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New inequalities for (p,h)-convex functions for τ-measurable operators

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Abstract. The main goal of this article is to present new inequalities for (p,h)-convex and (p,h) log-convex functions for a non-negative super-multiplicative and super-additive function h. Our first main result will be

$$h^{\lambda}\left(\frac{v}{\mu}\right) \leq \frac{(h(1-v)f(a)+h(v)f(b))^{\lambda} - f^{\lambda}\left[((1-v)a^p+vb^p)^{\frac{1}{p}}\right]}{(h(1-\mu)f(a)+h(\mu)f(b))^{\lambda} - f^{\lambda}\left[((1-\mu)a^p+\mu b^p)^{\frac{1}{p}}\right]} \leq h^{\lambda}\left(\frac{1-v}{1-\mu}\right),$$

for the positive (p,h)-convex function f, when $\lambda \geq 1$, $p \in \mathbb{R}\backslash\{0\}$ and $0 \leq v \leq \mu \leq 1$. This gives a generalization of an important result due to M. Sababheh [Linear Algebra Appl. **506** (2016), 588–602]. As applications of our results, we present many inequalities for the trace, and the symmetric norms for τ-measurable operators.

1. Introduction and preliminaries

Convex functions and their inequalities have played a major role in the study of various topics in Mathematics; including applied Mathematics, Mathematical Analysis, and Mathematical Physics. Recall that a function $f : I \to \mathbb{R}$ is said to be convex on the interval I if

$$f((1-v)a+vb) \leq (1-v)f(a)+vf(b), \tag{1}$$

for all $a, b \in I$ and $v \in [0,1]$. If this inequality is reversed, then f is said to be concave.
Recent studies of this topic have investigated possible refinements of the above inequality, where adding a positive term to the left side becomes possible. This idea has been treated in [3, 13–19], where not only refinements have been investigated, but reversed versions and much more have been discussed.

For example M. Sababheh in [13], presents the following nice result about the convexity, which presents a generalized refinement and reversed of the inequality (1).

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Theorem 1.1. *Let $f : [a, b] \to [0, \infty)$ be convex. Then*

$$\left(\frac{v}{\mu}\right)^{\lambda} \leq \frac{((1-v)f(a)+vf(b))^{\lambda} - f^{\lambda}((1-v)a+vb)}{((1-\mu)f(a)+\mu f(b))^{\lambda} - f^{\lambda}((1-\mu)a+\mu b)} \leq \left(\frac{1-v}{1-\mu}\right)^{\lambda} \tag{2}$$

for $\lambda \geq 1$ and $0 \leq v \leq \mu \leq 1$.

When $a, b > 0$, the functions $f(x) = a\sharp_x b := a^{1-x}b^x$ and $f(x) = a!_x b := ((1-x)a^{-1}+xb^{-1})^{-1}$, are convex functions-on $[0, 1]$. Applying Theorem 1.1 using these functions implies the results in [1] and [12].
The notion of convexity has been expanded and generalized in numerous ways utilizing new and modern methods in recent years. Before starting our analysis, let us recall the definitions of some special classes of functions. Let I be a p-convex subset of $\mathbb{R}$ (That means, $[(1-v)a^p + vb^p]^{\frac{1}{p}} \in I$ for all $a, b \in I$ and $v \in [0, 1]$).

Definition 1.2 ([23]). *A function $f : I \to \mathbb{R}$ is said to be a p-convex or belongs to the class $PC(I)$, if*

$$f\left([(1-v)a^p + vb^p]^{\frac{1}{p}}\right) \leq (1-v)f(a) + vf(b) \tag{3}$$

for all $a, b \in I$, $p \in \mathbb{R}\backslash\{0\}$ and $v \in [0, 1]$.

Definition 1.3 ([21]). *Let $h : J \to \mathbb{R}$ be a non-negative and non-zero function. We say that $f : I \to \mathbb{R}$ is an h-convex function or that f belongs to the class $SX(I)$, if f is non-negative and for all $a, b \in I$ and $v \in [0, 1]$ we have*

$$f((1-v)a + vb) \leq h(1-v)f(a) + h(v)f(b). \tag{4}$$

If this inequality is reversed, then f is said to be h-concave.

Definition 1.4 ([6]). *Let $h : J \to \mathbb{R}$ be a non-negative and non-zero function. We say that $f : I \to \mathbb{R}$ is a (p, h)-convex function or that f belongs to the class $ghx(h, p, I)$, if f is non-negative and*

$$f\left([(1-v)a^p + vb^p]^{\frac{1}{p}}\right) \leq h(1-v)f(a) + h(v)f(b) \tag{5}$$

for all $a, b \in I$, $v \in [0, 1]$ and $p \in \mathbb{R}\backslash\{0\}$. Similarly, if the inequality sign in (5) is reversed, then f is said to be a (p, h)-concave function or belong to the class $ghv(h, p, I)$.

Definition 1.5 ([9]). *Let $h : J \to \mathbb{R}$ be a non-negative and non-zero function. We say that $f : I \to \mathbb{R}$ is a (p, h) log-convex function, if f is non-negative and*

$$f\left([(1-v)a^p + vb^p]^{\frac{1}{p}}\right) \leq f(a)^{h(1-v)}f(b)^{h(v)} \tag{6}$$

for all $a, b \in I$, $v \in [0, 1]$ and $p \in \mathbb{R}\backslash\{0\}$. Similarly, if this inequality is reversed, then f is said to be a (p, h) log-concave function.

Definition 1.6. *Let $h : J \to \mathbb{R}$ be a non-negative and non-zero function. We say that $f : I \to \mathbb{R}$ is a (G-A)-h-convex function if f is non-negative and*

$$f\left(a^{1-v}b^v\right) \leq h(1-v)f(a) + h(v)f(b) \tag{7}$$

for all $a, b \in I$, and $v \in [0, 1]$. Similarly, if the inequality sign in (7) is reversed, then f is said to be a (G-A) h-concave function.

Example 1.7. *Let $h : [0, 1] \to \mathbb{R}^+$ be a non-negative and non-zero function such that $h(t) \geq t$ for all $t \in [0, 1]$. Then the function $\ln(t)$ is (G-A)-h-convex on $(0, +\infty)$.*

Definition 1.8. *Let $h : J \to \mathbb{R}$ be a non-negative and non-zero function. We say that $f : I \to \mathbb{R}$ is a (G-G)-h-convex function if f is non-negative and*

$$f\left(a^{1-v}b^{v}\right) \leq f^{h(1-v)}(a)f^{h(v)}(b) \tag{8}$$

for all $a, b \in I$, and $v \in [0,1]$. Similarly, if the inequality sign in (8) is reversed, then f is said to be a (G-G) h-concave function.

Example 1.9. *Let $h : [0,1] \to \mathbb{R}^{+}$ be a non-negative and non-zero function such that $h(t) \geq t$ for all $t \in [0,1]$. Then the function $\exp(t)$ is (G-G)-h-convex on $(0,+\infty)$.*

Definition 1.10 ([7]). *Let $h : J \to \mathbb{R}$. If for all $x, y \in J$, we have*

$$h(x)h(y) \leq h(xy) \tag{9}$$

then h is said to be a super-multiplicative function. If the inequality (9) is reversed, then h is said to be a sub-multiplicative function. If the equality holds in (9), then h is said to be a multiplicative function.

Definition 1.11 ([7]). *Let $h : J \to \mathbb{R}$. If for all $x, y \in J$, we have*

$$h(x) + h(y) \leq h(x+y), \tag{10}$$

then h is said to be a super-additive function. If inequality (10) is reversed, we say that h is a sub-additive function. If the equality (10) holds, we say that h is an additive function.

Example 1.12. *Let $h : I \to (0,\infty)$ be given by $h(x) = x^{k}, x > 0$. Then h is*

(1) additive if $k = 1$,
(2) sub-additive if $k \in (-\infty,-1] \cup [0,1)$,
(3) super-additive if $k \in (-1,0) \cup (1,\infty)$.

Let $h[1,+\infty) \mapsto \mathbb{R}^{+}$ be given by $h(x) = x^{3} - x^{2} + x$. We have

(4) $h(xy) - h(x)h(y) = xy(x+y)(1-x)(1-y) \geq 0$
(5) $h(x+y) - h(x) - h(y) = xy(x+y+(x-1)+(y-1)) \geq 0$.

Then h is a super-multiplicative and super-additive function.

(6) Let h be a convex function with $h(0) = 0$. Then h is a super-additive function. In particular the following function $h(x) = \exp(x^{k}) - 1$ for $k > 0$ is super-additive.

The purposes of this paper are manifold. First, we develop new techniques in order to extend Theorem 1.1 to the notion of the (p,h)-convex and (G-A)-h-convex functions for a non-negative super-multiplicative and super-additive function h. In other word, we drive some new refinements and reversed of the (p,h) log-convex and (G-G)-h-convex functions. Second, we further extended our presented refinements via the operator (p,h)-convex functions of several variables on the von Neumann algebra. At the end of this paper, we give some application using the new refinement established in the previous sections to the trace, and the symmetric norms for τ-measurable operators.

2. New inequalities for (p,h)-convex functions

In the following theorem, we state our first main result concerning (p,h)-convex functions, which presents one refining term of the inequality (5).

Theorem 2.1. *Let h be a non-negative super-multiplicative and super-additive function on J, f be a positive (p, h)-convex function on $[a, b]$ and $0 \leq v \leq \mu \leq 1$. Then*

$$\begin{aligned} h(1-v)f(a) &+ h(v)f(b) \geq f\left[((1-v)a^p + vb^p)^{\frac{1}{p}}\right] \\ &+ h\left(\frac{v}{\mu}\right)\left[h(1-\mu)f(a) + h(\mu)f(b) - f\left[((1-\mu)a^p + \mu b^p)^{\frac{1}{p}}\right]\right] \end{aligned}$$

Proof. Since h is super-multiplicative and super-additive, we have

$$\begin{aligned} h(1-v)f(a) &+ h(v)f(b) \\ &- h\left(\frac{v}{\mu}\right)\left[h(1-\mu)f(a) + h(\mu)f(b) - f\left[((1-\mu)a^p + \mu b^p)^{\frac{1}{p}}\right]\right] \\ &= \left(h(1-v) - h\left(\frac{v}{\mu}\right)h(1-\mu)\right)f(a) + \left(h(v) - h\left(\frac{v}{\mu}\right)h(\mu)\right)f(b) \\ &+ h\left(\frac{v}{\mu}\right)f\left[((1-\mu)a^p + \mu b^p)^{\frac{1}{p}}\right] \\ &\geq h\left(1 - \frac{v}{\mu}\right)f(a) + h\left(\frac{v}{\mu}\right)f\left[((1-\mu)a^p + \mu b^p)^{\frac{1}{p}}\right] \\ &\geq f\left[\left[\left(1 - \frac{v}{\mu}\right)a^p + \left(\frac{v}{\mu}\right)((1-\mu)a^p + \mu b^p)\right]^{\frac{1}{p}}\right] \\ &= f\left[((1-v)a^p + vb^p)^{\frac{1}{p}}\right], \end{aligned}$$

where the last inequality follows by the (p, h)-convexity of the function f. □

As a consequence of Theorem 2.1, we have the following corollary about the (G-A)-h-convex functions, which gives one refining term of (7).

Corollary 2.2. *Let h be a non-negative super-multiplicative and super-additive function on J, f be a positive (G-A)-h-convex function on $[a, b]$ and $0 \leq v \leq \mu \leq 1$. Then*

$$\begin{aligned} h(1-v)f(a) + h(v)f(b) &\geq f\left(a^{1-v}b^v\right) \\ &+ h\left(\frac{v}{\mu}\right)\left[h(1-\mu)f(a) + h(\mu)f(b) - f\left(a^{1-\mu}b^\mu\right)\right]. \end{aligned}$$

Proof. First, we prove that $f : I \subset (0, \infty) \to \mathbb{R}$ is (G-A) h-convex on $I \iff f \circ \exp : \ln I \to \mathbb{R}$ is h-convex on the interval $\ln I = \{\ln x \mid x \in I\}$.

($\Rightarrow$) Suppose that the function f is (G-A)-h-convex function on I. Then, we get

$$\begin{aligned} (f \circ \exp)((1-v)\ln a + v \ln b) &= f \circ \exp\left(\ln(a^{1-v}b^v)\right) \\ &= f\left(a^{1-v}b^v\right) \\ &\leq h(1-v)f(a) + h(v)f(b) \\ &= h(1-v)f \circ \exp(\ln a) + h(v)f \circ \exp(\ln b) \end{aligned}$$

Therefore, the function $f \circ \exp : \ln I \to \mathbb{R}$ is h-convex function on $\ln I$.

($\Leftarrow$) Let $f \circ \exp : \ln I \to \mathbb{R}$, be an h-convex function on $\ln I$. Then, we have

$$\begin{aligned} f\left(a^{1-v}b^{v}\right) &= f\left(e^{(1-v)\ln a + v\ln b}\right) \\ &= (f \circ \exp)\left((1-v)\ln a + v\ln b\right) \\ &\leq h(1-v) f\left(e^{\ln a}\right) + h(v) f\left(e^{\ln b}\right) \\ &= h(1-v)f(a) + h(v)f(b). \end{aligned}$$

Hence, the function $f : I \subset (0,\infty) \to \mathbb{R}$ is (G-A)-h-convex on I. Now, using Theorem 2.1 for $p = 1$, we get the desired result. □

Our second main result is the following theorem, which presents a reverse of Theorem 2.1.

Theorem 2.3. *Let h be a non-negative multiplicative and super-additive function on J, f be a positive (p,h)-convex function on $[a,b]$ and $0 \leq v \leq \mu \leq 1$. Then*

$$\begin{aligned} h(1-v)f(a) \quad &+ \quad h(v)f(b) \leq f\left[\left((1-v)a^{p} + vb^{p}\right)^{\frac{1}{p}}\right] \\ &+ \quad h\left(\frac{1-v}{1-\mu}\right)\left[h(1-\mu)f(a) + h(\mu)f(b) - f\left[\left((1-\mu)a^{p} + \mu b^{p}\right)^{\frac{1}{p}}\right]\right]. \end{aligned}$$

Proof. Since h is multiplicative and super-additive, we have

$$\begin{aligned} h(1-\mu)f(a) \quad &+ \quad h(\mu)f(b) - \frac{h(1-v)}{h\left(\frac{1-v}{1-\mu}\right)} f(a) - \frac{h(v)}{h\left(\frac{1-v}{1-\mu}\right)} f(b) \\ &+ \quad \frac{1}{h\left(\frac{1-v}{1-\mu}\right)} f\left[\left((1-v)a^{p} + vb^{p}\right)^{\frac{1}{p}}\right] \\ &= \quad \left(h(1-\mu) - h(1-\mu)\right) f(a) + \left(h(\mu) - h\left(\frac{v(1-\mu)}{1-v}\right)\right) f(b) \\ &+ \quad h\left(\frac{1-\mu}{1-v}\right) f\left[\left((1-v)a^{p} + vb^{p}\right)^{\frac{1}{p}}\right] \\ &\geq \quad h\left(1 - \frac{1-\mu}{1-v}\right) f(b) + h\left(\frac{1-\mu}{1-v}\right) f\left[\left((1-v)a^{p} + vb^{p}\right)^{\frac{1}{p}}\right] \\ &\geq \quad f\left[\left[\left(1 - \frac{1-\mu}{1-v}\right) b^{p} + \left(\frac{1-\mu}{1-v}\right)\left((1-v)a^{p} + vb^{p}\right)\right]^{\frac{1}{p}}\right] \\ &= \quad f\left[\left((1-\mu)a^{p} + \mu b^{p}\right)^{\frac{1}{p}}\right]. \end{aligned}$$

Multiplying the last inequality by $h\left(\frac{1-v}{1-\mu}\right)$, the desired inequality is obtained. □

As a consequence of Theorem 2.3, we have the following corollary, which gives a reverse of Corollary 2.2.

Corollary 2.4. *Let h be a non-negative multiplicative and super-additive function on J, f be a positive (G-A)-h-convex function on $[a,b]$ and $0 \leq v \leq \mu \leq 1$. Then*

$$\begin{aligned} h(1-v)f(a) \quad &+ \quad h(v)f(b) \leq f\left(a^{1-v}b^{v}\right) \\ &+ \quad h\left(\frac{1-v}{1-\mu}\right)\left[h(1-\mu)f(a) + h(\mu)f(b) - f\left(a^{1-\mu}b^{\mu}\right)\right]. \end{aligned}$$

The method used in [13] to prove Theorem 1.1 has a differential calculus approach, that we cannot use her to prove the general case. To prove the general case for the (p,h)-convex functions we will need the following lemma, which will enable use it here to prove a more general results for the (p,h)-convex functions.

Lemma 2.5 ([2]). *Let ϕ be a strictly increasing convex function defined on an interval K. If x, y, z and w are points in K such that*

$$z - w \leq x - y$$

where $w \leq z \leq x$ and $y \leq x$, then

$$(0 \leq) \quad \phi(z) - \phi(w) \leq \phi(x) - \phi(y).$$

By combining the preceding results with Lemma 2.5, we get the following theorem.

Theorem 2.6. *Let ϕ be a strictly increasing convex function defined on an interval K, f be a positive (p, h)-convex function on $[a, b]$ and $0 \leq v \leq \mu \leq 1$.*

1. *If h is a non-negative super-multiplicative and super-additive function, then*

$$\begin{aligned}&\phi\left(h\left(\frac{v}{\mu}\right)(h(1-\mu)f(a) + h(\mu)f(b))\right) - \phi\left[h\left(\frac{v}{\mu}\right)f\left[((1-\mu)a^p + \mu b^p)^{\frac{1}{p}}\right]\right]\\ &\leq \phi(h(1-v)f(a) + h(v)f(b)) - \phi \circ f\left[((1-v)a^p + vb^p)^{\frac{1}{p}}\right].\end{aligned}$$

2. *If h is a non-negative multiplicative and super-additive function, then*

$$\begin{aligned}&\phi(h(1-v)f(a) + h(v)f(b)) - \phi \circ f\left[((1-v)a^p + vb^p)^{\frac{1}{p}}\right]\\ &\leq \phi\left(h\left(\frac{1-v}{1-\mu}\right)(h(1-\mu)f(a) + h(\mu)f(b))\right)\\ &- \phi\left[h\left(\frac{1-v}{1-\mu}\right)f\left[((1-\mu)a^p + \mu b^p)^{\frac{1}{p}}\right]\right].\end{aligned}$$

Proof. Let

$$x = h(1-v)f(a) + h(v)f(b), \quad y = f\left(((1-v)a^p + vb^p)^{\frac{1}{p}}\right),$$

$$z = h\left(\frac{v}{\mu}\right)(h(1-\mu)f(a) + h(\mu)f(b)), \quad w = h\left(\frac{v}{\mu}\right)f\left(\left((1-\mu)a^p + \mu b^p\right)^{\frac{1}{p}}\right),$$

$$z' = h\left(\frac{1-v}{1-\mu}\right)(h(1-\mu)f(a) + h(\mu)f(b))$$

and

$$w' = h\left(\frac{1-v}{1-\mu}\right)f\left(\left((1-\mu)a^p + \mu b^p\right)^{\frac{1}{p}}\right).$$

Based on Theorems 2.1 and 2.3, we have

$$z - w \leq x - y \leq z' - w'.$$

The first and the second inequalities in Theorem 2.6 follows directly by applying Lemma 2.5, to the inequalities $z - w \leq x - y$, with $w \leq z \leq x$, $y \leq x$ and $x - y \leq z' - w'$ with $y \leq x \leq z'$, $w' \leq z'$, respectively. This completes the proof.

□

Replacing f by $\log f$, we obtain the following inequalities for (p, h) log-convex functions.

Theorem 2.7. *Let f be a (p, h)-convex function on $[a, b]$ and let $0 \leq v \leq \mu \leq 1$.*

1. *If h is a non-negative super-multiplicative and super-additive function, then*

$$\begin{aligned}&\phi\left(\log\left(f^{h(1-\mu)}(a)f^{h(\mu)}(b)\right)^{h\left(\frac{v}{\mu}\right)}\right) - \phi\left[\log f^{h\left(\frac{v}{\mu}\right)}\left[((1-\mu)a^p + \mu b^p)^{\frac{1}{p}}\right]\right]\\ &\leq \phi\left(\log\left(f^{h(1-v)}(a)f^{h(v)}(b)\right)\right) - \phi\left[\log f\left[((1-v)a^p + vb^p)^{\frac{1}{p}}\right]\right]. \qquad (11)\end{aligned}$$

2. *If h is a non-negative multiplicative and super-additive function, then*

$$\begin{aligned}&\phi\left(\log\left(f^{h(1-v)}(a)f^{h(v)}(b)\right)\right)-\phi\left[\log f\left[((1-v)a^p+vb^p)^{\frac{1}{p}}\right]\right]\\&\leq\phi\left(\log\left(f^{h(1-\mu)}(a)f^{h(\mu)}(b)\right)^{h\left(\frac{1-v}{1-\mu}\right)}\right)-\phi\left[\log f^{h\left(\frac{1-v}{1-\mu}\right)}\left[((1-\mu)a^p+\mu b^p)^{\frac{1}{p}}\right]\right].\end{aligned}\tag{12}$$

Now by selecting $\phi(x)=x^{\lambda}$ for $\lambda\geq 1$, in Theorem 2.6, we get the following corollary.

Corollary 2.8. *Let f be a (p,h)-convex function on $[a,b]$, $0\leq v\leq\mu\leq 1$ and $\lambda\geq 1$.*

1. *If h is a non-negative super-multiplicative and super-additive function, then*

$$h^{\lambda}\left(\frac{v}{\mu}\right)\leq\frac{(h(1-v)f(a)+h(v)f(b))^{\lambda}-f^{\lambda}\left[((1-v)a^p+vb^p)^{\frac{1}{p}}\right]}{(h(1-\mu)f(a)+h(\mu)f(b))^{\lambda}-f^{\lambda}\left[((1-\mu)a^p+\mu b^p)^{\frac{1}{p}}\right]}.\tag{13}$$

2. *If h is a non-negative multiplicative and super-additive function, then*

$$\frac{(h(1-v)f(a)+h(v)f(b))^{\lambda}-f^{\lambda}\left[((1-v)a^p+vb^p)^{\frac{1}{p}}\right]}{(h(1-\mu)f(a)+h(\mu)f(b))^{\lambda}-f^{\lambda}\left[((1-\mu)a^p+\mu b^p)^{\frac{1}{p}}\right]}\leq h^{\lambda}\left(\frac{1-v}{1-\mu}\right).\tag{14}$$

Remark 2.9. *Notice that, if we take $h(x)=x$ and $p=1$, in Corollary 2.8. Then we recapture Theorem 2.1 in [13].*

Now, by selecting $\phi(x)=\exp(x)$, in Theorem 2.6, we get the following new and important refinement and reverse for (p,h) log-convex functions.

Theorem 2.10. *Let f be a positive (p,h) log-convex function on $[a,b]$ and $0\leq v\leq\mu\leq 1$.*

1. *If h is a non-negative super-multiplicative and super-additive function, then*

$$\begin{aligned}&\left(f^{h(1-\mu)}(a)f^{h(\mu)}(b)\right)^{h\left(\frac{v}{\mu}\right)}-f^{h\left(\frac{v}{\mu}\right)}\left[((1-\mu)a^p+\mu b^p)^{\frac{1}{p}}\right]\\&\leq\left(f^{h(1-v)}(a)f^{h(v)}(b)\right)-f\left[((1-v)a^p+vb^p)^{\frac{1}{p}}\right].\end{aligned}$$

2. *If h is a non-negative multiplicative and super-additive function, then*

$$\begin{aligned}&\left(f^{h(1-v)}(a)f^{h(v)}(b)\right)-f\left[((1-v)a^p+vb^p)^{\frac{1}{p}}\right]\\&\leq\left(f^{h(1-\mu)}(a)f^{h(\mu)}(b)\right)^{h\left(\frac{1-v}{1-\mu}\right)}-f^{h\left(\frac{1-v}{1-\mu}\right)}\left[((1-\mu)a^p+\mu b^p)^{\frac{1}{p}}\right].\end{aligned}$$

In the following theorem we present a new generalized refinement and reversed inequality for the (G-A)-h-convex functions.

Theorem 2.11. *Let ϕ be a strictly increasing convex function defined on an interval K, f be a positive(G-A)-h-convex function on $[a,b]$ and $0\leq v\leq\mu\leq 1$.*

1. *If h is a non-negative super-multiplicative and super-additive function, then*

$$\begin{aligned}&\phi\left(h\left(\frac{v}{\mu}\right)(h(1-\mu)f(a)+h(\mu)f(b))\right)-\phi\left[h\left(\frac{v}{\mu}\right)f\left(a^{1-\mu}b^{\mu}\right)\right]\\&\leq\phi(h(1-v)f(a)+h(v)f(b))-\phi\circ f\left(a^{1-v}b^{v}\right).\end{aligned}$$

2. *If h is a non-negative multiplicative and super-additive function, then*

$$\begin{aligned}&\phi(h(1-v)f(a)+h(v)f(b))-\phi\circ f\left(a^{1-v}b^{v}\right)\\&\leq\phi\left(h\left(\frac{1-v}{1-\mu}\right)(h(1-\mu)f(a)+h(\mu)f(b))\right)-\phi\left[h\left(\frac{1-v}{1-\mu}\right)f\left(a^{1-\mu}b^{\mu}\right)\right].\end{aligned}$$

Proof. Let

$$x=h(1-v)f(a)+h(v)f(b),\quad y=f\left(a^{1-v}b^{v}\right),$$
$$z=h\left(\frac{v}{\mu}\right)(h(1-\mu)f(a)+h(\mu)f(b)),\quad w=h\left(\frac{v}{\mu}\right)f\left(a^{1-\mu}b^{\mu}\right),$$
$$z'=h\left(\frac{1-v}{1-\mu}\right)(h(1-\mu)f(a)+h(\mu)f(b)),\ \text{ and } w'=h\left(\frac{1-v}{1-\mu}\right)f\left(a^{1-\mu}b^{\mu}\right).$$

Based on Corollaries 2.2 and 2.4, we have

$$z-w\leq x-y\leq z'-w'.$$

The first and the second inequalities in Theorem 2.11 follows directly by applying Lemma 2.5, to the inequalities $z-w\leq x-y$, with $w\leq z\leq x$, $y\leq x$ and $x-y\leq z'-w'$ with $y\leq x\leq z'$, $w'\leq z'$, respectively. This completes the proof.

□

Now by selecting $\phi(x)=x^{\lambda}$ for $\lambda\geq 1$, in Theorem 2.11, we get the following corollary.

Corollary 2.12. *Let f be a (G-A)-h-convex function on $[a,b]$, $0\leq v\leq\mu\leq 1$ and $\lambda\geq 1$.*

1. *If h is a non-negative super-multiplicative and super-additive function, then*

$$h^{\lambda}\left(\frac{v}{\mu}\right)\leq\frac{(h(1-v)f(a)+h(v)f(b))^{\lambda}-f^{\lambda}\left(a^{1-v}b^{v}\right)}{(h(1-\mu)f(a)+h(\mu)f(b))^{\lambda}-f^{\lambda}\left(a^{1-\mu}b^{\mu}\right)}.\tag{15}$$

2. *If h is a non-negative multiplicative and super-additive function, then*

$$\frac{(h(1-v)f(a)+h(v)f(b))^{\lambda}-f^{\lambda}\left(a^{1-v}b^{v}\right)}{(h(1-\mu)f(a)+h(\mu)f(b))^{\lambda}-f^{\lambda}\left(a^{1-\mu}b^{\mu}\right)}\leq h^{\lambda}\left(\frac{1-v}{1-\mu}\right).\tag{16}$$

Remark 2.13. *Notice that, if we take $f(x)=h(x)=x$ and $p=1$, in Corollary 2.12. Then we recapture Theorem 2.1 in [1].*

Now, by selecting $\phi(x)=\exp(x)$, in Theorem 2.11 we get the following new and important refinement and reversed for (G-G)-h-convex functions.

Theorem 2.14. *Let f be a(G-G)-h-convex function on $[a,b]$, and $0\leq v\leq\mu\leq 1$.*

1. *If h is a non-negative super-multiplicative and super-additive function, then*

$$\begin{aligned}&\left(f^{h(1-\mu)}(a)f^{h(\mu)}(b)\right)^{h\left(\frac{v}{\mu}\right)}-f^{h\left(\frac{v}{\mu}\right)}\left(a^{1-\mu}b^{\mu}\right)\\&\leq\left(f^{h(1-v)}(a)f^{h(v)}(b)\right)-f\left(a^{1-v}b^{v}\right).\end{aligned}$$

2. *If h is a non-negative multiplicative and super-additive function, then*

$$\begin{aligned}&\left(f^{h(1-v)}(a)f^{h(v)}(b)\right)-f\left(a^{1-v}b^{v}\right)\\&\leq\left(f^{h(1-\mu)}(a)f^{h(\mu)}(b)\right)^{h\left(\frac{1-v}{1-\mu}\right)}-f^{h\left(\frac{1-v}{1-\mu}\right)}\left(a^{1-\mu}b^{\mu}\right).\end{aligned}$$

3. New inequalities for τ-measurable operators

Let $\mathcal{M} \subset B(\mathcal{H})$ be a finite von Neumann algebra on the separable Hilbert space $\mathcal{H}$, namely, $\mathcal{M}$ is a $*$-sub-algebra of $B(\mathcal{H})$ containing the identity I, which is closed for the weak operator topology. A trace τ on the von Neumann algebra $\mathcal{M}$ is a map $\tau : \mathcal{M}^+ \mapsto [0, +\infty)$ which is additive, positively homogeneous and unitarily invariant, that is, $\tau(A) = \tau(U^*AU)$ for all $A \in \mathcal{M}^+$ and unitary $U \in \mathcal{M}$, where $\mathcal{M}^+ = \{A \in \mathcal{M}, A \geq 0\}$. A trace τ is called

1. faithful if for all $A \in \mathcal{M}^+$, $\tau(A) = 0$ implies that $A = 0$,
2. semi-finite if for every $A \in \mathcal{M}^+$, with $\tau(A) > 0$, there exists $0 \leq B \leq A$, such that $0 < \tau(B) < \infty$,
3. normal if $A_i \uparrow_i A \in \mathcal{M}^+$, implies that $\tau(A_i) \uparrow_i \tau(A)$,
4. finite if $\tau(I) < \infty$.

We say that an operator $A : \mathcal{D}(A) \to \mathcal{H}$ is τ-measurable if A affiliated with $\mathcal{M}$ (that is $AU = UA$ for all unitary $U \in \mathcal{M}$) and there exists $\delta > 0$ such that $\tau(e^{|A|}(\delta, \infty)) < \infty$.

For $0 < p < +\infty$, $L_p(\mathcal{M}, \tau)$ is defined as the set of all τ-measurable operators A affiliated with $\mathcal{M}$ such that

$$\|A\|_p = \tau(|A|^p)^{\frac{1}{p}} < +\infty.$$

Note that $L_p(\mathcal{M}, \tau)$ is a Banach space under $\|.\|_p$ for $1 \leq p < +\infty$, see [11] for more information. From now on, we denote by E a symmetric Banach space on $(0, \infty)$. In the following we consider the non-commutative symmetric Banach space $(E(\mathcal{M}), \|.\|_{E(\mathcal{M})})$ (see [22]), defined by

$$E(\mathcal{M}) := \{A \in L_0(\mathcal{M}) \ : \ \mu(A) \in E\} \text{ and } \|A\|_{E(\mathcal{M})} = \|\mu(A)\|_E,$$

where $\mu(A)$ is the function $t \mapsto \inf\{\delta > 0 \ : \ \tau(e^{|A|}(\delta, \infty)) \leq t\}$ called the decreasing rearrangement of A (cf. [5]). As known $(L_p(\mathcal{M}), \|.\|_p)$, $0 < p < \infty$ becomes a special case of the previous construction and the same for $L_\infty(\mathcal{M}) = \mathcal{M}$. Moreover, for $0 < r < \infty$, define

$$E(\mathcal{M})^{(r)} := \{A \in L_0(\mathcal{M}) \ : \ |A|^r \in E\} \text{ and } \|A\|_{E(\mathcal{M})^{(r)}} = \||A|^r\|^{\frac{1}{r}}_{E(\mathcal{M})}.$$

We know from [4, Proposition 3.1], that if E is a symetric (quasi) Banach space, then it is the same for $E(\mathcal{M})^{(r)}$. Recall that a norm $\|.\|$ on $\mathcal{M}$ is symmetric if $\|UAV\| = \|A\|$ for all $A \in \mathcal{M}$ and all unitary $U, V \in \mathcal{M}$.

3.1. *Operator (p,h)-convex function inequalities for τ-measurable operators*

We define operator (p, h)-convex functions of several variables, which generalize the known definition of the operator (p, h)-convexity. In the following we suppose that $I_i \subset \mathbb{R}^+$ for $i = 1, \ldots, n$ and $p > 0$.

Definition 3.1. *Let $A_1, A_2, \ldots, A_n \in \mathcal{M}^+$ and $B_1, B_2, \ldots, B_n \in \mathcal{M}^+$ be $2n$ τ-measurable operators where $n \geq 1$ is an integer. Let $p \in \mathbb{R} \setminus \{0\}$, $v \in [0, 1]$ and $h : J \to \mathbb{R}$ be a non-negative and non-zero function. A real function $f : I_1 \times \cdots \times I_n \to \mathbb{R}$ with $\sigma(A_i) \cup \sigma(B_i) \subset I_i$ for $i = 1, \ldots, n$ is said operator (p, h)-convex function of order n if*

$$\begin{aligned} f\Big(((1-v)A_1^p + vB_1^p)^{\frac{1}{p}}, \ldots, ((1-v)A_n^p + vB_n^p)^{\frac{1}{p}}\Big) \\ \leq h(1-v)f(A_1, \ldots, A_n) + h(v)f(B_1, \ldots, B_n). \end{aligned}$$

By taking $n = 1, p = 1$ and $h = I$ we get the classical notion of operator convexity and by taking $n = 1$ and $p = 1$ we get the notion of operator h-convexity.

Definition 3.2. *Let $A_1, A_2, \ldots, A_n \in \mathcal{M}^+$ be n τ-measurable positive operators where $n \geq 1$ is an integer. A real function $f : I_1 \times \cdots \times I_n \to \mathbb{R}$ with $\sigma(A_i) \cup \sigma(B_i) \subset I_i$ for $i = 1, \ldots, n$ is said operator positive of order $2n$ if*

$$f(A_1, \ldots, A_n) \geq 0.$$

Theorem 3.3. *Let $A_1, A_2, \ldots, A_n \in \mathcal{M}^+$ and $B_1, B_2, \ldots, B_n \in \mathcal{M}^+$ be $2n$ τ-measurable operators where $n \geq 1$ is an integer. Let h be a non-negative super-multiplicative and super-additive function on J, f be positive operator (p, h)-convex function on $I_1 \times \cdots \times I_n$ such that $\sigma(A_i) \cup \sigma(B_i) \subset I_i$ for all $i \in \{1, \ldots, n\}$ and $0 < v \leq \mu < 1$. Then we have*

$$\begin{aligned} h(1-v)f(A_1,\ldots,A_n) &+ h(v)f(B_1,\ldots,B_n) \\ &\geq f\left[((1-v)A_1^p + vB_1^p)^{\frac{1}{p}},\ldots,((1-v)A_n^p + vB_n^p)^{\frac{1}{p}}\right] \\ &+ h\Big(\frac{v}{\mu}\Big)\Big[h(1-v)f(A_1,\ldots,A_n) + h(v)f(B_1,\ldots,B_n) \\ &- f\left[((1-v)A_1^p + vB_1^p)^{\frac{1}{p}},\ldots,((1-v)A_n^p + vB_n^p)^{\frac{1}{p}}\right]\Big]. \end{aligned}$$

Proof. By hypothesis h is super-multiplicative and super-additive. Let $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ be $2n$ τ-measurable operators, we have

$$\begin{aligned} & h(1-v)f(A_1,\ldots,A_n) + h(v)f(B_1,\ldots,B_n) \\ & -h\Big(\frac{v}{\mu}\Big)\Big[h(1-\mu)f(A_1,\ldots,A_n) + h(\mu)(B_1,\ldots,B_n)\Big] \\ & -h\Big(\frac{v}{\mu}\Big)f\left[((1-\mu)A_1^p + \mu B_1^p)^{\frac{1}{p}},\ldots,((1-\mu)A_n^p + \mu B_n^p)^{\frac{1}{p}}\right] \\ & = \Big(h(1-v) - h\Big(\frac{v}{\mu}\Big)h(1-\mu)\Big)f(A_1,\ldots,A_n) \\ & +\Big(h(v) - h\Big(\frac{v}{\mu}\Big)h(\mu)\Big)f(B_1,\ldots,B_n) \\ & +h\Big(\frac{v}{\mu}\Big)f\Big[((1-\mu)A_1^p + \mu B_1^p)^{\frac{1}{p}},\ldots,((1-\mu)A_n^p + \mu B_n^p)^{\frac{1}{p}}\Big] \\ & \geq h\Big(1 - \frac{v}{\mu}\Big)f(A_1,\ldots,A_n) \\ & +h\Big(\frac{v}{\mu}\Big)f\Big[((1-\mu)A_1^p + \mu B_1^p)^{\frac{1}{p}},\ldots,((1-\mu)A_1^p + \mu B_1^p)^{\frac{1}{p}}\Big] \\ & \geq f\left[\Big((1-\frac{v}{\mu})A_1^p + \frac{v}{\mu}(1-\mu)A_1^p + vB_1^p\Big)^{\frac{1}{p}},\ldots,\Big((1-\frac{v}{\mu})A_n^p + \frac{v}{\mu}(1-\mu)A_n^p + vB_n^p\Big)^{\frac{1}{p}}\right] \\ & = f\Big[((1-\mu)A_1^p + \mu B_1^p)^{\frac{1}{p}},\ldots,((1-\mu)A_n^p + \mu B_n^p)^{\frac{1}{p}}\Big]. \end{aligned}$$

The proof is thus completed. □

Theorem 3.4. *Let $A_1, A_2, \ldots, A_n \in \mathcal{M}^+$ and $B_1, B_2, \ldots, B_n \in \mathcal{M}^+$ be $2n$ τ-measurable operators where $n \geq 1$ is an integer. Let h be a non-negative multiplicative and super-additive function on J and f be a positive operator (p, h)-convex function on $I_1 \times \cdots \times I_n$ such that $\sigma(A_i) \cup \sigma(B_i) \subset I_i$ for all $i \in \{1, \ldots, n\}$ and $0 \leq v \leq \mu \leq 1$. Then we have*

$$\begin{aligned} h(1-v)f(A_1,\ldots,A_n) &+ h(v)f(B_1,\ldots,B_n) \\ &\leq f\Big[((1-v)A_1^p + vB_1^p)^{\frac{1}{p}},\ldots,((1-v)A_n^p + vB_n^p)^{\frac{1}{p}}\Big] \\ &+ h\Big(\frac{1-v}{1-\mu}\Big)\Big[h(1-\mu)f(A_1,\ldots,A_n) + h(\mu)f(B_1,\ldots,B_n) \\ &- f\Big[((1-\mu)A_1^p + \mu B_1^p)^{\frac{1}{p}},\ldots,((1-\mu)A_n^p + \mu B_n^p)^{\frac{1}{p}}\Big]\Big]. \end{aligned}$$

Proof. Since, h is multiplicative and super-additive and f be a positive operator function, we have

$$\begin{aligned}
h(1-\mu)f(A_1,\dots,A_n) &+ h(\mu)f(B_1,\dots,B_n)\\
&-\frac{h(1-v)}{h\big(\frac{1-v}{1-\mu}\big)}f(A_1,\dots,A_n)-\frac{h(v)}{h\big(\frac{1-v}{1-\mu}\big)}f(B_1,\dots,B_n)\\
&+\frac{1}{h\big(\frac{1-v}{1-\mu}\big)}f\Big[((1-v)A_1^p+vB_1^p)^{\frac{1}{p}},\dots,((1-v)A_n^p+vB_n^p)^{\frac{1}{p}})\Big]\\
&=\Big(h(1-\mu)-h(1-\mu)\Big)f(A_1,\dots,A_n)\\
&+\Big(h(\mu)-h\Big(\frac{v(1-\mu)}{1-v}\Big)\Big)f(B_1,\dots,B_n)\\
&+h\Big(\frac{1-\mu}{1-v}\Big)f\Big[((1-v)A_1^p+vB_1^p)^{\frac{1}{p}},\dots,((1-v)A_n^p+vB_n^p)^{\frac{1}{p}}\Big]\\
&\geq h\Big(1-\frac{1-\mu}{1-v}\Big)f(B_1,\dots,B_n)\\
&+h\Big(\frac{1-\mu}{1-v}\Big)f\Big[((1-v)A_1^p+vB_1^p)^{\frac{1}{p}},\dots,((1-v)A_n^p+vB_n^p)^{\frac{1}{p}}\Big]\\
&\geq f\Big[\Big(\Big(1-\frac{1-\mu}{1-v}\Big)B_1^p+\Big(\frac{1-\mu}{1-v}\Big)((1-v)A_1^p+vB_1^p)\Big)^{\frac{1}{p}},\dots\\
&,\Big(\Big(1-\frac{1-\mu}{1-v}\Big)B_n^p+\Big(\frac{1-\mu}{1-v}\Big)((1-v)A_n^p+vB_n^p)\Big)^{\frac{1}{p}}\Big]\\
&=f\Big[((1-\mu)A_1^p+\mu B_1^p)^{\frac{1}{p}},\dots,((1-\mu)A_n^p+\mu B_n^p)^{\frac{1}{p}}\Big].
\end{aligned}$$

At the end, it suffice to multiply the last inequality by $h\left(\frac{1-v}{1-\mu}\right)$, this end the proof. □

3.2. *Hölder's type inequalities for τ-measurable operators*

In this part of the paper, by selecting some appropriate log-convex functions, we obtain refinement and reverse of some Hölder's type inequalities for τ-measurable operators.
The celebrated Hölder's inequality for τ-measurable operators is as follows. For $r > 0, A, B \in \mathcal{M}^+$ $X \in E(\mathcal{M})^{(r)}$ and $0 < v < 1$, we have

$$\left\|A^{1-v}XB^v\right\|_{E(\mathcal{M})^{(r)}} \leq \|AX\|_{E(\mathcal{M})^{(r)}}^{1-v}\|XB\|_{E(\mathcal{M})^{(r)}}^{v}. \tag{17}$$

It is known that when $A, B \in \mathcal{M}^+$ and $X \in E(\mathcal{M})^{(r)}$ the function

$$f(v) = \|A^{1-v}XB^v\|_{E(\mathcal{M})^{(r)}}$$

is log-convex on $[0,1]$, (see [20]) for any symmetric norm $\|.\|_{E(\mathcal{M})^{(r)}}$. By using Theorem 2.10, for $h(x) = x$ and $p = 1$, we have the following theorem which presents a refinement and reversed of the corresponding Hölder's type inequality (17) for τ-measurable operators.

Theorem 3.5. *Let $r > 0, A, B \in \mathcal{M}^+$ and $X \in E(\mathcal{M})^{(r)}$. Then, for $0 \leq v \leq \mu \leq 1$,*

$$\begin{aligned}
&\Big(\|AX\|_{E(\mathcal{M})^{(r)}}^{1-\mu}\|XB\|_{E(\mathcal{M})^{(r)}}^{\mu}\Big)^{\frac{v}{\mu}} - \left\|A^{1-\mu}XB^{\mu}\right\|_{E(\mathcal{M})^{(r)}}^{\frac{v}{\mu}}\\
&\leq \|AX\|_{E(\mathcal{M})^{(r)}}^{v}\|XB\|_{E(\mathcal{M})^{(r)}}^{1-v} - \left\|A^{v}XB^{1-v}\right\|_{E(\mathcal{M})^{(r)}}\\
&\leq \Big(\|AX\|_{E(\mathcal{M})^{(r)}}^{1-\mu}\|XB\|_{E(\mathcal{M})^{(r)}}^{\mu}\Big)^{\frac{1-v}{1-\mu}} - \left\|A^{1-\mu}XB^{\mu}\right\|_{E(\mathcal{M})^{(r)}}^{\frac{1-v}{1-\mu}}.
\end{aligned} \tag{18}$$

In particular, if $\mathcal{M}$ is a finite von Neumann algebra, then

$$\begin{aligned}\left(\tau(A)^{1-\mu}\tau(B)^{\mu}\right)^{\frac{v}{\mu}} - \tau^{\frac{v}{\mu}}\left(A^{1-\mu}B^{\mu}\right) &\leq \tau(A)^{v}\tau(B)^{1-v} - \tau\left(A^{v}B^{1-v}\right) \\ &\leq \left(\tau(A)^{1-\mu}\tau(B)^{\mu}\right)^{\frac{1-v}{1-\mu}} - \tau^{\frac{1-v}{1-\mu}}\left(A^{1-\mu}B^{\mu}\right).\end{aligned}$$

Also, it is known that when $A, B \in \mathcal{M}^{+}$ and $X \in E(\mathcal{M})^{(r)}$ the function $f(v) = \|A^{v}XB^{v}\|_{E(\mathcal{M})^{(r)}}$ is log-convex on $[0,1]$, (see [20]) for any symmetric norm $\|.\|_{E(\mathcal{M})^{(r)}}$. In view of Theorem 2.10, for $h(x) = x$ and $p = 1$, we have the following theorem.

Theorem 3.6. *Let $r > 0, A, B \in \mathcal{M}^{+}$and $X \in E(\mathcal{M})^{(r)}$. Then, for $0 \leq v \leq \mu \leq 1$,*

$$\begin{aligned}&\left(\|X\|_{E(\mathcal{M})^{(r)}}^{1-\mu}\|AXB\|_{E(\mathcal{M})^{(r)}}^{\mu}\right)^{\frac{v}{\mu}} - \|A^{\mu}XB^{\mu}\|_{E(\mathcal{M})^{(r)}}^{\frac{v}{\mu}} \\ &\leq \|X\|_{E(\mathcal{M})^{(r)}}^{1-v}\|AXB\|_{E(\mathcal{M})^{(r)}}^{v} - \|A^{v}XB^{v}\|_{E(\mathcal{M})^{(r)}} \\ &\leq \left(\|X\|_{E(\mathcal{M})^{(r)}}^{1-\mu}\|AXB\|_{E(\mathcal{M})^{(r)}}^{\mu}\right)^{\frac{1-v}{1-\mu}} - \|A^{\mu}XB^{\mu}\|_{E(\mathcal{M})^{(r)}}^{\frac{1-v}{1-\mu}}.\end{aligned} \tag{19}$$

In particular, if $X = I$, we get

$$\begin{aligned}&\left(\|AB\|_{E(\mathcal{M})^{(r)}}^{\mu}\right)^{\frac{v}{\mu}} - \|A^{\mu}B^{\mu}\|_{E(\mathcal{M})^{(r)}}^{\frac{v}{\mu}} \\ &\leq \|AB\|_{E(\mathcal{M})^{(r)}}^{v} - \|A^{v}B^{v}\|_{E(\mathcal{M})^{(r)}} \\ &\leq \left(\|AB\|_{E(\mathcal{M})^{(r)}}^{\mu}\right)^{\frac{1-v}{1-\mu}} - \|A^{\mu}B^{\mu}\|_{E(\mathcal{M})^{(r)}}^{\frac{1-v}{1-\mu}}.\end{aligned} \tag{20}$$

For $A, B \in \mathcal{M}^{+}$ and $X \in E(\mathcal{M})^{(r)}$ the function

$$f(v) = \|A^{1-v}XB^{v}\|_{E(\mathcal{M})^{(r)}}\|A^{v}XB^{1-v}\|_{E(\mathcal{M})^{(r)}},$$

is log-convex on $[0,1]$, (see [20]) for any symmetric norm $\|.\|_{E(\mathcal{M})^{(r)}}$. Then by using Theorem 2.10, for $h(x) = x$ and $p = 1$, we have the following theorem.

Theorem 3.7. *Let $r > 0, A, B \in \mathcal{M}^{+}$and $X \in E(\mathcal{M})^{(r)}$. Then, for $0 \leq v \leq \mu \leq 1$, we have*

$$\begin{aligned}&\left(\|AX\|_{E(\mathcal{M})^{(r)}}\|XB\|_{E(\mathcal{M})^{(r)}}\right)^{\frac{v}{\mu}} - \left(\|A^{1-\mu}XB^{\mu}\|_{E(\mathcal{M})^{(r)}}\|A^{\mu}XB^{1-\mu}\|_{E(\mathcal{M})^{(r)}}\right)^{\frac{v}{\mu}} \\ &\leq \left(\|AX\|_{E(\mathcal{M})^{(r)}}\|XB\|_{E(\mathcal{M})^{(r)}}\right)^{\frac{v}{\mu}} - \left(\|A^{1-v}XB^{v}\|_{E(\mathcal{M})^{(r)}}\|A^{v}XB^{1-v}\|_{E(\mathcal{M})^{(r)}}\right)^{\frac{v}{\mu}} \\ &\leq \left(\|AX\|_{E(\mathcal{M})^{(r)}}\|XB\|_{E(\mathcal{M})^{(r)}}\right)^{\frac{v}{\mu}} - \left(\|A^{1-\mu}XB^{\mu}\|_{E(\mathcal{M})^{(r)}}\|A^{\mu}XB^{1-\mu}\|_{E(\mathcal{M})^{(r)}}\right)^{\frac{v}{\mu}}.\end{aligned} \tag{21}$$

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