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Conformal vector fields on f-cosymplectic manifolds

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Abstract. In this paper, at first we characterize f-cosymplectic manifolds admitting conformal vector fields. Next, we establish that if a 3-dimensional f-cosymplectic manifold admits a homothetic vector field $\mathbf{V}$, then either the manifold is of constant sectional curvature $-\tilde{f}$ or, $\mathbf{V}$ is an infinitesimal contact transformation. Furthermore, we also investigate Ricci-Yamabe solitons with conformal vector fields on f-cosymplectic manifolds. At last, two examples are constructed to validate our outcomes.

1. Introduction

A vector field $\mathbf{V}$ on a Riemannian manifold satisfying the equation

$$\pounds_{\mathbf{V}} g = 2\sigma g, \tag{1}$$

σ being a smooth function and £ is the Lie-derivative, is called a conformal vector field. If $\mathbf{V}$ is not Killing, it is termed as non-trivial. If σ vanishes, then the conformal vector field $\mathbf{V}$ is named Killing. $\mathbf{V}$ is called homothetic, if σ is constant. A finite dimensional Lie algebra is formed by the set of all proper conformal vector field and all Killing vector fields on a manifold. Although homothetic vector fields form a group, the Lie algebra structure does not. conformal vector field have been studied by many authors such as ([10]-[13], [17], [21]-[23]) and many others.

Killing, conformal and homothetic vector fields have wide applications in differential geometry as well as in mathematical physics.

If r, $\boldsymbol{R}$, $\boldsymbol{S}$ indicate the scalar curvature, the curvature tensor and the Ricci tensor, respectively, then the conformal vector field $\mathbf{V}$ satisfies the following relations [32]:

$$(\pounds_{\mathbf{V}}\nabla)(U_1, V_1) = (U_1\sigma)V_1 - (V_1\sigma)U_1 - g(U_1, V_1)D\sigma, \tag{2}$$

$$\begin{aligned}(\pounds_{\mathbf{V}}\boldsymbol{R})(U_1, V_1)W_1 \quad = \quad & g(\nabla_{U_1}D\sigma, W_1)V_1 - g(\nabla_{V_1}D\sigma, W_1)U_1 \\ & + g(U_1, W_1)\nabla_{V_1}D\sigma - g(V_1, W_1)\nabla_{U_1}D\sigma,\end{aligned} \tag{3}$$

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$$(\pounds_{\mathbf{V}}\boldsymbol{S})(U_1, V_1) = -(2m-1)g(\nabla_{U_1}D\sigma, V_1) - (\triangle\sigma)g(U_1, V_1), \tag{4}$$

$$\pounds_{\mathbf{V}}r = -4m(\triangle\sigma) - 2r\sigma \tag{5}$$

for all vector fields U_1, V_1, W_1 on $\mathbf{N}^{2m+1}$, where $D\sigma$ and $\triangle\sigma = divD\sigma$ respectively denote the gradient and Laplacian of σ.

A vector field $\mathbf{V}$ satisfying the relation

$$\pounds_{\mathbf{V}}\boldsymbol{\eta} = \rho\boldsymbol{\eta}, \tag{6}$$

ρ being a scalar function, is named an infinitesimal contact transformation. It is named as infinitesimal strict contact transformation, if ρ vanishes identically.

In [21], Sharma and Blair characterized $(k, 0)$-contact manifolds admitting a non-Killing conformal vector field. Also in 2010, Sharma and Vrancken[23] investigated $(k, \boldsymbol{\mu})$-contact metric manifolds admitting non-Killing conformal vector field. Very recently, De, Suh and Chaubey[7] studied conformal vector field on almost co-Kähler manifolds. In 2022 [27], Wang investigated almost Kenmotsu $(\mathbf{k}, \boldsymbol{\mu})'$-manifolds with conformal vector field in dimension three.

Guler and Crasmareanu [15] presented the Ricci-Yamabe flow of type (α_1, β_1), which is a scalar combination of Ricci and Yamabe flow[16]. The Ricci-Yamabe flow is an evolution for the metrics on a semi-Riemannian manifold defined as [15]

$$\frac{\partial}{\partial t}g(t) = -2\alpha_1\mathbf{S}(t) + \beta_1 r(t)g(t), \quad g_0 = g(0). \tag{7}$$

A Ricci-Yamabe soliton (in short, RYS) on a Riemannian manifold $(\mathbf{N}, g)$ is defined by

$$\pounds_{\mathbf{V}}g + 2\alpha_1\boldsymbol{S} + (2\boldsymbol{\lambda}_1 - \beta_1 r)g = 0, \tag{8}$$

where £ being Lie-derivative and $\alpha_1, \beta_1, \boldsymbol{\lambda}_1 \in \mathbb{R}$.

This soliton turns into

(i) Ricci soliton if $\alpha_1 = 1, \beta_1 = 0$,

(ii) Yamabe soliton if $\alpha_1 = 0, \beta_1 = 1$,

(iii) Einstein soliton if $\alpha_1 = 1, \beta_1 = -1$.

Several authors have studied Ricci solitons, Yamabe solitons and Ricci-Yamabe solitons, including ([8], [9], [24] – [26], [28] – [31]) and many others.

Because of their link to general relativity, there has also been a significant surge in interest in investigating Ricci solitons and their generalizations in many geometrical situations. Recently, in perfect fluid spacetimes, many authors investigated many type of solitons like Ricci solitons [6], gradient Ricci solitons [6], η-Ricci solitons [2], Yamabe solitons [5], gradient η-Einstein solitons([6]), gradient Schouten solitons [6], Ricci-Yamabe solitons ([20], [25]), respectively.

The above studies encourage us to investigate conformal vector field on f-cosymplectic manifolds. Precisely, we establish the following results:

Theorem 1.1. *If the Reeb vector field ζ of $\mathbf{N}^{2m+1}$ is a conformal vector field, then $\mathbf{N}^{2m+1}$ is locally the product of a Kähler manifold and an interval or unit circle S^1 and the Reeb vector field ζ is Killing.*

Theorem 1.2. *If a conformal vector field $\mathbf{V}$ in $\mathbf{N}^{2m+1}$ is pointwise collinear with the Reeb vector field ζ, then grad f is pointwise collinear with ζ.*

Theorem 1.3. *If a 3-dimensional f-**cm** admits a homothetic vector field $\mathbf{V}$, then either the manifold is of constant sectional curvature $-\tilde{f}$ or, $\mathbf{V}$ is an infinitesimal contact transformation.*

As a corollary of the above theorem, we have:

Corollary 1.4. *If a compact 3-dimensional f-**cm** without boundary admits a homothetic vector field* $\mathbf{V}$*, then either the manifold is of constant sectional curvature* $-\tilde{f}$ *or,* $\mathbf{V}$ *is an infinitesimal strict contact transformation.*

Theorem 1.5. *If a f-**cm** admits a Ricci-Yamabe soliton, then the soliton vector field is conformal if and only if the manifold is an Einstein manifold.*

2. Preliminaries

Let $\mathbf{N}^{2m+1}$ be an almost contact manifold (in short, acm) endowed with a triplet of almost contact structure $(\phi, \zeta, \boldsymbol{\eta})$, where ζ is the reeb vector field, ϕ is a $(1,1)$-type tensor and $\boldsymbol{\eta}$ is 1-form, satisfying [3]

$$\phi^2 V_1 = -V_1 + \boldsymbol{\eta}(V_1)\zeta, \quad \boldsymbol{\eta}(\zeta) = 1 \tag{9}$$

for any vector field V_1 and equation (9) immediately reveals that rank$(\phi) = 2m$, $\phi(\zeta) = 0$ and $\boldsymbol{\eta} \circ \phi = 0$.

If $\mathbf{N}^{2m+1}$ admits a Riemannian metric g such that

$$g(\phi U_1, \phi V_1) = g(U_1, V_1) - \boldsymbol{\eta}(U_1)\boldsymbol{\eta}(V_1), \quad g(V_1, \zeta) = \boldsymbol{\eta}(V_1) \tag{10}$$

for any vector fields U_1, V_1, then $\mathbf{N}^{2m+1}$ is named as an almost contact metric manifold (briefly, acmm).

A structure, named *almost complex structure* $\mathcal{J}$ on $\mathbf{N} \times \mathbb{R}$ is given as

$$\mathcal{J}(V_1, b\frac{d}{ds}) = (\phi V_1 - b\zeta, \boldsymbol{\eta}(V_1)\frac{d}{ds}),$$

where $(V_1, b\frac{d}{ds})$ indicates a tangent vector on $\mathbf{N} \times \mathbb{R}$, V_1 and $b\frac{d}{ds}$ being tangent to $\mathbf{N}$ and $\mathbb{R}$ respectively. An acmm becomes normal if the structure $\mathcal{J}$ is integrable [19].

Let us define $\Phi(U_1, V_1) = g(\phi U_1, V_1)$ for all $U_1, V_1 \in \chi(\mathbf{N})$. Then Φ is called the fundamental 2-form on $\mathbf{N}$. If the 1-form $\boldsymbol{\eta}$ and the fundamental 2-form Φ are closed, then an acmm is said to be almost cosymplectic and if the acmm is normal then it is said to be cosymplectic. For a non-zero constant β, an acmm is said to be an almost β-Kenmotsu if $\boldsymbol{\eta}$ is closed and $d\Phi = 2\beta\boldsymbol{\eta} \wedge \Phi$. If $\beta \in \mathbb{R}$, then an acmm is called an almost β-cosymplectic[18]. In 2014, Aktan et. al.[1] extended the notion of almost β-cosymplectic manifold and introduced an almost f-cosymplectic manifold as an acmm such that $d\Phi = 2f\boldsymbol{\eta} \wedge \Phi$ and $d\boldsymbol{\eta} = 0$ for a smooth function f. If an almost f-cosymplectic manifold is normal, then it is said to be f-cosymplectic manifold (in short, f-**cm**).

For an acmm we define $h = \frac{1}{2}\pounds_\zeta \phi$. For a normal f-**cm**, $h = 0$. The Levi-Civita connection ∇ is given by [1]

$$(\nabla_{U_1}\phi)V_1 = f[g(\phi U_1, V_1)\zeta - \boldsymbol{\eta}(V_1)\phi U_1]. \tag{11}$$

On a f-**cm** $\mathbf{N}^{2m+1}$, the following relations hold[1]:

$$\nabla_{V_1}\zeta = -f\phi^2 V_1, \tag{12}$$

$$\boldsymbol{R}(U_1, V_1)\zeta = \tilde{f}[\boldsymbol{\eta}(U_1)V_1 - \boldsymbol{\eta}(V_1)U_1], \tag{13}$$

$$\boldsymbol{Q}\zeta = -2m\tilde{f}\zeta, \tag{14}$$

the Ricci operator $\boldsymbol{Q}$ is defined by $\boldsymbol{S}(U_1, V_1) = g(\boldsymbol{Q}U_1, V_1)$ and $\tilde{f} = \zeta f + f^2$.

Lemma 2.1 ([4]). *If $\zeta(\tilde{f}) = 0$ in a f-**cm**, then $\tilde{f}$ = constant.*

Lemma 2.2 ([4]). *If a f-**cm** with $\zeta(\tilde{f}) = 0$ is compact, then it becomes a β-cosymplectic manifold. In particular, if $\tilde{f} = 0$, then **N** is cosymplectic.*

Remark 2.3 ([3]). *A **cm** is locally the product of a Kahler manifold and an interval or unit circle S^1.*

Lemma 2.4 ([4]). *For a three-dimensional f-**cm**, we have*

$$QV_1 = (\tilde{f} + \frac{r}{2})V_1 + (-3\tilde{f} - \frac{r}{2})\eta(V_1)\zeta \tag{15}$$

and hence

$$S(U_1, V_1) = (\tilde{f} + \frac{r}{2})g(U_1, V_1) - (3\tilde{f} + \frac{r}{2})\eta(U_1)\eta(V_1). \tag{16}$$

3. Proof of the Main Results

Proof of the Theorem 1.1.
Let the Reeb vector field ζ be a conformal vector field on $\mathbf{N}^{2m+1}$. Then equation (1) implies

$$(£_\zeta g)(U_1, V_1) = 2\sigma g(U_1, V_1), \tag{17}$$

which means that

$$g(\nabla_{U_1}\zeta, V_1) + g(U_1, \nabla_{V_1}\zeta) = 2\sigma g(U_1, V_1). \tag{18}$$

Using (9) and (12) in (18), we have

$$f[g(U_1, V_1) - \eta(U_1)\eta(V_1)] = \sigma g(U_1, V_1). \tag{19}$$

Setting $U_1 = V_1 = \zeta$ in the above equation implies

$$\sigma = 0. \tag{20}$$

Making use of (20) and (19), we get

$$f[g(U_1, V_1) - \eta(U_1)\eta(V_1)] = 0, \tag{21}$$

which means that $f = 0$. Therefore the manifold becomes a cosymplectic manifold. Hence from Remark 1, we get the result.
Thus the proof is finished.

Proof of the Theorem 1.2.
Suppose $\mathbf{V} = b\zeta$, where b is smooth function on $\mathbf{N}^{2m+1}$. Then from (1), we get

$$(£_{b\zeta} g)(U_1, V_1) = 2\sigma g(U_1, V_1), \tag{22}$$

which implies

$$g(\nabla_{U_1} b\zeta, V_1) + g(U_1, \nabla_{V_1} b\zeta) = 2\sigma g(U_1, V_1). \tag{23}$$

Using (12) in the above equation gives

$$(U_1 b)\eta(V_1) + (V_1 b)\eta(U_1) + 2fb[g(U_1, V_1) - \eta(U_1)\eta(V_1)] = 2\sigma g(U_1, V_1). \tag{24}$$

Putting $U_1 = V_1 = \zeta$ in (24) provides

$$\zeta b = \sigma. \tag{25}$$

Contracting (24) entails that

$$bf = \sigma. \tag{26}$$

Again, putting $V_1 = \zeta$ in (24) and using (25) and (26), we get

$$U_1 b = bf\boldsymbol{\eta}(U_1), \tag{27}$$

which implies

$$db = bf\boldsymbol{\eta}. \tag{28}$$

Operating d on both sides of the previous equation and using Poincare Lemma($d^2 \equiv 0$), we obtain

$$d(bf) \wedge \boldsymbol{\eta} = 0, \tag{29}$$

which means that

$$\frac{b}{2}[(U_1 f)\boldsymbol{\eta}(V_1) - (V_1 f)\boldsymbol{\eta}(U_1)] + \frac{f}{2}[(U_1 b)\boldsymbol{\eta}(V_1) - (V_1 b)\boldsymbol{\eta}(U_1)] = 0. \tag{30}$$

Using (27) in (30) gives

$$b[(U_1 f)\boldsymbol{\eta}(V_1) - (V_1 f)\boldsymbol{\eta}(U_1)] = 0, \tag{31}$$

which implies

$$(U_1 f)\boldsymbol{\eta}(V_1) = (V_1 f)\boldsymbol{\eta}(U_1). \tag{32}$$

Hence the above equation implies

$$U_1 f = (\zeta f)\boldsymbol{\eta}(U_1), \tag{33}$$

which means that *grad f* is pointwise collinear with ζ.
Hence the result follows.

Proof of the Theorem 1.3. Let the vector field $\mathbf{V}$ in $\mathbf{N}^3$ is homothetic. Then

$$(\pounds_{\mathbf{V}} g)(U_1, V_1) = 2\sigma g(U_1, V_1), \tag{34}$$

where σ is a constant, and from (4) and (5) we get

$$(\pounds_{\mathbf{V}} \boldsymbol{S})(U_1, V_1) = 0 \ \ \textit{and} \ \ \pounds_{\mathbf{V}} r = -2r\sigma. \tag{35}$$

Definition of Lie-derivative infers that

$$(\pounds_{\mathbf{V}}\boldsymbol{\eta})U_1 = \pounds_{\mathbf{V}}\boldsymbol{\eta}(U_1) - \boldsymbol{\eta}(\pounds_{\mathbf{V}} U_1). \tag{36}$$

Equation (34) and (36) together imply

$$\boldsymbol{\eta}(\pounds_{\mathbf{V}}\zeta) = -\sigma \quad \textit{and} \quad (\pounds_{\mathbf{V}}\boldsymbol{\eta})\zeta = \sigma. \tag{37}$$

From (15), we obtain

$$\boldsymbol{S}(U_1, V_1) = (\tilde{f} + \frac{r}{2})g(U_1, V_1) - (3\tilde{f} + \frac{r}{2})\boldsymbol{\eta}(U_1)\boldsymbol{\eta}(V_1). \tag{38}$$

Now, we take Lie-derivative of the equation (38) along the homothetic vector field $\mathbf{V}$ entails that

$$\begin{aligned}(\pounds_{\mathbf{V}}\boldsymbol{S})(U_1,V_1) &= (\mathbf{V}\tilde{f})[g(U_1,V_1)-3\boldsymbol{\eta}(U_1)\boldsymbol{\eta}(V_1)] \\ &+\frac{1}{2}(\pounds_{\mathbf{V}}r)[g(U_1,V_1)-\boldsymbol{\eta}(U_1)\boldsymbol{\eta}(V_1)] \\ &+(\tilde{f}+\frac{r}{2})(\pounds_{\mathbf{V}}g)(U_1,V_1) \\ &-(3\tilde{f}+\frac{r}{2})[(\pounds_{\mathbf{V}}\boldsymbol{\eta})U_1\boldsymbol{\eta}(V_1)+(\pounds_{\mathbf{V}}\boldsymbol{\eta})V_1\boldsymbol{\eta}(U_1)].\end{aligned} \tag{39}$$

Using (34) and (35) in (39), we infer

$$\begin{aligned}&-(\mathbf{V}\tilde{f})[g(U_1,V_1)-3\boldsymbol{\eta}(U_1)\boldsymbol{\eta}(V_1)] \\ &+(3\tilde{f}+\frac{r}{2})[(\pounds_{\mathbf{V}}\boldsymbol{\eta})U_1\boldsymbol{\eta}(V_1)+(\pounds_{\mathbf{V}}\boldsymbol{\eta})V_1\boldsymbol{\eta}(U_1)] \\ &-2\sigma(\tilde{f}+\frac{r}{2})g(U_1,V_1)+r\sigma[g(U_1,V_1)-\boldsymbol{\eta}(U_1)\boldsymbol{\eta}(V_1)]=0.\end{aligned} \tag{40}$$

Setting $V_1=\zeta$ in (40) and using (37), we get

$$2(\mathbf{V}\tilde{f})\boldsymbol{\eta}(U_1)-2\sigma(\tilde{f}+\frac{r}{2})\boldsymbol{\eta}(U_1)+(3\tilde{f}+\frac{r}{2})[(\pounds_{\mathbf{V}}\boldsymbol{\eta})U_1+\sigma\boldsymbol{\eta}(U_1)]=0. \tag{41}$$

Putting $U_1=\zeta$ in (41) and using (37) entails that

$$\mathbf{V}\tilde{f}=-2\sigma\tilde{f}. \tag{42}$$

From the above two equations, we provide

$$(3\tilde{f}+\frac{r}{2})[(\pounds_{\mathbf{V}}\boldsymbol{\eta})U_1-\sigma\boldsymbol{\eta}(U_1)]=0, \tag{43}$$

which implies either $3\tilde{f}+\frac{r}{2}=0$ or, $3\tilde{f}+\frac{r}{2}\neq 0$.

Case I: If $3\tilde{f}+\frac{r}{2}=0$, which means $r=-6\tilde{f}$. Hence (38) implies

$$\boldsymbol{S}(U_1,V_1)=-2\tilde{f}g(U_1,V_1), \tag{44}$$

which is an Einstein manifold. In 3-dimension,

$$\begin{aligned}\boldsymbol{R}(U_1,V_1)W_1 &= \boldsymbol{S}(V_1,W_1)U_1-\boldsymbol{S}(U_1,W_1)V_1+g(V_1,W_1)\boldsymbol{Q}U_1 \\ &-g(U_1,W_1)\boldsymbol{Q}V_1-\frac{r}{2}[g(V_1,W_1)U_1-g(U_1,W_1)V_1].\end{aligned} \tag{45}$$

In view of (44) and (45), we get

$$\boldsymbol{R}(U_1,V_1)W=-\tilde{f}[g(V_1,W_1)U_1-g(U_1,W_1)V_1], \tag{46}$$

which means that the manifold is of constant sectional curvature $-\tilde{f}$.

Case II: If $3\tilde{f}+\frac{r}{2}\neq 0$, then $(\pounds_{\mathbf{V}}\boldsymbol{\eta})U_1=\sigma\boldsymbol{\eta}(U_1)$. Hence $\mathbf{V}$ is an infinitesimal contact transformation. Hence the proof is completed.

Proof of the Corollary 1.1. It is well known that a homothetic vector field on a compact manifold with out boundary is Killing[14]. Hence from (41) and (42), and using $\sigma=0$, we get

$$(3\tilde{f}+\frac{r}{2})(\pounds_{\mathbf{V}}\boldsymbol{\eta})U_1=0,$$

which implies either $3\tilde{f} + \frac{r}{2} = 0$ or $(£_{\mathbf{V}}\boldsymbol{\eta})U_1 = 0$.
Therefore the result follows.

Proof of the Theorem 1.4.
Assume that the f-**cm** $\mathbf{N}^{2m+1}$ admits a RYS with conformal vector field. Then from (8) we have

$$(£_{\mathbf{V}}g)(U_1, V_1) + 2\alpha_1 S(U_1, V_1) + (2\lambda - \beta_1 r)g(U_1, V_1) = 0. \tag{47}$$

If we take the soliton vector field is conformal, then using (1) in (47), we get

$$\sigma g(U_1, V_1) + \alpha_1 S(U_1, V_1) + (\lambda - \frac{\beta_1}{2}r)g(U_1, V_1) = 0, \tag{48}$$

which implies

$$\alpha_1 S(U_1, V_1) = -(\sigma + \lambda - \frac{\beta_1}{2}r)g(U_1, V_1). \tag{49}$$

Thus, $\mathbf{N}^{2m+1}$ is an Einstein manifold.
Again, if we take $\alpha_1 S(U_1, V_1) = -(\alpha_1 + \lambda - \frac{\beta_1}{2}r)g(U_1, V_1)$, then from (47), we get

$$(£_{\mathbf{V}}g)(U_1, V_1) = -2(\psi + \lambda - \frac{\beta_1}{2}r)g(U_1, V_1), \tag{50}$$

where $\psi = \frac{(\alpha_1+\lambda-\frac{\beta}{2}r)}{\alpha_1}$.
This completes the proof.

4. Examples

Example 1. We figure out the manifold $\mathbf{N}^3 = \{(x, y, z) \in \mathbb{R}^3\}$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. Let

$$z_1 = e^{z^2}\frac{\partial}{\partial x}, \quad z_2 = e^{z^2}\frac{\partial}{\partial y}, \quad z_3 = \frac{\partial}{\partial z} \tag{51}$$

are the linearly independent vector fields of $\mathbf{N}^3$[1].
Then

$$[z_1, z_2] = 0, \ [z_1, z_3] = -2zz_1, \ [z_2, z_3] = -2zz_2. \tag{52}$$

Let g be the Riemannian metric identified by

$$g(z_1, z_1) = g(z_2, z_2) = g(z_3, z_3) = 1$$

and

$$g(z_1, z_2) = g(z_2, z_3) = g(z_1, z_3) = 0.$$

Let $\boldsymbol{\eta}$ be the one-form defined by $\boldsymbol{\eta}(V_1) = g(V_1, z_3)$ for any vector field V_1 on $\mathbf{N}^3$ and ϕ be the (1,1)-tensor field defined by

$$\phi z_1 = z_2, \ \phi z_2 = -z_1, \ \phi z_3 = 0.$$

Using the above relations, we acquire

$$\begin{aligned} &\phi^2 V_1 = -V_1 + \boldsymbol{\eta}(V_1)z_3, \ \boldsymbol{\eta}(z_3) = 1, \\ &g(\phi U_1, \phi V_1) = g(U_1, V_1) - \boldsymbol{\eta}(U_1)\boldsymbol{\eta}(V_1) \end{aligned} \tag{53}$$

for any $U_1,\ V_1 \in \chi(\mathbf{N}^3)$. In [1], the authors proved that $\mathbf{N}^3$ is a f-**cm**.
Using Koszul's formula we get

$$\nabla_{z_1} z_1 = 2z z_3,\ \nabla_{z_1} z_2 = 0,\ \nabla_{z_1} z_3 = -2z z_1,$$
$$\nabla_{z_2} z_1 = 0,\ \nabla_{z_2} z_2 = 2z z_3,\ \nabla_{z_2} z_3 = -2z z_2,$$
$$\nabla_{z_3} z_1 = 0,\ \nabla_{z_3} z_2 = 0,\ \nabla_{z_3} z_3 = 0.$$

We can easily reach with the help of the above results

$$R(z_1, z_2)z_3 = 0, \quad R(z_2, z_3)z_3 = (2 - 4z^2)z_2, \quad R(z_1, z_3)z_3 = (2 - 4z^2)z_1,$$
$$R(z_1, z_2)z_2 = -4z^2 z_1, \quad R(z_2, z_3)z_2 = (-2 + 4z^2)z_3, \quad R(z_1, z_3)z_2 = 0,$$
$$R(z_1, z_2)z_1 = 4z^2 z_2, \quad R(z_2, z_3)z_1 = 0, \quad R(z_1, z_3)z_1 = (-2 + 4z^2)z_3$$

and

$$S(z_1, z_1) = S(z_2, z_2) = 2 - 8z^2, \quad S(z_3, z_3) = 4 - 8z^2.$$

We find $r = 8(1 - 3z^2)$, from the above results.

Let $\mathbf{V} = (x + y)e^{-z^2} z_1 + (-x + y)e^{-z^2} z_2$, $\lambda = -\frac{2}{3}$, $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{3}$. By direct computations equation (47) holds. Hence $\mathbf{N}^3$ defines a Ricci-Yamabe soliton.

Example 2. We figure out the manifold $\mathbf{N}^5 = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5\}$, where $(x_1, x_2, x_3, x_4, x_5)$ are the standard coordinates in $\mathbb{R}^5$. Let

$$z_1 = x_5 \frac{\partial}{\partial x_1}, \quad z_2 = x_5 \frac{\partial}{\partial x_2}, \quad z_3 = -\frac{1}{x_5^3} \frac{\partial}{\partial x_3}, \quad z_4 = -\frac{1}{x_5^3} \frac{\partial}{\partial x_4}, \quad z_5 = \frac{\partial}{\partial x_5}$$

are the linearly independent vector fields of $\mathbf{N}^5$[1]. Therefore,

$$[z_5, z_1] = \frac{1}{x_5} z_1,\ [z_5, z_2] = \frac{1}{x_5} z_2,\ [z_5, z_3] = -\frac{3}{x_5} z_3,\ [z_5, z_4] = -\frac{3}{x_5} z_4.$$

The Riemannian metric g is defined by

$$g(z_i, z_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

Let η be the one-form defined by $\eta(V_1) = g(V_1, z_5)$ for any vector field V_1 on $\mathbf{N}^5$ and ϕ be the (1,1)-tensor field defined by

$$\phi z_1 = z_3,\ \phi z_2 = z_4,\ \phi z_3 = -z_1,\ \phi z_4 = -z_2,\ \phi z_5 = 0.$$

Using the above relations, we acquire

$$\begin{aligned} &\phi^2 V_1 = -V_1 + \eta(V_1) z_5,\ \eta(z_5) = 1, \\ &g(\phi U_1, \phi V_1) = g(U_1, V_1) - \eta(U_1)\eta(V_1) \end{aligned} \tag{54}$$

for any $U_1,\ V_1 \in \chi(\mathbf{N}^5)$. In [1], the authors proved that $\mathbf{N}^5$ is a f-**cm** with $f = \frac{1}{x_5}$.
Using Koszul's formula we get

$$\nabla_{z_1} z_1 = \frac{1}{x_5} z_5,\ \nabla_{z_1} z_2 = 0,\ \nabla_{z_1} z_3 = 0,\ \nabla_{z_1} z_4 = 0,\ \nabla_{z_1} z_5 = -\frac{1}{x_5} z_1,$$

$$\nabla_{z_2}z_1 = 0,\ \nabla_{z_2}z_2 = \frac{1}{x_5}z_5,\ \nabla_{z_2}z_3 = 0,\ \nabla_{z_2}z_4 = 0,\ \nabla_{z_2}z_5 = -\frac{1}{x_5}z_2,$$

$$\nabla_{z_3}z_1 = 0,\ \nabla_{z_3}z_2 = 0,\ \nabla_{z_3}z_3 = -\frac{3}{x_5}z_5,\ \nabla_{z_3}z_4 = 0,\ \nabla_{z_3}z_5 = \frac{3}{x_5}z_3,$$

$$\nabla_{z_4}z_1 = 0,\ \nabla_{z_4}z_2 = 0,\ \nabla_{z_4}z_3 = 0,\ \nabla_{z_4}z_4 = -\frac{3}{x_5}z_5,\ \nabla_{z_4}z_5 = \frac{3}{x_5}z_4,$$

$$\nabla_{z_5}z_1 = 0,\ \nabla_{z_5}z_2 = 0,\ \nabla_{z_5}z_3 = 0,\ \nabla_{z_5}z_4 = 0,\ \nabla_{z_5}z_5 = 0.$$

We can easily reach with the help of the above results

$$R(z_1,z_2)z_2 = -\frac{1}{x_5^2}z_1,\ R(z_1,z_3)z_3 = \frac{3}{x_5^2}z_1,\ R(z_1,z_4)z_4 = \frac{3}{x_5^2}z_1,\ R(z_1,z_5)z_5 = -\frac{2}{x_5^2}z_1,$$

$$R(z_1,z_2)z_1 = \frac{1}{x_5^2}z_2,\ R(z_1,z_3)z_1 = -\frac{3}{x_5^2}z_3,\ R(z_1,z_4)z_1 = -\frac{3}{x_5^2}z_4,\ R(z_1,z_5)z_1 = \frac{2}{x_5^2}z_5,$$

$$R(z_2,z_3)z_3 = \frac{3}{x_5^2}z_2,\ R(z_2,z_4)z_4 = \frac{3}{x_5^2}z_2,\ R(z_2,z_5)z_5 = -\frac{2}{x_5^2}z_2,\ R(z_3,z_4)z_4 = -\frac{9}{x_5^2}z_3,$$

$$R(z_3,z_5)z_5 = -\frac{6}{x_5^2}z_3,\ R(z_4,z_5)z_5 = -\frac{6}{x_5^2}z_4,\ R(z_2,z_5)z_2 = \frac{2}{x_5^2}z_5,\ R(z_4,z_5)z_4 = \frac{6}{x_5^2}z_5,$$

$$R(z_3,z_5)z_3 = \frac{6}{x_5^2}z_5,\ R(z_5,z_3)z_5 = \frac{6}{x_5^2}z_3,\ R(z_2,z_4)z_2 = -\frac{3}{x_5^2}z_4,\ R(z_2,z_3)z_2 = -\frac{3}{x_5^2}z_3$$

and

$$S(z_1,z_1) = S(z_2,z_2) = \frac{3}{x_5^2},\ S(z_3,z_3) = S(z_4,z_4) = -\frac{9}{x_5^2},\ S(z_5,z_5) = -\frac{16}{x_5^2}.$$

Hence,

$$r = S(z_1,z_1) + S(z_2,z_2) + S(z_3,z_3) + S(z_4,z_4) + S(z_5,z_5) = -\frac{28}{x_5^2}.$$

Let $\mathbf{V} = 3x_1z_1 + 3x_2z_2 + x_5^4x_3z_3 + x_5^4x_4z_4 + x_5^2z_5$ and $\sigma = 2x_5$. By direct computations equation (1) holds. Hence $\mathbf{N}^5$ defines a conformal vector field.

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