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Comparative index and Hörmander index in finite dimension and their connections

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Abstract. In this paper we prove new relations between the comparative index and the Hörmander index (and the Maslov index) in the finite dimensional case. As a main result we derive an algebraic formula for calculating the Hörmander index of four given Lagrangian planes as a difference of two comparative indices involving certain transformed Lagrangian planes, or as a combination of four comparative indices. This result is based on a generalization of the comparison theorem for the Maslov index involving three Lagrangian paths. In this way we contribute to the recent efforts in the literature (by Zhou, Wu, Zhu in 2018 and by Howard in 2021) devoted to an efficient calculation of the Hörmander index in this finite dimensional case.

1. Introduction

Recently, there has been an intensive research activity in the study of the Maslov index $\operatorname{Mas}(Y, \hat{Y}, [a, b])$ of two Lagrangian paths Y and $\hat{Y}$ or in the study of the Hörmander index $s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2)$ associated with four Lagrangian planes $Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2 \in \Lambda(n)$, see [2, 16, 17, 19, 20, 23, 30] and the references given therein. Recall that for a fixed dimension $n \in \mathbb{N}$ the space of *Lagrangian planes* is defined as

$$\Lambda(n) := \{Y \in \mathbb{R}^{2n\times n},\ W(Y, Y) = 0,\ \operatorname{rank} Y = n\},$$

where $W(Y, \hat{Y}) \in \mathbb{R}^{n\times n}$ denotes the Wronskian of the two Lagrangian planes Y and $\hat{Y}$, i.e.,

$$W(Y, \hat{Y}) := Y^T \mathcal{J} \hat{Y}, \quad \mathcal{J} := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{1.1}$$

Here I and 0 denote the identity and zero matrices and $\mathcal{J} \in \mathbb{R}^{2n\times 2n}$ is the canonical skew-symmetric matrix. Each matrix $Y \in \Lambda(n)$ can be identified via its image with a Lagrangian subspace of $\mathbb{R}^{2n}$, which is spanned by the columns of Y. The matrix Y is then sometimes referred to as a *frame* of the Lagrangian subspace generated by Y, or as a *conjoined* or *isotropic* basis, see [5, 24]. Furthermore, a continuous function $Y : [a, b] \to \Lambda(n)$ is called a *Lagrangian path*.

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The Maslov index for the Lagrangian paths and the Hörmander index $s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2)$ are defined in a geometric way (see below), with the Hörmander index given as a difference of two Maslov indices involving a Lagrangian path $Y(t)$ connecting Y_1 and Y_2, which intersects with $\tilde{Y}_1$ or with $\tilde{Y}_2$. The efforts in the recent papers [16, 30] are directed to an efficient calculation of the Hörmander index in this finite dimensional case. In particular, in [30, Theorem 1.1] the authors present the Hörmander index, denoted here by $s_Z(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2)$, in terms of the triple index defined by [6, Eq. (5)], as

$$s_Z(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = i(Y_1, Y_2, \tilde{Y}_2) - i(Y_1, Y_2, \tilde{Y}_1) = i(Y_1, \tilde{Y}_1, \tilde{Y}_2) - i(Y_2, \tilde{Y}_1, \tilde{Y}_2). \tag{1.2}$$

Recall (see [6]) that the definition of the triple index $i(\alpha, \beta, \gamma)$ with $\alpha, \beta, \gamma \subseteq \mathbb{R}^{2n}$ being Lagrangian subspaces uses the bilinear form $Q(\alpha, \beta, \gamma)$ defined on the subspace $\alpha \cap (\beta + \gamma)$, as well as it uses the information about the dimensions of the intersections $\alpha \cap \gamma$ and $\alpha \cap \beta \cap \gamma$, see [30, Lemma 3.13] for more details. However, taking in mind the recent applications of the Maslov index in the oscillation and spectral theory of linear Hamiltonian differential systems (see [11, 15, 16, 18–20]), where the main results are formulated in terms of the frames of Lagrangian paths, it seems natural to present connections between the Maslov indices for different paths, and in particular the Hörmander index, in terms of the frames Y_1, Y_2, $\tilde{Y}_1$, $\tilde{Y}_2$. According to our knowledge, the representations of the Hörmander index in terms of the frames Y_1, Y_2, $\tilde{Y}_1$, $\tilde{Y}_2$ are known in this situation only for special cases associated with different transversality conditions for the Lagrangian planes, meaning that some blocks of Y_1, Y_2, $\tilde{Y}_1$, $\tilde{Y}_2$ or/and their Wronskians (1.1) are nonsingular, see [20, Lemma 2.3, Corollary 1] and [16, Section 3].

The aim of this paper is to offer a convenient algebraic tool, which we call the *comparative index* (see [9] or [7, Chapter 3]), presenting connections between the Maslov indices for three Lagrangian paths Y_1, Y_2, Y_3 in terms of the frames $Y_1(a)$, $Y_2(a)$, $Y_3(a)$ and $Y_1(b)$, $Y_2(b)$, $Y_3(b)$ defined by their endpoint values. More precisely, for the Lagrangian paths Y_1, Y_2, Y_3 on $[a, b]$ we consider a continuous symplectic matrix $Z_1(t)$ satisfying $Y_1(t) = Z_1(t)\,(0\; I)^T$ on $[a, b]$. Then we prove the formula, see Theorem 2.2,

$$\begin{aligned} &\operatorname{Mas}(Y_1, Y_2, [a, b]) + \operatorname{Mas}(Y_2, Y_3, [a, b]) - \operatorname{Mas}(Y_1, Y_3, [a, b]) \\ &\quad = \mu\Big(Z_1^{-1}(a)\, Y_3(a),\, Z_1^{-1}(a)\, Y_2(a)\Big) - \mu\Big(Z_1^{-1}(b)\, Y_3(b),\, Z_1^{-1}(b)\, Y_2(b)\Big). \end{aligned}$$

The numbers $\mu\Big(Z_1^{-1}(t) Y_3(t),\, Z_1^{-1}(t) Y_2(t)\Big)$ for $t \in \{a, b\}$ are defined by the Wronskians involving $Y_1(t)$, $Y_2(t)$, $Y_3(t)$ for $t \in \{a, b\}$ and they do not depend on the choice of the matrix $Z_1(t)$, for which $Y_1(t)$ forms its second block column according to the above definition. The number $\mu(Y, \hat{Y})$ is defined for arbitrary Lagrangian planes Y and $\hat{Y}$ and it is called the *comparative index*, see [9] or [7, Chapter 3] and Section 2.1 for more details. It has useful applications in the oscillation and spectral theory of linear Hamiltonian systems and their discrete analogs – symplectic difference systems, see [7, 10–12, 26, 27] and the references therein.

For the special case when $Y_1(t) := Y(t)$, $Y_2(t) := \tilde{Y}(b)$, $Y_3(t) := \tilde{Y}(a)$ and using the formula for the Hörmander index $s(Y(a), Y(b), \tilde{Y}(a), \tilde{Y}(b))$ as the difference of the Maslov indices $\operatorname{Mas}(Y, \tilde{Y}(b), [a, b])$ and $\operatorname{Mas}(Y, \tilde{Y}(a), [a, b])$ (see [16, Section 3], [30, Section 3], and equation (3.5) below) we derive from Theorem 2.2 the representation of the Hörmander index in terms of the frames $Y(a)$, $Y(b)$, $\tilde{Y}(a)$, $\tilde{Y}(b)$ as

$$s(Y(a), Y(b), \tilde{Y}(a), \tilde{Y}(b)) = \mu\Big(Z^{-1}(a)\, \tilde{Y}(a),\, Z^{-1}(a)\, \tilde{Y}(b)\Big) - \mu\Big(Z^{-1}(b)\, \tilde{Y}(a),\, Z^{-1}(b)\, \tilde{Y}(b)\Big),$$

where $Z(t)$ is a continuous symplectic matrix on $[a, b]$ having $Y(t)$ as its second block column (see Theorem 3.2). Alternatively, we obtain the formula

$$s(Y(a), Y(b), \tilde{Y}(a), \tilde{Y}(b)) = \mu(\tilde{Y}(b), Y(a)) - \mu(\tilde{Y}(b), Y(b)) - \mu(\tilde{Y}(a), Y(a)) + \mu(\tilde{Y}(a), Y(b)).$$

We also consider several applications of Theorem 3.2, which are based on the properties of the comparative index. These include general estimates for the Hörmander index, explicit conditions for its extreme values or its sign, or connections of the comparative index with the triple index (Corollaries 3.7, 3.9, and 3.13). Therefore, we trust that these results are a useful complement to the geometric approach to the Hörmander index in [6, 16, 30].

The organization of the paper is the following. In Section 2 we present the connections of the comparative index with the Maslov index, including the proof of the above mentioned Theorem 2.2. In Section 3 we study the relations between the comparative index and the Hörmander index and present several applications of Theorem 3.2, including a connection between the comparative index and the triple index. Finally, in Section 4 we make some additional comments about the results of this paper and their further development.

2. Comparative index and Maslov index in finite dimension

2.1. Comparative index

By [8, 9] or [7, Chapter 3], for two Lagrangian planes $Y_1, Y_2 \in \Lambda(n)$ we define their *comparative index* $\mu(Y_1, Y_2)$ and the *dual comparative index* $\mu^*(Y_1, Y_2)$ by

$$\mu(Y_1, Y_2) := \operatorname{rank} \mathcal{M} + \operatorname{ind} \mathcal{P}, \quad 0 \le \mu(Y_1, Y_2) \le n, \tag{2.1}$$
$$\mu^*(Y_1, Y_2) := \operatorname{rank} \mathcal{M} + \operatorname{ind} (-\mathcal{P}), \quad 0 \le \mu^*(Y_1, Y_2) \le n, \tag{2.2}$$

where the matrices $\mathcal{M}$ and $\mathcal{P}$ are defined by

$$\mathcal{M} := (I - X_1^\dagger X_1)\, W(Y_1, Y_2), \quad \mathcal{P} := V\, [W(Y_1, Y_2)]^T X_1^\dagger X_2\, V, \quad V := I - \mathcal{M}^\dagger \mathcal{M}, \tag{2.3}$$

and where X_1 and X_2 are the upper $n \times n$ blocks of Y_1 and Y_2. Here we use the partitions

$$Y_1 = (X_1^T,\, U_1^T)^T, \quad Y_2 = (X_2^T,\, U_2^T)^T. \tag{2.4}$$

Note that the matrix $\mathcal{P}$ is symmetric according to [9, Theorem 2.1] or [7, Theorem 3.2(iii)]. The dagger in (2.3) denotes the Moore–Penrose pseudoinverse, see e.g. [1, 3].

The comparative index and the dual comparative index defined in (2.1) and (2.2) satisfy, among other properties, the relations

$$\mu(Y_1, Y_2) + \mu(Y_2, Y_1) = \operatorname{rank} W(Y_1, Y_2) = \mu^*(Y_1, Y_2) + \mu^*(Y_2, Y_1), \tag{2.5}$$
$$\mu(Y_1, Y_2) + \mu^*(Y_1, Y_2) = \operatorname{rank} W(Y_1, Y_2) - \operatorname{rank} X_1 + \operatorname{rank} X_2, \tag{2.6}$$
$$\mu(Z_1 (0\ I)^T, Z_2 (0\ I)^T) = \mu^*(Z_1^{-1} (0\ I)^T, Z_1^{-1} Z_2 (0\ I)^T), \tag{2.7}$$

where Z_1 and Z_2 are arbitrary $2n \times 2n$ symplectic matrices. These properties are proven in [9, pg. 448] or in [7, Theorem 3.5]. In addition, if the upper blocks in (2.4) are invertible, then the comparative index of Y_1 and Y_2 reduces to the index of the difference of the associated Riccati quotients, i.e.,

$$\mu(Y_1, Y_2) = \operatorname{ind} (Q_2 - Q_1), \quad \mu^*(Y_1, Y_2) = \operatorname{ind} (Q_1 - Q_2), \quad Q_j := U_j X_j^{-1}. \tag{2.8}$$

These formulas are easily obtained from (2.1), (2.2), and (2.3).

If we denote the vertical Lagrangian plane (also called the Dirichlet Lagrangian plane) and the horizontal Lagrangian plane (also called the Neuman Lagrangian plane) by

$$E := \begin{pmatrix} 0 & I \end{pmatrix}^T, \quad N := \begin{pmatrix} I & 0 \end{pmatrix}^T, \tag{2.9}$$

then we obtain for every Lagrangian plane $Y \in \Lambda(n)$ the expressions

$$\mu(Y, E) = 0, \quad \mu^*(Y, E) = 0, \tag{2.10}$$
$$\mu(Y, N) = n - \operatorname{rank} X + \operatorname{ind} (-X^T U), \quad \mu^*(Y, N) = n - \operatorname{rank} X + \operatorname{ind} (X^T U), \tag{2.11}$$

which are easily obtained from (2.1), (2.2), (2.5), and (2.6).

2.2. Maslov index

For the definition of the Maslov index of two Lagrangian paths Y and $\hat{Y}$ on $[a,b]$ we will utilize the continuous angles $\varphi_j(t)$ of the eigenvalues $\gamma_j(t) = \exp(i\varphi_j(t))$ for $j \in \{1,\dots,n\}$ of the unitary matrix

$$\Gamma(t) := [X(t) + i\,U(t)]\,[X(t) - i\,U(t)]^{-1}\,[\hat{X}(t) - i\,\hat{U}(t)]\,[\hat{X}(t) + i\,\hat{U}(t)]^{-1}$$

for $t \in [a,b]$. The matrices X, U and $\hat{X}$, $\hat{U}$ are the $n\times n$ blocks of Y and $\hat{Y}$, which are defined on $[a,b]$ according to the notation introduced in (2.4). This approach to the Maslov index is known e.g. in [2, 16, 18, 19, 30], see also [22, 23, 25]. Equivalently we may use the Lidskii angles of the symplectic orthogonal matrix $S(t) := Z_Y^T(t)\,Z_{\hat{Y}}(t)$, where $Z_Y(t)$ and $Z_{\hat{Y}}(t)$ are the symplectic and orthogonal matrices associated with $Y(t)$ and $\hat{Y}(t)$ through the formula

$$Z_Y(t) := \begin{pmatrix} \mathcal{J}\,Y(t)K_Y(t) & Y(t)K_Y(t) \end{pmatrix}, \quad K_Y(t) := [Y^T(t)\,Y(t)]^{-1/2}, \quad t \in [a,b], \tag{2.12}$$

see [28, 29] for the notion of Lidskii angles of a symplectic matrix. In particular, we know that the numbers $\gamma_j(t) = \exp(i\varphi_j(t))$ are equal to the eigenvalues of the unitary matrix

$$W(t) := K_{\hat{Y}}^{-1}(t)\,[\,Y^T(t)\,\hat{Y}(t) + i\,W(Y(t),\hat{Y}(t))]^{-1}\,[\,Y^T(t)\,\hat{Y}(t) - i\,W(Y(t),\hat{Y}(t))]\,K_{\hat{Y}}(t),$$

as the matrices $\Gamma(t)$ and $W(t)$ are similar by [15, Lemma 4.1]. Then the *Maslov index* of the Lagrangian paths Y and $\hat{Y}$ is defined by

$$\operatorname{Mas}(Y,\hat{Y},[a,b]) := \sum_{j=1}^{n} \left(\left\lfloor \frac{\varphi_j(b)}{2\pi} \right\rfloor - \left\lfloor \frac{\varphi_j(a)}{2\pi} \right\rfloor \right), \tag{2.13}$$

where for $x \in \mathbb{R}$ the notation $\lfloor x \rfloor$ stands for the greatest integer which is smaller or equal to x (the floor function), see [2, Section 2.2], [30, Definition 2.2], and [15, Theorem 4.2]. Similarly, the *dual Maslov index* of the Lagrangian paths Y and $\hat{Y}$ is defined by

$$\operatorname{Mas}^*(Y,\hat{Y},[a,b]) := \sum_{j=1}^{n} \left(\left\lceil \frac{\varphi_j(b)}{2\pi} \right\rceil - \left\lceil \frac{\varphi_j(a)}{2\pi} \right\rceil \right), \tag{2.14}$$

where $\lceil x \rceil$ stands for the smallest integer which is greater or equal to x (the ceiling function), see also [15, Remark 4.5]. Moreover, by [15, Eq. (4.21)] we have the duality relation

$$\operatorname{Mas}^*(Y,\hat{Y},[a,b]) = -\operatorname{Mas}(\hat{Y},Y,[a,b]). \tag{2.15}$$

We note that the Maslov indices Mas and Mas^* in (2.13) and (2.14) coincide, respectively, with the Maslov indices Mas_- and Mas_+ considered in [30, Definition 2.2] and [2, Section 2.2].

In the special case, when the Lagrangian path Y is constant and equal to the vertical plane E defined in (2.9), the above Maslov index $\operatorname{Mas}(E,\hat{Y},[a,b])$ and the dual Maslov index $\operatorname{Mas}^*(E,\hat{Y},[a,b])$ reduce respectively to the oscillation number $\mathcal{N}(\hat{Y},[a,b])$ and the dual oscillation number $\mathcal{N}^*(\hat{Y},[a,b])$ of the Lagrangian path $\hat{Y}$. These notions are developed in [11–13] and in [15, Sections 3 and 4] by means of the comparative index theory. Consequently, we derived in [15, Corollary 5.4 and Remark 5.5] the following comparison results for the Maslov index of two Lagrangian paths Y and $\hat{Y}$, which involve the comparative index of Y and $\hat{Y}$ evaluated at the endpoints of the interval $[a,b]$.

Proposition 2.1. *Let Y and $\hat{Y}$ be given Lagrangian paths on $[a,b]$. Then we have*

$$\left.\begin{aligned} \operatorname{Mas}(Y,\hat{Y},[a,b]) &= \operatorname{Mas}(E,\hat{Y},[a,b]) - \operatorname{Mas}(E,Y,[a,b]) \\ &\quad + \mu(\hat{Y}(a),Y(a)) - \mu(\hat{Y}(b),Y(b)), \end{aligned}\right\} \tag{2.16}$$

$$\left.\begin{aligned} \operatorname{Mas}^*(Y,\hat{Y},[a,b]) &= \operatorname{Mas}^*(E,\hat{Y},[a,b]) - \operatorname{Mas}^*(E,Y,[a,b]) \\ &\quad + \mu^*(\hat{Y}(b),Y(b)) - \mu^*(\hat{Y}(a),Y(a)). \end{aligned}\right\} \tag{2.17}$$

By the symplectic invariance of the Maslov index, see e.g. [4, Property V in Section 1], for an arbitrary continuous symplectic matrix-valued function S on $[a,b]$ we have the relation

$$\left.\begin{aligned} \operatorname{Mas}(SY, S\hat{Y}, [a,b]) &= \operatorname{Mas}(Y, \hat{Y}, [a,b]), \\ \operatorname{Mas}^*(SY, S\hat{Y}, [a,b]) &= \operatorname{Mas}^*(Y, \hat{Y}, [a,b]). \end{aligned}\right\} \tag{2.18}$$

By using identity (2.18) we derive the following generalization of Proposition 2.1 to three Lagrangian paths Y_1, Y_2, Y_3 on $[a,b]$. For a given Lagrangian path Y_j on $[a,b]$ we consider a continuous symplectic matrix Z_j defined on $[a,b]$ with the property $Y_j(t) = Z_j(t)E$ on $[a,b]$, i.e., the matrix $Y_j(t)$ forms the second blocks column of $Z_j(t)$. Such a matrix function $Z = (\bar{Y},\ Y)$ always exists, in particular we can complete any Lagrangian path Y by another Lagrangian path $\bar{Y} := \mathcal{J} Y K_Y^2$ to the so-called normalized pair of Lagrangian paths satisfying $W(\bar{Y}(t), Y(t)) = I$ on $[a,b]$, where the invertible matrix $K_Y(t)$ is defined in (2.12). In this way the continuous symplectic matrix

$$Z(t) = \left(\mathcal{J}\, Y(t) K_Y^2(t), \quad Y(t)\right), \quad t \in [a,b], \tag{2.19}$$

can be determined only by the Lagrangian path Y, see e.g. [21, Corollary 3.3.9]. Moreover, by (2.19) and the formula for the inverse of a symplectic matrix we obtain for another Lagrangian path $\hat{Y}$ that

$$Z^{-1}(t)\,\hat{Y}(t) = -\mathcal{J} Z^T(t) \mathcal{J}\,\hat{Y}(t) \overset{(2.19)}{=} \begin{pmatrix} -W(Y(t), \hat{Y}(t)) \\ K_Y^2(t)\, Y^T(t)\, \hat{Y}(t) \end{pmatrix}, \quad t \in [a,b]. \tag{2.20}$$

It can be shown (see [15, Remark 2.2 and Theorem 3.12]) that the results below are invariant with respect to the choices of the matrices $Z_j(t)$ satisfying $Y_j(t) = Z_j(t)E$ for $j \in \{1,2,3\}$. In particular, the matrices $Z_j(t)$ can be chosen in the form of (2.19). Expressions of the form (2.20) can be used in the following comparison result.

Proposition 2.2. *Let Y_1, Y_2, Y_3 be given Lagrangian paths on $[a,b]$, with their associated continuous symplectic matrices Z_j satisfying $Y_j(t) = Z_j(t)E$ on $[a,b]$ for $j \in \{1,2,3\}$. Then*

$$\left.\begin{aligned} &\operatorname{Mas}(Y_1, Y_2, [a,b]) + \operatorname{Mas}(Y_2, Y_3, [a,b]) - \operatorname{Mas}(Y_1, Y_3, [a,b]) \\ &\quad = \mu\Big(Z_1^{-1}(a)\, Y_3(a), Z_1^{-1}(a)\, Y_2(a)\Big) - \mu\Big(Z_1^{-1}(b)\, Y_3(b), Z_1^{-1}(b)\, Y_2(b)\Big), \end{aligned}\right\} \tag{2.21}$$

$$\left.\begin{aligned} &\operatorname{Mas}^*(Y_1, Y_2, [a,b]) + \operatorname{Mas}^*(Y_2, Y_3, [a,b]) - \operatorname{Mas}^*(Y_1, Y_3, [a,b]) \\ &\quad = \mu^*\Big(Z_1^{-1}(b)\, Y_3(b), Z_1^{-1}(b)\, Y_2(b)\Big) - \mu^*\Big(Z_1^{-1}(a)\, Y_3(a), Z_1^{-1}(a)\, Y_2(a)\Big). \end{aligned}\right\} \tag{2.22}$$

Proof. We consider the Lagrangian paths $Y := Z_1^{-1} Y_2$ and $\hat{Y} := Z_1^{-1} Y_3$ on $[a,b]$. Then by formula (2.16) in Proposition 2.1 we have

$$\begin{aligned} \operatorname{Mas}(Z_1^{-1}Y_2, Z_1^{-1}Y_3, [a,b]) &= \operatorname{Mas}(E, Z_1^{-1}Y_3, [a,b]) - \operatorname{Mas}(E, Z_1^{-1}Y_2, [a,b]) \\ &\quad + \mu\Big(Z_1^{-1}(a)Y_3(a), Z_1^{-1}(a)Y_2(a)\Big) - \mu\Big(Z_1^{-1}(b)Y_3(b), Z_1^{-1}(b)Y_2(b)\Big). \end{aligned}$$

Applying the symplectic invariance, i.e., formula (2.18) with the matrix $S := Z_1^{-1}$, to all Maslov indices in the above formula and incorporating that $Y_1(t) = Z_1(t)E$ on $[a,b]$ we derive equation (2.21). The proof of equation (2.22) is based on formulas (2.17) and (2.18) in a similar way. □

Remark 2.3. (i) The comparative indices on the right-hand sides of (2.21) and (2.22) are uniquely defined by the Wronskians $W(Y_1, Y_2)$, $W(Y_1, Y_3)$, and $W(Y_2, Y_3)$ and they do not depend on the choice of the matrices $Z_j(t)$ with $Y_j(t) = Z_j(t)E$ for $j \in \{1,2,3\}$. Indeed, by using the representation (compare with (2.20))

$$W(Y_k, Y_j) = -(I,\ 0)\, Z_k^{-1} Y_j, \quad Y_k = Z_k E,$$

and by (2.1), (2.2), and (2.3) we have (suppressing the argument $t \in \{a, b\}$)

$$\mu(Z_1^{-1}Y_3, Z_1^{-1}Y_2) = \mu_1 + \mu_2, \quad \mu^*(Z_1^{-1}Y_3, Z_1^{-1}Y_2) = \mu_1 + \mu_2^*,$$

where

$$\mu_1 = \mu_1(Z_1^{-1}Y_3, Z_1^{-1}Y_2) := \operatorname{rank} \mathcal{M}, \quad \mathcal{M} = \big(I - [W(Y_1, Y_3)]^\dagger W(Y_1, Y_3)\big) W(Y_3, Y_2),$$

while with $V = I - \mathcal{M}^\dagger \mathcal{M}$ we have

$$\mu_2 = \mu_2(Z_1^{-1}Y_3, Z_1^{-1}Y_2) := \operatorname{ind} \mathcal{P}, \quad \mu_2^* = \mu_2^*(Z_1^{-1}Y_3, Z_1^{-1}Y_2) := \operatorname{ind}(-\mathcal{P}),$$
$$\mathcal{P} = V [W(Y_3, Y_2)]^T [W(Y_1, Y_3)]^\dagger W(Y_1, Y_2) V.$$

In the above expressions we also simplified the Wronskian $W(Z_1^{-1}Y_3, Z_1^{-1}Y_2) = W(Y_3, Y_2)$.

(ii) The comparative indices on the right-hand sides of (2.21) and (2.22) can be presented respectively in terms of $\mu(Y_3, Y_2)$, $\mu(Y_2, Y_1)$, $\mu(Y_3, Y_1)$ and $\mu^*(Y_3, Y_2)$, $\mu^*(Y_2, Y_1)$, $\mu^*(Y_3, Y_1)$ by the main theorem of the comparative index theory, see [9, Theorem 2.2, Eq. (2.14), (2.15)] or [7, Theorem 3.5, Corollary 3.12, Eq. (3.17), (3.26)]). More precisely, for an arbitrary $2n \times 2n$ symplectic matrix S and for any Lagrangian planes Y and $\hat{Y}$ we have

$$\mu(S^{-1}Y, S^{-1}\hat{Y}) = \mu(Y, \hat{Y}) + \mu(\hat{Y}, SE) - \mu(Y, SE), \tag{2.23}$$
$$\mu^*(S^{-1}Y, S^{-1}\hat{Y}) = \mu^*(Y, \hat{Y}) + \mu^*(\hat{Y}, SE) - \mu^*(Y, SE). \tag{2.24}$$

For example, by taking $Y := Y_3$, $\hat{Y} := Y_2$, and $S := Z_1$ in (2.23) and (2.24) we deduce that

$$\mu(Z_1^{-1}Y_3, Z_1^{-1}Y_2) = \mu(Y_3, Y_2) + \mu(Y_2, Y_1) - \mu(Y_3, Y_1), \tag{2.25}$$
$$\mu^*(Z_1^{-1}Y_3, Z_1^{-1}Y_2) = \mu^*(Y_3, Y_2) + \mu^*(Y_2, Y_1) - \mu^*(Y_3, Y_1). \tag{2.26}$$

The sums of the comparative indices on the right-hand sides of (2.25) and (2.26) are special cases of *cyclic sums* of comparative indices (of the second kind). Such cyclic sums are investigated in the recent paper [14], where they are denoted by $\nu_c^-(Y_3, Y_2, Y_1)$ for (2.25), resp. by $\nu_c^+(Y_3, Y_2, Y_1)$ for (2.26). We can see from (2.25) and (2.26) that these cyclic sums possess the symplectic invariance property $\nu_c^\mp(SY_3, SY_2, SY_1) = \nu_c^\mp(Y_3, Y_2, Y_1)$, compare with (2.18).

The results in Propositions 2.1 and 2.2 are fundamental for the connections of the comparative index with the Maslov index and with the Hörmander index presented below and in the next section. Observe that if $Y_1 := E$ is the constant vertical Lagrangian path (i.e., the matrix $Z_1(t) \equiv I$), then the result in Proposition 2.2 reduces exactly to that in Proposition 2.1.

Remark 2.4. Based on Proposition 2.2 we can provide useful estimates of the expressions on left-hand side of (2.21) and (2.22), namely

$$\big|\operatorname{Mas}(Y_1, Y_2, [a, b]) + \operatorname{Mas}(Y_2, Y_3, [a, b]) - \operatorname{Mas}(Y_1, Y_3, [a, b])\big| \le n, \tag{2.27}$$
$$\big|\operatorname{Mas}^*(Y_1, Y_2, [a, b]) + \operatorname{Mas}^*(Y_2, Y_3, [a, b]) - \operatorname{Mas}^*(Y_1, Y_3, [a, b])\big| \le n. \tag{2.28}$$

These general estimates are based on the simplest bounds for the comparative index and the dual comparative index shown in (2.1) and (2.2), and they are independent on the chosen Lagrangian paths Y_1, Y_2, Y_3. We note that more precise estimates for the lower and upper bounds for $\mu(Y_1, Y_2)$ and $\mu^*(Y_1, Y_2)$ are presented in [9, Property 7, pg. 449] and [7, Theorem 3.5(vii) and Remark 3.10] in terms of the quantities $\operatorname{rank} X_1$, $\operatorname{rank} X_2$, and $\operatorname{rank} W(Y_1, Y_2)$. Therefore, the estimates in (2.27) and (2.28) can be improved in the same spirit.

3. Comparative index and Hörmander index in finite dimension

3.1. Hörmander index

The following definition of the Hörmander index is motivated by [16, Section 3] and [30, Section 3]. Let us fix four Lagrangian planes $Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2 \in \Lambda(n)$. We consider two Lagrangian paths Y and $\tilde{Y}$ on $[a,b]$ connecting the Lagrangian plane Y_1 with Y_2, and the Lagrangian plane $\tilde{Y}_1$ with $\tilde{Y}_2$. That is, we have

$$Y(a) = Y_1, \quad Y(b) = Y_2, \quad \tilde{Y}(a) = \tilde{Y}_1, \quad \tilde{Y}(b) = \tilde{Y}_2. \tag{3.1}$$

Then, by using formula (2.16), we can easily derive the relation

$$\operatorname{Mas}(Y, \tilde{Y}_1, [a,b]) + \operatorname{Mas}(Y_2, \tilde{Y}, [a,b]) - \operatorname{Mas}(Y, \tilde{Y}_2, [a,b]) - \operatorname{Mas}(Y_1, \tilde{Y}, [a,b]) = 0, \tag{3.2}$$

where we used that $\operatorname{Mas}(E, C, [a,b]) = 0$ for every $C \in \Lambda(n)$, as the Maslov index of two constant Lagrangian paths (i.e., of two Lagrangian planes) is zero. Formula (3.2) can be interpreted by two combined Lagrangian paths in the arguments $t \in [a,b]$ and $\tilde{t} \in [a,b]$, whose Maslov index is zero by the homotopy invariance, see e.g. the Maslov box in [16, Figure 2]. Equation (3.2) implies that

$$\operatorname{Mas}(Y, \tilde{Y}_2, [a,b]) - \operatorname{Mas}(Y, \tilde{Y}_1, [a,b]) = \operatorname{Mas}(Y_2, \tilde{Y}, [a,b]) - \operatorname{Mas}(Y_1, \tilde{Y}, [a,b]), \tag{3.3}$$

where the left-hand side depends on the endpoint values of $\tilde{Y}$ (but not on $\tilde{Y}$ itself) and the right-hand side depends on the endpoint values of Y (but not on Y itself). Equation (3.3) then shows that the difference $\operatorname{Mas}(Y, \tilde{Y}_2, [a,b]) - \operatorname{Mas}(Y, \tilde{Y}_1, [a,b])$ does not depend on the choice of the Lagrangian path Y with $Y(a) = Y_1$ and $Y(b) = Y_2$, and at the same time the difference $\operatorname{Mas}(Y_2, \tilde{Y}, [a,b]) - \operatorname{Mas}(Y_1, \tilde{Y}, [a,b])$ does not depend on the choice of the Lagrangian path $\tilde{Y}$ with $\tilde{Y}(a) = \tilde{Y}_1$ and $\tilde{Y}(b) = \tilde{Y}_2$, and that these two differences are equal.

The *Hörmander index* of the Lagrangian planes $Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2$ is now defined as the integer

$$s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) := \operatorname{Mas}(Y, \tilde{Y}_2, [a,b]) - \operatorname{Mas}(Y, \tilde{Y}_1, [a,b]) \tag{3.4}$$

$$= \operatorname{Mas}(Y_2, \tilde{Y}, [a,b]) - \operatorname{Mas}(Y_1, \tilde{Y}, [a,b]), \tag{3.5}$$

where the definitions in (3.4) and (3.5) do not depend on the choice of the Lagrangian paths Y and $\tilde{Y}$ with (3.1), as we discussed above. Similarly, by using formula (2.17) for the dual Maslov index and the dual comparative index we easily obtain the equality

$$\operatorname{Mas}^*(Y, \tilde{Y}_2, [a,b]) - \operatorname{Mas}^*(Y, \tilde{Y}_1, [a,b]) = \operatorname{Mas}^*(Y_2, \tilde{Y}, [a,b]) - \operatorname{Mas}^*(Y_1, \tilde{Y}, [a,b]),$$

which leads to the definition of the *dual Hörmander index* as the integer

$$s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) := \operatorname{Mas}^*(Y, \tilde{Y}_2, [a,b]) - \operatorname{Mas}^*(Y, \tilde{Y}_1, [a,b]) \tag{3.6}$$

$$= \operatorname{Mas}^*(Y_2, \tilde{Y}, [a,b]) - \operatorname{Mas}^*(Y_1, \tilde{Y}, [a,b]). \tag{3.7}$$

We note that it may seem artificial to introduce two Hörmander indices by the above equations. But in some situations we will take advantage of working with both Hörmander index $s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2)$ and dual Hörmander index $s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2)$, see e.g. the proof of Corollary 3.13.

Remark 3.1. Observe that the Hörmander index considered in [30, Definition 3.9], denote it by s_Z, is equal to the dual Hörmander index s^* defined in (3.6) and (3.7), since the ceiling function is used in the definition of the corresponding dual Maslov index in (2.14) as well as in [30, Definition 3.9] with the same angles $\varphi_j(t)$. On the other hand, the Hörmander index considered in [16, Section 3], denote it by s_H, is equal to $-s^*$, since it is defined by the Maslov index in (2.13) with roles of the pairs Y_1, Y_2 and $\tilde{Y}_1, \tilde{Y}_2$ interchanged. More precisely, according to [16, Eq. (3.2)–(3.3)] we have

$$s_H(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) := \operatorname{Mas}(\tilde{Y}, Y_2, [a,b])) - \operatorname{Mas}(\tilde{Y}, Y_1, [a,b])$$

$$\overset{(2.15)}{=} -\operatorname{Mas}^*(Y_2, \tilde{Y}, [a,b]) + \operatorname{Mas}^*(Y_1, \tilde{Y}, [a,b]) \overset{(3.7)}{=} -s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2),$$

that is, the Hörmander index s_H from [16, Section 3] satisfies $s_H = -s^* = -s_Z$.

Formulas (3.4) and (3.6), resp. (3.5) and (3.7), suggest the interpretation the Hörmander index as the correction term in the target exchange on the second position in the Maslov index, resp. on the first position in the Maslov index. Moreover, the symplectic invariance of the Maslov index in (2.18) implies the same property for the Hörmander index, i.e., for any $2n \times 2n$ symplectic matrix S we have

$$\left.\begin{aligned} s(SY_1, SY_2, S\tilde{Y}_1, S\tilde{Y}_2) &= s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2), \\ s^*(SY_1, SY_2, S\tilde{Y}_1, S\tilde{Y}_2) &= s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2). \end{aligned}\right\} \tag{3.8}$$

In the following theorem we show that the Hörmander index and the dual Hörmander index can be calculated from the data $Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2$ by means of linear algebra (matrix analysis) by evaluating the involved comparative indices and the dual comparative indices through (2.1) and (2.2) with the corresponding matrices in (2.3). It is the main result of this section.

Theorem 3.2. *Let $Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2 \in \Lambda(n)$ be given Lagrangian planes with the associated symplectic matrices Z_1, Z_2 such that $Y_j = Z_j E$ for $j \in \{1, 2\}$. Then the Hörmander index defined in* (3.4)– (3.5) *is equal to*

$$s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = \mu(Z_1^{-1}\tilde{Y}_1, Z_1^{-1}\tilde{Y}_2) - \mu(Z_2^{-1}\tilde{Y}_1, Z_2^{-1}\tilde{Y}_2), \tag{3.9}$$

$$s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = \mu(Z_2^{-1}\tilde{Y}_2, Z_2^{-1}\tilde{Y}_1) - \mu(Z_1^{-1}\tilde{Y}_2, Z_1^{-1}\tilde{Y}_1), \tag{3.10}$$

$$s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = \mu(\tilde{Y}_2, Y_1) - \mu(\tilde{Y}_2, Y_2) - \mu(\tilde{Y}_1, Y_1) + \mu(\tilde{Y}_1, Y_2), \tag{3.11}$$

and the dual Hörmander index defined in (3.6)– (3.7) *is equal to*

$$s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = \mu^*(Z_2^{-1}\tilde{Y}_1, Z_2^{-1}\tilde{Y}_2) - \mu^*(Z_1^{-1}\tilde{Y}_1, Z_1^{-1}\tilde{Y}_2), \tag{3.12}$$

$$s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = \mu^*(Z_1^{-1}\tilde{Y}_2, Z_1^{-1}\tilde{Y}_1) - \mu^*(Z_2^{-1}\tilde{Y}_2, Z_2^{-1}\tilde{Y}_1), \tag{3.13}$$

$$s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = \mu^*(\tilde{Y}_2, Y_2) - \mu^*(\tilde{Y}_2, Y_1) - \mu^*(\tilde{Y}_1, Y_2) + \mu^*(\tilde{Y}_1, Y_1). \tag{3.14}$$

Moreover, the Hörmander index and the dual Hörmander index are related by the formula

$$s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = -s(\tilde{Y}_1, \tilde{Y}_2, Y_1, Y_2). \tag{3.15}$$

Proof. Let the matrices $Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2$ and Z_1, Z_2 be as in the theorem. Let $Z(t)$ be a continuous symplectic matrix on $[a, b]$ connecting the matrices Z_1 and Z_2, i.e., $Z(a) = Z_1$ and $Z(b) = Z_2$. Then we set $Y(t) := Z(t)E$ on $[a, b]$, so that $Y(a) = Y_1$ and $Y(b) = Y_2$. We now apply Proposition 2.2 with the Lagrangian paths $Y_1(t) := Y(t)$, $Y_2(t) \equiv \tilde{Y}_2$, and $Y_3(t) \equiv \tilde{Y}_1$. Then by using that $Y_2(t)$ and $Y_3(t)$ are constant on $[a, b]$ we obtain that

$$\operatorname{Mas}(Y, \tilde{Y}_2, [a, b]) - \operatorname{Mas}(Y, \tilde{Y}_1, [a, b]) \overset{(2.21)}{=} \mu(Z_1^{-1}\tilde{Y}_1, Z_1^{-1}\tilde{Y}_2) - \mu(Z_2^{-1}\tilde{Y}_1, Z_2^{-1}\tilde{Y}_2),$$

where we also used that $\operatorname{Mas}(\tilde{Y}_2, \tilde{Y}_1, [a, b]) = 0$. By the definition of $s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2)$ in (3.4) we now obtain formula (3.9). For the proof of (3.10) we apply the first formula in (2.5) to the two comparative indices on the right-hand side of (3.9). Then by using that the Wronskians $W(Z_j^{-1}\tilde{Y}_1, Z_j^{-1}\tilde{Y}_2) = W(\tilde{Y}_1, \tilde{Y}_2)$ for $j \in \{1, 2\}$ for both terms are the same we derive from (3.9) the formula in (3.10). Equation (3.11) follows from the definition in (3.4) by expanding the two Maslov indices with the comparison formula (2.16). In more details, we have

$$\begin{aligned} s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) &\overset{(3.4)}{=} \operatorname{Mas}(Y, \tilde{Y}_2, [a, b]) - \operatorname{Mas}(Y, \tilde{Y}_1, [a, b]) \\ &\overset{(2.16)}{=} \Big\{\operatorname{Mas}(E, \tilde{Y}_2, [a, b]) - \operatorname{Mas}(E, Y, [a, b]) + \mu(\tilde{Y}_2, Y(a)) - \mu(\tilde{Y}_2, Y(b))\Big\} \\ &\qquad - \Big\{\operatorname{Mas}(E, \tilde{Y}_1, [a, b]) - \operatorname{Mas}(E, Y, [a, b]) + \mu(\tilde{Y}_1, Y(a)) - \mu(\tilde{Y}_1, Y(b))\Big\} \\ &\overset{(3.1)}{=} \mu(\tilde{Y}_2, Y_1) - \mu(\tilde{Y}_2, Y_2) - \mu(\tilde{Y}_1, Y_1) + \mu(\tilde{Y}_1, Y_2), \end{aligned}$$

where we used that $\operatorname{Mas}(E, \tilde{Y}_j, [a,b]) = 0$. The proof of (3.12) follows from equation (2.22) in Proposition 2.2 (with the Lagrangian paths $Y_1 := Y$, $Y_2 := \tilde{Y}_2$, and $Y_3 := \tilde{Y}_1$) and from the definition of $s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2)$ in (3.6). For the proof of (3.13) we apply the second formula in (2.5) to the two comparative indices on the right-hand side of (3.12). Equation (3.14) follows by expanding the two Maslov indices in (3.6) with the comparison formula (2.17) for the dual Maslov index. Finally, by using the relationship between the Maslov index and the dual Maslov index in (2.15) we get

$$\begin{aligned} s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) &\overset{(3.6)}{=} \operatorname{Mas}^*(Y, \tilde{Y}_2, [a,b]) - \operatorname{Mas}^*(Y, \tilde{Y}_1, [a,b]) \\ &\overset{(2.15)}{=} -\operatorname{Mas}(\tilde{Y}_2, Y, [a,b]) + \operatorname{Mas}(\tilde{Y}_1, Y, [a,b]) \overset{(3.5)}{=} -s(\tilde{Y}_1, \tilde{Y}_2, Y_1, Y_2), \end{aligned}$$

which proves formula (3.15). The proof is complete. □

Remark 3.3. (i) Recall that by Remark 2.3(i) the comparative indices on the right-hand sides of (3.9)–(3.10) and (3.12)–(3.13) are uniquely defined by the Wronskians $W(Y_j, \tilde{Y}_1)$, $W(Y_j, \tilde{Y}_2)$, and $W(\tilde{Y}_1, \tilde{Y}_2)$ for $j \in \{1,2\}$, and then these comparative indices do not depend on the choice of the matrices Z_j. In particular, in view of (2.20) the transformed Lagrangian planes appearing in (3.9)–(3.10) and (3.12)–(3.13) can be taken in the form (for $j,k \in \{1,2\}$)

$$Z_j^{-1}\tilde{Y}_k = -\mathcal{J} Z_j^T \mathcal{J} \tilde{Y}_k \overset{(2.20)}{=} \begin{pmatrix} -W(Y_j, \tilde{Y}_k) \\ K_j^2 Y_j^T \tilde{Y}_k \end{pmatrix}, \quad K_j := K_{Y_j} = (Y_j^T Y_j)^{-1/2}. \tag{3.16}$$

(ii) By using the duality relation in (3.15) we can derive from equations (3.12)–(3.14) alternative expressions for the Hörmander index $s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2)$ in terms of the dual comparative index in the form

$$s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = \mu^*(\tilde{Z}_1^{-1} Y_1, \tilde{Z}_1^{-1} Y_2) - \mu^*(\tilde{Z}_2^{-1} Y_1, \tilde{Z}_2^{-1} Y_2), \tag{3.17}$$

$$s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = \mu^*(\tilde{Z}_2^{-1} Y_2, \tilde{Z}_2^{-1} Y_1) - \mu^*(\tilde{Z}_1^{-1} Y_2, \tilde{Z}_1^{-1} Y_1), \tag{3.18}$$

$$s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = \mu^*(Y_2, \tilde{Y}_1) - \mu^*(Y_2, \tilde{Y}_2) - \mu^*(Y_1, \tilde{Y}_1) + \mu^*(Y_1, \tilde{Y}_2), \tag{3.19}$$

where $\tilde{Z}_1, \tilde{Z}_2$ are symplectic matrices such that $\tilde{Y}_j = \tilde{Z}_j E$ for $j \in \{1,2\}$. Similarly, from equations (3.9)–(3.11) we can derive alternative expressions for the dual Hörmander index $s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2)$ in terms of the comparative index in the form

$$s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = \mu(\tilde{Z}_2^{-1} Y_1, \tilde{Z}_2^{-1} Y_2) - \mu(\tilde{Z}_1^{-1} Y_1, \tilde{Z}_1^{-1} Y_2), \tag{3.20}$$

$$s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = \mu(\tilde{Z}_1^{-1} Y_2, \tilde{Z}_1^{-1} Y_1) - \mu(\tilde{Z}_2^{-1} Y_2, \tilde{Z}_2^{-1} Y_1), \tag{3.21}$$

$$s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = \mu(Y_2, \tilde{Y}_2) - \mu(Y_2, \tilde{Y}_1) - \mu(Y_1, \tilde{Y}_2) + \mu(Y_1, \tilde{Y}_1). \tag{3.22}$$

(iii) We note that equalities (3.11) and (3.14) also follow from Remark 2.3(ii) by applying formulas (2.25) and (2.26) to the comparative indices on the right-hand sides of (3.9)–(3.10) and (3.12)–(3.13).

As an application of Theorem 3.2 we derive a simple geometric interpretation of the comparative index of two Lagrangian planes Y_1 and Y_2. Namely, it is the difference of the Maslov indices (i.e., the intersection numbers) of an arbitrarily chosen Lagrangian path Y connecting Y_1 and Y_2 with the Lagrangian planes Y_2 and E. In other words, the comparative index is a special Hörmander index involving the Lagrangian planes Y_1, Y_2, and E.

Theorem 3.4. *Let $Y_1, Y_2 \in \Lambda(n)$ be given Lagrangian planes. Then we have the formulas*

$$\mu(Y_1, Y_2) = s(E, Y_2, Y_1, Y_2) = \operatorname{Mas}(Y_2, Y, [a,b]) - \operatorname{Mas}(E, Y, [a,b]), \tag{3.23}$$

$$\mu^*(Y_1, Y_2) = -s^*(E, Y_2, Y_1, Y_2) = \operatorname{Mas}^*(E, Y, [a,b]) - \operatorname{Mas}^*(Y_2, Y, [a,b]), \tag{3.24}$$

where Y is an arbitrary Lagrangian path on $[a,b]$ with $Y(a) = Y_1$ and $Y(b) = Y_2$. Alternatively,

$$\mu(Y_1, Y_2) = -s^*(Y_1, Y_2, E, Y_2), \quad \mu^*(Y_1, Y_2) = s(Y_1, Y_2, E, Y_2). \tag{3.25}$$

Proof. By calculating the values $s(E, Y_2, Y_1, Y_2)$ and $s^*(E, Y_2, Y_1, Y_2)$ according to equations (3.11) and (3.14) we obtain

$$s(E, Y_2, Y_1, Y_2) = \mu(Y_2, E) - \mu(Y_2, Y_2) - \mu(Y_1, E) + \mu(Y_1, Y_2) = \mu(Y_1, Y_2),$$
$$s^*(E, Y_2, Y_1, Y_2) = \mu^*(Y_2, Y_2) - \mu^*(Y_2, E) - \mu^*(Y_1, Y_2) + \mu^*(Y_1, E) = -\mu^*(Y_1, Y_2),$$

where we used the basic properties in (2.10) of the comparative index. This proves the first equations in (3.23) and (3.24). The second equations in (3.23) and (3.24) then follow from the definition of the Hörmander index and the dual Hörmander index in (3.5) and (3.7). Finally, the equalities in (3.25) follow from (3.23) and (3.24) by the relationship between the Hörmander index and the dual Hörmander index in (3.15). □

3.2. Several applications

Next we present several applications of Theorem 3.2. First we consider the special case when the upper blocks of the Lagrangian planes in (3.9)–(3.14) or in (3.17)–(3.22) are invertible matrices. In this case we can calculate the Hörmander index by using formula (2.8).

Remark 3.5. (i) Let $Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2 \in \Lambda(n)$ be given Lagrangian planes with the associated symplectic matrices Z_1, Z_2 in (3.16) such that $Y_j = Z_j E$ for $j \in \{1, 2\}$. Assume that the Wronskians $W(Y_j, \tilde{Y}_k)$ are invertible for $j, k \in \{1, 2\}$, i.e., the subspaces generated by Y_j and $\tilde{Y}_k$ are transversal. Such an assumption is common in some references, such as in [30, Corollary 3.11]. Then we have the equality

$$\left.\begin{aligned} s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) &= \operatorname{ind}(M_{11} - M_{12}) - \operatorname{ind}(M_{21} - M_{22}), \\ s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) &= s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2), \\ M_{jk} &:= K_j^2\, Y_j^T \tilde{Y}_k\, [W(Y_j, \tilde{Y}_k)]^{-1}, \end{aligned}\right\} \tag{3.26}$$

where we applied formula (2.8) to the comparative indices in (3.9) and to the the dual comparative indices in (3.12). Analogous results can be obtained by applying formula (2.8) to the other comparative indices and dual comparative indices appearing in Theorem 3.2 or in Remark 3.3(ii). Formulas of the type (3.26) were derived in [16, pp. 23–24]. Note also that the second formula in (3.26) is consistent with [30, Corollary 3.11].

(ii) Let $Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2 \in \Lambda(n)$ be given Lagrangian planes, whose upper blocks X_j and $\tilde{X}_k$ are invertible matrices for $j, k \in \{1, 2\}$, i.e., the Lagrangian planes Y_j and $\tilde{Y}_k$ do not intersect the vertical plane E. Then we have the equalities

$$s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = \operatorname{ind}(Q_1 - \tilde{Q}_2) - \operatorname{ind}(Q_2 - \tilde{Q}_2) - \operatorname{ind}(Q_1 - \tilde{Q}_1) + \operatorname{ind}(Q_2 - \tilde{Q}_1), \tag{3.27}$$
$$s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = \operatorname{ind}(\tilde{Q}_2 - Q_2) - \operatorname{ind}(\tilde{Q}_2 - Q_1) - \operatorname{ind}(\tilde{Q}_1 - Q_2) + \operatorname{ind}(\tilde{Q}_1 - Q_1), \tag{3.28}$$

where for $j, k \in \{1, 2\}$ we have

$$Q_j := U_j X_j^{-1}, \quad \tilde{Q}_k := \tilde{U}_k \tilde{X}_k^{-1}, \quad Q_j - \tilde{Q}_k = -X_j^{T-1}\, W(Y_j, \tilde{Y}_k)\, \tilde{X}_k^{-1}. \tag{3.29}$$

For equality (3.27) we applied formula (2.8) to the comparative indices appearing in (3.11) or equivalently in (3.19), while for equality (3.28) we applied formula (2.8) to the dual comparative indices appearing in (3.14) or equivalently in (3.22).

The result in Remark 3.5(ii) can be generalized by using the symplectic invariance of the Hörmander index as follows. Note that with the choice of the symplectic matrix $R = I$ the equalities in (3.30)–(3.32) below reduce to (3.27)–(3.29).

Corollary 3.6. *Let $Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2 \in \Lambda(n)$ be given Lagrangian planes. Let R be an arbitrary symplectic matrix, i.e., $R = \begin{pmatrix} R_1, & R_2 \end{pmatrix}$ with $R_1, R_2 \in \Lambda(n)$ and $W(R_1, R_2) = I$. Assume that the Lagrangian plane $R_2 = RE$ is transversal* *to $Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2$, i.e., the Wronskians $W(R_2, Y_j)$ and $W(R_2, \tilde{Y}_k)$ are invertible for $j, k \in \{1, 2\}$. Then we have the equalities*

$$s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = \operatorname{ind}(\tilde{N}_2 - N_1) - \operatorname{ind}(\tilde{N}_2 - N_2) - \operatorname{ind}(\tilde{N}_1 - N_1) + \operatorname{ind}(\tilde{N}_1 - N_2), \tag{3.30}$$

$$s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = \operatorname{ind}(N_2 - \tilde{N}_2) - \operatorname{ind}(N_1 - \tilde{N}_2) - \operatorname{ind}(N_2 - \tilde{N}_1) + \operatorname{ind}(N_1 - \tilde{N}_1), \tag{3.31}$$

where for $j, k \in \{1, 2\}$ the Riccati quotients N_j and $\tilde{N}_k$ are defined by

$$N_j := W(R_1, Y_j)\,[W(R_2, Y_j)]^{-1}, \quad \tilde{N}_j := W(R_1, \tilde{Y}_j)\,[W(R_2, \tilde{Y}_j)]^{-1}. \tag{3.32}$$

Proof. By the symplectic invariance of the Hörmander index, i.e., equation (3.8) with the symplectic matrix $S := R^{-1}$, we know that

$$s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = s(R^{-1}Y_1, R^{-1}Y_2, R^{-1}\tilde{Y}_1, R^{-1}\tilde{Y}_2), \tag{3.33}$$

$$s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = s^*(R^{-1}Y_1, R^{-1}Y_2, R^{-1}\tilde{Y}_1, R^{-1}\tilde{Y}_2), \tag{3.34}$$

where the upper blocks of the transformed Lagrangian planes $R^{-1}Y_j$ and $R^{-1}\tilde{Y}_k$ are equal respectively to the Wronskians $-W(R_2, Y_j)$ and $-W(R_2, \tilde{Y}_k)$, i.e., they are invertible by our assumptions. Therefore, equations (3.30) and (3.31) follow by the application of (3.27) and (3.28) to the right-hand side of (3.33) and (3.34), i.e., we take $Q_j := -N_j$ and $\tilde{Q}_k := -\tilde{N}_k$. □

Next we obtain the following universal estimates for the values of the Hörmander index of four arbitrary Lagrangian planes. This result also implies that for a given Lagrangian path on $[a, b]$ its Maslov indices (i.e., the intersection numbers) with respect to any two fixed Lagrangian planes can differ by at most n.

Corollary 3.7. *Let $Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2 \in \Lambda(n)$ be given Lagrangian planes. Then we have*

$$\left|s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2)\right| \le n, \quad \left|s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2)\right| \le n. \tag{3.35}$$

In addition, the Hörmander index attains its maximal value $s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = n$ if and only if

$$\mu(Z_1^{-1}\tilde{Y}_1, Z_1^{-1}\tilde{Y}_2) = n \quad \text{and} \quad \mu(Z_2^{-1}\tilde{Y}_1, Z_2^{-1}\tilde{Y}_2) = 0, \tag{3.36}$$

while the Hörmander index attains its minimal value $s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = -n$ if and only if

$$\mu(Z_1^{-1}\tilde{Y}_1, Z_1^{-1}\tilde{Y}_2) = 0 \quad \text{and} \quad \mu(Z_2^{-1}\tilde{Y}_1, Z_2^{-1}\tilde{Y}_2) = n, \tag{3.37}$$

where Z_1, Z_2 are associated symplectic matrices such that $Y_j = Z_j E$ for $j \in \{1, 2\}$.

Proof. The proof of the estimates in (3.35) follows from (3.9) and (3.12) by using the lower and upper bounds for the comparative index in (2.1) and (2.2). The statements in (3.36) and (3.37) about the maximal and minimal values of $s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2)$ follow from equation (3.9). □

Remark 3.8. (i) Further equivalent conditions for the extreme value $s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = n$ or for the extreme value $s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = -n$ in the spirit of (3.36) and (3.37) can be obtained from equations (3.10), (3.17), and (3.18). Similarly, we may formulate equivalent conditions for the extreme values of the dual Hörmander index $s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = n$ or for $s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = -n$ via equations (3.12), (3.13), (3.20), and (3.21).

(ii) The conditions on $\mu(Y, \hat{Y}) = 0$ in (3.36) and (3.37) or in part (i) of this remark can be efficiently verified by checking the validity of the equivalent conditions presented in [9, Eq. (1.13), (1.14)] or [7, Theorem 3.14(iv)]. For this purpose we recall that the transformed Lagrangian planes appearing in (3.36) and (3.37) can have the form (3.16).

The third application of Theorem 3.2 is based on the additional assumptions on the Wronskians $W(Y_j, \tilde{Y}_k) = 0$, or equivalently on $\dim(\operatorname{Im} Y_j \cap \operatorname{Im} \tilde{Y}_k) = n$ for the associated Lagrangian subspaces. In this case we can investigate the sign of the Hörmander index and the dual Hörmander index.

Corollary 3.9. *Let $Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2 \in \Lambda(n)$ be given Lagrangian planes with the associated symplectic matrices Z_1, Z_2 such that $Y_j = Z_j E$ for $j \in \{1,2\}$. If $W(Y_1, \tilde{Y}_1) = 0$, then*

$$\left.\begin{aligned} s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) &= \mu(Z_2^{-1}\tilde{Y}_2, Z_2^{-1}\tilde{Y}_1) \geq 0, \\ s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) &= -\mu^*(Z_2^{-1}\tilde{Y}_2, Z_2^{-1}\tilde{Y}_1) \leq 0. \end{aligned}\right\} \tag{3.38}$$

Similarly, if $W(Y_2, \tilde{Y}_2) = 0$, then

$$\left.\begin{aligned} s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) &= \mu(Z_1^{-1}\tilde{Y}_1, Z_1^{-1}\tilde{Y}_2) \geq 0, \\ s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) &= -\mu^*(Z_1^{-1}\tilde{Y}_1, Z_1^{-1}\tilde{Y}_2) \leq 0. \end{aligned}\right\} \tag{3.39}$$

Moreover, if $W(Y_2, \tilde{Y}_1) = 0$, then

$$\left.\begin{aligned} s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) &= -\mu(Z_1^{-1}\tilde{Y}_2, Z_1^{-1}\tilde{Y}_1) \leq 0, \\ s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) &= \mu^*(Z_1^{-1}\tilde{Y}_2, Z_1^{-1}\tilde{Y}_1) \geq 0. \end{aligned}\right\} \tag{3.40}$$

Similarly, if $W(Y_1, \tilde{Y}_2) = 0$, then

$$\left.\begin{aligned} s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) &= -\mu(Z_2^{-1}\tilde{Y}_1, Z_2^{-1}\tilde{Y}_2) \leq 0, \\ s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) &= \mu^*(Z_2^{-1}\tilde{Y}_1, Z_2^{-1}\tilde{Y}_2) \geq 0. \end{aligned}\right\} \tag{3.41}$$

Proof. The proof is based on the fact that $\mu(Y_1, Y_2) = 0 = \mu^*(Y_1, Y_2)$ under the assumption that the upper block of Y_2 satisfies $X_2 = 0$, see (2.3) or (2.10). In our case the upper block of $Z_j^{-1}\tilde{Y}_k$ is equal to $-W(Y_j, \tilde{Y}_k)$ according to (3.16). Then (3.38) follows from (3.10) and (3.13) under the assumption $W(Y_1, \tilde{Y}_1) = 0$, while (3.39) follows from (3.9) and (3.12) under the assumption $W(Y_2, \tilde{Y}_2) = 0$. And similarly, (3.40) follows from (3.10) and (3.13) under the assumption $W(Y_2, \tilde{Y}_1) = 0$, while (3.41) follows from (3.9) and (3.12) under the assumption $W(Y_1, \tilde{Y}_2) = 0$. The proof is complete. □

As the fourth application of Theorem 3.2 we obtain the expression of the comparative index of two Lagrangian planes as a special Hörmandex index, indicating that the Hörmander index plays a balancing role in exchanging the first Lagrangian plane in the comparative index by the vertical plane E.

Corollary 3.10. *Let $Y_1, Y_2 \in \Lambda(n)$ be given Lagrangian planes with partitions* (2.4)*. Then*

$$\mu(Y_1, Y_2) = \mu(E, Y_2) - s(E, Y_2, E, Y_1), \quad \mu(E, Y_2) = \operatorname{rank} X_2, \tag{3.42}$$

$$\mu^*(Y_1, Y_2) = \mu^*(E, Y_2) + s^*(E, Y_2, E, Y_1), \quad \mu^*(E, Y_2) = \operatorname{rank} X_2. \tag{3.43}$$

Proof. Formulas (3.42) and (3.43) follow from (3.11) and (3.14) by using property (2.10) of the comparative index. □

The result in Theorem 3.2 also allows to prove in a direct way the following well known exchange formulas, see [30, Eq. (18)–(20)] in combination with Remark 3.1.

Proposition 3.11. *Let $Y_1, Y_2, Y_3, \tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3 \in \Lambda(n)$ be given Lagrangian planes. Then we have*

$$s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = s(Y_1, Y_3, \tilde{Y}_1, \tilde{Y}_2) + s(Y_3, Y_2, \tilde{Y}_1, \tilde{Y}_2), \tag{3.44}$$

$$s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_3) + s(Y_1, Y_2, \tilde{Y}_3, \tilde{Y}_2), \tag{3.45}$$

$$s(Y_2, Y_1, \tilde{Y}_1, \tilde{Y}_2) = -s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2), \tag{3.46}$$

$$s(Y_1, Y_2, \tilde{Y}_2, \tilde{Y}_1) = -s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2), \tag{3.47}$$

$$s(\tilde{Y}_1, \tilde{Y}_2, Y_1, Y_2) = -s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) - \sum_{j,k\in\{1,2\}} (-1)^{j+k} \operatorname{rank} W(Y_j, \tilde{Y}_k). \tag{3.48}$$

Formulas (3.44)–(3.47) *hold also with the dual Hörmander index s^*, while* (3.48) *is replaced by*

$$s^*(\tilde{Y}_1, \tilde{Y}_2, Y_1, Y_2) = -s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) + \sum_{j,k\in\{1,2\}} (-1)^{j+k} \operatorname{rank} W(Y_j, \tilde{Y}_k). \tag{3.49}$$

Finally, we provide an example illustrating the applicability of Theorem 3.2 for special cases considered in [16, Section 3.3].

Example 3.12. Let $Y_1, \tilde{Y}_1, \tilde{Y}_2 \in \Lambda(n)$ be given and assume that $Y_2 = E$ is the vertical Lagrangian plane. Then by (3.11) and (3.14) together with (2.10) we derive

$$s(Y_1, E, \tilde{Y}_1, \tilde{Y}_2) = \mu(\tilde{Y}_2, Y_1) - \mu(\tilde{Y}_1, Y_1), \quad s^*(Y_1, E, \tilde{Y}_1, \tilde{Y}_2) = \mu^*(\tilde{Y}_1, Y_1) - \mu^*(\tilde{Y}_2, Y_1). \tag{3.50}$$

If in addition $Y_1 = N$ is the horizontal Lagrangian plane, then by (2.11) and (3.50) we get

$$s(N, E, \tilde{Y}_1, \tilde{Y}_2) = \operatorname{rank} \tilde{X}_1 - \operatorname{rank} \tilde{X}_2 + \operatorname{ind}(-\tilde{X}_2^T \tilde{U}_2) - \operatorname{ind}(-\tilde{X}_1^T \tilde{U}_1), \tag{3.51}$$

$$s^*(N, E, \tilde{Y}_1, \tilde{Y}_2) = \operatorname{rank} \tilde{X}_1 - \operatorname{rank} \tilde{X}_2 + \operatorname{ind}(\tilde{X}_2^T \tilde{U}_2) - \operatorname{ind}(\tilde{X}_1^T \tilde{U}_1), \tag{3.52}$$

Alternatively, consider another special case of (3.50) when the upper blocks of Y_1 and $\tilde{Y}_1$ are invertible, then the comparative indices $\mu(\tilde{Y}_1, Y_1)$ and $\mu^*(\tilde{Y}_1, Y_1)$ in (3.50) reduce to the index of the difference of the associated Riccati quotients, as we show in (2.8). We also note that the case $Y_2 = N = \mathcal{J}E$ can be reduced to $Y_1 = E$ by the symplectic invariance property (3.8) with the matrix $S := -\mathcal{J}$.

3.3. Triple index

As the last result in this section we discuss the connection of the comparative index with the triple index $i(Y_1, Y_2, Y_3)$, which was first defined in [6, Eq. (2.16)], see also [30, Corollary 3.12]. In the present discussion we again identify the Lagrangian subspace $\operatorname{Im} Y_j \subseteq \mathbb{R}^{2n}$ with the matrix $Y_j \in \Lambda(n)$ itself. For three Lagrangian planes $Y_1, Y_2, Y_3 \in \Lambda(n)$ the *triple index* $i(Y_1, Y_2, Y_3)$ is the integer defined as

$$i(Y_1, Y_2, Y_3) := \operatorname{ind} Q(Y_1, Y_0, Y_2) + \operatorname{ind} Q(Y_2, Y_0, Y_3) - \operatorname{ind} Q(Y_1, Y_0, Y_3), \tag{3.53}$$

where $Y_0 \in \Lambda(n)$ is a Lagrangian plane for which the Wronskians $W(Y_j, Y_0)$ for $j \in \{1, 2, 3\}$ are invertible. Here $Q(\alpha, \beta, \gamma)$ with $\alpha, \beta, \gamma \subseteq \mathbb{R}^{2n}$ being Lagrangian subspaces is a bilinear form defined on the subspace $\alpha \cap (\beta + \gamma)$. We refer to [30, Section 3.1] for more details regarding the form $Q(\alpha, \beta, \gamma)$. The definition in (3.53) does not depend on the choice of the Lagrangian plane $Y_0 \in \Lambda(n)$, as long at it has the required property regarding the invertibility of the Wronskians $W(Y_j, Y_0)$. Note that the number $i(Y_1, Y_2, Y_3)$ is nonnegative, which follows e.g. from [30, Lemma 3.13].

A difference of two triple indices is used in [30, Theorem 1.1] for the expression of the (dual) Hörmander index, see Remark 3.1 and equation (1.2), which in our context reads as

$$s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = i(Y_1, Y_2, \tilde{Y}_2) - i(Y_1, Y_2, \tilde{Y}_1) = i(Y_1, \tilde{Y}_1, \tilde{Y}_2) - i(Y_2, \tilde{Y}_1, \tilde{Y}_2). \tag{3.54}$$

Note that these results are proven in [30] on the basis of [6, Lemma 2.5]. In addition, by using the duality principle in (3.15) with the aid of (3.54) we get the alternative formulas

$$s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = i(\tilde{Y}_1, \tilde{Y}_2, Y_1) - i(\tilde{Y}_1, \tilde{Y}_2, Y_2) = i(\tilde{Y}_2, Y_1, Y_2) - i(\tilde{Y}_1, Y_1, Y_2). \tag{3.55}$$

By using Theorem 3.2 we shall connect the triple index with the comparative index. We recall that for the evaluation of the obtained comparative indices we may use the symplectic matrices Z_j considered in (3.16) in Remark 3.3(i).

Corollary 3.13. *Let $Y_1, Y_2, Y_3 \in \Lambda(n)$ be given Lagrangian planes with the associated symplectic matrices Z_1, Z_2, Z_3 such that $Y_j = Z_j E$ for $j \in \{1, 2, 3\}$. Then we have*

$$i(Y_1, Y_2, Y_3) = \mu(Z_3^{-1} Y_1, Z_3^{-1} Y_2) = \mu(Y_1, Y_2) + \mu(Y_2, Y_3) - \mu(Y_1, Y_3), \tag{3.56}$$

$$i(Y_1, Y_2, Y_3) = \mu^*(Z_1^{-1} Y_3, Z_1^{-1} Y_2) = \mu^*(Y_3, Y_2) + \mu^*(Y_2, Y_1) - \mu^*(Y_3, Y_1). \tag{3.57}$$

In particular, we can express the comparative index and the dual comparative index as

$$\mu(Y_1, Y_2) = i(Y_1, Y_2, E), \quad \mu^*(Y_1, Y_2) = i(E, Y_2, Y_1). \tag{3.58}$$

Proof. For the proof we utilize [30, Corollary 3.16], where the authors formulate their result in terms of the endpoint values $Y(a)$ and $Y(b)$ of a Lagrangian path Y on $[a, b]$. By using the notation with fixed Lagrangian planes $Y_1, Y_2, Y_3 \in \Lambda(n)$ and taking Remark 3.1 into account, we can reformulate the second equality in [30, Corollary 3.16] as

$$i(Y_1, Y_2, Y_3) = s^*(Y_1, Y_2, Y_2, Y_3). \tag{3.59}$$

Then we proceed by applying Theorems 3.2 and 3.4 and Proposition 3.11 together with the symplectic invariance of the Hörmander index (with the matrix $S := Z_3^{-1}$). Namely, we obtain

$$\begin{aligned} i(Y_1, Y_2, Y_3) &\overset{(3.59)}{=} s^*(Y_1, Y_2, Y_2, Y_3) \overset{(3.15)}{=} -s(Y_2, Y_3, Y_1, Y_2) \overset{(3.46)}{=} s(Y_3, Y_2, Y_1, Y_2) \\ &\overset{(3.8)}{=} s(Z_3^{-1}Y_3, Z_3^{-1}Y_2, Z_3^{-1}Y_1, Z_3^{-1}Y_2) = s(E, Z_3^{-1}Y_2, Z_3^{-1}Y_1, Z_3^{-1}Y_2) \\ &\overset{(3.23)}{=} \mu(Z_3^{-1}Y_1, Z_3^{-1}Y_2), \end{aligned}$$

which proves the first equality in (3.56), and then by Remark 2.3(ii) we get the second equality in (3.56). The equations in (3.57) follow from (3.56) by using property (2.7) of the comparative index and from (2.26). Finally, the choice of $Y_3 = E$ (i.e., $Z_3 = I$) in (3.56) proves the validity of the first equation in (3.58), while the choice of $Y_1 = E$ (i.e., $Z_1 = I$) in (3.57) yields that $\mu^*(Y_3, Y_2) = i(E, Y_2, Y_3)$. By relabeling Y_3 as Y_1 we then obtain the second equation in (3.58). The proof is complete. □

4. Conclusions

In this paper we investigated the relations between the comparative index and the Hörmander index (including the Maslov index and the triple index) in the finite dimensional case. As a main result we derived an algebraic expression for the Hörmander index of four given Lagrangian planes as a difference of two comparative indices involving certain transformed Lagrangian planes, or as a combination of four comparative indices (Theorem 3.2). This result is based on a generalization of the comparison theorem for the Maslov index from [15] involving three Lagrangian paths (Proposition 2.2), hence it is based entirely on the comparative index theory. Our approach allows to present a geometric interpretation of the comparative index as a special Hörmander index or as a special triple index involving the vertical Lagrangian plane (Theorem 3.4 and Corollary 3.13). We also derived estimates for the values of the Hörmander index and presented conditions allowing to determine its extreme values and its sign (Corollaries 3.7 and 3.9). In this way we contribute to the recent efforts in [16, 30] devoted to efficient calculation of the Hörmander index in the finite dimensional case.

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