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On non-adjointable semi-C*-Fredholm operators and semi-C*-Weyl operators

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Abstract. We extend the results from semi-Fredholm theory of adjointable, bounded C*-operators on the standard C*-module, presented in [3], to the case of general bounded C*-operators on arbitrary Hilbert C*-modules. Next, in the special case of the standard C*-module, we show that the set of those semi-C*-Fredholm operators that are not semi-C*-Weyl operators is open in the norm topology, and that the set of non-adjointable semi-C*-Weyl operators is invariant under perturbations by general compact operators. Moreover, we provide an extended Schechter characterization and a generalized Fredholm alternative in the case of adjointable C*-operators on the standard C*-module. Finally, we provide examples of semi-C*-Fredholm operators.

1. Introduction

The Fredholm and semi-Fredholm theory on Hilbert and Banach spaces started by studying the certain integral equations introduced in the pioneering work by Fredholm in 1903 in [1]. After that, the abstract theory of Fredholm and semi-Fredholm operators on Banach spaces was further developed in numerous papers.

A special part of semi-Fredholm theory is semi-Weyl theory. Semi-Weyl operators have been considered in several papers. We recall that an operator on a Banach space is called upper semi-Weyl if the operator is an upper semi-Fredholm operator with negative index, whereas an operator is called lower semi-Weyl if the operator is lower semi-Fredholm with positive index. A Weyl operator is a Fredholm operator with zero index.

Now, Hilbert C^*-modules are natural generalization of Hilbert spaces when the field of scalars is replaced by an arbitrary C^*-algebra.
Fredholm theory on Hilbert C^*-modules as a generalization of Fredholm theory on Hilbert spaces was started by Mishchenko and Fomenko in [9]. They have introduced the notion of a Fredholm C^*-operator on the standard module over a unital C^*-algebra. Moreover, they have shown that the set of these generalized Fredholm operators is open in the norm topology, that it is invariant under compact perturbation, and they have proved the generalization of the Atkinson theorem and of the index theorem. The interest for studying such operators comes from operators that arise from natural cases, e.g. (pseudo) differential

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operators acting on manifolds. The classical theory works nice for compact manifolds, but not for general ones. Even operators on Euclidean spaces are hard to study, e.g. Laplacian is not Fredholm. However, they can become Fredholm when we look at them as operators on a torus with coefficients in the group C^*-algebra of the integers (as the torus is the quotient of the Euclidean space modulo the action of integers). Kernels and cokernels of many operators are infinite-dimensional as Banach spaces, but become finitely generated viewed as Hilbert modules. This is the most important reason for studying semi-C^*-Fredholm operators.

In [3] we went further in this direction and defined adjointable semi-C^*-Fredholm and adjointable semi-C^*-Weyl operators on Hilbert C^*-modules. We investigated then and proved several properties of these generalized semi-Fredholm and semi-Weyl operators on Hilbert C^*-modules as an analogue or a generalization of the well-known properties of the classical semi-Fredholm and semi-Weyl operators on Hilbert and Banach spaces.

The main aim of this paper is to extend the results from [3] in several directions, as listed below.
In Section 3 of this paper we extend the results in [3] to arbitrary Hilbert C^*-modules. One of the limitations of several results in [3] is that they are proved only for the standard module case. The proofs of these results can not be applied to the case of arbitrary Hilbert C^*-modules because they rely on the fact that semi-C^*-Fredholm operators on the standard module are exactly those operators on the standard module that are one-sided invertible modulo compact operators, the fact which has so far been proved only for the standard module case and not for the case of general modules. Thanks to Lemma 3.1 and Lemma 3.7 aa in this paper, we provide new proofs of these results that allow us to extend these results to the case of arbitrary Hilbert C^*-modules.
In Section 4, we work with non-adjointable semi-C^*-Weyl operators and prove that the set of upper semi-C^*-Weyl operators is invariant under perturbations by compact operators where we consider compact operators in the sense of Irmatov and Mischenko as defined in [2]. We recall that not all bounded C^*-operators admit an adjoint. In [3] we consider only adjointable C^*-operators, however, in this paper we consider additionally non-adjointable C^*-operators. In addition, in Section 4 we prove that set consisting of those semi-C^*-Fredholm operators that are not semi-C^*-Weyl operators is open in the norm topology and we deduce various corollaries from this result.
In Section 5, we introduce in terms of equivalent conditions an improved version of generalized Schechter' s characterization of upper semi-C*-Fredholm operators given in [3]. Moreover, we provide a generalization of the Fredholm alternative in the setting of operators on the standard module over a C^*-algebra whose K-group satisfies the cancellation property. Also, we show in a counterexample that this generalized Fredholm alternative does not hold if we consider the standard module over $B(H)$ where H is a separable, infinite-dimensional Hilbert space.

At the end, in Section 6 we provide concrete examples of semi-C^*-Fredholm operators. We use the structure of the C^*-algebra itself in order to construct these new examples different from the classical examples of semi-Fredholm operators on Hilbert spaces.

The paper contains the unpublished results from the PhD thesis by the author, see [5].

2. Preliminaries

In this paper we let $\mathcal{A}$ denote a unital C^*-algebra. For a right Hilbert $\mathcal{A}$-module M we let $B(M)$ denote the Banach algebra of $\mathcal{A}$-linear bounded operators on M, whereas we will denote by $B^a(M)$ the C^*-algebra of all $\mathcal{A}$-linear, bounded, adjointable operators on M. In this paper we will only consider right Hilbert $\mathcal{A}$-modules.
Next, we let $\mathcal{K}^*(M)$ denote the norm closure of the linear span of elementary operators on M. We recall from [8] that $\mathcal{K}^*(M)$ is a closed, two-sided ideal in $B^a(M)$.

By the symbol $\tilde{\oplus}$ we denote the direct sum of modules as given in [8].

Thus, if M is a Hilbert C^*-module and M_1, M_2 are two closed submodules of M, we write $M = M_1 \tilde{\oplus} M_2$ if $M_1 \cap M_2 = \{0\}$ and $M_1 + M_2 = M$. If, in addition M_1 and M_2 are mutually orthogonal, then we write $M = M_1 \oplus M_2$.

We recall some examples of Hilbert C^*-modules.

Example 2.1. *[8, Example 1.3.3] If $J \subset \mathcal{A}$ is a closed right ideal, then the pre-Hilbert module J is complete with respect to the norm $\| \cdot \|_J = \| \cdot \|$. In particulaar, the unital C^*-algebra $\mathcal{A}$ itself is a free Hilbert $\mathcal{A}$-module with one generator.*

Example 2.2. *[8, Example 1.3.4] If $\{\mathcal{M}_i\}$ is a finite set of Hilbert $\mathcal{A}$-modules, then one can define the direct sum $\oplus \mathcal{M}_i$. The inner product on $\oplus \mathcal{M}_i$ is given by the formula $\langle x, y \rangle := \sum_i \langle x_i, y_i \rangle$ where $x = (x_i), y = (y_i) \in \oplus \mathcal{M}_i$. We denote the direct sum of n copies of a Hilbert module $\mathcal{M}$ by $\mathcal{M}^n$ or $L_n(\mathcal{M})$.*

In the case when $\mathcal{M} = \mathcal{A}$, we will simply denote $L_n(\mathcal{A})$ by L_n in the rest of the paper.

Example 2.3. *[8, Example 1.3.5] If $\{\mathcal{M}_i\}, i \in \mathbb{N}$, is a countable set of Hilbert $\mathcal{A}$-modules, then one can define their direct sum $\oplus \mathcal{M}_i$ to be the set of all sequences $x = (x_i) : x_i \in \mathcal{M}_i$, such that the series $\sum_i \langle x_i, y_i \rangle$ is norm-convergent in the C^*-algebra $\mathcal{A}$. Then we define the inner product by*

$$\langle x, y \rangle := \sum_i \langle x_i, y_i \rangle \text{ for } x, y \in \oplus \mathcal{M}_i.$$

With respect to this inner product $\oplus \mathcal{M}_i$ is a Hilbert $\mathcal{A}$-module. If each $\mathcal{M}_i = \mathcal{A}$, then we will denote $\oplus \mathcal{M}_i$ by $H_{\mathcal{A}}$. ***This module is called the standard module over $\mathcal{A}$. So, in other words $H_{\mathcal{A}} = l^2(\mathcal{A})$.*** *If $\mathcal{A}$ is unital, then $H_{\mathcal{A}} = l^2(\mathcal{A})$ has a natural orthonormal basis $\{e_j\}_{j \in \mathbb{N}}$.*

Definition 2.4. *[2, Definition 1] An $\mathcal{A}$-operator $K : H_{\mathcal{A}} \to H_{\mathcal{A}}$ is called a finitely generated $\mathcal{A}$-operator if it can be represented as a composition of bounded $\mathcal{A}$-operators f_1 and f_2:*

$$K : H_{\mathcal{A}} \xrightarrow{f_1} M \xrightarrow{f_2} H_{\mathcal{A}},$$

where M is a finitely generated Hilbert C^*-module. The set $\mathcal{FG}(\mathcal{A}) \subset B(H_{\mathcal{A}})$ of all finitely generated $\mathcal{A}$-operators forms a two-sided ideal. By definition, an $\mathcal{A}$-operator K is called compact if it belongs to the closure

$$\mathcal{K}(H_{\mathcal{A}}) = \overline{\mathcal{FG}(\mathcal{A})} \subset B(H_{\mathcal{A}}),$$

which also forms two-sided ideal.

As observed in [2], in general, the set $\mathcal{FG}(\mathcal{A}) \subset B(H_{\mathcal{A}})$ is not a closed subset. For example, in classical case, when $\mathcal{A} = \mathbb{C}$ the set $\mathcal{FG}(\mathcal{A})$ consists of all finite rank operators, while not all compact operators are finite rank operators if the space is infinite-dimensional.

Definition 2.5. *Let M be a Hilbert $\mathcal{A}$-module and $F \in B(M)$. We say that F is an upper semi-$\mathcal{A}$-Fredholm operator if there exists a decomposition*

$$M = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = M$$

with respect to which F has the matrix

$$\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix},$$

where F_1 is an isomorphism, M_1, M_2, N_1, N_2 are closed submodules of M and N_1 is finitely generated. Similarly, we say that F is a lower semi-$\mathcal{A}$-Fredholm operator if all the above conditions hold except that in this case we assume that N_2 (and not N_1) is finitely generated. If both N_1 and N_2 are finitely generated, then F is $\mathcal{A}$-Fredholm operator.

Set

$$\widehat{\mathcal{M}\Phi}_l(M) = \{F \in B(M) \mid F \text{ is upper semi-}\mathcal{A}\text{-Fredholm }\},$$

$$\widehat{\mathcal{M}\Phi}_r(M) = \{F \in B(M) \mid F \text{ is lower semi-}\mathcal{A}\text{-Fredholm }\},$$

$$\widehat{\mathcal{M}\Phi}(M) = \{F \in B(M) \mid F \text{ is } \mathcal{A}\text{-Fredholm operator on } M\}.$$

Then we put

$$\mathcal{M}\Phi_+(M) = \widehat{\mathcal{M}\Phi}_l(M) \cap B^a(M),$$
$$\mathcal{M}\Phi_-(M) = \widehat{\mathcal{M}\Phi}_r(M) \cap B^a(M)$$

and

$$\mathcal{M}\Phi(M) = \widehat{\mathcal{M}\Phi}(M) \cap B^a(M).$$

Remark 2.6. *It is not hard to see that F is $\mathcal{A}$-Fredholm operator in the sense of Definition 2.8 if and only if F is $\mathcal{A}$-Fredholm in the sense of [2].*

Definition 2.7. *[3, Definition 5.6] Let M be a Hilbert $\mathcal{A}$-module and $F \in \widehat{\mathcal{M}\Phi}_l(M)$. We say that $F \in \widehat{\mathcal{M}\Phi}_l^{-'}(M)$ if there exists a decomposition*

$$M = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = M$$

with respect to which

$$F = \begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix},$$

where F_1 is an isomorphism, N_1 is closed, finitely generated and $N_1 \preceq N_2$. Similarly, we define the class $\widehat{\mathcal{M}\Phi}_r^{+'}(M)$, only in this case $F \in \widehat{\mathcal{M}\Phi}_r(M)$, N_2 is finitely generated and $N_2 \preceq N_1$.
Such operators will be called semi-$\mathcal{A}$-Weyl operators throughout the paper.

Further, we define $\widehat{\mathcal{M}\Phi}_0(M)$ to be the set of all $F \in \widehat{\mathcal{M}\Phi}(M)$ for which there exists an $\widehat{\mathcal{M}\Phi}$-decomposition

$$M = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = M,$$

where $N_1 \cong N_2$.
Such operators will be called $\mathcal{A}$-Weyl operators throughout the paper.

Definition 2.8. *[7] [8, Definition 2.7.1] Let M be an abelian monoid. Consider the Cartesian product $M \times M$ and its quotient monoid with respect to the equivalence relation*

$$(m, n) \sim (m', n') \Leftrightarrow \exists p, q : (m, n) + (p, p) = (m', n') + (q, q).$$

This quotient monoid is a group, which is denoted by $S(M)$ and is called the symmetrization of M. Consider now the additive category $\mathcal{P}(\mathcal{A})$ of projective modules over a unital C^*-algebra $\mathcal{A}$ and denoted by $[\mathcal{M}]$ the isomorphism class of an object $\mathcal{M}$ from $\mathcal{P}(\mathcal{A})$. The set $\phi(\mathcal{P}(\mathcal{A}))$ of these classes has the structure of an Abelian monoid with respect to the operation $[\mathcal{M}] + [\mathcal{N}] = [\mathcal{M} \oplus \mathcal{N}]$. In this case the group $S(\phi(\mathcal{P}(\mathcal{A})))$ is denoted by $K(\mathcal{A})$ or $K_0(\mathcal{A})$ and is called the K-group of $\mathcal{A}$ or the Grothendieck group of the category $\mathcal{P}(\mathcal{A})$.

As regards the K-group $K_0(\mathcal{A})$, it is worth mentioning that it is not true in general that $[M] = [N]$ implies that $M \cong N$ for two finitely generated Hilbert modules M, N over $\mathcal{A}$. If $K_0(\mathcal{A})$ satisfies the property that $[N] = [M]$ implies that $N \cong M$ for any two finitely generated, Hilbert modules M, N over $\mathcal{A}$, then $K_0(\mathcal{A})$ is said to satisfy "the cancellation property". For more details about this property, see [10, Section 6.2] and [13].

Definition 2.9. *[2, Definition 4] We put by definition* index $F = [N_2] - [N_1] \in K_0(\mathcal{A})$.

By [2, Corollary 2] the index is well-defined.

Remark 2.10. *If follows that if $F \in \widehat{\mathcal{M}\Phi}_0(H_{\mathcal{A}})$, then index$F = 0$.*

3. Non-adjointable semi-C^*-Fredholm operators on general Hilbert C^*-modules

The main aim of this section is to extend the results given in [3] from the case of adjointable bounded C^*-operators on the standard Hilbert C^*-module to the case of general bounded C^*-operators on arbitrary Hilbert C^*-modules. To this end, we present first the following lemma.

Lemma 3.1. *Let M be a Hilbert C^*-module and $F \in B(M)$. Suppose that there are decompositions*

$$M = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = M,$$

$$M = M_1' \tilde{\oplus} N_1' \xrightarrow{F} M_2' \tilde{\oplus} N_2' = M,$$

with respect to which F has matrices $\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}$ and $\begin{bmatrix} F_1' & 0 \\ 0 & F_4' \end{bmatrix}$, respectively, where F_1, F_1' are isomorphisms and N_1, N_2' are finitely generated. Then N_2 and N_1' are finitely generated as well.

Proof. We show first that N_2 is finitely generated. Let $\sqcap$ denote the projection onto N_2 along M_2 and consider the direct sum of modules $N_1 \oplus N_2'$ in the sense of [8, Example 1.3.4]. We claim that the map $\iota : N_1 \oplus N_2' \to N_2$ given by $\iota(x, y') = Fx + \sqcap y'$ is an epimorphism. To see this, let $y \in N_2$. Then $y = y_1' + y_2'$ for some $y_1' \in M_2'$ and $y_2' \in N_2'$. Since $F_{|_{M_1'}}$ is an isomorphism onto M_2', there exists an $m_1' \in M_1'$ such that $Fm_1' = y_1'$. We can write m_1' as $m_1' = m_1 + n_1$ for some $m_1 \in M_1$ and $n_1 \in N_1$. Then we obtain $y = Fm_1 + Fn_1 + y_2'$. Hence we get $y = \sqcap y = \sqcap Fm_1 + \sqcap Fn_1 + \sqcap y_2' = Fn_1 + \sqcap y_2'$. Since $y \in N_2$ was chosen arbitrary, it follows that ι is an epimorphism. However, $N_1 \oplus N_2'$ is finitely generated since both N_1 and N_2' are so by assumption, hence we must have that N_2 is finitely generated as well.

Next we show that N_1' is finitely generated. Let $\sqcap_{M_2}, \sqcap_{M_2'}, \sqcap_{N_1'}$ and $\sqcap_{N_2'}$ denote the projections onto M_2 along N_2, onto M_2' along N_2', onto N_1' along M_1' and onto N_2' along M_2', respectively. We claim that the map $\iota' : N_2' \oplus N_1 \longrightarrow N_1'$ given by

$$\iota'(n_2', n_1) = \sqcap_{N_1'} F_1^{-1} \sqcap_{M_2} (n_2' - \sqcap_{M_2'} Fn_1) + \sqcap_{N_1'} n_1$$

is an epimorphism. In order to show this, let $y = N_1'$. Then $y = m_1 + n_1$ for some $m_1 \in M_1$ and $n_1 \in N_1$. Set $m_2 = Fm_1$, then $m_1 = F_1^{-1} m_2$. We get $Fy = m_2 + Fn_1$. Now, since $\sqcap_{N_1'} y = y$ and $F \sqcap_{N_1'} = \sqcap_{N_2'} F$, we get

$$Fy = F \sqcap_{N_1'} y = \sqcap_{N_2'} Fy = \sqcap_{N_2'} m_2 + \sqcap_{N_2'} Fn_1.$$

Hence $m_2 + Fn_1 = \sqcap_{N_2'}(m_2 + Fn_1)$ which gives $\sqcap_{M_2'}(m_2 + Fn_1) = 0$, so $\sqcap_{M_2'} m_2 = - \sqcap_{M_2'} Fn_1$. Therefore, we get

$$m_2 = \sqcap_{N_2'} m_2 + \sqcap_{M_2'} m_2 = \sqcap_{N_2'} m_2 - \sqcap_{M_2'} Fn_1.$$

So we derive that

$$y = m_1 + n_1 = F_1^{-1} m_2 + n_1 = F_1^{-1}(\sqcap_{N_2'} m_2 - \sqcap_{M_2'} Fn_1) + n_1$$

$$= F_1^{-1} \sqcap_{M_2} (\sqcap_{N_2'} m_2 - \sqcap_{M_2'} Fn_1) + n_1 = F_1^{-1} \sqcap_{M_2} (n_2' - \sqcap_{M_2'} Fn_1) + n_1,$$

where we put $n_2' = \sqcap_{N_2'} m_2$. Recalling that $\sqcap_{N_1'} y = y$, we obtain that y can be written as

$$y = \sqcap_{N_1'} F_1^{-1} \sqcap_{M_2} (n_2' - \sqcap_{M_2'} Fn_1) + \sqcap_{N_1'} n_1,$$

where $n_2' \in N_2'$ and $n_1 \in N_1$. Since $y \in N_1'$ was chosen arbitrary, it follows that ι' is an epimorphism from $N_2' \oplus N_1$ onto N_1', hence N_1' is finitely generated. □

Remark 3.2. *From the proof of Lemma 3.1 it follows that there exist epimorphisms from $N_1 \oplus N_2'$ onto N_2 and onto N_1' also in the case when N_1 and N_2' are not finitely generated. Moreover, this holds in the case of arbitrary Banach spaces and not just Hilbert C^*-modules.*

Corollary 3.3. *For any Hilbert C^*-module M, we have*

$$\widehat{\mathcal{M}\Phi}(M) = \widehat{\mathcal{M}\Phi}_l(M) \cap \widehat{\mathcal{M}\Phi}_r(M).$$

Proof. It suffices to show " $\supseteq$ ". However, if $F \in \widehat{\mathcal{M}\Phi}_l(M) \cap \widehat{\mathcal{M}\Phi}_r(M)$ and

$$M = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = M,$$

$$M = M_1' \tilde{\oplus} N_1' \xrightarrow{F} M_2' \tilde{\oplus} N_2' = M$$

are an $\widehat{\mathcal{M}\Phi}_l$-decomposition and an $\widehat{\mathcal{M}\Phi}_r$-decomposition for F, respectively, then from Lemma 3.1 it follows that both these decompositions are $\widehat{\mathcal{M}\Phi}$-decompositions for F. □

The following proposition is a generalization of [3, Lemma 2.16].

Proposition 3.4. *Let M be a Hilbert C^*-module and $F \in \widehat{\mathcal{M}\Phi}(M)$. Then any $\widehat{\mathcal{M}\Phi}_l$-decomposition or $\widehat{\mathcal{M}\Phi}_r$-decomposition for F is an $\widehat{\mathcal{M}\Phi}$-decomposition for F.*

Proof. Let

$$M = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = M$$

be an $\widehat{\mathcal{M}\Phi}_l$-decomposition for F. Since $F \in \widehat{\mathcal{M}\Phi}(M)$ by assumption, there exists an $\widehat{\mathcal{M}\Phi}$- decomposition for F

$$M = M_1' \tilde{\oplus} N_1' \xrightarrow{F} M_2' \tilde{\oplus} N_2' = M.$$

In particular, N_1 and N_2' are finitely generated. We may hence apply Lemma 3.1 on these two decompositions for F and deduce that N_2 is finitely generated. The proof of the second statement is similar. □

Remark 3.5. *By applying Proposition 3.4 instead of [3, Lemma 2.16] we can extend [3, Proposition 5.7] and the results from [3, Section 4] from the standard module case to the case of arbitrary Hilbert C^*-modules.*

Set

$$\widehat{\mathcal{M}\Phi}_-(H_{\mathcal{A}}) = \{G \in B(H_{\mathcal{A}}) \mid \text{ there exist closed submodules } M, N, M' \text{ of } H_{\mathcal{A}}$$
$$\text{such that } H_{\mathcal{A}} = M \tilde{\oplus} N, N \text{ is finitely generated and } G_{|_{M'}} \text{ is an isomorphism onto } M\}.$$

We have the following lemma.

Lemma 3.6. *It holds that $\widehat{\mathcal{M}\Phi}_-(H_{\mathcal{A}}) = \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}})$.*

Proof. Obviously, we have $\widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}}) \subseteq \widehat{\mathcal{M}\Phi}_-(H_{\mathcal{A}})$, so it suffices to prove the opposite inclusion. Let $G \in \widehat{\mathcal{M}\Phi}_-(H_{\mathcal{A}})$ and choose Hilbert submodules M, N and M' such that $H_{\mathcal{A}} = M \tilde{\oplus} N$, N is finitely generated and $G_{|_{M'}}$ is an isomorphism onto M. We wish to show that

$$H_{\mathcal{A}} = M' \tilde{\oplus} G^{-1}(N).$$

To this end, choose an $x \in H_{\mathcal{A}}$. Since $H_{\mathcal{A}} = M \tilde{\oplus} N$, there exist some $m \in M$ and $n \in N$ such that $Gx = m + n$. Now, since $G_{|_{M'}}$ is an isomorphism onto M, there exists an $m' \in M'$ such that $Gm' = m$. So, we have $Gx = Gm' + n$. On the other hand, $Gx = Gm' + G(x - m')$, hence $n = G(x - m')$. It follows that $x - m' \in G^{-1}(N)$ and $x = m' + (x - m')$, which gives $H_{\mathcal{A}} = M' + G^{-1}(N)$. Finally, $M' \cap G^{-1}(N) = \{0\}$ because $G(M') = M$, $M \cap N = \{0\}$ and $G_{|_{M'}}$ is an isomorphism, thus injective.
Therefore, G has the matrix $\begin{bmatrix} G_1 & 0 \\ 0 & G_4 \end{bmatrix}$ with respect to the decomposition

$$H_{\mathcal{A}} = M' \tilde{\oplus} G^{-1}(N) \xrightarrow{G} M \tilde{\oplus} N = H_{\mathcal{A}},$$

where G_1 is an isomorphism, hence $G \in \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}})$. □

Lemma 3.7. *Let M be a Hilbert C^*-module and $F, G \in B(M)$. Suppose that there exists a decomposition*

$$M = M_1 \tilde{\oplus} N_1 \xrightarrow{GF} M_2 \tilde{\oplus} N_2 = M$$

with respect to which GF has the matrix $\begin{bmatrix} (GF)_1 & 0 \\ 0 & (GF)_4 \end{bmatrix}$, where $(GF)_1$ is an isomorphism. Then we have $M = F(M_1) \tilde{\oplus} G^{-1}(N_2)$ and moreover, with respect to the decompositions

$$M = M_1 \tilde{\oplus} N_1 \xrightarrow{F} F(M_1) \tilde{\oplus} G^{-1}(N_2) = M,$$

$$M = F(M_1) \tilde{\oplus} G^{-1}(N_2) \xrightarrow{G} M_2 \tilde{\oplus} N_2 = M,$$

the operators F and G have the matrices $\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}$ and $\begin{bmatrix} G_1 & 0 \\ 0 & G_4 \end{bmatrix}$, respectively, where F_1 and G_1 are isomorphisms.

Notice that Lemma 3.7 is also valid in the case of general bounded linear operators on arbitrary Banach spaces.

The next proposition is a generalization of [3, Corollary 2.6].

Proposition 3.8. *Let $F, D \in B(M)$. If $DF \in \widehat{\mathcal{M}\Phi}_l(M)$, then $F \in \widehat{\mathcal{M}\Phi}_l(M)$. If $DF \in \widehat{\mathcal{M}\Phi}_r(M)$, then $D \in \widehat{\mathcal{M}\Phi}_r(M)$.*

Proof. Suppose that M is a Hilbert C^*-module and $DF \in \widehat{\mathcal{M}\Phi}_l(M)$. If

$$M = M_1 \tilde{\oplus} N_1 \xrightarrow{DF} M_2 \tilde{\oplus} N_2 = M$$

is an $\widehat{\mathcal{M}\Phi}_l$-decomposition for DF, then, by Lemma 3.7 , F has the matrix $\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}$ with respect to the decomposition

$$M = M_1 \tilde{\oplus} N_1 \xrightarrow{F} F(M_1) \tilde{\oplus} D^{-1}(N_2) = M,$$

whereas D has the matrix $\begin{bmatrix} D_1 & 0 \\ 0 & D_4 \end{bmatrix}$ with respect to the decomposition

$$M = F(M_1) \tilde{\oplus} D^{-1}(N_2) \xrightarrow{D} M_2 \tilde{\oplus} N_2 = M,$$

where F_1 and D_1 are isomorphisms. Since N_1 is finitely generated, the first statement follows. The proof of the second statement is similar. □

The next proposition is a generalization of [3, Corollary 2.7].

Proposition 3.9. *Let $F, D \in B(M)$. If $DF \in \widehat{\mathcal{M}\Phi}_l(M)$ and $F \in \widehat{\mathcal{M}\Phi}(M)$, then $D \in \widehat{\mathcal{M}\Phi}_l(M)$. If $DF \in \widehat{\mathcal{M}\Phi}_r(M)$ and $D \in \widehat{\mathcal{M}\Phi}(M)$, then $F \in \widehat{\mathcal{M}\Phi}_r(M)$.*

Proof. Let M be a Hilbert C^*-module and $DF \in \widehat{\mathcal{M}\Phi}_l(M)$. Suppose that $F \in \widehat{\mathcal{M}\Phi}(M)$ and let

$$M = M_1 \tilde{\oplus} N_1 \xrightarrow{DF} M_2 \tilde{\oplus} N_2 = M$$

be an $\widehat{\mathcal{M}\Phi}_l$-decomposition for DF. By Lemma 3.7 we have that

$$M = M_1 \tilde{\oplus} N_1 \xrightarrow{F} F(M_2) \tilde{\oplus} D^{-1}(N_2) = M$$

is an $\widehat{\mathcal{M}\Phi}_l$-decomposition for F and D has the matrix $\begin{bmatrix} D_1 & 0 \\ 0 & D_4 \end{bmatrix}$ with respect to the decomposition

$$M = F(M_1) \tilde{\oplus} D^{-1}(N_1) \xrightarrow{D} M_2 \tilde{\oplus} N_2 = M,$$

where D_1 is an isomorphism. Now, since

$$M = M_1 \tilde{\oplus} N_1 \xrightarrow{F} F(M_1) \tilde{\oplus} D^{-1}(N_2) = M$$

is an $\widehat{\mathcal{M}\Phi}_l$-decomposition for F, from Proposition 3.4 it follows that $D^{-1}(N_2)$ must be finitely generated since $F \in \widehat{\mathcal{M}\Phi}(M)$. Hence,

$$M = F(M_1) \tilde{\oplus} D^{-1}(N_2) \xrightarrow{D} M_2 \tilde{\oplus} N_2 = M$$

is an $\widehat{\mathcal{M}\Phi}_l$-decomposition for D, so $D \in \widehat{\mathcal{M}\Phi}_l(M)$. By applying Proposition 3.4 on the operator D instead of F and using the similar arguments, we obtain the second statement in the corollary. □

The next proposition is a generalization of [3, Corollary 2.8].

Proposition 3.10. *Let $F, D \in B(M)$. If $D \in \widehat{\mathcal{M}\Phi}_l(M)$ and $DF \in \widehat{\mathcal{M}\Phi}(M)$, then $D \in \widehat{\mathcal{M}\Phi}(M)$. If $F \in \widehat{\mathcal{M}\Phi}_r(M)$ and $DF \in \widehat{\mathcal{M}\Phi}(M)$, then $F \in \widehat{\mathcal{M}\Phi}(M)$.*

Proof. Let M be a Hilbert C^*-module and $D, F \in B(M)$. Suppose that $D \in \widehat{\mathcal{M}\Phi}_l(M)$ and $DF \in \widehat{\mathcal{M}\Phi}(M)$. If

$$M = M_1 \tilde{\oplus} N_1 \xrightarrow{DF} M_2 \tilde{\oplus} N_2 = M$$

is an $\widehat{\mathcal{M}\Phi}$-decomposition for DF, then, by Lemma 3.7, we have that

$$M = F(M_1) \tilde{\oplus} D^{-1}(N_2) \xrightarrow{D} M_2 \tilde{\oplus} N_2 = M$$

is an $\widehat{\mathcal{M}\Phi}_r$-decomposition for D. Hence, by Corollary 3.3 we get that

$$D \in \widehat{\mathcal{M}\Phi}_r(M) \cap \widehat{\mathcal{M}\Phi}_l(M) = \widehat{\mathcal{M}\Phi}(M).$$

In the similar way we can deduce the second statement of the corollary. □

The next proposition is a generalization of [3, Corollary 2.9].

Proposition 3.11. *If $D \in \widehat{\mathcal{M}\Phi}(M)$ and $DF \in \widehat{\mathcal{M}\Phi}(M)$, then $F \in \widehat{\mathcal{M}\Phi}(M)$. If $F \in \widehat{\mathcal{M}\Phi}(M)$ and $DF \in \widehat{\mathcal{M}\Phi}(M)$, then $D \in \widehat{\mathcal{M}\Phi}(M)$.*

Proof. Let M be a Hilbert C^*-module. Suppose that $D \in \widehat{\mathcal{M}\Phi}(M)$ and $DF \in \widehat{\mathcal{M}\Phi}(M)$. If

$$M = M_1 \tilde{\oplus} N_1 \xrightarrow{DF} M_2 \tilde{\oplus} N_2 = M$$

is an $\widehat{\mathcal{M}\Phi}$-decomposition for DF, then, by Lemma 3.7,

$$M = F(M_1) \tilde{\oplus} D^{-1}(N_2) \xrightarrow{D} M_2 \tilde{\oplus} N_2 = M$$

is an $\widehat{\mathcal{M}\Phi}_r$-decomposition for D. Since $D \in \widehat{\mathcal{M}\Phi}(M)$, by Proposition 3.4 we have that $D^{-1}(N_2)$ is finitely generated. It follows by Lemma 3.7 that

$$M = M_1 \tilde{\oplus} N_1 \xrightarrow{F} F(M_1) \tilde{\oplus} D^{-1}(N_2) = M$$

is an $\widehat{\mathcal{M}\Phi}$-decomposition for F, so $F \in \widehat{\mathcal{M}\Phi}(M)$.

The case when $F \in \widehat{\mathcal{M}\Phi}(M)$ and $DF \in \widehat{\mathcal{M}\Phi}(M)$ can be treated similarly. □

4. Non-adjointable semi-C^*-Weyl operators

In this section we are going to present some new results concerning semi-C^*-Weyl operators. We start with the following lemmas.

Lemma 4.1. *Let M be a Hilbert C^*-module and $F \in \widehat{\mathcal{M}\Phi}_l^{-'}(M)$. If*

$$M = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = M$$

is an $\widehat{\mathcal{M}\Phi}_l^{-'}$-decomposition for F and $D \in B(M)$ is such that $\sqcap(D+F)_{|_{M_1}} \in \widehat{\mathcal{M}\Phi}_l^{-'}(M_1, M_2)$, where $\sqcap$ stands for the projection onto M_2 along N_2, then $D + F \in \widehat{\mathcal{M}\Phi}_l^{-'}(M)$. Similar statements hold for the classes $\widehat{\mathcal{M}\Phi}_r^{+'}$, $\widehat{\mathcal{M}\Phi}_l$, $\widehat{\mathcal{M}\Phi}_r$, $\widehat{\mathcal{M}\Phi}$, $\widehat{\mathcal{M}\Phi}_0$, $\widehat{\widetilde{\mathcal{M}\Phi}}_l^{-}$, and $\widehat{\widetilde{\mathcal{M}\Phi}}_r^{+}$.

Proof. Let

$$M_1 = \tilde{M}_1 \tilde{\oplus} \tilde{N}_1 \xrightarrow{F} \tilde{M}_2 \tilde{\oplus} \tilde{N}_2 = M_2$$

be an $\widehat{\mathcal{M}\Phi}_l^{-'}$-decomposition for $\sqcap(D+F)_{|_{M_1}}$. Then $\tilde{N}_1$ is finitely generated, $\tilde{N}_1 \preceq \tilde{N}_2$ and $\sqcap(D+F)_{|_{\tilde{M}_1}}$ is an isomorphism onto $\tilde{M}_2$. If we let $\tilde{\sqcap}$ denote the projection onto $\tilde{M}_2$ along $\tilde{N}_2 \tilde{\oplus} N_2$, then $\tilde{\sqcap}(D+F)_{|_{\tilde{M}_1}} = \sqcap(D+F)_{|_{\tilde{M}_1}}$. Hence $D + F$ has the matrix $\begin{bmatrix} (D+F)_1 & (D+F)_2 \\ (D+F)_3 & (D+F)_4 \end{bmatrix}$ with respect to the decomposition

$$M = \tilde{M}_1 \tilde{\oplus} (\tilde{N}_1 \tilde{\oplus} N_1) \xrightarrow{D+F} \tilde{M}_2 \tilde{\oplus} (\tilde{N}_2 \tilde{\oplus} N_2) = M,$$

where $(D+F)_1$ is an isomorphism. Moreover, since $N_1 \preceq N_2$, $\tilde{N}_1 \preceq \tilde{N}_2$ and N_1, N_2 are finitely generated, it follows that $N_1 \tilde{\oplus} \tilde{N}_1$ is finitely generated and $N_1 \tilde{\oplus} \tilde{N}_1 \preceq N_2 \tilde{\oplus} \tilde{N}_2$. Then we can proceed in the same way as in the proof of [8, Lemma 2.7.10] to deduce that there exist isomorphisms U and V such that

$$M = \tilde{M}_1 \tilde{\oplus} U(\tilde{N}_1 \tilde{\oplus} N_1) \xrightarrow{D+F} V(\tilde{M}_2) \tilde{\oplus} (\tilde{N}_2 \tilde{\oplus} N_2) = M$$

is an $\widehat{\mathcal{M}\Phi}_l^{-'}$-decomposition for $D + F$.

The proofs for the other cases are similar.

□

Theorem 4.2. *The sets $\widehat{\mathcal{M}\Phi}_l(H_\mathcal{A}) \setminus \widehat{\mathcal{M}\Phi}_l^{-'}(H_\mathcal{A})$, $\widehat{\mathcal{M}\Phi}_r(H_\mathcal{A}) \setminus \widehat{\mathcal{M}\Phi}_r^{+'}(H_\mathcal{A})$, $\widehat{\mathcal{M}\Phi}(H_\mathcal{A}) \setminus \widehat{\mathcal{M}\Phi}_0(H_\mathcal{A})$ are open.*

Proof. Let $F \in \widehat{\mathcal{M}\Phi}_l(H_\mathcal{A}) \setminus \widehat{\mathcal{M}\Phi}_l^{-'}(H_\mathcal{A})$ and

$$H_\mathcal{A} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_\mathcal{A}$$

be an $\widehat{\mathcal{M}\Phi}_l$-decomposition for F. By the proof of [8, Lemma 2.7.10] there exists an $\epsilon > 0$ such that if $\| F - D \| < \epsilon$, then D has an $\widehat{\mathcal{M}\Phi}_l$-decomposition

$$H_\mathcal{A} = M_1' \tilde{\oplus} N_1{}' \xrightarrow{D} M_2' \tilde{\oplus} N_2' = H_\mathcal{A},$$

where $M_1 \cong M_1', N_1 \cong N_1', M_2 \cong M_2'$ and $N_2 \cong N_2'$. Suppose that $D \in \widehat{\mathcal{M}\Phi}_l^{-'}(H_\mathcal{A})$. Then there exists an $\widehat{\mathcal{M}\Phi}_l^{-'}$-decomposition for D,

$$H_\mathcal{A} = M_1'' \tilde{\oplus} N_1'' \xrightarrow{D} M_2'' \tilde{\oplus} N_2'' = H_\mathcal{A},$$

which means in particular that N_1'' is finitely generated and $N_1'' \preceq N_2''$. By the proof of [8, Lemma 2.7.11] there exists an $n \in \mathbb{N}$ and finitely generated Hilbert submodules P', P'' such that

$$H_{\mathcal{A}} = L_n^{\perp} \tilde{\oplus} (P' \tilde{\oplus} N_1') \xrightarrow{D} D(L_n^{\perp}) \tilde{\oplus} (D(P') \tilde{\oplus} V'(N_2')) = H_{\mathcal{A}}$$

and

$$H_{\mathcal{A}} = L_n^{\perp} \tilde{\oplus} (P'' \tilde{\oplus} N_1'') \xrightarrow{D} D(L_n^{\perp}) \tilde{\oplus} (D(P'') \tilde{\oplus} V''(N_2'')) = H_{\mathcal{A}}$$

are two $\widehat{\mathcal{M}\Phi}_l$–decompositions for D, where V and V'' are isomorphisms. It follows that

$$P' \tilde{\oplus} N_1' \cong P'' \tilde{\oplus} N_1'' \text{ and } D(P') \tilde{\oplus} V'(N_2') \cong D(P'') \tilde{\oplus} V''(N_2'').$$

Moreover, $M_1' \cong L_n^{\perp} \tilde{\oplus} P', M_1'' \cong L_n^{\perp} \tilde{\oplus} P'', M_2' \cong D(L_n^{\perp}) \tilde{\oplus} D(P'), M_2'' \cong D(L_n^{\perp}) \tilde{\oplus} D(P''), D(P') \cong P'$ and $D(P'') \cong P''$. Since $N_1'' \preceq N_2''$, we get that

$$P'' \tilde{\oplus} N_1'' \preceq D(P'') \tilde{\oplus} V''(N_2'').$$

Hence we obtain that

$$P' \tilde{\oplus} N_1' \cong P'' \tilde{\oplus} N_1'' \preceq D(P'') \tilde{\oplus} V''(N_2'') \cong D(P') \tilde{\oplus} V'(N_2').$$

Now, we have $M_1 \cong M_1' \cong L_n^{\perp} \oplus P'$ and $M_2 \cong M_2' \cong D(L_n^{\perp}) \tilde{\oplus} D(P') \cong L_n^{\perp} \oplus P'$. Therefore, there exist isomorphisms U_1 and U_2 such that

$$M_1 = U_1(L_n^{\perp}) \tilde{\oplus} U_1(P'),\ M_2 = U_2(L_n^{\perp}) \tilde{\oplus} U_2(P').$$

With respect to the decomposition

$$H_{\mathcal{A}} = U_1(L_n^{\perp}) \tilde{\oplus} (U_1(P') \tilde{\oplus} N_1) \xrightarrow{F} F(U_1(L_n^{\perp})) \tilde{\oplus} (F(U_1(P'))) \tilde{\oplus} N_2) = H_{\mathcal{A}},$$

the operator F has the matrix $\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}$, where F_1 is an isomorphism and $F(U_1(P')) \cong P'$.

Hence, $(F(U_1(P') \tilde{\oplus} N_2)) \cong D(P') \tilde{\oplus} V'(N_2')$ since

$$F(U_1(P')) \cong P' \cong D(P') \text{ and } N_2 \cong N_2' \cong V'(N_2').$$

Moreover, $U_1(P') \tilde{\oplus} N_1 \cong P' \tilde{\oplus} N_1'$ since $N_1 \cong N_1'$ and U_1 is an isomorphism. Since we have from above that $P' \tilde{\oplus} N_1' \preceq D(P') \tilde{\oplus} V'(N_2')$, we deduce that $U_1(P') \tilde{\oplus} N_1 \preceq F(U_1(P')) \tilde{\oplus} N_2$. So

$$H_{\mathcal{A}} = U_1(L_n^{\perp}) \tilde{\oplus} (U_1(P') \tilde{\oplus} N_1) \xrightarrow{F} F(U_1(L_n^{\perp})) \tilde{\oplus} (F(U_1(P')) \tilde{\oplus} N_2) = H_{\mathcal{A}}$$

is an $\widehat{\mathcal{M}\Phi}_l^{-'}$-decomposition for F. We get a contradiction since we assumed that $F \notin \widehat{\mathcal{M}\Phi}_l^{-'}(H_{\mathcal{A}})$. Thus, we must have that $D \notin \widehat{\mathcal{M}\Phi}_l^{-'}(H_{\mathcal{A}})$, which means that $\widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}}) \setminus \widehat{\mathcal{M}\Phi}_l^{-'}(H_{\mathcal{A}})$ is open. The proofs of the other statements are similar. □

Corollary 4.3. *Let $f : [0,1] \to B^a(H_{\mathcal{A}})$ be a continuous map such that $f([0,1]) \subseteq \widehat{\mathcal{M}\Phi}_{\pm}(H_{\mathcal{A}})$. Then*
1) If $f(0) \in \widehat{\mathcal{M}\Phi}_l^{-'}(H_{\mathcal{A}})$, then $f(1) \in \widehat{\mathcal{M}\Phi}_l^{-'}(H_{\mathcal{A}})$.
2) If $f(0) \in \widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}}) \setminus \widehat{\mathcal{M}\Phi}_l^{-'}(H_{\mathcal{A}})$, then $f(1) \in \widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}}) \setminus \widehat{\mathcal{M}\Phi}_l^{-'}(H_{\mathcal{A}})$.
3) If $f(0) \in \widehat{\mathcal{M}\Phi}_l^{-'}(H_{\mathcal{A}})$, then $f(1) \in \widehat{\mathcal{M}\Phi}_l^{-'}(H_{\mathcal{A}})$.
4) If $f(0) \in \widehat{\mathcal{M}\Phi}_r^{+'}(H_{\mathcal{A}})$, then $f(1) \in \widehat{\mathcal{M}\Phi}_r^{+'}(H_{\mathcal{A}})$.
5) If $f(0) \in \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}}) \setminus \widehat{\mathcal{M}\Phi}_r^{+'}(H_{\mathcal{A}})$, then $f(1) \in \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}}) \setminus \widehat{\mathcal{M}\Phi}_r^{+'}(H_{\mathcal{A}})$.
6) If $f(0) \in \widehat{\mathcal{M}\Phi}_0(H_{\mathcal{A}})$, then $f(1) \in \widehat{\mathcal{M}\Phi}_0(H_{\mathcal{A}})$.
7) If $f(0) \in \widehat{\mathcal{M}\Phi}(H_{\mathcal{A}}) \setminus \widehat{\mathcal{M}\Phi}_0(H_{\mathcal{A}})$, then $f(1) \in \widehat{\mathcal{M}\Phi}(H_{\mathcal{A}}) \setminus \widehat{\mathcal{M}\Phi}_0(H_{\mathcal{A}})$.

Proof. By applying Theorem 4.2 we can proceed in the same way as in the proof of [3, Corollary 4.3]. □

Proposition 4.4. *Let $F \in \widehat{\mathcal{M}\Phi}_l^{-'}(H_{\mathcal{A}}) \cap \widehat{\mathcal{M}\Phi}_r^{+'}(H_{\mathcal{A}})$. Then there exists an $\widehat{\mathcal{M}\Phi}$-decomposition*

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

for F with the property that $N_1 \preceq N_2$ and $N_2 \preceq N_1$.

Proof. Let

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}},$$

$$H_{\mathcal{A}} = M_1' \tilde{\oplus} N_1' \xrightarrow{F} M_2' \tilde{\oplus} N_2' = H_{\mathcal{A}}$$

be an $\widehat{\mathcal{M}\Phi}_l^{-'}$ and an $\widehat{\mathcal{M}\Phi}_r^{+'}$-decomposition for F, respectively. By Proposition 3.4 it follows that both these decompositions are actually $\mathcal{M}\Phi$-decompositions for F. Hence, both N_1 and N_1' are finitely generated. Therefore, by [8, Theorem 2.7.5] there exists an $n \in \mathbb{N}$ such that $H_{\mathcal{A}} = L_n^{\perp} \tilde{\oplus} P \tilde{\oplus} N_1 = L_n^{\perp} \tilde{\oplus} P' \tilde{\oplus} N_1'$. By the proof of [8, Lemma 2.7.11], there exists then isomorphisms V and V' such that

$$H_{\mathcal{A}} = L_n^{\perp} \tilde{\oplus} (P \tilde{\oplus} N_1) \xrightarrow{F} F(L_n^{\perp}) \tilde{\oplus} (F(P) \tilde{\oplus} V(N_2) = H_{\mathcal{A}},$$

$$H_{\mathcal{A}} = L_n^{\perp} \tilde{\oplus} (P' \tilde{\oplus} N_1') \xrightarrow{F} F(L_n^{\perp}) \tilde{\oplus} (F(P') \tilde{\oplus} V'(N_2') = H_{\mathcal{A}}$$

are two $\widehat{\mathcal{M}\Phi}$-decompositions for F and moreover, $P \cong F(P)$, $P' \cong F(P')$. Since $N_1 \preceq N_2$, we get that $(P \tilde{\oplus} N_1) \preceq (F(P) \tilde{\oplus} V(N_2))$. Similarly, we have $(F(P') \tilde{\oplus} V'(N_2')) \preceq (P' \tilde{\oplus} N_1')$ since $N_2' \preceq N_1'$. Finally,

$$P \tilde{\oplus} N_1 \cong P' \tilde{\oplus} N_1', \; F(P) \tilde{\oplus} V(N_2) \cong F(P') \tilde{\oplus} V'(N_2').$$

Hence, $(F(P) \tilde{\oplus} V(N_2)) \preceq (P \tilde{\oplus} N_1)$. □

In [4, Lemma 11] it has been proved that $\widehat{\mathcal{M}\Phi}_r^{+'}(H_{\mathcal{A}})$ is invariant under compact perturbations. Now we are going to show $\widehat{\mathcal{M}\Phi}_l^{-'}(H_{\mathcal{A}})$ has the same property. To this end, we give first the following auxiliary lemma.

Lemma 4.5. *Let M be a Hilbert C^*-module and $F \in B(M)$. Suppose that*

$$M = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = M$$

is a decomposition with respect to which F has the matrix $\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix}$, where F_1 is an isomorphism. Then $N_1 = F^{-1}(N_2)$.

Proof. Obviously, $N_1 \subseteq F^{-1}(N_2)$. Assume now that $x \in F^{-1}(N_2)$. Then $x = m_1 + n_1$ for some $m_1 \in M_1$ and $n_1 \in N_1$. We get $Fx = Fm_1 + Fn_1 \in N_2$. Since $Fm_1 \in M_2$ and $Fn_1 \in N_2$, we must have $Fm_1 = 0$. As $F_{|_{M_1}}$ is an isomorphism, we deduce that $m_1 = 0$. Hence $x = n_1 \in N_1$. □

Remark 4.6. *Lemma 4.5 also holds if we consider arbitrary Banach spaces and not just Hilbert C^*-modules.*

Now we are ready to prove that $\widehat{\mathcal{M}\Phi}_l^{-'}(H_{\mathcal{A}})$ is invariant under compact perturbations.

Theorem 4.7. *Let $F \in \widehat{\mathcal{M}\Phi}_l^{-'}(H_{\mathcal{A}})$ and $K \in \mathcal{K}(H_{\mathcal{A}})$. Then $F + K \in \widehat{\mathcal{M}\Phi}_l^{-'}(H_{\mathcal{A}})$.*

Proof. Let $F \in \widehat{\mathcal{M}\Phi}_l^{-'}(H_{\mathcal{A}}), K \in \mathcal{K}(H_{\mathcal{A}})$ and

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

be an $\widehat{\mathcal{M}\Phi}_l^{-'}$-decomposition for F. Set $F_1 = F_{|_{M_1}}$ and consider the operator G given by the operator matrix $\begin{bmatrix} F_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$ with respect to the decomposition

$$H_{\mathcal{A}} = M_2 \tilde{\oplus} N_2 \longrightarrow M_1 \tilde{\oplus} N_1 = H_{\mathcal{A}}.$$

Then GF has the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ with respect to the decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{GF} M_1 \tilde{\oplus} N_1 = H_{\mathcal{A}}.$$

Now, as in the proof of [8, Lemma 2.7.13], we may without loss of generality assume that there exists some $m \in \mathbb{N}$ such that for all $k \geq m$ we have $M_1 = L_k^{\perp} \oplus P$ and $L_k = P \tilde{\oplus} N_1$, since N_1 is finitely generated.
Let now $K \in \mathcal{K}(H_{\mathcal{A}})$. Again, since $\mathcal{K}(H_{\mathcal{A}})$ is a two-sided ideal in $B(H_{\mathcal{A}})$, we have $GK \in \mathcal{K}(H_{\mathcal{A}})$. By [2, Theorem 2] there exists some $k \geq m$ such that $\| q_k GK \| < 1$. Then we observe that $M_1 = L_m^{\perp} \oplus P = L_k^{\perp} \oplus \tilde{P}$, where $\tilde{P} = P \oplus (L_m^{\perp} \setminus L_k^{\perp})$. It follows that GF has the matrix $\begin{bmatrix} 1 & 0 \\ 0 & \sqcap \end{bmatrix}$ with respect to the decomposition $L_k^{\perp} \oplus L_k \xrightarrow{GF} L_k^{\perp} \oplus L_k$, where $\sqcap$ denotes the projection onto $\tilde{P}$ along N_1. Then, with respect to the decomposition

$$H_{\mathcal{A}} = L_k^{\perp} \tilde{\oplus} L_k \xrightarrow{GF+GK} L_k^{\perp} \tilde{\oplus} L_k = H_{\mathcal{A}},$$

the operator $GF + GK$ has the matrix $\begin{bmatrix} (GF+GK)_1 & (GF+GK)_2 \\ (GF+GK)_3 & (GF+GK)_4 \end{bmatrix}$, where $(GF+GK)_1$ is an isomorphism, since $\| q_k GK_{|_{L_k^{\perp}}} \| \leq \| q_k GK \| < 1$. Hence $GF + GK$ has the matrix

$$\begin{bmatrix} \overline{(GF+GK)_1} & 0 \\ 0 & \underline{(GF+GK)_4} \end{bmatrix}$$

with respect to the decomposition

$$H_{\mathcal{A}} = L_k^{\perp} \tilde{\oplus} U(L_k) \xrightarrow{GF+GK} V^{-1}(L_k^{\perp}) \tilde{\oplus} L_k = H_{\mathcal{A}},$$

where $\overline{(GF+GK)_1}, U, V$ are isomorphisms. From this and by Lemma 3.7 we obtain that G has the matrix $\begin{bmatrix} G_1 & 0 \\ 0 & G_4 \end{bmatrix}$ with respect to the decomposition

$$H_{\mathcal{A}} = (F+K)L_k^{\perp} \tilde{\oplus} N \xrightarrow{G} V^{-1}(L_k^{\perp}) \tilde{\oplus} L_k = H_{\mathcal{A}},$$

where $N = G^{-1}(L_k)$ and G_1 is an isomorphism. Also, we obtain that $F + K$ has the matrix

$$\begin{bmatrix} (F+K)_1 & 0 \\ 0 & (F+K)_4 \end{bmatrix}$$

with respect to the decomposition

$$H_{\mathcal{A}} = L_k^{\perp} \tilde{\oplus} U(L_k) \xrightarrow{F+K} (F+K)L_k^{\perp} \tilde{\oplus} N = H_{\mathcal{A}},$$

where $(F+K)_1$ is an isomorphism.

However, since G has the matrix $\begin{bmatrix} F_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$ with respect to the decomposition

$$H_{\mathcal{A}} = M_2 \tilde{\oplus} N_2 \xrightarrow{G} M_1 \tilde{\oplus} N_1 = H_{\mathcal{A}},$$

it follows that G has the matrix $\begin{bmatrix} \tilde{\tilde{G}}_1 & 0 \\ 0 & \tilde{\tilde{G}}_4 \end{bmatrix}$ with respect to the decomposition

$$H_{\mathcal{A}} = F(L_k^{\perp}) \tilde{\oplus} (F(\tilde{P}) \tilde{\oplus} N_2) \xrightarrow{G} L_k^{\perp} \tilde{\oplus} L_k = H_{\mathcal{A}},$$

where $\tilde{\tilde{G}}_1 = F_1^{-1}{}_{|_{F(L_k^{\perp})}}$ is an isomorphism (observe that $M_2 = F(L_k^{\perp}) \tilde{\oplus} F(\tilde{P})$ since $M_1 = L_k^{\perp} \oplus \tilde{P}$). From Lemma 4.5 it follows that $F(\tilde{P}) \tilde{\oplus} N_2 = N = G^{-1}(L_k)$. Since $N_1 \preceq N_2$ and $F_{|_{\tilde{P}}}$ is an isomorphism, we get that

$$L_k = \tilde{P} \tilde{\oplus} N_1 \preceq F(\tilde{P}) \tilde{\oplus} N_2 = N.$$

Moreover, $L_k \cong U(L_k)$ and, as we have seen above, $F+K$ has the matrix $\begin{bmatrix} (F+K)_1 & 0 \\ 0 & (F+K)_4 \end{bmatrix}$ with respect to the decomposition

$$H_{\mathcal{A}} = L_k^{\perp} \tilde{\oplus} U(L_k) \xrightarrow{F+K} (F+K) L_k^{\perp} \tilde{\oplus} N = H_{\mathcal{A}},$$

where $(F+K)_1$ is an isomorphism. $\square$

5. Extended Schechter's characterization and generalized Fredholm alternative for adjointable C^*-operators

In this section we extend the results from [3, Section 3] by describing $\mathcal{M}\Phi_+$-operators in terms of some equivalent conditions that generalize Schechter's characterization of the classical upper semi-Fredholm operators. First we give the following version of [3, Lemma 3.2].

Lemma 5.1. *[3, Lemma 3.2] Let $F \in B^a(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi_+(H_{\mathcal{A}})$. Then there exists a sequence $\{x_k\} \subseteq H_{\mathcal{A}}$ and an increasing sequence $\{n_k\} \subseteq \mathbb{N}$ such that*

$$x_k \in L_{n_k} \cap L_{n_{k-1}}^{\perp}, \quad \| x_k \| = 1$$

and

$$\| F x_k \| \leq 2^{1-2k} \text{ for all } k \in \mathbb{N}.$$

Lemma 5.2. *Let $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$. Then there is no sequence of unit vectors $\{x_n\}$ in $H_{\mathcal{A}}$ such that $\langle e_k, x_n \rangle \to 0$ as $n \to \infty$ for all $k \in \mathbb{N}$ and $\lim_{n \to \infty} \| F x_n \| = 0$.*

Proof. Let $D \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$ and $K \in \mathcal{K}^*(H_{\mathcal{A}})$ be such that $DF = I + K$. Such operators D and K exist by [3, Theorem 2.2]. If $K = 0$, then $DF = I$, which in particularly means that F is bounded below. Since $\| x_n \| = 1$ for all $n \in \mathbb{N}$, it follows that $F x_n \nrightarrow 0$ as $n \to \infty$.

Suppose next that $K \neq 0$. Then

$$|1 - \| DFx_n \| \,| = |\, \| x_n \| - \| DFx_n \| \,| \leq \| (I - DF) x_n \| = \| K x_n \| .$$

Here we have applied the same arguments as in the proof of [6, Chapter XI, Theorem 2.3] part $(a) \Rightarrow (d)$. Given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $\| K_{|_{L_n^{\perp}}} \| < \dfrac{\epsilon}{2}$ for all $n \geq N$, since $K \in \mathcal{K}^*(H_{\mathcal{A}})$. This follows from [8, Proposition 2.2.1]. If $\langle e_k, x_n \rangle \overset{n \to \infty}{\longrightarrow} 0$ for all $k \in \{1, 2, \cdots, N\}$, then we may choose an $M \in \mathbb{N}$ such that

$\| \langle e_k, x_n \rangle \| < \dfrac{\epsilon}{2 \| K \| N}$ for all $n \geq M$ and for all $k \in \{1, \dots, N\}$. Let P_N denote the orthogonal projection onto $L_{N+1}^{\perp}$. Then, for all $n \geq M$, we have

$$\| Kx_n \| \leq \| KP_N x_n \| + \sum_{k=1}^{N} \| Ke_k \cdot \langle e_k, x_n \rangle \| \leq \frac{\epsilon}{2} + \sum_{k=1}^{N} \| K \| \ \ \| \langle e_k, x_n \rangle \| < \epsilon.$$

Thus, $\| Kx_n \| \to 0$, so from the above calculations it follows that $\| DFx_n \| \to 1$ as $n \to \infty$. Therefore, we can not have that $\| Fx_n \| \to 0$ as $n \to \infty$. □

Corollary 5.3. *If $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$, then $Fe_n \nrightarrow 0$ as $n \to \infty$.*

The next proposition is a generalization of Schechter's characterization of the classical upper semi-Fredholm operators.

Proposition 5.4. *Let $F \in B^a(H_{\mathcal{A}})$. Then $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$ if and only if there is no sequence of unit vectors $\{x_k\}_{k \in \mathbb{N}}$ in $H_{\mathcal{A}}$ satisfying the conditions of Lemma 5.1.*

Proof. The implication in one direction follows from Lemma 5.1. Let us prove the implication in the other direction. To this end, suppose that $F \in B^a(H_{\mathcal{A}})$ and that there exists a sequence of unit vectors $\{x_n\}_{n \in \mathbb{N}} \subseteq H_{\mathcal{A}}$ satisfying the conditions of Lemma 5.1. By these conditions, it follows then that $\lim_{n\to\infty} \langle e_k, x_n \rangle = 0$ for all $k \in \mathbb{N}$ and moreover, $\lim_{n\to\infty} \| Fx_n \| = 0$. Hence, by Lemma 5.2, we deduce that $F \in B^a(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi_+(H_{\mathcal{A}})$, which shows the implication in the other direction. □

Example 5.5. *If we consider $\mathcal{A}$ as a Hilbert module over itself, then, in general, we can find closed submodules of $\mathcal{A}$ that are not finitely generated. As an example, if $\mathcal{A} = C([0,1])$, then $C_0([0,1])$ is a Hilbert submodule of $\mathcal{A}$ that is not finitely generated. Similarly, if $\mathcal{A} = B(H)$ where H is a Hilbert space, then the closed ideal of compact operators on H is an example of a Hilbert submodule that is not finitely generated. Let P denote the orthogonal projection onto $L_1^{\perp}$. Then $P \in \mathcal{M}\Phi(H_{\mathcal{A}})$ and $\ker P = L_1$. It follows that $\ker P$ contains a Hilbert submodule that is not finitely generated in the case when $\mathcal{A} = C([0,1])$ or when $\mathcal{A} = B(H)$. Compared to [6, Chapter XI, Theorem 2.3], this illustrates that $\mathcal{A}$-Fredholm operators may behave differently from the classical Fredholm operators on Hilbert spaces. In general, suppose, a one-sided maximal norm-closed ideal I of a fixed C^*-algebra $\mathcal{A}$ is considered as a Hilbert $\mathcal{A}$-submodule of $\mathcal{A}$ in the natural way and could be divided out of $\mathcal{A}$ as a direct orthogonal summand. If I is a finitely generated projective $\mathcal{A}$-module, as supposed, this has to happen. Then it is supported by a maximal projection $(1_{\mathcal{A}} - p_I)$ such that p_I is an atomic projection from the type I part of the bidual von Neumann algebra $\mathcal{A}^{**}$ of $\mathcal{A}$ which belongs to $\mathcal{A}$ itself. Consequently, if $\mathcal{A}$ does only contain finitely generated projective maximal norm-closed ideals, then $\mathcal{A}$ has to be a compact C^*-algebra in the sense of [11], see also [12]. As a consequence, all non-compact C^*-algebras contain a one-sided maximal norm-closed ideal which cannot be finitely generated. Resorting to unital C^*-algebras $\mathcal{A}$ we arrive at all non-matrix C^*-algebras with this discomfort.*

Remark 5.6. *The second part of Example 5.5 regarding generalizations has been suggested by the reviewer.*

Next we present the generalization of the well known Fredholm alternative in the setting of adjointable bounded C^*-operators on the standard Hilbert C^*-module. To this end we give first the following proposition.

Proposition 5.7. *Let $K \in \mathcal{K}^*(H_{\mathcal{A}})$ and $T \in B^a(H_{\mathcal{A}})$. Suppose that T is invertible and that $K_0(\mathcal{A})$ satisfies the cancellation property. Then the equation $(T+K)x = y$ has a solution for every $y \in H_{\mathcal{A}}$ if and only if $T+K$ is bounded below. In this case the solution of the above equation is unique.*

Proof. Since T is invertible, by [8, Lemma 2.7.13] it follows that index $(T+K) = 0$. Now, if the equation $(T+K)x = y$ has a solution for each $y \in H_{\mathcal{A}}$, this simply means that $T+K$ is surjective. Then, by [8, Theorem 2.3.3], $\ker(T+K)$ is orthogonally complementable in $H_{\mathcal{A}}$. Therefore, by [4, Lemma 12] we have that

$$H_{\mathcal{A}} = \ker(T+K)^{\perp} \oplus \ker(T+K) \xrightarrow{T+K} H_{\mathcal{A}} \oplus \{0\} = H_{\mathcal{A}}$$

is also an $\mathcal{M}\Phi$-decomposition for $T+K$ and, thus, index $(T+K) = [\ker(T+K)]$. However, index $(T+K) = 0$. Since $K_0(\mathcal{A})$ satisfies the cancellation property by assumption, it follows that $\ker(T+K) = \{0\}$, so $T+K$ is invertible, thus bounded below.
Conversely, if $T+K$ bounded below, then, by [8, Theorem 2.3.3], $Im(T+K)$ is orthogonally complementable in $H_{\mathcal{A}}$. Thus, again by [4, Lemma 12] we have that

$$H_{\mathcal{A}} \oplus \{0\} \xrightarrow{T+K} Im(T+K) \oplus Im(T+K)^{\perp} = H_{\mathcal{A}}$$

is an $\mathcal{M}\Phi$-decomposition for $T+K$. By the same argument as above, since index $(T+K) = 0$ and $K_0(\mathcal{A})$ satisfies the cancellation property, it follows that $Im(T+K)^{\perp} = \{0\}$. □

For $\alpha \in \mathcal{A}$ we may let αI be the operator on $H_{\mathcal{A}}$ given by

$$\alpha I(x_1, x_2, \dots) = (\alpha x_1, \alpha x_2, \dots).$$

It is straightforward to check that αI is an $\mathcal{A}$-linear operator on $H_{\mathcal{A}}$ since we consider $H_{\mathcal{A}}$ as a right Hilbert $\mathcal{A}$-module. Moreover, αI is bounded and we have $\| \alpha I \| = \| \alpha \|$. Finally, αI is adjointable and its adjoint is given by $(\alpha I)^* = \alpha^* I$.

We give then the following generalization of the well known Fredholm alternative stated in [6, Chapter VII, Corollary 7.10].

Theorem 5.8. *Let $K \in \mathcal{K}^*(H_{\mathcal{A}})$ and $\alpha \in G(\mathcal{A})$. Suppose that $K_0(\mathcal{A})$ satisfies the cancellation property. Then the equation $(K - \alpha I)\, x = y$ has a solution for every $y \in H_{\mathcal{A}}$ if and only if $K - \alpha I$ is bounded below. In this case the solution of the above equation is unique.*

Example 5.9. *Let $\mathcal{A} = B(H)$, where H is an infinite-dimensional, separable Hilbert space. If H_1 is any infinite-dimensional subspace of H, then there exists an isometric isomorphism U of H onto H_1. Set $\tilde{U}$ to be the operator on $\mathcal{A}$ given by $\tilde{U}(F) = JUF$ for all $F \in \mathcal{A}$ where J is the inclusion of H_1 into H. Then $\tilde{U} \in B^a(\mathcal{A})$ and moreover, $\tilde{U}$ is an isometry. Put T to be the operator with the matrix $\begin{bmatrix} 1 & 0 \\ 0 & \tilde{U} \end{bmatrix}$ with respect to the decomposition*

$$H_{\mathcal{A}} = L_1^{\perp} \oplus L_1 \xrightarrow{T} L_1^{\perp} \oplus L_1 = H_{\mathcal{A}}.$$

Then $T \in B^a(H_{\mathcal{A}})$ and T is bounded below. Moreover,

$$ImT^{\perp} = Span_{\mathcal{A}}\{(P, 0, 0, 0, \dots)\},$$

where P is the orthogonal projection of H onto $H_1^{\perp}$. However, $T = I + K$ where $K = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{U} - 1 \end{bmatrix}$ with respect to the decomposition $L_1^{\perp} \oplus L_1 \to L_1^{\perp} \oplus L_1$, hence $K \in \mathcal{K}^*(H_{\mathcal{A}})$. This shows that the assumption that $K_0(\mathcal{A})$ satisfies the cancellation property in Proposition 5.7 and Theorem 5.8 is indeed necessary.

6. Examples of semi-C^*-Fredholm operators

In this section we introduce some examples of semi-$\mathcal{A}$-Fredholm operators.

Example 6.1. *Let $F, D \in B^a(H_{\mathcal{A}})$ satisfying that $F(e_k) = e_{2k}$, $D(e_{2k-1}) = 0$ and $D(e_{2k}) = e_k$ for all $k \in \mathbb{N}$. Then $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$ and $D \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$. Indeed, since*

$$F(x_1, x_2, \dots) = (0, x_1, 0x_2, \dots) \text{ for all } (x_1, x_2, \dots) \in H_{\mathcal{A}},$$

it is not hard to see that $ImF = \overline{Span_{\mathcal{A}}}\{e_{2k} \mid k \in \mathbb{N}\}$ where $Span_{\mathcal{A}}$ denotes the $\mathcal{A}$-linear span. Moreover, F is obviously an isometry, so F is an isomorphism onto its image. It is easy to check that $ImF^{\perp} = \overline{Span_{\mathcal{A}}}\{e_{2k-1} \mid k \in \mathbb{N}\}$, hence we have

$H_{\mathcal{A}} = ImF \oplus ImF^{\perp}$. Therefore, $\widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}})$ and $H_{\mathcal{A}} \oplus \{0\} \xrightarrow{F} ImF \oplus ImF^{\perp}$ is an $\widehat{\mathcal{M}\Phi}_l$-decomposition for F. It remains to show that F is adjointable. However, for all $x, y \in H_{\mathcal{A}}$ we have that $< Fx, y >= \sum_{k=1}^{\infty} x_k^* y_{2k} =< x, Dy >$ (where $x = (x_1, x_2, \dots)$, and $y = (y_1, y_2, \dots)$), hence $D = F^*$. It follows that $KerD = ImF^{\perp}$. Moreover, it is straightforward to check that $ImD = H_{\mathcal{A}}$. Hence, we have that $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$, $D \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$, and $KerD^{\perp} \oplus KerD \xrightarrow{D} H_{\mathcal{A}} \oplus \{0\}$ is $\mathcal{M}\Phi_-$-decomposition for D.

Example 6.2. *In general, let $\iota : \mathbb{N} \to \iota(\mathbb{N})$ be a bijection such that $\iota(\mathbb{N}) \subseteq \mathbb{N}$ and $\mathbb{N} \setminus \iota(\mathbb{N})$ is infinite. Moreover, we may define ι in a such way that $\iota(1) < \iota(2) < \iota(3) < \dots$. Then, we define an $\mathcal{A}$-linear bounded operator F on $H_{\mathcal{A}}$ as $F(e_k) = e_{\iota(k)}$ for all k and we define an $\mathcal{A}$-linear operator D on $H_{\mathcal{A}}$ as*

$$D(e_k) = \begin{cases} e_{\iota^{-1}(k)}, & \text{for } k \in \iota(\mathbb{N}), \\ 0, & \text{else.} \end{cases}$$

In a similar way as in Example 6.1 it can be shown that $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$ and $D \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$.

Those examples are also valid in the case when $\mathcal{A} = \mathbb{C}$, that is when $H_{\mathcal{A}} = H$ is a Hilbert space. We will now introduce examples where we use the structure of $\mathcal{A}$ itself in the case when $\mathcal{A} \neq \mathbb{C}$.

Example 6.3. *Let $\mathcal{A} = L^{\infty}([0, 1], \mu)$, where μ is the Lebesgue measure. Set*

$$F(f_1, f_2, f_3, \dots) = (\mathcal{X}_{[0,\frac{1}{2}]} f_1, \mathcal{X}_{[\frac{1}{2},1]} f_1, \mathcal{X}_{[0,\frac{1}{2}]} f_2, \mathcal{X}_{[\frac{1}{2},1]} f_2, \dots) .$$

Then F is a bounded $\mathcal{A}-$ linear operator, $\ker F = \{0\}$,

$$ImF = Span_{\mathcal{A}}\{\mathcal{X}_{[0,\frac{1}{2}]} e_1, \mathcal{X}_{[\frac{1}{2},1]} e_2, \mathcal{X}_{[0,\frac{1}{2}]} e_3, \mathcal{X}_{[\frac{1}{2},1]} e_4, \dots\},$$

and, clearly, $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$. Actually, F is an isometry onto its image.

Example 6.4. *Let again $\mathcal{A} = (L^{\infty}([0, 1]), \mu)$. Set*

$$D(g_1, g_2, g_3, \dots) = (\mathcal{X}_{[0,\frac{1}{2}]} g_1 + \mathcal{X}_{[\frac{1}{2},1]} g_2, \mathcal{X}_{[0,\frac{1}{2}]} g_3 + \mathcal{X}_{[\frac{1}{2},1]} g_4, \dots) .$$

Then $\ker D = ImF^{\perp}$, D is an $\mathcal{A}$-linear, bounded operator and $ImD = H_{\mathcal{A}}$. Thus, $D \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$. Indeed, $D = F^*$, where F is the operator from Example 6.3.

Example 6.5. *Let $\mathcal{A} = B(H)$, where H is a Hilbert space and let P be an orthogonal projection on H. Set*

$$F(T_1, T_2, \dots) = (PT_1, (I - P)T_1, PT_2, (I - P)T_2, \dots),$$

$$D(S_1, S_2, \dots) = (PS_1 + (I - P)S_2, PS_3 + (I - P)S_4, \dots).$$

Then, by the similar arguments as in Example 6.3 and Example 6.4, we have $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$ and $D \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$. Moreover, $D = F^*$.

Example 6.6. *In general, suppose that $\{p_j^i\}_{j,i\in\mathbb{N}}$ is a family of projections in $\mathcal{A}$ such that $p_{j_1}^i p_{j_2}^i = 0$ for all i, whenever $j_1 \neq j_2$, and $\sum_{j=1}^{k} p_j^i = 1$ for all i and some $k \in \mathbb{N}$.*
Set

$$F'(\alpha_1, \dots, \alpha_n, \dots) = (p_1^1\alpha_1, p_2^1\alpha_1, \dots p_k^1\alpha_1, p_2^1\alpha_2, p_2^2\alpha_2, \dots p_k^2\alpha_2, \dots),$$

$$D'(\beta_1, \dots, \beta_n, \dots) = (\sum_{i=1}^{k} p_i^1\beta_i, \sum_{i=1}^{k} p_i^2\beta_{i+k}, \dots).$$

Then $F' \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$ and $D' \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$.

Recalling that a composition of two $\mathcal{M}\Phi_+$ operators on $H_{\mathcal{A}}$ is again an $\mathcal{M}\Phi_+$ operator on $H_{\mathcal{A}}$ and that the same is true for $\mathcal{M}\Phi_-$ operators, we may take suitable compositions of operators from these examples in order to construct more $\mathcal{M}\Phi_\pm$ operators.
Even more $\mathcal{M}\Phi_\pm$ operators can be obtained by composing these operators with isomorphisms of $H_{\mathcal{A}}$. We will present here also some isomorphisms of $H_{\mathcal{A}}$.

Example 6.7. *Let $j : \mathbb{N} \to \mathbb{N}$ be a bijection. Then the operator U given by $U(e_k) = e_{j(k)}$ for all k is an isomorphism of $H_{\mathcal{A}}$. This is a classical well known example of an isomorphism.*

Remark 6.8. *Example 6.7 is in fact equivalent to the statement that sequences from $H_{\mathcal{A}}$ are unconditionally convergent.*

Example 6.9. *Let $(\alpha_1, \dots, \alpha_n, \dots) \in \mathcal{A}^{\mathbb{N}}$ be a sequence of invertible elements in $\mathcal{A}$ such that $\| \alpha_k \|, \| \alpha_k^{-1} \| \leq M$ for all $k \in \mathbb{N}$ and some $M > 0$. If the operator V is given by*

$$V(x_1, \cdots, x_n, \cdots) = (\alpha_1 x_1 \cdots, \alpha_n x_n, \cdots) \text{ for all } (x_1, \cdots, x_n, \cdots) \in H_{\mathcal{A}},$$

then V is an isomorphism of $H_{\mathcal{A}}$.

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