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Golden Riemannian submersions

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Abstract. In this paper, we study a Golden Riemannian submersion between Golden Riemannian manifolds. Here, we investigate the geometric properties of such a submersion and obtain some results. Also, we study the relations between the Ricci curvatures of any fibre, base and target manifolds of Golden Riemannian submersion and using these relations obtain two sharp inequalities. Moreover, we give some characterizations of Golden Riemannian submersion whose total space admits a Ricci soliton with the horizontal or vertical potential field. Finally, we construct some examples of Golden Riemannian submersions.

The theory of Riemannian submersions is very interesting topic in Differential Geometry and Theoretical Physics, since such theory has many applications in Kaluza-Klein theory, Yang-Mills theory, supergravity and superstring theories (see [9]).

The Riemannian submersions goes back to five decades ago, when B. O'Neill and A. Gray studied the basis of such theory, independently (see [12, 14]). From the geometric point of view, Riemannian submersions are important tools in Riemannian geometry since the total space of such submersions carry additional structures (of contact, complex, quaternionic type, etc.) Hence, the geometry of Riemannian submersions have been studied and developed in the last three decades (see papers, [3, 9, 18, 20, 21]).

On the other hand, the number $\phi = \frac{(1+\sqrt{5})}{2}$ is a solution of the equation $x^2 - x - 1 = 0$ which is called a Golden ratio. The notion of Golden ratio has occupied in many different areas such as arts, architecture, music, philosophies and besides it is also appears in Nature. Being inspired by the Golden ratio ϕ, the concept of Golden manifold was introduced in [7] with a $(1, 1)-$tensor field Φ on such a manifold satisfies $\Phi^2 - \Phi - I = 0$ and they obtained the eigenvalues of Φ are Golden ratios ϕ and $1 - \phi$. In [19], the authors defined Golden maps between Golden Riemannian manifolds and gave some properties of the induced structure on their submanifolds. Moreover, the authors studied two types of submersions whose total space is an almost trans-1-Golden manifold (see [22]). Nowadays, there are several works on Golden Riemannian manifolds in literature and they are still in progress (see [1, 4, 10, 11, 17]).

The notion of Ricci soliton is firstly appeared after Hamilton was introduced the Ricci flows and showed that the self-similar solutions of such flows are Ricci solitons (see [13]).

From a mathematical point of view, a Riemannian manifold (M, g) is called a Ricci soliton, if

$$\mathcal{L}_\sigma g + Ric + kg = 0 \tag{1}$$

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is satisfied, where $\mathcal{L}_\sigma$ denotes the Lie-derivative with respect to the vector field σ which is called potential field, Ric is the Ricci tensor on M and k is a constant. Such a Ricci soliton is denoted by (M, g, σ, k). Moreover, if the constant k is positive, zero or negative, then the Ricci soliton (M, g, σ, k) is said to be expanding, steady or shrinking, respectively. Ricci solitons became popular after Perelman used them to solve the Poincare conjecture in [16] and since then, the theory of Ricci solitons has been studied intensively in many different areas (more details, see [5, 6, 8, 15]).

Our purpose is to study in the present paper a class of submersions between Golden Riemannian manifolds which is called Golden Riemannian submersion. Here, we characterize the Golden Riemannian submersions between such manifolds by investigating the $\Phi-$invariance of horizontal and vertical distributions and the fundamental tensor fields $\mathcal{T}$ and $\mathcal{A}$ of such distributions in Sect. 3. Also, we investigate the Ricci curvatures of any fibre, base and target manifolds of Golden Riemannian submersion and present the relations among them. Using these relations, we obtain sharp inequalities for Golden Riemannian submersion. Moreover, we deal with the total space of such a submersion admits a Ricci soliton and study here the necessary conditions for Ricci soliton to be either shrinking or expanding, depending on whether the potential field is horizontal or vertical. In Sect. 4, we give some examples of Golden Riemannian submersion.

1. Preliminaries

In this section, we recall the following notations:

Let (M, g) be a Riemannian manifold. A non-null tensor field Φ of type $(1, 1)$ on M is called a Golden structure, if it satifies

$$\Phi^2 = \Phi + I, \tag{2}$$

where I is the identity transformation on the Lie algebra $\Gamma(TM)$ of the differentiable vector fields on M.

Here, we note that the metric g is Φ compatible if

$$g(\Phi X, Y) = g(X, \Phi Y) \tag{3}$$

is satisfied, for any X, Y vector fields on M. If we substitute ΦX into X in (3), it becomes

$$\begin{aligned} g(\Phi X, \Phi Y) = g(\Phi^2 X, Y) &= g((\Phi + I)X, Y)) \\ &= g(\Phi X, Y) + g(X, Y), \end{aligned}$$

for any X, Y vector fields on M. Then, the Riemannian metric g in Eq. (3) is called $\Phi-$compatible and (M, Φ, g) is called an almost Golden Riemannian manifold. Also, M is called a Golden manifold if it has an integrable almost Golden structure. Recall that the structure Φ is integrable if the Nijenhuis tensor N_Φ vanishes, where

$$N_\Phi(X, Y) = \Phi^2[X, Y] + [\Phi X, \Phi Y] - \Phi[\Phi X, Y] - \Phi[X, \Phi Y],$$

for all X, Y vector fields on M (see [2]).

It is known that the integrability of Φ is equivalent to the existence of a torsion-free affine connection with respect to which the equation $\nabla\Phi = 0$ holds (see [7, 10]).

On the other hand, some basic notations about Riemannian submersions from [9] as follows:

A map $\mathcal{F} : (M, g) \to (\bar{M}, \bar{g})$ is called a C^∞-submersion between Riemannian manifolds (M^m, g) and $(\bar{M}^n, \bar{g})$, if $\mathcal{F}$ has a maximal rank at any point of M. For any $x \in \bar{M}$, $\mathcal{F}^{-1}(x)$ is closed r-dimensional submanifold of M, such that $r = m - n$. For any $p \in M$, the distribution $\ker \mathcal{F}_{*p}$ which is integrable. Also, $T_p\mathcal{F}^{-1}(x)$ are $r-$dimensional subspaces of $\ker \mathcal{F}_{*p}$ and it follows that $\ker \mathcal{F}_{*p} = T_p\mathcal{F}^{-1}(x)$. Hence, $\ker \mathcal{F}_{*p}$ is called the

vertical space of any point $p \in M$.
Denote the complementary distribution of $\ker \mathcal{F}_*$ by $(\ker \mathcal{F}_*)^\perp$, the one has

$$T_p(M) = \ker \mathcal{F}_{*p} \oplus (\ker \mathcal{F}_{*p})^\perp$$

where $(\ker \mathcal{F}_{*p})^\perp$ is called the horizontal space of any point $p \in M$.

Let $\mathcal{F} : (M, g) \to (\bar{M}, \bar{g})$ be a submersion between Riemannian manifolds. At any point $p \in M$, we say that $\mathcal{F}$ is a Riemannian submersion if $\mathcal{F}_{*p}$ preserves the length of the horizontal vectors.

Let $\mathcal{F} : (M, g) \to (\bar{M}, \bar{g})$ be a Riemannian submersion, and denote by ∇ and $\bar{\nabla}$ the Levi-Civita connections of M and $\bar{M}$, respectively. If X, Y are the basic vector fields, $\mathcal{F}$-related to $\bar{X}, \bar{Y}$, one has:

(*i*) $g(X, Y) = \bar{g}(\bar{X}, \bar{Y}) \circ \mathcal{F}$,
(*ii*) $h[X, Y]$ is the basic vector field $\mathcal{F}$-related to $[\bar{X}, \bar{Y}]$,
(*iii*) $h(\nabla_X Y)$ is the basic vector field $\mathcal{F}$-related to $\bar{\nabla}_{\bar{X}} \bar{Y}$,
(*iv*) for any vertical vector field V, $[X, V]$ is the vertical.

A Riemannian submersion $\mathcal{F} : (M, g) \to (\bar{M}, \bar{g})$ determines two tensor fields $\mathcal{T}$ and $\mathcal{A}$ on M, which are called the fundamental tensor fields or the invariants of Riemannian submersion $\mathcal{F}$ and they are defined by

$$\mathcal{T}(E, F) = \mathcal{T}_E F = h\nabla_{vE} vF + v\nabla_{vE} hF,$$
$$\mathcal{A}(E, F) = \mathcal{A}_E F = v\nabla_{hE} hF + h\nabla_{hE} vF,$$

where v and h are the vertical and horizontal projections, respectively and ∇ is a Levi-Civita connection of M, for any $E, F \in \Gamma(TM)$. Indeed, the fundamental tensors $\mathcal{T}$ and $\mathcal{A}$ satisfy the followings:

$$\mathcal{T}_V W = \mathcal{T}_W V, \tag{4}$$
$$\mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2} v[X, Y], \tag{5}$$

for any $V, W \in \ker \mathcal{F}_{*p}$ and $X, Y \in (\ker \mathcal{F}_{*p})^\perp$.

We here note that the vanishing of the tensor field $\mathcal{A}$ means the horizontal distribution $(\ker \mathcal{F}_*)^\perp$ is integrable. On the other hand, the vanishing of the tensor field $\mathcal{T}$ means any fibre of Riemannian submersion $\mathcal{F}$ is totally geodesic submanifold of M. Also, any fibre of Riemannian submersion $\mathcal{F}$ is totally umbilical if and only if

$$\mathcal{T}_V W = g(V, W) H, \tag{6}$$

where H denotes the mean curvature vector field of any fibre in M, for any $V, W \in \ker \mathcal{F}_*$.

Also, for any $E, F, G \in \Gamma(TM)$ one has

$$g(\mathcal{T}_E F, G) + g(\mathcal{T}_E G, F) = 0, \tag{7}$$
$$g(\mathcal{A}_E F, G) + g(\mathcal{A}_E G, F) = 0. \tag{8}$$

Using fundamental tensor fields $\mathcal{T}$ and $\mathcal{A}$, the following formulas are given as

$$\nabla_V W = \mathcal{T}_V W + \hat{\nabla}_V W, \tag{9}$$
$$\nabla_V X = h\nabla_V X + \mathcal{T}_V X, \tag{10}$$
$$\nabla_X V = \mathcal{A}_X V + v\nabla_X V, \tag{11}$$
$$\nabla_X Y = h\nabla_X Y + \mathcal{A}_X Y \tag{12}$$

for any $V, W \in \ker \mathcal{F}_*$ *and* $X, Y \in (\ker \mathcal{F}_*)^\perp$.

Moreover, by using the properties of the fundamental tensor fields $\mathcal{T}$ and $\mathcal{A}$, we have

$$\sum_{j=1}^{r} g(\mathcal{T}_U U_j, \mathcal{T}_U U_j) = \sum_{i=1}^{n} g(\mathcal{T}_U X_i, \mathcal{T}_U X_i), \tag{13}$$

$$\sum_{i=1}^{n} g(\mathcal{A}_X X_i, \mathcal{A}_X X_i) = \sum_{j=1}^{r} g(\mathcal{A}_X U_j, \mathcal{A}_X U_j), \tag{14}$$

where $\{U_j, X_i\}_{1\leq j\leq r,\ 1\leq i\leq n}$ is an orthonormal frame of M, for any horizontal and vertical vector fields X and U, respectively.

On the other hand, one has the folloiwng formulas:

$$R(U,V,F,W) = \hat{R}(U,V,F,W) - g(\mathcal{T}_U W, \mathcal{T}_V F) + g(\mathcal{T}_V W, \mathcal{T}_U F), \tag{15}$$

$$\begin{aligned} R(X,Y,Z,H) &= \bar{R}(\bar{X},\bar{Y},\bar{Z},\bar{H}) \circ \mathcal{F} + 2g(\mathcal{A}_X Y, \mathcal{A}_Z H) \\ &\quad - g(\mathcal{A}_Y Z, \mathcal{A}_X H) + g(\mathcal{A}_X Z, \mathcal{A}_Y H), \end{aligned} \tag{16}$$

for any $X, Y, Z, H \in (\ker \mathcal{F}_*)^\perp$ and $U, V, F, W \in \ker \mathcal{F}_*$.

2. Golden Riemannian Submersions between Golden Riemannian Manifolds

We give the following notion:

Definition 2.1. *Let (M, Φ, g) and $(\bar{M}, \bar{\Phi}, \bar{g})$ be Golden Riemannian manifolds. A surjective map $\mathcal{F} : M \to \bar{M}$ is called a Golden Riemannian submersion if the followings are hold:*

i) $\mathcal{F}$ is a Riemannian submersion,

ii) $\mathcal{F}$ is a $(\Phi, \bar{\Phi})-$holomorphic, i.e. $\mathcal{F}_* \circ \Phi = \bar{\Phi} \circ \mathcal{F}_*$.

Proposition 2.2. *Let $\mathcal{F} : (M, \Phi, g) \to (\bar{M}, \bar{\Phi}, \bar{g})$ be a Golden Riemannian submersion between Golden Riemannian manifolds. Then the horizontal and vertical distributions are $\Phi-$invariant, i.e.*

$$\Phi(\ker \mathcal{F}_*) = \ker \mathcal{F}_*, \quad \Phi(\ker \mathcal{F}_*)^\perp = (\ker \mathcal{F}_*)^\perp.$$

Proof. Let U_1 be a vertical vector field. Since $\mathcal{F}$ is a $(\Phi, \bar{\Phi})-$holomorphic, one has

$$\mathcal{F}_*(\Phi U_1) = \bar{\Phi}(\mathcal{F}_* U_1) = 0,$$

which gives ΦU_1 is vertical vector field.

On the other hand, let X_1 be a horizontal vector field. Then, using (3), we can write

$$g(\Phi X_1, U_1) = g(X, \Phi U_1) = 0,$$

which gives ΦX_1 is also horizontal vector field. Hence, we obtain that $\Phi(\ker \mathcal{F}_*) \subset \ker \mathcal{F}_*$ and $\Phi(\ker \mathcal{F}_*)^\perp \subset (\ker \mathcal{F}_*)^\perp$, and so

$$\Phi(\ker \mathcal{F}_*) = \ker \mathcal{F}_* \quad \text{and} \quad \Phi(\ker \mathcal{F}_*)^\perp = (\ker \mathcal{F}_*)^\perp.$$

□

Proposition 2.3. *Let $\mathcal{F} : (M, \Phi, g) \to (\bar{M}, \bar{\Phi}, \bar{g})$ be a Golden Riemannian submersion between Golden Riemannian manifolds and X_1, X_2 be basic vector fields on M which are $\mathcal{F}-$related to $\bar{X}_1, \bar{X}_2$ on $\bar{M}$, respectively. Then, one has*

1. *ΦX_1 is the basic vector field which is $\mathcal{F}-$related to $\bar{\Phi}\bar{X}_1$,*
2. *$h(N_\Phi(X_1, X_2))$ is the basic vector field which is $\mathcal{F}-$related to $\bar{N}_{\bar{\Phi}}(\bar{X}_1, \bar{X}_1)$,*
3. *$h((\nabla_{X_1}\Phi)X_2)$ is the basic vector field which is $\mathcal{F}-$related to $(\bar{\nabla}_{\bar{X}_1}\bar{\Phi})\bar{X}_2$.*

Remark 2.4. *Let $\mathcal{F} : (M, \Phi, g) \to (\bar{M}, \bar{\Phi}, \bar{g})$ be a Golden Riemannian submersion between Golden Riemannian manifolds. From the Proposition 2.2, one has the vertical distribution $\ker\mathcal{F}_*$ is a $\Phi-$invariant and so any fibre inherits a Golden structure Φ from M. Hence, any fibre is a closed invariant submanifold of M.*

Then, one has the following:

Proposition 2.5. *Let $\mathcal{F} : (M, \Phi, g) \to (\bar{M}, \bar{\Phi}, \bar{g})$ be a Golden Riemannian submersion between Golden Riemannian manifolds. If the Golden structure Φ is parallel on M, then $\bar{\Phi}$ is also parallel on $\bar{M}$.*

Proof. If the Golden structure Φ is parallel with respect to ∇ on M, one has $\nabla\Phi = 0$, where ∇ is the Levi-Civita connection on M. For any horizontal vector fields X_1 and X_2, we can write $(\nabla_{X_1}\Phi)X_2 = 0$. Then, it follows

$$\nabla_{X_1}\Phi X_2 = \Phi\nabla_{X_1}X_2,$$
$$h(\nabla_{X_1}\Phi X_2) = h(\Phi\nabla_{X_1}X_2).$$

Using Proposition 2.3, the last statement is equivalent to

$$(\bar{\nabla}_{\bar{X}_1}\bar{\Phi}\bar{X}_2) \circ \mathcal{F} = (\bar{\Phi}\bar{\nabla}_{\bar{X}_1}\bar{X}_2) \circ \mathcal{F},$$

which gives $(\bar{\nabla}_{\bar{X}_1}\bar{\Phi})\bar{X}_2 = 0$, for any vector fields $\bar{X}_1, \bar{X}_2$ on M. Hence, we get the Golden structure $\bar{\Phi}$ is parallel with respect to the Levi-Civita connection $\bar{\nabla}$ on $\bar{M}$. □

Assumption: From now on, we assume that the Golden structures Φ and $\bar{\Phi}$ are parallel on M and $\bar{M}$, respectively.

Theorem 2.6. *Let $\mathcal{F} : (M, \Phi, g) \to (\bar{M}, \bar{\Phi}, \bar{g})$ be a Golden Riemannian submersion between Golden Riemannian manifolds. For any horizontal and vertical vector fields X_1 and U_1, respectively, one has:*

(i) $\mathcal{A}_{\Phi X_1} = \mathcal{A}_{X_1} \circ \Phi$ *on* $\ker\mathcal{F}_*$,

(ii) $\mathcal{T}_{\Phi U_1} = \mathcal{T}_{U_1} \circ \Phi$ *on* $(\ker\mathcal{F}_*)^\perp$.

Proof. Let X_2 be a horizontal vector field. If we use the equalities (8) and (5), respectively in the following, we have

$$\begin{aligned} g(\mathcal{A}_{\Phi X_1}U_1, X_2) &= -g(\mathcal{A}_{\Phi X_1}X_2, U_1) = g(\mathcal{A}_{X_2}\Phi X_1, U_1) = g(\Phi\mathcal{A}_{X_2}X_1, U_1) \\ &= -g(\Phi\mathcal{A}_{X_1}X_2, U_1) = -g(\mathcal{A}_{X_1}X_2, \Phi U_1) = g(\mathcal{A}_{X_1}\Phi U_1, X_2), \end{aligned}$$

which follows

$$\mathcal{A}_{\Phi X_1}U_1 = \mathcal{A}_{X_1}\Phi U_1.$$

Similarly, for any vertical vector field U_2 and using the equalities (7) and (4), respectively, we can write

$$\begin{aligned} g(\mathcal{T}_{\Phi U_1}X_1, U_2) &= -g(\mathcal{T}_{\Phi U_1}U_2, X_1) = -g(\mathcal{T}_{U_2}\Phi U_1, X_1) = -g(\Phi\mathcal{T}_{U_2}U_1, X_1) \\ &= -g(\mathcal{T}_{U_2}U_1, \Phi X_1) = -g(\mathcal{T}_{U_1}U_2, \Phi X_1) = g(\mathcal{T}_{U_1}\Phi X_1, U_2), \end{aligned}$$

which gives

$$\mathcal{T}_{\Phi U_1}X_1 = \mathcal{T}_{U_1}\Phi X_1.$$

Therefore, (i) and (ii) are obtained. □

The next lemma gives the relations the Ricci curvatures of any fibre, target manifold M and base manifold $\bar{M}$, as follows:

Lemma 2.7. *Let $\mathcal{F} : (M, \Phi, g) \to (\bar{M}, \bar{\Phi}, \bar{g})$ be a Golden Riemannian submersion between Golden Riemannian manifolds with totally umbilical fibres. Then, the Ricci curvature on M is given as*

$$\begin{aligned} ric(\Phi U_1) &= \hat{ric}(\Phi U_1) + r\|H\|^2 + rg(\Phi H, H) - \|\Phi \mathcal{A}_{X_i} U_1\|^2 \\ &\quad - div(H)\|\Phi U_1\|^2, \end{aligned} \tag{17}$$

$$\begin{aligned} ric(\Phi X_1) &= \bar{ric}(\bar{\Phi}\bar{X}_1) \circ \mathcal{F} - rg(\nabla_{\Phi X_1} H, \Phi X_1) + \|\Phi \mathcal{T}_{U_j} X_1\|^2 \\ &\quad + 2\|\Phi \mathcal{A}_{X_1} X_i\|^2, \end{aligned} \tag{18}$$

where $\hat{ric}$ and $\bar{ric}$ denote the Ricci curvatures of any fibre and $\bar{M}$, for any unit vertical and horizontal vector fields U_1 and X_1, respectively.

Proof. For any unit vertical vector field U_1 on M, one has ΦU_1 is vertical. Considering the equality (15), we can write

$$\begin{aligned} ric(\Phi U_1) &= \hat{ric}(\Phi U_1) + \sum_j \Big\{ g(\mathcal{T}_{\Phi U_1} \Phi U_1, \mathcal{T}_{U_j} U_j) - g(\mathcal{T}_{\Phi U_1} U_j, \mathcal{T}_{\Phi U_1} U_j) \Big\} \\ &\quad + \sum_i \Big\{ g(\mathcal{T}_{\Phi U_1} X_i, \mathcal{T}_{\Phi U_1} X_i) - g(\mathcal{A}_{X_i} \Phi U_1, \mathcal{A}_{X_i} \Phi U_1) \\ &\quad - g((\nabla_{X_i} \mathcal{T})(\Phi U_1, \Phi U_1), X_i) \Big\}, \end{aligned} \tag{19}$$

where $\{U_j, X_i\}_{1 \le j \le r, 1 \le i \le n}$ is an orthonormal frame of M.

Taking into account of the equality (13), above (19) becomes

$$\begin{aligned} ric(\Phi U_1) &= \hat{ric}(\Phi U_1) + \sum_j g(\mathcal{T}_{\Phi U_1} \Phi U_1, \mathcal{T}_{U_j} U_j) - \sum_i \Big\{ g(\mathcal{A}_{X_i} \Phi U_1, \mathcal{A}_{X_i} \Phi U_1) \\ &\quad + g\Big((\nabla_{X_i} \mathcal{T})(\Phi U_1, \Phi U_1), X_i\Big) \Big\}. \end{aligned} \tag{20}$$

On the other hand, since a Golden Riemannian submersion $\mathcal{F}$ has totally umbilical fibre and using the equalities (2)-(4) and (6), one has

$$\begin{aligned} \sum_j g(\mathcal{T}_{\Phi U_1} \Phi U_1, \mathcal{T}_{U_j} U_j) &= rg(\mathcal{T}_{\Phi U_1} \Phi U_1, H) = rg(\Phi \mathcal{T}_{\Phi U_1} U_1, H) = rg(\mathcal{T}_{\Phi U_1} U_1, \Phi H) \\ &= rg(\mathcal{T}_{U_1} \Phi U_1, \Phi H) = rg(\Phi \mathcal{T}_{U_1} U_1, \Phi H) = rg(\mathcal{T}_{U_1} U_1, \Phi^2 H) \\ &= rg(\mathcal{T}_{U_1} U_1, \Phi H) + rg(\mathcal{T}_{U_1} U_1, H), \\ &= r\|U_1\|^2 g(H, \Phi H) + r\|U_1\|^2 \|H\|^2, \end{aligned} \tag{21}$$

where H is the mean curvature vector field of any fibre in M. Also, since U_1 is unit vertical vector field, above (21) gives

$$\sum_j g(\mathcal{T}_{\Phi U_1} \Phi U_1, \mathcal{T}_{U_j} U_j) = rg(H, \Phi H) + r\|H\|^2. \tag{22}$$

Moreover, using the condition of totally umbilical, we get

$$\begin{aligned} \sum_i g\Big((\nabla_{X_i} \mathcal{T})(\Phi U_1, \Phi U_1), X_i\Big) &= \sum_i \Big\{ (\nabla_{X_i} g)(\Phi U_1, \Phi U_1) g(H, X_i) \\ &\quad + g(\nabla_{X_i} H, X_i) g(\Phi U_1, \Phi U_1) \Big\}, \end{aligned}$$

which gives

$$\sum_i g\Big((\nabla_{X_i} \mathcal{T})(\Phi U_1, \Phi U_1), X_i\Big) = \sum_i g(\nabla_{X_i} H, X_i) \|\Phi U_1\|^2. \tag{23}$$

Putting (22) and (23) in (20), we get

$$\begin{aligned} ric(\Phi U_1) &= \hat{ric}(\Phi U_1) + r\|H\|^2 + rg(\Phi H, H) - \sum_i \Big\{ g(\Phi \mathcal{A}_{X_i} U_1, \Phi \mathcal{A}_{X_i} U_1) \\ &\quad + g(\nabla_{X_i} H, X_i) \|\Phi U_1\|^2 \Big\}, \end{aligned}$$

where $\{X_i\}_{1\le i\le n}$ is an orthonormal basis of the horizontal distribution $(\ker \mathcal{F}_*)^\perp$.

Hence, the last equality is equivalent to (17) .

Similarly, for any unit horizontal vector field X_1 on M, we get

$$\begin{aligned} ric(\Phi X_1) &= \bar{ric}(\bar{\Phi}\bar{X}_1) \circ \mathcal{F} + \sum_j \Big\{ g(\mathcal{T}_{U_j} \Phi X_1, \mathcal{T}_{U_j} \Phi X_1) - g(\mathcal{A}_{\Phi X_1} U_j, \mathcal{A}_{\Phi X_1} U_j) \\ &\quad - g((\nabla_{\Phi X} \mathcal{T})(U_j, U_j), \Phi X_1) \Big\} + 3 \sum_i g(\mathcal{A}_{\Phi X_1} X_j, \mathcal{A}_{\Phi X_1} X_j), \end{aligned} \tag{24}$$

where $\{U_j, X_i\}_{1\le j\le r, 1\le i\le n}$ is an orthonormal frame of the target manifold M.
Using the equality (14) in (24), it gives

$$\begin{aligned} ric(\Phi X_1) &= \bar{ric}(\bar{\Phi}\bar{X}_1) \circ \mathcal{F} + \sum_j \Big\{ g(\mathcal{T}_{U_j} \Phi X_1, \mathcal{T}_{U_j} \Phi X_1) \\ &\quad - g((\nabla_{\Phi X_1} \mathcal{T})(U_j, U_j), \Phi X_1) \Big\} + 2 \sum_i g(\mathcal{A}_{\Phi X_1} X_j, \mathcal{A}_{\Phi X_1} X_j). \end{aligned} \tag{25}$$

Since $\mathcal{F}$ has totally umbilical fibres and using (6) in (25), it gives

$$\begin{aligned} ric(\Phi X_1) &= \bar{ric}(\bar{\Phi}\bar{X}_1) \circ \mathcal{F} + \sum_j g(\mathcal{T}_{U_j} \Phi X_1, \mathcal{T}_{U_j} \Phi X_1) \\ &\quad - rg(\nabla_{\Phi X_1} H, \Phi X_1) + 2 \sum_i g(\mathcal{A}_{\Phi X_1} X_j, \mathcal{A}_{\Phi X_1} X_j). \end{aligned}$$

Then, using the parallelism of Golden structure Φ on M, it follows

$$\begin{aligned} ric(\Phi X_1) &= \bar{ric}(\bar{\Phi}\bar{X}_1) \circ \mathcal{F} - rg(\nabla_{\Phi X_1} H, \Phi X_1) + \sum_j \Big\{ g(\Phi \mathcal{T}_{U_j} X_1, \Phi \mathcal{T}_{U_j} X_1) \\ &\quad + 2 \sum_i g(\Phi \mathcal{A}_{X_1} X_i, \Phi \mathcal{A}_{X_1} X_i), \end{aligned}$$

which is nothing but (18). □

Using above Lemma 2.7, we have sharp inequalities with the following:

Theorem 2.8. *Let $\mathcal{F} : (M, \Phi, g) \to (\bar{M}, \bar{\Phi}, \bar{g})$ be a Golden Riemannian submersion between Golden Riemannian manifolds with totally umbilical fibres. Then, one has the following inequalities:*

$$\begin{aligned} ric(\Phi U_1) &\le \hat{ric}(\Phi U_1) + r\|H\|^2 + rg(\Phi H, H) - div(H) \|\Phi U_1\|^2, \\ ric(\Phi X_1) &\ge \bar{ric}(\bar{\Phi}\bar{X}_1) \circ \mathcal{F} - rg(\nabla_{\Phi X_1} H, \Phi X_1) + \|\Phi \mathcal{T}_{U_j} X_1\|^2, \end{aligned}$$

for any vertical vector field U_1 and horizontal vector field X_1. The equality cases of both above inequalities are satisfied if and only if the horizontal distribution $(\ker \mathcal{F}_*)^\perp$ is integrable.

The next lemma is about a Golden Riemannian submersion whose total manifold admits a Ricci soliton with vertical potential field.

Lemma 2.9. *Let $\mathcal{F} : (M, \Phi, g) \to (\bar{M}, \bar{\Phi}, \bar{g})$ be a Golden Riemannian submersion with totally geodesic fibres. If M admits a Ricci soliton (M, g, σ, k) with vertical potential field, then the relation between the Ricci tensor $\bar{Ric}$ and Ricci curvature $\bar{ric}$ on $\bar{M}$ is given as*

$$\bar{Ric}(\bar{\Phi}\bar{X}_1, \bar{X}_1) \circ \mathcal{F} + \bar{ric}(\bar{X}_1) \circ \mathcal{F} + 2\sum_i \|\Phi\mathcal{A}_{X_i}X_1\|^2 + \lambda\|\Phi X_1\|^2 = 0, \tag{26}$$

where $\{X_i\}_{1\le i\le n}$ is an orthonormal frame on $(\ker\mathcal{F}_*)^\perp$ and X_1 is any horizontal vector field which is $\mathcal{F}-$related to $\bar{X}_1$.

Proof. If the Golden manifold M admits a Ricci soliton with vertical potential field σ, from the eq. (1) we can write

$$\frac{1}{2}\Big(\mathcal{L}_\sigma g\Big)(\Phi X_1, \Phi X_2) + Ric(\Phi X_1, \Phi X_2) + kg(\Phi X_1, \Phi X_2) \quad = \quad 0, \tag{27}$$

for any horizontal vector fields X_1, X_2 on $(\ker\mathcal{F}_*)^\perp$.

Also, from the definition of Lie-derivative and using the equalities (5), (8) and (11) one has

$$\begin{aligned}
\frac{1}{2}\Big(\mathcal{L}_\sigma g\Big)(\Phi X_1, \Phi X_2) &= \frac{1}{2}\Big(g(\nabla_{\Phi X_1}\sigma, \Phi X_2) + g(\nabla_{\Phi X_2}\sigma, \Phi X_1)\Big) \\
&= \frac{1}{2}\Big\{g(\mathcal{A}_{\Phi X_1}\sigma, \Phi X_2) + g(\mathcal{A}_{\Phi X_2}\sigma, \Phi X_1)\Big\} \\
&= \frac{1}{2}\Big\{-g(\mathcal{A}_{\Phi X_1}\Phi X_2, \sigma) - g(\mathcal{A}_{\Phi X_2}\Phi X_1, \sigma)\Big\} \\
&= \frac{1}{2}\Big\{-g(\mathcal{A}_{\Phi X_1}\Phi X_2, \sigma) + g(\mathcal{A}_{\Phi X_1}\Phi X_2, \sigma)\Big\} \\
&= 0.
\end{aligned}$$

Then, the eq. (27) is equivalent to

$$Ric(\Phi X_1, \Phi X_2) + kg(\Phi X_1, \Phi X_2) = 0 \tag{28}$$

for any horizontal vector fields X_1, X_2.

Considering the eq. (16) and putting in (28), it gives

$$\begin{aligned}
&\bar{Ric}(\bar{\Phi}\bar{X}_1, \bar{\Phi}\bar{X}_2) \circ \mathcal{F} - \frac{1}{2}\Big\{g(\nabla_{\Phi X_1}N, \Phi X_2) + g(\nabla_{\Phi X_2}N, \Phi X_1)\Big\} \\
&+2\sum_i g(\mathcal{A}_{\Phi X_1}X_i, \mathcal{A}_{\Phi X_2}X_i) + \sum_j g(\mathcal{T}_{U_j}\Phi X_1, \mathcal{T}_{U_j}\Phi X_2) \\
&+kg(\Phi X_1, \Phi X_2) \;=\; 0,
\end{aligned} \tag{29}$$

where $\{X_i, U_j\}_{1\le i\le n; 1\le j\le r}$ is an orthonormal frame on M.

Since $\mathcal{F}$ has totally geodesic fibres, above (29) is equivalent to

$$\bar{Ric}(\bar{\Phi}\bar{X}_1, \bar{\Phi}\bar{X}_2) \circ \mathcal{F} + 2\sum_i g(\mathcal{A}_{\Phi X_1}X_i, \mathcal{A}_{\Phi X_2}X_i) + kg(\Phi X_1, \Phi X_2) \;=\; 0. \tag{30}$$

Particularly, choosing $X_1 = X_2$ in (30), it becomes

$$\bar{Ric}(\bar{\Phi}\bar{X}_1, \bar{\Phi}\bar{X}_1) \circ \mathcal{F} + 2\sum_i g(\mathcal{A}_{\Phi X_1}X_i, \mathcal{A}_{\Phi X_1}X_i) + k\|\Phi X_1\|^2 \;=\; 0. \tag{31}$$

On the other hand, considering (5) we can write

$$\begin{aligned}\sum_i g(\mathcal{A}_{\Phi X_1}X_i, \mathcal{A}_{\Phi X_1}X_i) &= \sum_i g(\mathcal{A}_{X_i}\Phi X_1, \mathcal{A}_{X_i}\Phi X_1)\\ &= \sum_i g(\Phi\mathcal{A}_{X_i}X_1, \Phi\mathcal{A}_{X_i}X_1)\\ &= \sum_i \|\Phi\mathcal{A}_{X_i}X_1\|^2.\end{aligned}$$

Putting the last equality in (31), one has

$$\bar{Ric}(\bar{\Phi}\bar{X}_1, \bar{\Phi}\bar{X}_1)\circ\mathcal{F} + 2\sum_i \|\Phi\mathcal{A}_{X_i}X_1\|^2 + k\|\Phi X_1\|^2 = 0.$$

Since $\bar{M}$ is a Golden Riemannian manifold and using (2), it follows

$$\bar{Ric}(\bar{\Phi}\bar{X}_1, \bar{X}_1)\circ\mathcal{F} + \bar{ric}(\bar{X}_1)\circ\mathcal{F} + 2\sum_i \|\Phi\mathcal{A}_{X_i}X_1\|^2 + k\|\Phi X_1\|^2 = 0,$$

is obtained. □

Using Lemma 2.9, we have the following:

Theorem 2.10. *Let $\mathcal{F} : (M, \Phi, g) \to (\bar{M}, \bar{\Phi}, \bar{g})$ be a Golden Riemannian submersion with totally geodesic fibres and the Golden manifold M admits a Ricci soliton (M, g, σ, k) with vertical potential field. If $\bar{M}$ has positive or zero Ricci tensor, then the Ricci soliton (M, g, σ, k) is shrinking.*

The next lemma provides a Golden Riemannian submersion whose total manifold admits a Ricci soliton with horizontal potential field.

Lemma 2.11. *Let $\mathcal{F} : (M, \Phi, g) \to (\bar{M}, \bar{\Phi}, \bar{g})$ be a Golden Riemannian submersion with totally geodesic fibres and M admits a Ricci soliton (M, g, σ, k) with horizontal potential field. Then, the relation between Ricci tensor $\hat{Ric}$ and Ricci curvature $\hat{ric}$ on any fibre of $\mathcal{F}$ is given as*

$$\hat{Ric}(\Phi U_1, U_1) + \hat{ric}(U_1) - \sum_i \|\Phi\mathcal{A}_{X_i}U_1\|^2 + k\|\Phi U_1\|^2 = 0,$$

where $\{X_i\}_{1\le i\le n}$ is an orthonormal frame on $(\ker\mathcal{F}_*)^\perp$ and U_1 is any vertical vector field.

Proof. If the Golden manifold M admits a Ricci soliton with horizontal potential field σ, from the eq. (1) one has

$$\frac{1}{2}\Big(\mathcal{L}_\sigma g\Big)(\Phi U_1, \Phi U_2) + Ric(\Phi U_1, \Phi U_2) + kg(\Phi U_1, \Phi U_2) = 0 \tag{32}$$

for any $U_1, U_2 \in \ker\mathcal{F}$. Also, from the calculation of Lie-derivative, we can write

$$\begin{aligned}\frac{1}{2}\Big(\mathcal{L}_\sigma g\Big)(\Phi U_1, \Phi U_2) &= \frac{1}{2}\Big\{g(\nabla_{\Phi U_1}\sigma, \Phi U_2) + g(\nabla_{\Phi U_2}\sigma, \Phi U_1)\Big\}\\ &= \frac{1}{2}\Big\{g(\mathcal{T}_{\Phi U_1}\sigma, \Phi U_2) + g(\mathcal{T}_{\Phi U_2}\sigma, \Phi U_1)\Big\}\\ &= -\frac{1}{2}\Big\{g(\mathcal{T}_{\Phi U_1}\Phi U_2, \sigma) + g(\mathcal{T}_{\Phi U_2}\Phi U_1, \sigma)\Big\}\\ &= -\frac{1}{2}\Big\{g(\mathcal{T}_{\Phi U_1}\Phi U_2, \sigma) + g(\mathcal{T}_{\Phi U_1}\Phi U_2, \sigma)\Big\}\\ &= g(\mathcal{T}_{\Phi U_1}\sigma, \Phi U_2).\end{aligned}$$

Then, the eq. (32) gives

$$g(\mathcal{T}_{\Phi U_1}\sigma, \Phi U_2) + Ric(\Phi U_1, \Phi U_2) + kg(\Phi U_1, \Phi U_2) = 0, \tag{33}$$

for any vertical vector fields U_1, U_2.

Using the eq. (15) in (33), we get

$$\begin{aligned} &g(\mathcal{T}_{\Phi U_1}\sigma, \Phi U_2) + \hat{Ric}(\Phi U_1, \Phi U_2) + \sum_j g(\mathcal{T}_{U_j}U_j, \mathcal{T}_{\Phi U_1}\Phi U_2) \\ &- \sum_i \left\{ g((\nabla_{X_i}\mathcal{T})(\Phi U_1, \Phi U_2), X_i) + g(\mathcal{A}_{X_i}\Phi U_1, \mathcal{A}_{X_i}\Phi U_2) \right\} \\ &+ kg(\Phi U_1, \Phi U_2) = 0, \end{aligned} \tag{34}$$

where $\{X_i, U_j\}_{1\leq i\leq n; 1\leq j\leq r}$ is an orthonormal frame on M.

Since $\mathcal{F}$ has totally geodesic fibres, above (34) is equivalent to

$$\hat{Ric}(\Phi U_1, \Phi U_2) - \sum_i g(\mathcal{A}_{X_i}\Phi U_1, \mathcal{A}_{X_i}\Phi U_2) + kg(\Phi U_1, \Phi U_2) = 0. \tag{35}$$

In particular case, by choosing $U_1 = U_2$ in (35), it follows

$$\hat{Ric}(\Phi U_1, \Phi U_1) - \sum_i ||\mathcal{A}_{X_i}\Phi U_1||^2 + k||\Phi U_1||^2 = 0. \tag{36}$$

On the other hand, we can write

$$\begin{aligned} \sum_i ||\mathcal{A}_{X_i}\Phi U_1||^2 &= \sum_i g(\mathcal{A}_{X_i}\Phi U_1, \mathcal{A}_{X_i}\Phi U_1) = \sum_i g(\Phi\mathcal{A}_{X_i}U_1, \Phi\mathcal{A}_{X_i}U_1) \\ &= ||\Phi\mathcal{A}_{X_i}U_1||^2. \end{aligned}$$

Putting the last equality in (36), we have

$$\hat{Ric}(\Phi U_1, \Phi U_1) - \sum_i ||\Phi\mathcal{A}_{X_i}U_1||^2 + k||\Phi U_1||^2 = 0,$$

which gives

$$\hat{Ric}(\Phi U_1, U_1) + \hat{ric}(U_1) - \sum_i ||\Phi\mathcal{A}_{X_i}U_1||^2 + k||\Phi U_1||^2 = 0.$$

Hence the proof is completed. □

Theorem 2.12. *Let $\mathcal{F} : (M, \Phi, g) \to (\bar{M}, \bar{\Phi}, \bar{g})$ be a Golden Riemannian submersion with totally geodesic fibres and M admits a Ricci soliton (M, g, σ, k) with horizontal potential field. If any fibre has zero Ricci tensor, then the Ricci soliton (M, g, σ, k) is expanding.*

3. Examples of Golden Riemannian Submersions

In this part, we construct some examples of Golden Riemannian submersions with the following:

Example 3.1. *We consider tensor fields Φ and $\bar{\Phi}$ of type-(1,1) on Euclidean spaces $\mathbb{R}^4$ and $\mathbb{R}^2$ with the local coordinates*

$$\Phi\Big(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3},\frac{\partial}{\partial x_4}\Big) = \Big(\phi\frac{\partial}{\partial x_2},\phi\frac{\partial}{\partial x_1},\phi\frac{\partial}{\partial x_4},\phi\frac{\partial}{\partial x_3}\Big)$$

and

$$\bar{\Phi}\Big(\frac{\partial}{\partial y_1},\frac{\partial}{\partial y_2}\Big) = \Big(\bar{\phi}\frac{\partial}{\partial y_1},\bar{\phi}\frac{\partial}{\partial y_2}\Big)$$

respectively. Here, ϕ and $\bar{\phi}$ are the roots of the algebraic equation $x^2 - x - 1 = 0$. Hence, one can see that (M,Φ,g) and $(\bar{M},\bar{\Phi},\bar{g})$ are Golden Riemannian manifolds.

Let $\mathcal{F} : (\mathbb{R}^4,\Phi,g) \to (\mathbb{R}^2,\bar{\Phi},\bar{g})$ be a map which is defined by

$$\mathcal{F}(x_1,x_2,x_3,x_4) = \Big(\frac{1}{\sqrt{2}}(x_1+x_2),\frac{1}{\sqrt{2}}(x_3+x_4)\Big).$$

By the direct computations, we get

$$\ker\mathcal{F}_* = Span\Big\{U_1 = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2}\Big),\ U_2 = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial x_3}-\frac{\partial}{\partial x_4}\Big)\Big\}$$

and

$$(\ker\mathcal{F}_*)^{\perp} = Span\Big\{X_1 = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\Big),\ X_2 = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial x_3}+\frac{\partial}{\partial x_4}\Big\}.$$

Also, it is easy to see that $\mathcal{F}_{*p}$ preserves the length of the horizontal vectors at each point $p\in M$, that is

$$g(X_1,X_1) = \bar{g}(\mathcal{F}_*X_1,\mathcal{F}_*X_1) \text{ and } g(X_2,X_2) = \bar{g}(\mathcal{F}_*X_2,\mathcal{F}_*X_2),$$

which gives $\mathcal{F}$ is a Riemannian submersion.

On the other hand, we get

$$\mathcal{F}_*(\Phi X_1) = \mathcal{F}_*(\phi X_1) = (\bar{\phi}\bar{X}_1)\circ\mathcal{F} = (\bar{\Phi}\bar{X}_1)\circ\mathcal{F} = \bar{\Phi}(\mathcal{F}_*X_1)$$

and similarly,

$$\mathcal{F}_*(\Phi X_2) = \mathcal{F}_*(\phi X_2) = (\bar{\phi}\bar{X}_2)\circ\mathcal{F} = (\bar{\Phi}\bar{X}_2)\circ\mathcal{F} = \bar{\Phi}(\mathcal{F}_*X_2)$$

where ϕX_1 and ϕX_2 are the basic vector fields, $\mathcal{F}-$related to $\bar{\phi}\bar{X}_1,\bar{\phi}\bar{X}_2$, respectively. Hence, $\mathcal{F}$ is a Golden Riemannian submersion.

Example 3.2. *We consider tensor fields Φ and $\bar{\Phi}$ of type-(1,1) on Euclidean spaces $\mathbb{R}^4$ and $\mathbb{R}^2$ with the local coordinates*

$$\Phi\Big(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3},\frac{\partial}{\partial x_4}\Big) = \Big(\phi\frac{\partial}{\partial x_2},(1-\phi)\frac{\partial}{\partial x_1},\phi\frac{\partial}{\partial x_4},(1-\phi)\frac{\partial}{\partial x_3}\Big)$$

and

$$\bar{\Phi}\Big(\frac{\partial}{\partial y_1},\frac{\partial}{\partial y_2}\Big) = \Big(\bar{\phi}\frac{\partial}{\partial y_2},(1-\bar{\phi})\frac{\partial}{\partial y_1}\Big)$$

respectively. Here, ϕ and $\bar{\phi}$ are the roots of the algebraic equation $x^2 - x - 1 = 0$. Hence, one can see that (M, Φ, g) and $(\bar{M}, \bar{\Phi}, \bar{g})$ are Golden Riemannian manifolds.

Let $\mathcal{F} : (\mathbb{R}^4, \Phi, g) \to (\mathbb{R}^2, \bar{\Phi}, \bar{g})$ be a map which is defined by

$$\mathcal{F}(x_1, x_2, x_3, x_4) = \Big(x_1 sin\alpha - x_3 cos\alpha, x_2 sin\alpha - x_4 cos\alpha\Big).$$

By the direct computations, we get

$$\ker \mathcal{F}_* = Span\Big\{U_1 = cos\alpha\frac{\partial}{\partial x_1} + sin\alpha\frac{\partial}{\partial x_3},\ U_2 = cos\alpha\frac{\partial}{\partial x_2} + sin\alpha\frac{\partial}{\partial x_4}\Big)\Big\}$$

and

$$(\ker \mathcal{F}_*)^{\perp} = Span\Big\{X_1 = sin\alpha\frac{\partial}{\partial x_1} - cos\alpha\frac{\partial}{\partial x_3},\ X_2 = sin\alpha\frac{\partial}{\partial x_2} - cos\alpha\frac{\partial}{\partial x_4}\Big\}.$$

Also, it is easy to see that $\mathcal{F}_{*p}$ preserves the length of the horizontal vectors at each point $p \in M$, that is

$$g(X_1, X_1) = \bar{g}(\mathcal{F}_* X_1, \mathcal{F}_* X_1) \ \text{and}\ g(X_2, X_2) = \bar{g}(\mathcal{F}_* X_2, \mathcal{F}_* X_2),$$

which gives $\mathcal{F}$ is a Riemannian submersion.

On the other hand, we get

$$\mathcal{F}_*(\Phi X_1) = \mathcal{F}_*(\phi X_1) = (\bar{\phi}\bar{X}_1) \circ \mathcal{F} = (\bar{\Phi}\bar{X}_1) \circ \mathcal{F} = \bar{\Phi}(\mathcal{F}_* X_1)$$

and similarly,

$$\mathcal{F}_*(\Phi X_2) = \mathcal{F}_*(\phi X_2) = (\bar{\phi}\bar{X}_2) \circ \mathcal{F} = (\bar{\Phi}\bar{X}_2) \circ \mathcal{F} = \bar{\Phi}(\mathcal{F}_* X_2)$$

where ϕX_1 and ϕX_2 are the basic vector fields, $\mathcal{F}-$related to $\bar{\phi}\bar{X}_1$, $\bar{\phi}\bar{X}_2$, respectively. Hence, $\mathcal{F}$ is a Golden Riemannian submersion.

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