^2}{(k+1)^{\alpha-1}(k+2)} \\ &\leq \sum_{k=0}^{\infty} |c_k|^2 \left(|a|b \frac{k+2m}{(k+m)(k+m+1)^{\alpha-1}(k+m+2)} + \frac{|a|k}{b(k+m)(k+1)^{\alpha-1}(k+2)} \right). \end{aligned}$$

Hence, we get

$$\begin{aligned} \|f\|_{\mathcal{D}_\alpha}^2 - \|g\|_{\mathcal{D}_\alpha}^2 \leq & (\alpha - 1) \cdot \sum_{k=0}^{\infty} \frac{|c_k|^2}{(k+1)^{\alpha-1}(k+m+2)} \cdot \left[(m+1)|a|^2 \right. \\ & - \frac{(m-1)(k+1)^{\alpha-1}(k+m+2)}{(k+2)(k+m+1)^{\alpha-1}} + \frac{|a|b(k+1)^{\alpha-1}(k+2m)}{(k+m)(k+m+1)^{\alpha-1}} \\ & \left. + \frac{|a|k(k+m+2)}{b(k+2)(k+m)} \right]. \end{aligned}$$

We denote the expression in the bracket above by d_k . Then, we are ready to show that $d_k \leq 0$.

(i) For $m \geq 3$.

Suppose $|a| \leq \sqrt{\frac{m-2}{2(m-1)}} \cdot \frac{1}{(k+m+1)^{\mu-2}}$, $k = 0, 1, \dots$, and $b = \sqrt{\frac{2}{(m-1)(m-2)}}$.

Then

$$|a|b \leq \frac{1}{(m-1)(k+m+1)^{\alpha-2}}, \quad \frac{|a|}{b} \leq \frac{m-2}{2} \frac{1}{(k+m+1)^{\alpha-2}}.$$

Thus,

$$\begin{aligned} d_k \leq & (m+1) \frac{m-2}{2(m-1)} \frac{1}{(k+m+1)^{\alpha-2}} - \frac{(m-1)}{(k+m+1)^{\alpha-2}} \left[1 - \frac{m}{(k+2)(k+m+1)} \right] \\ & + \frac{(k+1)^{\alpha-2}}{(m-1)(k+m+1)^{\alpha-2}} \left[1 - \frac{m(m-1)}{(k+m)(k+m+1)} \right] \\ & + \frac{m-2}{2} \frac{1}{(k+m+1)^{\alpha-2}} \left[1 - \frac{2m}{(k+m)(k+2)} \right] \\ \leq & \frac{(k+1)^{\alpha-2}}{(k+m+1)^{\alpha-2}} \left\{ (m+1) \frac{m-2}{2(m-1)} - (m-1) \left[1 - \frac{m}{(k+2)(k+m+1)} \right] \right. \\ & \left. + \frac{1}{m-1} \left[1 - \frac{m(m-1)}{(k+m)(k+m+1)} \right] + \frac{m-2}{2} \left[1 - \frac{2m}{(k+m)(k+2)} \right] \right\} \\ = & 0. \end{aligned}$$

(ii) For $m = 2$.

Choose $b = 1$. Consider

$$\begin{aligned} d_k \leq & 3|a|^2 - \frac{1}{(k+3)^{\alpha-2}} \left[1 - \frac{2}{(k+2)(k+3)} \right] \\ & + |a|b(k+1)^{\alpha-2} \left[1 - \frac{2}{(k+2)(k+3)} \right] + \frac{|a|}{b} \left[1 - \frac{4}{(k+2)^2} \right] \\ = & 3|a|^2 + \frac{(k+4) \left[(k+1)^{\alpha-1}(k+2) + k(k+3) \right]}{(k+2)^2(k+3)} |a| - \frac{(k+1)(k+4)}{(k+2)(k+3)^{\alpha-1}}. \end{aligned}$$

Let

$$F(|a|) = 3|a|^2 + \frac{(k+4) \left[(k+1)^{\mu-1}(k+2) + k(k+3) \right]}{(k+2)^2(k+3)} |a| - \frac{(k+1)(k+4)}{(k+2)(k+3)^{\alpha-1}}, \tag{49}$$

with $|a| \in [0, 1)$. Then $d_k \leq 0$ if and only if the above binary linear equation of $|a|$ (that is, $F(|a|)$) have solutions. Of course, it does have. Suppose $F(|a^*|) = 0$, hence $|a| \leq |a^*|$. \square

References

- [1] V. Božin, and B. Karapetrović, *Failure of Korenblum's maximum principle in Bergman spaces with small exponents*, Proc. Amer. Math. Soc. 146 (2018), no. 6, 2577–2584.
- [2] L. Brown, and A. Shields, *Cyclic vectors in the Dirichlet space*, Trans. Amer. Math. Soc. 285 (1984) 269–304.
- [3] P. Duren, *Theory of H^p spaces*, Academic Press, New York, 1970.
- [4] P. Duren, A. Schuster, *Bergman Spaces*, Math. Surveys Monogr. vol. 100, Amer. Math. Soc. Providence, RI (2004).
- [5] D. S. Fan, S. Z. Lu, D. C. Yang, *Boundedness of operators in Morrey spaces on homogeneous spaces and its applications*, (Chinese) Acta Math. Sinica (Chin. Ser.) 42 (1999), no. 4, 583–590.
- [6] G. D. Fazio, S. Z. Lu, D. C. Yang, *On Dirichlet problem in Morrey spaces*, Differential Integral Equations 6 (1993), no. 2, 383–391.
- [7] G. D. Fazio, M. A. Ragusa, *Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients*, J. Funct. Anal. 112 (1993), no. 2, 241–256.
- [8] T. M. Flett, *The dual of an inequality of Hardy and Littlewood and some related inequalities*, J. Math. Anal. Appl. 38, 746–765 (1972).
- [9] X. Gao, C. Wang, *Domination in certain spaces associated with Q_p spaces*, Analysis (Munich) 31 (2011), no. 3, 293–298.
- [10] A. Gogatishvili, R. C. Mustafayev, *New characterization of Morrey spaces*, Eurasian Math. J. 4 (2013), no. 1, 54–64.
- [11] A. Hinkkanen, *On a maximum principle in Bergman space*, J. Anal. Math. 79 (1999), 335–344.
- [12] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, 2nd ed., Cambridge Univ. Press, Cambridge, 1952.
- [13] J. Hu, Z. Lou, *The Korenblum's maximum principle in Fock spaces with small exponents*, J. Math. Anal. Appl. 470 (2019), no. 2, 770–776.
- [14] W. K. Hayman, *On a conjecture of Korenblum*, Analysis (Munich) 19 (1999), 195–205.
- [15] B. Korenblum, *Transformation of zero sets by contractive operators in the Bergman spaces*, Bull. Sci. Math., 114: 385–394, 1990.
- [16] B. Korenblum, *A maximum principle for the Bergman space*, Publ. Mat., Barc., 35: 479–486, 1991.
- [17] B. Korenblum, R. O'Neil, K. Richards and K. Zhu, *Totally monotone functions with applications to the Bergman space*, Trans. Am. Math. Soc., 337: 795–806, 1993.
- [18] B. Korenblum, and K. Richards, *Majorization and domination in the Bergman space*, Am. Math. Soc., 117: 153–158, 1993.
- [19] B. Karapetrović, *Korenblum maximum principle in mixed norm spaces*, Archiv der Mathematik, 118 (2022), 497–507.
- [20] P. Li, J. Liu, Z. Lou, *Integral operators on analytic Morrey spaces*, Sci. China Math. 57 (2014), no. 9, 1961–1974.
- [21] J. E. Littlewood, R. E. A. C. Paley, *Theorems on Fourier series and power series*, II. Proc. Lond. Math. Soc 42, 52–89 (1936).
- [22] N. N. Lebedev, *Special Functions and Their Applications*, Dover Publications, Inc. New York, 1972.
- [23] S. A. Vinogradov, *Multiplication and division in the space of analytic functions with area integrable derivative, and in some related spaces*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 222, 45–77, 308 (1995) (in Russian). Issled. Linein. Oper. i Teor. Funktsii 23. Translated in: J. Math. Sci. (New York) 87 (5), 3806–3827 (1997).
- [24] H. Wulan, J. Zhou, *QK and Morrey type spaces*, Ann Acad Sci Fenn Math, 2013, 38: 193–207.
- [25] C. Wang, *Refining the constant in a maximum principle for the Bergman space*, Proc. Amer. Math. Soc. 132 (2004), 853–855.
- [26] C. Wang, *On Korenblum's constant*, J. Math. Anal. Appl. 296 (2004), 262–264.
- [27] C. Wang, *On Korenblum's maximum principle*, Proc. Amer. Math. Soc. 134 (2006), 2061–2066.
- [28] C. Wang, *On a maximum principle for Bergman spaces with small exponents*, Integral Equations Operator Theory 59 (2007) 597–601.
- [29] C. Wang, *Domination in the Bergman space and Korenblum constant*, Integral Equations Oper. Theory, 61: 423–432, 2008.
- [30] J. Wee, H. K. Le, *Korenblum constants for some function spaces*, Proc. Amer. Math. Soc. 148 (2020), no. 3, 1175–1185.
- [31] J. Xiao, *Geometric Q_p Functions*, Frontiers in Mathematics. Basel: Birkhauser Verlag, 2006.
- [32] K. Zhu, *Operator Theory in Function Spaces*, Marcel Dekker, New York, 1990.
- [33] K. Zhu, *Operator Theory in Function Spaces*, Second Edition. Mathematical Surveys and Monographs, 138. Amer. Math. Soc. Providence (2007).
- [34] R. Zhao, *On a general family of function spaces*, Ann. Acad. Sci. Fenn. Math. Diss. 105, 56 (1996).