Filomat 37:16 (2023), 5283–5286
https://doi.org/10.2298/FIL2316283C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Sendov's conjecture

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Abstract. We will give some sufficient conditions, which imply the conjecture of Sendov. We use convexity methods in order to prove the main result.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ be the closed unit disk in $\mathbb{C}$. Let $\mathbb{C}[z]$ denote the set of polynomials $P(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-3} + a_{n-1} z + a_n$, where $a_k \in \mathbb{C}$, $k \in \{0, 1, 2, \ldots, n\}$ and $n \in \mathbb{N}^*$.
We will prove sufficient conditions regarding the roots of a polynomial $P \in \mathbb{C}[z]$ which imply the following conjecture, attributed to the bulgarian mathematician Blagovest Sendov.

Conjecture 1.1. *If all the roots of a polynomial $P \in \mathbb{C}[z]$ lie in $\mathbb{D}$ and z^* is an arbitrary root of the polynomial P then the disk $\{z \in \mathbb{C} : |z - z^*| \leq 1\}$ contains at least one root of P'.*

In [6] it is proved the Conjecture 1.1 holds for sufficiently high degree polynomials. This result turns back our attention to the particular cases.
In [5] the author proved the following results:

Theorem 1.2. *Let $P \in \mathbb{C}[z]$, $P(z) = z^n + a_1 z^{n-1} + \ldots + a_n$. If $P(z_1) = 0$ and $|P'(z_1)| < n$, then the disk $|z - z_1| < 1$ contains at last one critical point of P.*

Theorem 1.3. *Let $P(z)$ be a polynomial whose zeros $z_1, z_2, z_3, \ldots, z_n$ $(n > 2)$ lie in $|z| \leq 1$ such that $|z_1| = 1$. Then the disk $|z - z_1| < 1$ always contains a zero of $P'(z) = 0$.*

This theorems imply the following interesting corollary.

Corollary 1.4. *Let z_k, $k \in \{1, 2, 3, \ldots, n-1\}$ be the affixes of the vertices of a regular n gone inscribed the unit circle $|z| = 1$.*
If z_0 is an arbitrary point in $\mathbb{D}$, then in case of polynomial $Q(z) = (z - z_0) \prod_{k=1}^{n-1} (z - z_k)$ the Sendov conjecture holds.

2020 *Mathematics Subject Classification*. Primary 33C10; Secondary 30C45.
Keywords. Polynomial, critical points, Sendov's conjecture.
Received: 28 September; Accepted: 12 January 2023
Communicated by Miodrag Mateljević
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Proof. Indeed, in case of z_k, $k \in \{1, 2, \ldots, n-1\}$ we have $|z_k| = 1$ and consequently Theorem 1.3 implies the assertion.
In case of $z_0 \in \mathbb{D}$ we have $|z_0| < 1$. Let z^* be the affixum of the n-th vertice of the regular n gon. Then the complex numbers $\overline{z^*}z_1, \overline{z^*}z_2, \overline{z^*}z_3, \ldots, \overline{z^*}z_{n-1}$ are the roots of the equation

$$z^{n-1} + z^{n-2} + z^{n-3} + \ldots + z + 1 = 0.$$

Since $|z^*| = 1$, we get

$$\begin{aligned}|Q'(z_0)| = \prod_{k=1}^{n-1} |z_0 - z_k| = \prod_{k=1}^{n-1} |\overline{z^*}z_0 - z_k\overline{z^*}| = \Big|\prod_{k=1}^{n-1}(\overline{z^*}z_0 - z_k\overline{z^*})\Big| = \\ \Big|(\overline{z^*}z_0)^{n-1} + (\overline{z^*}z_0)^{n-2} + \ldots + \overline{z^*}z_0 + 1\Big| \leq \\ |\overline{z^*}z_0|^{n-1} + |\overline{z^*}z_0|^{n-2} + \ldots + |\overline{z^*}z_0| + 1 = \\ |z_0|^{n-1} + |z_0|^{n-2} + \ldots + |z_0| + 1 < n. \end{aligned} \tag{1}$$

Thus Sendov's conjecture holds in case of the root z_0 too. □

Interesting results about Sendov conjecture are also obtained by Kumar, see [7].
The aim of this paper is to deduce new conditions regarding the roots of a polynomial P which imply the conjecture of Sendov like the previous theorems and corollary.
In order to prove the main result we need the following lemmas.

2. Preliminaries

Lemma 2.1 (Krein-Milman). *A compact convex subset of a Hausdorff locally convex topological vector space is equal to the closed convex hull of its extreme points.*

Lemma 2.2 (Gauss-Lucas). *If P is a (nonconstant) polynomial with complex coefficients, then all the zeros of the derivative P′ belong to the convex hull of the zeros of P.*

3. The Main Result

Theorem 3.1. *Let $P \in \mathbb{C}[z]$, $P(z) = z^n + a_1 z^{n-1} + \ldots + a_n$ be a complex polynomial. Suppose that all the roots of the polynomial P are in the unit disk $\mathbb{D}$. Suppose that z^* is a root of P and the circle $|z - z^*| = 1$ intersects $\partial\mathbb{D}$ at the points A and B. Let the closed set $\mathcal{K}$ be limited by the arc 5.0pt$\widehat{AB}$ of the circle $|z - z^*| = 1$, which does not belong to $\mathbb{D}$ and the line segment $[AB]$ and let the set Ω be defined by $\Omega = \mathbb{D} \setminus \mathcal{K}$.*
If in case of a fixed $k \in \{1, 2, 3, \ldots, n-1\}$ the equation $P^{(k)}(z) = 0$ has a root in $\mathcal{K}$, then the $|z - z^*| < 1$ disk contains a root of $P'(z) = 0$.

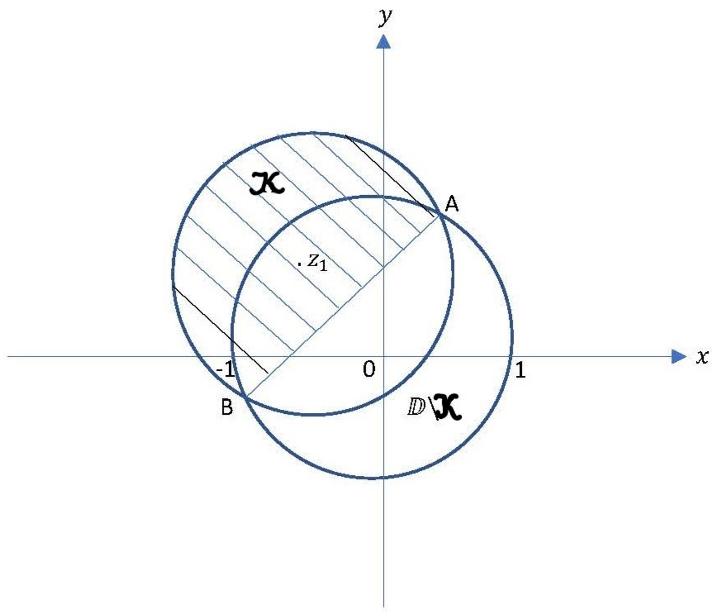
Proof. Let denote the closed convex hull of the roots of $P^{(k)}(z) = 0$ by C(k). The Gauss-Lucas theorem implies the inclusions:

$$C(n-1) \subset C(n-2) \subset \ldots \subset C(k) \subset \ldots \subset C(1) \subset C(0). \tag{2}$$

The sets $\mathcal{K}$ and Ω are convex.
According to the conditions of the theorem, we have $C(k) \cup \mathcal{K} \neq \emptyset$ for some $k \in \{1, 2, 3, \ldots, n-1\}$.

$$\text{The inclusions (2) imply } C(1) \cap \mathcal{K} \neq \emptyset. \tag{3}$$

The extreme points of $C(1)$ are between the roots of $P'(z) = 0$. Suppose all the extreme points are elements of Ω, then the convexity of Ω and the Krein-Milman theorem would imply $C(1) \subset \Omega$ and this contradicts (3). This contradiction shows that $\mathcal{K}$ contains extreme points of $C(1)$ and these extreme points are roots of $P'(z) = 0$. □



Taking particular cases of the proved result, we get interesting conditions regarding to the roots of a polynomial which imply the Sendov's conjecture.

Corollary 3.2. *Suppose that the degree of the polynomial $Q \in \mathbb{C}[z]$ is less than $n-2$ and all the roots of the polynomial*

$$P(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + Q(z)$$

are in the unit disk $\mathbb{D}$. If z^* is a root of the polynomial P which satisfies one of the following two inequalities

$$\left|\frac{-a_1 + \sqrt{a_1^2 - \frac{2n}{n-1}a_2}}{n} - z^*\right| < \frac{|z^*|}{2}, \tag{4}$$

or

$$\left|\frac{-a_1 - \sqrt{a_1^2 - \frac{2n}{n-1}a_2}}{n} - z^*\right| < \frac{|z^*|}{2}, \tag{5}$$

then the Sendov's conjecture holds in case of z^*, that is the disc $|z - z^*| < 1$ contains a critical point.

Proof. We have $P^{(n-2)}(z) = 0 \Leftrightarrow n(n-1)z^2 + 2(n-1)a_1 z + 2a_2 = 0$.
The conditions (4) and (5) imply that

$$C(n-2) \cap \mathcal{K} \neq \emptyset.$$

Thus the derivative of order $n-2$ of P has a root in $\mathcal{K}$ and Theorem 1.3 implies Sendov's conjecture in case of the root z^*. □

Corollary 3.3. *Suppose that the degree of the polynomial $Q \in \mathbb{C}[z]$ is less than $n-1$ and all the roots of the polynomial $P(z) = z^n - n\alpha z^{n-1} + Q(z)$ are in the unit disk $\mathbb{D}$. If z^* is a root of the polynomial P which satisfies $|\alpha - z^*| < \frac{|z^*|}{2}$, then the Sendov's conjecture holds in case of z^*, that is the disc $|z - z^*| < 1$ contains a critical point.*

Proof. We have $P^{(n-1)}(z) = n(n-1)(n-2)\ldots 2z - n!\alpha$ with the root $z_0 = \alpha$. The inequality $|\alpha - z^*| < \frac{|z^*|}{2}$, is equivalent to $|z_0 - z^*| < \frac{|z^*|}{2}$, which implies $z_0 \in \mathcal{K}$. Thus the derivative of order $n-1$ of P has a root in $\mathcal{K}$ and Theorem 1.3 implies Sendov's conjecture in case of the root z^*. □

Example 3.4. *Let $P(z) = z^3 + a_1 z^2 + a_2 z + a_3$ be the monic polynomial with the roots $z_1 = \frac{1}{2} + i\frac{1}{3}$, $z_2 = \frac{1}{3} + i\frac{1}{2}$, $z_3 = \frac{5}{6} + i\frac{1}{10}$.*
We use the notations of Corollary 1.4: $\alpha = \frac{z_1+z_2+z_3}{3} = \frac{5}{6} + i\frac{14}{45}$ and $z^* = \frac{5}{6} + i\frac{1}{10}$. We have $|\alpha - z^*| = \frac{19}{90} < \frac{1}{2}\sqrt{\frac{143}{180}} = \frac{|z^*|}{2}$, and consequently the conjecture of Sendov holds in case of $z^* = z_3$.
A simple calculation shows that $3 > \left|P(z_1)\right|$ and $3 > \left|P(z_2)\right|$, thus according to Theorem 1.2 Sendov's conjecture holds in case of z_1 and z_2.

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