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A Littlewood-type theorem for random weighted Bergman functions and random power series in $\mathcal{L}_0^{2,\lambda}$ spaces

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Abstract. The paper obtained some results about Littlewood-Type Theorem for random weighted Bergman functions and studied random power series in $\mathcal{L}_0^{2,\lambda}$ spaces based on Cheng, Fang, and Liu's work [4] and Li, Wu's work [11] respectively.

1. Introduction

Let $\mathbb{D}$ and $H(\mathbb{D})$ denote the open unit disk and the set of all analytic functions on $\mathbb{D}$ respectively.
For $-1 < \gamma$, $0 < p < \infty$, an analytic function f in $\mathbb{D}$ is said to be in the weighted Bergman spaces A_γ^p if

$$\|f\|_{A_\gamma^p} = \left(\int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\gamma dA(z)\right)^{\frac{1}{p}} < \infty,$$

where $dA(z) = \frac{1}{\pi} dxdy$ denotes the Lebesgue area measure on $\mathbb{D}$.

For $0 < p \leqslant \infty$, $0 < q < \infty$, $0 < \alpha < \infty$, the mixed norm spaces $H(p,q,\alpha)$ consists of these analytic functions $f \in H(\mathbb{D})$ satisfying

$$\|f\|_{H(p,q,\alpha)} = \left(\int_0^1 M_p^q(r,f)(1-r)^{\alpha q-1} dr\right)^{\frac{1}{q}} < \infty,$$

where $M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}}$ and $M_\infty(r,f) = \sup_{\theta\in[0,2\pi)} |f(re^{i\theta})|$. Moreover,

$$\|f\|_{H(p,\infty,\alpha)} = \sup_{0<r<1} (1-r)^\alpha M_p(r,f).$$

Clearly, the weighted Bergman norm $\|f\|_{A_\gamma^p}$ is comparable to $\|f\|_{H(p,p,\frac{\gamma+1}{p})}$.

Assume that all random variables $\{X_n\}_{n\geqslant 0}$ are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the expectation denoted by $\mathbb{E}(\cdot)$. We shall consider sequences of random variables $\{X_n\}_{n\geqslant 0}$, where $\{X_n\}_{n\geqslant 0}$ are independent

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identically distributed symmetric random variables. Then, we introduce

$$\mathcal{R}f(z) = \sum_{n=0}^{\infty} a_n X_n z^n$$

if $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$.

Let $\mathcal{X}$ be a Banach space of analytic functions over the unit disk. We introduce another analytic function space $(\mathcal{X})_*$ by

$$(\mathcal{X})_* = \{f \in H(\mathbb{D}) : \mathbb{P}(\mathcal{R}f \in \mathcal{X}) = 1\}.$$

This notion has been justified in [4, Lemma 4].

The coefficient multiplier spaces $(\mathcal{X}, \mathcal{Y})$ are our key tool in our arguments. Given two analytic function spaces $\mathcal{X}$ and $\mathcal{Y}$, the coefficient multiplier space $(\mathcal{X}, \mathcal{Y})$ consists of all complex sequences $\{\lambda_n\}_{n\geqslant 0}$ such that

$$\sum_{n=0}^{\infty} \lambda_n a_n z^n \in \mathcal{Y}$$

holds for all $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{X}$. The coefficient multiplier space $(\mathcal{X}, \mathcal{Y})$ has an obvious but useful property:

$$\mathcal{X} \subset \mathcal{Y} \Longleftrightarrow (1, 1, \cdots) \in (\mathcal{X}, \mathcal{Y}).$$

The sequential version of the mixed norm space $\ell(p, q)$ with $0 < p, q < \infty$ consists of complex sequences $\{a_n\}_{n\geqslant 0}$ such that

$$\|\{a_n\}\|^q_{\ell(p,q)} = \sum_{n=0}^{\infty} \left(\sum_{k\in I_n} |a_k|^p \right)^{\frac{q}{p}} < \infty.$$

For $0 < p < \infty$, $q = \infty$, the mixed norm space $\ell(p, \infty)$ contains of these sequences $\{a_n\}_{n\geqslant 0}$ satisfying

$$\|\{a_n\}\|_{\ell(p,\infty)} = \sup_{n\geqslant 0} \left(\sum_{k\in I_n} |a_k|^p \right)^{\frac{1}{p}} < \infty.$$

where we recall that $I_0 = \{0\}$ and $I_n = \{k \in \mathbb{N} : 2^{n-1} \leqslant k < 2^n\}$ when $n \geqslant 1$. When $p = q$, we use ℓ^p for $\ell(p, p)$ for convenience.

Define

$$D^t(f)(z) = \sum_{n=0}^{\infty} (n+1)^t a_n z^n, \quad t \in \mathbb{R}$$

for $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$. Also, the definition of $D^\alpha \ell(p, q)$ is given in the similar method by identifying an analytic function with its Taylor coefficient sequences.

Recall that, a functional $\|\cdot\| : \mathcal{X} \to [0, \infty)$ is called a $p-$norm with $0 < p < 1$, if $\mathcal{X}$ is a complex vector space and for any $f, g \in X$,

(i) $\|f\| > 0$ if $f \neq 0$;

(ii) $\|\lambda f\| = |\lambda|\|f\|$ for $\lambda \in \mathbb{C}$;

(iii) $\|f + g\|^p \leqslant \|f\|^p + \|g\|^p$.

If $(\mathcal{X}, d)$ with $d(f, g) = \|f - g\|^p$ is complete, then it is called a $p-$Banach space. Banach spaces are $p-$Banach spaces for each $0 < p < 1$. If $p \in (0, 1)$ or $q \in (0, 1)$, $H(p, q, \alpha)$ is an $s-$Banach space with $s = \min\{p, q\}$ [14, p.83], that is,

$$\|f + g\|^s_{H(p,q,\alpha)} \leqslant \|f\|^s_{H(p,q,\alpha)} + \|g\|^s_{H(p,q,\alpha)}.$$

In particular, the weighted Bergman space A^p_γ with $-1 < \gamma$, $0 < p < 1$ is a $p-$Banach space.

The abbreviation "a.s." means "almost surely" and the symbol " $\Leftrightarrow$ " denotes "if and only if". In addition, $A \simeq B$ stands that there exist positive constants $C_1, C_2 > 0$ such that $AC_1 \leqslant B \leqslant C_2 A$.

2. Some results about Littlewood-Type Theorem for generalized mixed norm functions by randomization

2.1 An improvement of Littlewood's Theorem

First, we need some auxiliary lemmas which will be used in the desired results.

Lemma 2.1. *([15]) Let $0 < q \leqslant \infty$, $\alpha > 0$. Then, $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(2, q, \alpha)$ if and only if $\{n^{-\alpha} a_n\}_{n\geqslant 1} \in \ell(2, q)$.*

Lemma 2.2. *([7, Theorem 12.3.1, p.253]) Let $0 < p \leqslant 1$, $p \leqslant u \leqslant \infty$, $0 < q, v \leqslant \infty$ and $0 < \alpha, \beta < \infty$. Then*

$$(H(p,q,\alpha), H(u,v,\beta)) = \left\{ g \in H(\mathbb{D}) : D^{\alpha+\frac{1}{p}-1} g \in H(u, q \ominus v, \beta) \right\}.$$

Here,

$$a \ominus b = \infty \ if \ a \leqslant b, \quad and \quad \frac{1}{a \ominus b} = \frac{1}{b} - \frac{1}{a} \ if \ a > b. \tag{1}$$

Lemma 2.3. *([7, Theorem 12.4.2, p.259]) If $2 \leqslant p \leqslant \infty$ and $0 < u \leqslant 2$, then*

$$(H(p,q,\alpha), H(u,v,\beta)) = D^{\beta-\alpha} \ell(\infty, q \ominus v). \tag{2}$$

Lemma 2 and Lemma 3 are the useful coefficient multiplier results provided by Jevtić, Vukotić, and Arsenović [7].

The classical Littlewood theorem is reformulated as

$$(H^p)_* = H^2$$

for $0 < p < \infty$. For $p = \infty$, Marcus and Pisier [13] give a description about $(H^\infty)_*$. Recently, Cheng, Fang and Liu [4] provided the following profound lemma, which indicates that the circular $p-$norm can be transformed into an orthogonal 2-norm and the radial parameter q matters nothing. Also, the randomization $\mathcal{R}(\cdot)$ can be seen as an operation of circular orthogonalization.

If $\mathbb{P}(X = 1) = \mathbb{P}(X = -1) = \frac{1}{2}$, the random variable X is called Bernoulli; If it is uniformly distributed on the unit circle, it is called Steinhaus, and denoted by $N(0, 1)$, as well as the Gaussian variable law with zero mean and unit variance. A standard sequence is a sequence of independent, identically distributed variables. $\{X_n\}_{n\geqslant 0}$ is said to be a standard random sequence, if it is either a standard Bernoulli, Steinhaus, or Gaussian sequence.

Lemma 2.4. *([4, Theorem 6]) Let $0 < p, q, \alpha < \infty$ and $\{X_n\}$ be a standard random sequence. Then $(H(p,q,\alpha))_* = H(2, q, \alpha)$.*

By Lemma 2.4, the embedding $H(p,q,\alpha)$ into $H(u,v,\beta)$ via $\mathcal{R}$ can be reduced to be the inclusion of $H(p,q,\alpha)$ to $H(2,v,\beta)$, which is characterized in [3].

Lemma 2.5. *([4, Lemma 11]) Let $\{e_n\}_{n\geqslant 1}$ be a sequence of elements in a $p-$Banach space $\mathcal{X}$ and $\{X_n\}_{n\geqslant 0}$ be a standard random sequence. Let $\mathcal{S} = \sum_{n=1}^{\infty} X_n e_n$ be an a.s. convergent series in $\mathcal{X}$. Then, $S \in L^q(\Omega; \mathcal{X})$ for all $0 < q < \infty$, and moreover,*

$$\|\mathcal{S}\|_{L^{q_1}(\Omega;\mathcal{X})} \simeq \|\mathcal{S}\|_{L^{q_2}(\Omega;\mathcal{X})}$$

for any $0 < q_1, q_2 < \infty$, where $\|\mathcal{S}\|^q_{L^q(\Omega;\mathcal{X})} = E(\|\mathcal{S}\|^q_{\mathcal{X}})$.

Proposition 2.6. *Let $0 < p, q < \infty$, $-1 < \alpha$, and $\{X_n\}_{n\geqslant 0}$ be a standard random sequence. Then, the following conditions hold:*

(i) $(A^p_\gamma)_* = H(2, p, \frac{\gamma+1}{p})$;

(ii) the map $\mathcal{R} : H(2, p, \frac{\gamma+1}{p}) \to L^q(\Omega; A^p_\gamma)$ is continuous.

Proof. Statement (i) follows from Lemma 2.4. By Lemma 2.5, it is sufficient to prove (ii) with $q = p$. Then, one gets that

$$\begin{aligned}
\|\sum_{k=0}^{\infty} a_k X_k z^k\|_{L^p(\Omega;A^p_\gamma)} &= \left(\int_{\mathbb{D}}\int_{\Omega}\left|\sum_{k=0}^{\infty} a_k X_k z^k\right|^p (1-|z|^2)^\gamma d\mathbb{P}dA(z)\right)^{\frac{1}{p}} \\
&\simeq \left(\int_{\mathbb{D}}\left(\int_{\Omega}\left|\sum_{k=0}^{\infty} a_k X_k z^k\right|^2 d\mathbb{P}\right)^{\frac{p}{2}} (1-|z|^2)^\gamma dA(z)\right)^{\frac{1}{p}} \\
&= \left(\int_0^1 \left(\sum_{k=0}^{\infty} |a_k|^2 r^{2k}\right)^{\frac{p}{2}} (1-r)^\gamma r dr\right)^{\frac{1}{p}} \\
&\leqslant \left(\int_0^1 \left(\sum_{k=0}^{\infty} |a_k|^2 r^{2k}\right)^{\frac{p}{2}} (1-r)^\gamma dr\right)^{\frac{1}{p}} \\
&= \|f\|_{H(2,p,\frac{\gamma+1}{p})}.
\end{aligned}$$

□

Lemma 2.7. *([5, p.87]) (Hardy-Littlewood) If $0 < p < q \leqslant \infty$, $f \in H^p$, $\lambda \geqslant p$, $\alpha = \frac{1}{p} - \frac{1}{q}$, then*

$$\int_0^1 M_q^\lambda(r,f)(1-r)^{\lambda\alpha-1}dr < \infty.$$

The following lemma is a special case of Theorem 11.2.2 in [7, p.240].

Lemma 2.8. *Let $0 < p < 1$. Then, $(H^p, \ell(u,v)) = D^{1-\frac{1}{p}}\ell(u, p \ominus v)$.*

The following Hardy-Littlewood in [5] will be used later.

Lemma 2.9. *Let $0 < p < q \leqslant \infty$, $\lambda \geqslant p$, and $\alpha = \frac{1}{p} - \frac{1}{q}$, then*

$$\int_0^1 (1-r)^{\lambda\alpha-1} M_q^\lambda(r,f)dr < \infty$$

for every $f \in H^p$.

Theorem 2.10. *Let $0 < p, u, v, \beta < \infty$, and $\{X_n\}_{n\geqslant 0}$ be a standard random sequence. Then, $\mathcal{R}f \in H(u,v,\beta)$ a.s. for all $f \in H^p$ if and only if p, β satisfy one of the following conditions:*
(i) $2 \leqslant p < \infty$;
(ii) $0 < p < 2$, $\beta - \frac{1}{p} + \frac{1}{2} \geqslant 0$.

Proof. Since $\cap_{p>0} H^p \subset H(u,v,\beta)$, the case $2 \leqslant p < \infty$ holds by an application of Lemma 2.9.

Now, we consider $0 < p < 2$. Employing Proposition 2.6, we have that $\mathcal{R}f \in H(u,v,\beta)$ a.s. for all $f \in H^p$ is equivalent to $H^p \subset (H(u,v,\beta))_* = H(2,v,\beta)$. By Lemma 2.1, $\sum_{n=1}^\infty a_n z^n \in H(2,v,\beta)$ if and only if $\{n^{-\beta}a_n\}_{n\geqslant 1} \in \ell(2,v)$. So, the problem is reduced to characterize the pairs p, v such that

$$D^{-\beta}H^p \subset \ell(2,v). \tag{3}$$

Next, we recast this problem as a coefficient multiplier problem.

Case 1: $0 < p < 1$. By Lemma 2.8, (3) can be reformulated as to characterizing the pairs p, v such that

$$\left\{\frac{1}{(n+1)^{\beta-\frac{1}{p}+1}}\right\} \in \ell(2, p \ominus v).$$

When $v \geqslant p$, then $p \ominus v = \infty$, and

$$\begin{aligned}\left\|\left\{\frac{1}{(n+1)^{\beta-\frac{1}{p}+1}}\right\}_{n\geqslant 0}\right\|_{\ell(2,\infty)} &= \sup_{n\geqslant 0}\left(\sum_{k\in I_n}\frac{1}{(k+1)^{2\beta-\frac{2}{p}+2}}\right)^{\frac{1}{2}} \\ &\simeq \sup_{n\geqslant 0}\frac{1}{2^{(2\beta-\frac{2}{p}+1)\cdot\frac{n}{2}}}. \end{aligned} \tag{4}$$

If $v < p$, then

$$\begin{aligned}\left\|\left\{\frac{1}{(n+1)^{\beta-\frac{1}{p}+1}}\right\}_{n\geqslant 0}\right\|_{\ell(2,p\ominus v)}^{p\ominus v} &= \sum_{n=0}^{\infty}\left(\sum_{k\in I_n}\frac{1}{(k+1)^{2\beta-\frac{2}{p}+2}}\right)^{\frac{p\ominus v}{2}} \\ &= \sum_{n=0}^{\infty}\frac{1}{2^{(2\beta-\frac{2}{p}+1)\cdot\frac{n(p\ominus v)}{2}}}. \end{aligned} \tag{5}$$

Inequalities (4) and (5) together imply that $\left\{\frac{1}{(n+1)^{\beta-\frac{1}{p}+1}}\right\}_{n\geqslant 0} \in \ell(2, p\ominus v)$ if and only if $\beta - \frac{1}{p} + \frac{1}{2} \geqslant 0$.

Case 2: $1 \leqslant p < 2$. The necessity is ensured by [4, Lemma 26, (iii.2), p23].

For sufficiency, by the Hausdorff-Young theorem in [5, Theorem 6.1], the inequality (3) are reduced to find the proper p, v such that

$$\left\{\frac{1}{(n+1)^{\beta}}\right\}_{n\geqslant 0} \in (\ell(p', p'), \ell(2, v)).$$

Also, $(\ell(p', p'), \ell(2, v)) = \ell(p' \ominus 2, \infty)$ by [7, Lemma 11.1.1].

For $p' \leqslant v$, we have $\ell(p' \ominus 2, p' \ominus v) = \ell(p' \ominus 2, \infty)$. So

$$\begin{aligned}\left\|\left\{\frac{1}{(n+1)^{\beta}}\right\}_{n\geqslant 0}\right\|_{\ell(p'\ominus 2,\infty)} &= \sup_{n\geqslant 0}\sum_{k\in I_n}\frac{1}{(k+1)^{\beta(p'\ominus 2)}} < \infty \simeq \sup_{n\geqslant 0}\frac{2^n}{2^{\beta(p'\ominus 2)n}} < \infty \\ &\simeq \beta - \frac{1}{p} + \frac{1}{2} \geqslant 0. \end{aligned} \tag{6}$$

For $p' > v$, consider

$$\begin{aligned}\left\|\left\{\frac{1}{(n+1)^{\beta}}\right\}_{n\geqslant 0}\right\|_{\ell(p'\ominus 2,p'\ominus v)}^{p'\ominus v} &= \sum_{n=0}^{\infty}\left(\sum_{k\in I_n}\frac{1}{(k+1)^{\beta(p'\ominus 2)}}\right)^{\frac{p'\ominus v}{p'\ominus 2}} < \infty \\ &\simeq \sum_{n=0}^{\infty}\left(\frac{1}{2^{(\beta(p'\ominus 2)-1)n}}\right)^{\frac{p'\ominus v}{p'\ominus 2}} < \infty \\ &\simeq \beta - \frac{1}{p} + \frac{1}{2} \geqslant 0. \end{aligned} \tag{7}$$

The sufficiency is finished by (6) and (7). □

2.2 Hadamard lacunary series

A Hadamard lacunary sequence is a subsequence $\{n_k\}_{k\geqslant 1}$ of $\mathbb{N}$ such that

$$\inf_{k\geqslant 1} \frac{n_{k+1}}{n_k} > 1.$$

Jevtić, Vukotić, and Arsenović [7, p.192] give the fact that, the Hadamard lacunary series in mixed norm space $H(p,q,\alpha)$ can be characterized by Taylor coefficients.

Lemma 2.11. *Let $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k} \in H(\mathbb{D})$, for which $\{n_k\}_{k\geqslant 1}$ is a Hadamard lacunary sequence. Then the following statements are equivalent:*

(i) $f \in H(p,q,\alpha)$;

(ii)$\{n_k^{-\alpha} a_k\}_{k\geqslant 1} \in \ell^q$.

Theorem 2.12. *Suppose $\{n_k\}_{k\geqslant 1}$ is a Hadamard lacunary sequence, $0 < p,q,\alpha,u,v,\beta < \infty$, and $\{X_n\}_{n\geqslant 1}$ is a sequence of independent, identically distributed symmetric random variables with $X_n \in L^2(\Omega)$. Then $\mathcal{R}f \in H(p,q,\alpha)$ a.s. for each Hadamard lacunary series $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k} \in H(u,v,\beta)$ if and only if v,q,α,β satisfy one of the following conditions:*

(i) $v > q$, $\alpha > \beta$;

(ii) $v \leqslant q$, $\alpha \geqslant \beta$.

Proof. Note that $\{n_k^{-\alpha} a_k\}_{k\geqslant 1} \in \ell^q$ is independent of p, and $M_p(r,f)$ is comparable to $M_2(r,f)$ for any Hadamard lacunary series by [7, Theorem 6.2.2]. Lemma 9 implies that $\mathcal{R}f \in H(p,q,\alpha)$ a.s. if and only if $\sum_{k=1}^{\infty} \frac{|a_k X_{n_k}|^q}{n_k^{\alpha q}} < \infty$ a.s. ,which is equivalent to $\sum_{k=1}^{\infty} \frac{|a_k|^q}{n_k^{\alpha q}} < \infty$ by [8, Theorem 5].

By Lemma 9, it follows that $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k} \in H(u,v,\beta)$ if and only if

$$\left\{n_k^{-\beta} a_k\right\}_{k\geqslant 1} \in \ell^v.$$

Thus, we need to consider p,q,α satisfying $\left\{n_k^{-\alpha+\beta}\right\}_{k\geqslant 1} \in (\ell^v, \ell^q)$, which is equal to $\ell^{v\ominus q}$ by [7, Lemma 11.1.1].

If $v > q$, then

$$\left\|\left\{n_k^{-\alpha+\beta}\right\}_{k\geqslant 1}\right\|_{\ell^{v\ominus q}}^{v\ominus q} = \sum_{k=1}^{\infty} \frac{1}{n_k^{(\alpha-\beta)\cdot v\ominus q}} < \infty$$

holds if and only if $\alpha > \beta$.

If $v \leqslant q$, $\ell^{v\ominus q} = \ell^{\infty}$, then $\left\{n_k^{-\alpha+\beta}\right\}_{k\geqslant 1} \in \ell^{v\ominus q}\ell^{\infty}$ if and only if $\alpha \geqslant \beta$. □

In view of Theorem 2.12, there is an interesting corollary.

Corollary 2.13. *Let $\{n_k\}_{k\geqslant 1}$ be a Hadamard lacunary sequence. Let $-1 < \alpha,\beta$, $0 < p,q < \infty$, and $\{X_n\}_{n\geqslant 1}$ be a sequence of independent, identically distributed symmetric random variables with $X_n \in L^2(\Omega)$. Then, $\mathcal{R}f \in A_\gamma^p$ a.s. for each Hadamard lacunary series $f(z) = \sum_{k=1}^{\infty} \in A_\beta^q$ if and only if $p \leqslant q$.*

2.3 Decreasing Taylor coefficients

Theorem 2.14. *Let $1 < u < \infty$, $0 < p,q,v,\alpha,\beta < \infty$, and $\{X_n\}_{n\geqslant 0}$ be a standard random sequence. Then $\mathcal{R}f \in H(p,q,\alpha)$ a.s. for each $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(u,v,\beta)$ with a_n being a sequence of real numbers decreasing to zero if and only if p,q,u,v,α,β satisfy one of the following conditions:*

(i) $v \leqslant q$, $\beta + \frac{1}{u} - \alpha - \frac{1}{2} \geqslant 0$;

(ii) $v > q$, $\beta + \frac{1}{u} - \alpha - \frac{1}{2} > 0$.

Proof. Based on [7, Theorem 8.1.2], $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(u,v,\beta)$ if and only if $\{a_n\}_{n\geqslant 0} \in D^{\beta+\frac{1}{u}-1}\ell(\infty, v)$, and $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(2,q,\alpha)$ if and only if $\{a_n\}_{n\geqslant 0} \in D^{\alpha-\frac{1}{2}}\ell(\infty, q)$.

Then, we only need to consider the embedding problem $D^{\beta+\frac{1}{u}-\alpha-\frac{1}{2}}\ell(\infty, v) \subset \ell(\infty, q)$. By [7, Lemma 11.1.1], we have

$$(\ell(\infty, v), \ell(\infty, q)) = \ell(\infty, v \ominus q).$$

If $v \leqslant q$, then $\ell(\infty, v\ominus q) = \ell^{\infty}$. Also,

$$\left\|\left\{\frac{1}{(n+1)^{\beta+\frac{1}{u}-\alpha-\frac{1}{2}}}\right\}_{n\geqslant 0}\right\|_{\ell^{\infty}} < \infty \; if\ and\ only\ if \; \beta + \frac{1}{u} - \alpha - \frac{1}{2} \geqslant 0.$$

If $v > q$, so

$$\begin{aligned}\left\|\left\{\frac{1}{(n+1)^{\beta+\frac{1}{u}-\alpha-\frac{1}{2}}}\right\}_{n\geqslant 0}\right\|_{\ell(\infty, v\ominus q)}^{v\ominus q} &= \sum_{n=0}^{\infty}\left(\sup_{k\in I_n}\frac{1}{(k+1)^{\beta+\frac{1}{u}-\alpha-\frac{1}{2}}}\right)^{v\ominus q}\\ &\simeq \sum_{n=0}^{\infty}\frac{1}{2^{n(\beta+\frac{1}{u}-\alpha-\frac{1}{2})v\ominus q}} < \infty\end{aligned}$$

if and only if $\beta + \frac{1}{u} - \alpha - \frac{1}{2} > 0$.

This theorem is completed. □

3. Random Power Series on $\mathcal{L}_0^{2,\lambda}$ Spaces

The Carleson type measure is an useful tool in the process of studying function space theory. Suppose I is an arc on $\mathbb{T} = \partial\mathbb{D}$ and $|I|$ is the normalized length of I. Based on I, the Carleson box is defined by

$$S(I) = \{z = re^{i\theta} : 1 - |I| \leqslant r < 1, e^{i\theta} \in I\}.$$

For $0 < s < \infty$, suppose μ defined on $\mathbb{D}$ is a positive Borel measure, μ is said to be a bounded s−Carleson measure, if there is a $C > 0$ satisfying

$$\mu(S(I)) \leqslant C|I|^p.$$

μ is called a vanishing s−Carleson measure, if the following

$$\mu(S(I)) = o(|I|^p)$$

holds as $|I| \to 0$. When $s = 1$, we obtain the classical Carleson measure.

Let $f \in H(\mathbb{D})$. It is well-known that $f \in \mathcal{L}^{2,\lambda}$ if and only if $d\mu(z) = |f'(z)|^2(1-|z|^2)dxdy$ is a bounded λ−Carleson measure, and $f \in \mathcal{L}_0^{2,\lambda}$ if and only if $d\mu(z) = |f'(z)|^2(1-|z|^2)dxdy$ is a vanishing λ−Carleson measure [12]. Remark that a bounded λ−Carleson measure is a finite measure; so for $0 < \lambda \leqslant 1$, one has

$$\sup_{a\in\mathbb{D}}(1-|a|^2)^{1-\lambda}\int_0^1(1-r^2)M_q^2(r, f')dr < \infty, 0 < q \leqslant 2. \tag{8}$$

On the reverse, if $f \in H(\mathbb{D})$, then the argument

$$\sup_{a\in\mathbb{D}}(1-|a|^2)^{1-\lambda}\int_0^1(1-r)M_\infty^2(r, f')dr < \infty \tag{9}$$

implies that $f \in \mathcal{L}^{2,\lambda}$ for $0 < \lambda < 1$, and

$$\int_0^1(1-r)M_\infty^2(r, f')dr < \infty \tag{10}$$

implies that $f \in BMOA$ [1].

The following lemma generalizes the general case for $0 < \lambda < 1$ of random power series in $\mathcal{L}^{2,\lambda}$. For $\lambda = 1$, it is given by Li and Wu [11].

Lemma 3.1. *Let $f(z) \in H(\mathbb{D})$ and $0 < \lambda < 1$. If*

$$\sup_{a\in\mathbb{D}}(1-|a|^2)^{1-\lambda}\int_0^1(1-r^2)M_\infty^2(r,f')dr < \infty. \tag{11}$$

Then $f \in \mathcal{L}_0^{2,\lambda}$.

Proof. For any $I \subset \partial\mathbb{D}$, one has

$$\int_I |f'(re^{i\theta})|^2 d\theta \leqslant M_\infty^2(r,f')|I|.$$

Let $d\mu(z) = |f'(z)|^2(1-|z|^2)dxdy$.

Consider

$$\begin{aligned}\sup_{a\in\mathbb{D}}(1-|a|^2)^{1-\lambda}\mu(S(I)) &\leqslant \sup_{a\in\mathbb{D}}(1-|a|^2)^{1-\lambda}\int_{1-|I|}^1\int_I(1-r^2)|f'(re^{i\theta})|^2d\theta dr\\ &\leqslant \sup_{a\in\mathbb{D}}(1-|a|^2)^{1-\lambda}|I|\int_{1-|I|}^1(1-r^2)M_\infty^2(r,f')dr.\end{aligned}$$

As we know, the inequality (11) implies that, for arbitrary $\varepsilon > 0$, there exists a $\delta, 0 < \delta < 1$ such that

$$\int_{1-\delta}^1(1-r^2)M_\infty^2(r,f')dr < \varepsilon.$$

Thus,

$$\sup_{a\in\mathbb{D}}(1-|a|^2)^{1-\lambda}\mu(S(I)) < |I|\varepsilon < |I|^\lambda\varepsilon$$

holds for any subarc I on $\partial\mathbb{D}$ with $|I| < \delta$.

This indicates that the measure $d\mu(z) = |f'(z)|^2(1-|z|^2)dxdy$ is a vanishing $\lambda-$Carleson measure, so we get $f \in \mathcal{L}_0^{2,\lambda}$. □

When $M_\infty^2(r,f')$ was replaced by $M_q^2(r,f')$ for $2 < q < \infty$, we have the following implication.

Lemma 3.2. *Let $0 < \lambda < 1$, $2 < q < \infty$, and $1 - \frac{2}{q} \geqslant \lambda$. For any $f(z) \in H(\mathbb{D})$, if*

$$\sup_{a\in\mathbb{D}}(1-|a|^2)^{1-\lambda}\int_0^1(1-r^2)M_q^2(r,f')dr < \infty, \tag{12}$$

Then $f \in \mathcal{L}_0^{2,\lambda}$.

Proof. For any $I \subset \partial\mathbb{D}$, employing Hölder's inequality, it follows that

$$\begin{aligned}\int_I\left|f'(re^{i\theta})\right|^2d\theta &\leqslant \left(\int_I\left|f'(re^{i\theta})\right|^q d\theta\right)^{\frac{2}{q}}\cdot\left(\int_I d\theta\right)^{1-\frac{2}{q}}\\ &\leqslant M_q^2(r,f')|I|^{1-\frac{2}{q}}\end{aligned}$$

for $2 < q < \infty$.

Let $d\mu(z) = |f'(z)|^2(1-|z|^2)dxdy$.

$$\begin{aligned}\sup_{a\in\mathbb{D}}(1-|a|^2)^{1-\lambda}\mu(S(I)) &\leqslant \sup_{a\in\mathbb{D}}(1-|a|^2)^{1-\lambda}\int_{1-|I|}^{1}\int_I(1-r^2)\left|f'(re^{i\theta})\right|^2 d\theta dr\\ &\leqslant \sup_{a\in\mathbb{D}}(1-|a|^2)^{1-\lambda}|I|^{1-\frac{2}{q}}\int_{1-|I|}^{1}(1-r^2)M_q^2(r,f')dr.\end{aligned}$$

The inequality (12) holds implies that there exists a positive number $\delta, 0<\delta<1$ satisfying

$$\sup_{a\in\mathbb{D}}(1-|a|^2)^{1-\lambda}\int_{1-\delta}^{1}(1-r^2)M_q^2(r,f')dr<\varepsilon.$$

for any $\varepsilon>0$. Hence, we get

$$\sup_{a\in\mathbb{D}}(1-|a|^2)^{1-\lambda}\mu(S(I))<|I|^{1-\frac{2}{q}}\varepsilon\leqslant|I|^{\lambda}\varepsilon$$

where $1-\frac{2}{q}\geqslant\lambda$ and the subarc I on $\partial\mathbb{D}$ with $|I|<\delta$ is arbitrary. We proof that the measure $d\mu(z) = (1-|z|^2)|f'(z)|^2dxdy$ is a vanishing $\lambda-$Carleson measure, so $f\in\mathcal{L}_0^{2,\lambda}$. □

remark. We must point out that the proof skills and methods of Propositions 4,6 of [11] are used in Lemmas 10 and 11.

The random power series in $\mathcal{L}_0^{2\lambda}$ for $0<\lambda<1$ are discussed in Theorem 3.1.

Theorem 3.3. *Let* $0<\lambda<1$. *If* $\sum_{n=0}^{\infty}|a_n|^2<\infty$, *then*

$$(\mathcal{R}f)(z)=\sum_{n=0}^{\infty}a_nX_nz^n\in\mathcal{L}_0^{2,\lambda}\ a.s.$$

where $f(z)=\sum_{n=0}^{\infty}a_nz^n\in H(\mathbb{D})$, *the standard random sequence* $\{X_n\}_{n\geqslant0}$ *is a standard Bernoulli.*

Proof. Assume that $\sum_{n=0}^{\infty}|a_n|^2<\infty$ for any $2<q<\infty$, using Fubini's theorem, Jensen's inequality and Khintchine's inequality, it follows that

$$\begin{aligned}&\mathbb{E}\left(\sup_{a\in\mathbb{D}}(1-|a|^2)^{1-\lambda}\int_0^1 M_q^2\left(r,(\mathcal{R}f)'\right)(1-r^2)dr\right)\\ &=\sup_{a\in\mathbb{D}}(1-|a|^2)^{1-\lambda}\int_0^1\mathbb{E}\left(\int_0^{2\pi}\left|\sum_{n=0}^{\infty}na_nX_nr^{n-1}e^{i(n-1)\theta}\right|^q\frac{d\theta}{2\pi}\right)^{\frac{2}{q}}(1-r^2)dr\\ &\leqslant\sup_{a\in\mathbb{D}}(1-|a|^2)^{1-\lambda}\int_0^1\left(\mathbb{E}\int_0^{2\pi}\left|\sum_{n=0}^{\infty}na_nX_nr^{n-1}e^{i(n-1)\theta}\right|^q\frac{d\theta}{2\pi}\right)^{\frac{2}{q}}(1-r^2)dr\\ &\lesssim\int_0^1\sum_{n=0}^{\infty}n^2|a_n|^2r^{2n-2}(1-r^2)dr\\ &=\sum_{n=0}^{\infty}n^2|a_n|^2\int_0^1 r^{2n-2}(1-r^2)dr.\end{aligned}$$

By [11, Lemma 3], one gets

$$\int_0^1 r^{2n-2}(1-r^2)dr=\int_0^1 r^{n-1}(1-r)dr\lesssim\frac{1}{n^2}$$

for any $n \geqslant 1$. Thus,

$$\mathbb{E}\left(\sup_{a\in\mathbb{D}}(1-|a|^2)^{1-\lambda}\int_0^1 M_q^2\left(r,(\mathcal{R}f)'\right)(1-r^2)dr\right)\lesssim\sum_{n=0}^{\infty}|a_n|^2<\infty.$$

This implies that

$$\sup_{a\in\mathbb{D}}(1-|a|^2)^{1-\lambda}\int_0^1 M_q^2\left(r,(\mathcal{R}f)'\right)(1-r^2)dr<\infty,\ a.s.$$

By Lemma 3.2, the implication $\mathcal{R}f\in\mathcal{L}_0^{2,\lambda}$ a.s. follows for fixed $q\geqslant\frac{2}{1-\lambda}$. □

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