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A classification of cyclic Ricci semi-symmetric hypersurfaces in the complex hyperbolic quadric

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Abstract. In this paper, the notion of cyclic Ricci semi-symmetric real hypersurfaces in the complex hyperbolic quadric $Q^{m*} = SO^0_{2,m}/SO_2SO_m$ is introduced. Under the assumption of singular normal vector field N, we have two cases, that is, normal vector field N is either $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic. Even though, in the case of $\mathfrak{A}$-principal, we proved that there does not exist a real hypersurface in the complex hyperbolic quadric $Q^{m*} = SO^0_{2,m}/SO_2SO_m$ satisfying the cyclic Ricci semi-symmetric. But on the other case, we proved existence of real hypersurfaces with the same condition.

1. Introduction

About the latter part of twentieth century, many geometers have investigated some real hypersurfaces in Hermitian symmetric spaces of rank 1 like the complex projective space $\mathbb{C}P^m$ or the complex hyperbolic space $\mathbb{C}H^m$. Some geometric characterizations of real hypersurfaces in the complex projective space $\mathbb{C}P^m$, the complex hyperbolic space $\mathbb{C}H^m$, or in the quaternionic projective space $\mathbb{H}P^m$ was obtained by Okumura [16], Montiel and Romero [13], Martinez and Pérez [12] and Pérez and Suh [18] respectively. In particular Okumura [16] proved that the Reeb flow on a real hypersurface in $\mathbb{C}P^m = SU_{m+1}/S(U_1U_m)$ is isometric if and only if M is an open part of a tube around a totally geodesic $\mathbb{C}P^k \subset \mathbb{C}P^m$ for some $k \in \{0, \dots, m-1\}$. Moreover, for the complex hyperbolic space $\mathbb{C}H^m$ Montiel and Romero [13] have proved that a real hypersurface M has an isometric Reeb flow if and only if M is an open part of a tube around a totally geodesic $\mathbb{C}H^k \subset \mathbb{C}H^m$ for some $k \in \{0, \dots, m-1\}$ or a part of horosphere.

Now let us introduce the complex hyperbolic quadric $Q^{m*} = SO^o_{m,2}/SO_2SO_m$, which can be regarded as a Hermitian symmetric space with rank 2 of noncompact type. Montiel and Romero [14] proved that the

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complex hyperbolic quadric Q^{m*} can be immersed in the indefinite complex hyperbolic space $\mathbb{C}H_1^{m+1}(-c)$, $c > 0$, by interchanging the Kähler metric by its opposite. From now on, the subscript represents the index of negative sign for the given manifold. Because, if we change the Kähler metric of $\mathbb{C}P_{m-s}^{m+1}$ by its opposite, we have that Q_{m-s}^m endowed with its opposite metric $g' = -g$ is also an Einstein hypersurface of $\mathbb{C}H_{s+1}^{m+1}(-c)$. When $s = 0$, we know that $(Q_m^m, g' = -g)$ can be regarded as the complex hyperbolic quadric $Q^{m*} = SO_{m,2}^o/SO_2SO_m$, which is immersed in the indefinite complex hyperbolic quadric $\mathbb{C}H_1^{m+1}(-c)$, $c > 0$ as a complex Einstein hypersurface (see Reckziegel [21], Romero [22], [23], and Smyth [24]). Accordingly, the complex hyperbolic quadric admits two important geometric structures a complex conjugation structure A and a Kähler structure J, which anti-commute with each other, that is, $AJ = -JA$. Then for $m\geq 2$ the triple (Q^{m*}, J, g) for the complex hyperbolic quadric which is one of the Hermitian symmetric space of noncompact type with rank 2 and its minimal sectional curvature is equal to -4 up to scaling(see Klein and Suh [5], and Reckziegel [21]).

Among the study of Hermitian symmetric spaces with rank 2, the classification problems of real hypersurfaces in the complex 2-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ with certain geometric conditions were mainly discussed in Berndt and Suh [1], Suh [26] and [28]. So, the classification of *contact hypersurfaces*, *parallel Ricci tensor* and *harmonic curvature* for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ were extensively studied. Moreover, in [27] we have asserted that the Reeb flow on a real hypersurface in complex hyperbolic 2-plane Grassmannian $SU_{2,m}/S(U_2U_m)$ is isometric if and only if M is an open part of a tube around a totally geodesic $SU_{2,m-1}/S(U_2U_{m-1}) \subset SU_{2,m}/S(U_2U_m)$. For real hypersurfaces in the complex hyperbolic quadric Q^{m*}, Suh [30] proved the following theorem for isometric Reeb flow:

Theorem A. Let M be a real hypersurface of the complex hyperbolic quadric Q^{m*}, $m \geq 3$. The Reeb flow on M is isometric if and only if m is even, say $m = 2k$, and M is an open part of a tube around a totally geodesic $\mathbb{C}H^k$ in the complex hyperbolic quadric Q^{2k^*}, $k\geq 2$, or a horosphere in Q^{2k^*} whose center at infinity is in the equivalence class of an $\mathfrak{A}$-isotropic singular geodesic in Q^{2k^*}.

A nonzero tangent vector $W \in T_{[z]}Q^{m*}$ is called *singular* if it is tangent to more than one maximal flat in Q^{m*}. Since the complex hyperbolic quadric Q^{m*} is a Hermitian symmetric space with rank 2, there are two types of singular tangent vectors as follows:

(a) If there exists a conjugation $A \in \mathfrak{A}$ such that $W \in V(A) := \{W \mid AW = W\}$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-*principal*.

(b) If there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $W/\|W\| = (X + JY)/\sqrt{2}$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-*isotropic*.

Motivated by the researches mentioned above, many geometers have considered the notions of *parallel Ricci tensor* $\nabla\mathrm{Ric} = 0$, *harmonic curvature* $(\nabla_X\mathrm{Ric})Y = (\nabla_Y\mathrm{Ric})X$ or *Killing Ricci tensor* $\mathfrak{S}_{X,Y,Z X,Y,Z}\,g((\nabla_X\mathrm{Ric})Y, Z) = 0$ for any vector fields X, Y and Z on M, where ∇ denotes the induced connection on M from the Levi-Civita connection $\bar\nabla$ on a Kähler manifold $\bar M$ (see Blair [2], Kimura [6], Lee and Suh [10], [11], Pérez [17], Pérez and Suh [18], [19], [20], and Suh [31], Yano [34]). In particular, for real hypersurfaces in the complex projective space $\mathbb{C}P^m$ the notion of cyclic parallel Ricci tensor was considered by Kwon and Nakagawa [9] and in the quaternionic projective space $\mathbb{Q}P^m$ by Pérez [17] respectively.

From such a view point, recently, in the complex quadric Q^m, Woo, Kim and Suh [33] considered Hopf real hypersurfaces with cyclic Ricci semi-symmetric operator defined by

$$\mathfrak{S}_{X,Y,Z}(R(X,Y)Ric)(Z) = 0$$

for any X, Y and $Z\in T_zM$, $z\in M$. It is a weaker notion than usual Ricci symmetric $\nabla Ric = 0$, that is, parallel Ricci tensor. They [33] proved the following:

Theorem B. Let M be a Hopf real hypersurface in the complex quadric Q^m, $m\geq 3$, with $\mathfrak{A}$-principal unit singular normal vector field. If it satisfies the cyclic Ricci semi-symmetric condition, then M is locally congruent to a tube of radius $r = \frac{1}{\sqrt{2}}\arctan\sqrt{\frac{m-2}{2}}$ over the totally real and totally geodesic m-dimensional sphere S^m whose 3-distinct constant principal curvatures are given by $\alpha = -\sqrt{m-2}$, $\lambda = 0$, and $\mu = \frac{2}{\sqrt{m-2}}$ with multiplicities 1, $m-1$ and $m-1$ respectively. Moreover, among them there exists a pseudo-Einstein real hypersurface ($S = ag + b\eta\otimes\xi$) in Q^m with $a = 2m$ and $b = -2m$, with distributions $[\xi]\oplus T_\lambda = JV(A)$ and $T_\mu\oplus N = V(A)$, where $T_zQ^m = V(A)\oplus JV(A)$, $z\in Q^m$.

In the complex hyperboilc quadric Q^{m*} with $\mathfrak{A}$-principal unit normal vector field, Berndt and Suh (see [1]) recently introduced the following :

Theorem C. Let M be a connected orientable real hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 3$. Then M is a contact hypersurface with constant mean curvature if and only if M is locally congruent to one of the following hypersurfaces:

(i) a tube of radius r around the Hermitian symmetric space $Q^{(m-1)^*}$ which is imbedded in Q^{m*} as a totally geodesic complex hypersurface,
(ii) a horosphere in Q^{m*} whose center at infinity is the equivalence class of an $\mathfrak{A}$-principal geodesic in Q^{m*},
(iii) a tube of radius r around the m-dimensional real hyperbolic space $\mathbb{R}H^m$ which is embedded in Q^{m*} as a real space form of Q^{m*}.

Motivated by Theorem B in the complex quadric Q^m and Theorem C for a Hopf real hypersurface M in the complex hyperbolic quadric Q^{m*}, we introduce the notion of cyclic Ricci semi-symmetric real hypersurfaces in the complex hyperbolic quadric Q^{m*} defined by

$$\mathfrak{S}_{X,Y,Z}(R(X,Y)Ric)(Z) = 0$$

for any X, Y and $Z\in T_zM$, $z\in M$. This is a natural generalization of the Ricci semi-symmetric. Then by the first Bianchi identity the cyclic Ricci semi-symmetric tensor can be given by

$$\mathfrak{S}_{X,Y,Z}R(X,Y)Ric(Z) = 0.$$

The notion of cyclic Ricci semi-symmetric tensor on Riemanian manifolds and its physical meaning is due to Chaubey, Suh and De [3]. As compared with the result in Theorem B, in the complex hyperbolic quadric Q^{m*} we give a non-existence result in the complex hyperbolic quadric Q^{m*} as follows:

Main Theorem 1. Let M be a Hopf real hypersurface in the complex hyperbolic quadric Q^{m*}, $m\geq 3$, with $\mathfrak{A}$-principal unit singular normal vector field. Then there does not exist a real hypersurface satisfying the cyclic Ricci semi-symmetric. Moreover, among them there does not exist a pseudo-Einstein real hypersurface in Q^{m*}.

Remark. In Suh [31], we have proved that there does not exists a Hopf pseudo-Einstein real hypersurface in the complex hyperbolic quadric Q^{m*}. But in the complex quadric Q^m there exists a complete classification of pseudo-Einstein real hypersurfaces such that a tube around S^m in Q^m, with $a = 2m$ and $b = -2m$ or a tube around $\mathbb{C}P^k$ in Q^{2k} with $a = 4k$ and $b = -4 + \frac{2}{k}$ (see [29]).

Here it can be easily checked that the vector fields $A\xi$ and AN are tangent to the space T_zM, $z \in M$ if the unit normal vector field N becomes $\mathfrak{A}$-isotropic. Then by virtue of Theorem A, in this paper we give a classification for Hopf real hypersurfaces in the complex hyperbolic quadric Q^{m*} with cyclic semi-symmetric Ricci tensor and $\mathfrak{A}$-isotropic unit normal as follows:

Main Theorem 2. Let M be a Hopf real hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 3$, with $\mathfrak{A}$-isotropic singular unit normal. If it satisfies the cyclic Ricci semi-symmetric, then M is locally congruent to one of the following:

(1) a tube of radius r over the k-dimensional complex hyperbolic space $\mathbb{C}H^k$ which can be immersed as a totally geodesic in Q^{2k^*}.
(2) a horosphere in Q^{m*} whose center at infinity is the equivalence class of an $\mathfrak{A}$-principal geodesic in Q^{m*},

(3) M, $m\geq 4$, has 4-distinct constant principal curvatures such that

$$\alpha, \quad \beta=\gamma=0, \quad \lambda=\frac{-\alpha+\sqrt{(2m-5)\alpha^2+4(m-3)}}{2(m-3)}$$

and

$$\mu=\frac{-\alpha-\sqrt{(2m-5)\alpha^2+4(m-3)}}{2(m-3)}$$

whose corresponding principal curvature spaces are $\xi\in T_\alpha$, $A\xi, AN\in T_{\beta=\gamma}$, T_λ and T_μ with multiplicities $1, 2, m-1$ and $m-1$ respectively.

(4) M, $m=3$, has 3-distinct principal curvatures given by

$$\alpha=0, \quad \beta=\gamma=0, \quad \lambda=\frac{h+\sqrt{h^2+4}}{2}, \quad \text{and} \quad \mu=\frac{h-\sqrt{h^2+4}}{2}$$

with multiplicities 1, 2, 1 and 1 respectively.

This paper is composed as follows: In section 2 we give some basic material about the complex hyperbolic quadric Q^{m*}, including its Riemannian curvature tensor and a description of its singular vectors for $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic unit normal vector field. Apart from the complex structure J there is another distinguished geometric structure on Q^{m*}, namely a parallel rank two vector bundle $\mathfrak{A}$ which covers an S^1-bundle of real structures, that is, complex conjugations A on the tangent spaces of Q^{m*}. A maximal $\mathfrak{A}$-invariant subbundle $\mathcal{Q}$ of the tangent bundle TM of a real hypersurface M in Q^{m*} is determined by one of these real structures A.

Accordingly, in section 3, we study the geometry of this subbundle $\mathcal{Q}$ for real hypersurfaces in Q^{m*} and the equation of Codazzi from the curvature tensor of the complex hyperbolic quadric Q^{m*} and some important formulas from the complex conjugation A of M in Q^{m*}.

In section 4, we give a valuable Theorem 4.2 which asserts that the *pseudo-Einstein* real hypersurface with $S=ag+b\eta\otimes\xi$ satisfies cyclic Ricci semi-symmetric. This gives a strong motivation in the proof of our Main Theorem 1 which includes the notion of *pseudo-Einstein*. Moreover, in section 4, in order to prove our Main Theorem 1 for an $\mathfrak{A}$-principal normal vector field, the first step is to get the Ricci tensor from the equation of Gauss for real hypersurfaces M in Q^{m*}, and next by using the assumption of cyclic Ricci semi-symmetric for an $\mathfrak{A}$-principal normal vector field we will get some useful formulas and a remarkable Theorem 4.2.

By virtue of Theorem 4.2 and Proposition 4.4, we give a complete proof of a non-existence property in our Main Theorem 1 that there does not exist a real hypersurface in Q^{m*} with cyclic Ricci semi-symmetric. Proposition 4.4 will play an important role in the proof of Main Theorem 1 and will be used to give a characterization of contact hypersurfaces in the complex hyperbolic quadric Q^{m*}. Consequently, we have proved that there does not exist a pseudo-Einstein real hypersurface in Q^{m*}.

In section 5, we give a complete classification of our Main Theorem 2. The first part of this proof is to give some crucial equations from the cyclic semi-symmetric Ricci tensor for an $\mathfrak{A}$-isotropic unit normal vector field. Then in the middle part of the proof we will concentrate ourselves on the study of valuable formulas which can be obtained from the cyclic semi-symmetric Ricci tensor. In the proof of our Main Theorem 2 we will use an important Lemma 5.1 frequently which assures that $SA\xi=0$ and $SAN=0$ on the distribution $\mathcal{Q}^\perp=\text{Span}\,\{A\xi, AN\}$ of T_zQ^{m*}, $z\in Q^{m*}$.

2. The complex hyperbolic quadric

In this section, we introduce the Riemanian hyperbolic structures of the complex hyperbolic quadric Q^{m*} in contrast to the complex quadric Q^m. This section is due to Klein and Suh [5], and Suh [30].

The m-dimensional complex hyperbolic quadric Q^{m*} is the non-compact dual of the m-dimensional complex quadric Q^m, i.e. the simply connected Riemannian symmetric space whose curvature tensor is the negative of the curvature tensor of Q^m.

The complex hyperbolic quadric Q^{m*} cannot be realized as a homogeneous complex hypersurface of the complex hyperbolic space $\mathbb{C}H^{m+1}$. In fact, Smyth [24, Theorem 3(ii)] has shown that every homogeneous complex hypersurface in $\mathbb{C}H^{m+1}$ is totally geodesic. This is in marked contrast to the situation for the complex quadric Q^m, which can be realized as a homogeneous complex hypersurface of the complex projective space $\mathbb{C}P^{m+1}$ in such a way that the shape operator for any unit normal vector to Q^m is a real structure on the corresponding tangent space of Q^m, see [5] and [21]. Another related result by Smyth, [25, Theorem 1], which states that any complex hypersurface $\mathbb{C}H^{m+1}$ for which the square of the shape operator has constant eigenvalues (counted with multiplicity) is totally geodesic, also precludes the possibility of a model of Q^{m*} as a complex hypersurface of $\mathbb{C}H^{m+1}$ with the analogous property for the shape operator.

Therefore we realize the complex hyperbolic quadric Q^{m*} as the quotient manifold $SO_{2,m}/SO_2SO_m$. As Q^{1^*} is isomorphic to the real hyperbolic space $\mathbb{R}H^2 = SO_{1,2}/SO_2$, and Q^{2^*} is isomorphic to the Hermitian product of complex hyperbolic spaces $\mathbb{C}H^1 \times \mathbb{C}H^1$, we suppose $m \geq 3$ in the sequel and throughout this paper. Let $G := SO_{2,m}$ be the transvection group of Q^{m*} and $K := SO_2SO_m$ be the isotropy group of Q^{m*} at the "origin" $p_0 := eK \in Q^{m*}$. Then

$$\sigma : G \to G,\ g \mapsto sgs^{-1} \quad \text{with} \quad s := \begin{pmatrix} -1 & & & & & \\ & -1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

is an involutive Lie group automorphism of G with $\mathrm{Fix}(\sigma)_0 = K$, and therefore $Q^{m*} = G/K$ is a Riemannian symmetric space. The center of the isotropy group K is isomorphic to SO_2, and therefore Q^{m*} is in fact a Hermitian symmetric space.

The Lie algebra $\mathfrak{g} := \mathfrak{so}_{2,m}$ of G is given by

$$\mathfrak{g} = \left\{ X \in \mathfrak{gl}(m+2, \mathbb{R}) \,\middle|\, X^t \cdot s = -s \cdot X \right\}$$

(see [7, p. 59]). In the sequel we will write members of $\mathfrak{g}$ as block matrices with respect to the decomposition $\mathbb{R}^{m+2} = \mathbb{R}^2 \oplus \mathbb{R}^m$, i.e. in the form

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix},$$

where X_{11}, X_{12}, X_{21}, X_{22} are real matrices of dimension 2×2, $2 \times m$, $m \times 2$ and $m \times m$, respectively. Then

$$\mathfrak{g} = \left\{ \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \,\middle|\, X_{11}^t = -X_{11},\ X_{12}^t = X_{21},\ X_{22}^t = -X_{22} \right\}.$$

The linearisation $\sigma_L = \mathrm{Ad}(s) : \mathfrak{g} \to \mathfrak{g}$ of the involutive Lie group automorphism σ induces the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where the Lie subalgebra

$$\begin{aligned} \mathfrak{k} &= \mathrm{Eig}(\sigma_*, 1) = \{X \in \mathfrak{g} | sXs^{-1} = X\} \\ &= \left\{ \begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix} \,\middle|\, X_{11}^t = -X_{11},\ X_{22}^t = -X_{22} \right\} \\ &\cong \mathfrak{so}_2 \oplus \mathfrak{so}_m \end{aligned}$$

is the Lie algebra of the isotropy group K, and the $2m$-dimensional linear subspace

$$\mathfrak{m} = \mathrm{Eig}(\sigma_*, -1) = \{X \in \mathfrak{g} | sXs^{-1} = -X\} = \left\{ \begin{pmatrix} 0 & X_{12} \\ X_{21} & 0 \end{pmatrix} \,\middle|\, X_{12}^t = X_{21} \right\}$$

is canonically isomorphic to the tangent space $T_{p_0}Q^{m*}$. Under the identification $T_{p_0}Q^{m*} \cong \mathfrak{m}$, the Riemannian metric g of Q^{m*} (where the constant factor of the metric is chosen so that the formulae become as simple as possible) is given by

$$g(X, Y) = \frac{1}{2}\operatorname{tr}(Y^t \cdot X) = \operatorname{tr}(Y_{12} \cdot X_{21}) \quad \text{for} \quad X, Y \in \mathfrak{m}\ .$$

g is clearly $\mathrm{Ad}(K)$-invariant, and therefore corresponds to an $\mathrm{Ad}(G)$-invariant Riemannian metric on Q^{m*}. The complex structure J of the Hermitian symmetric space is given by

$$JX = \mathrm{Ad}(j)X \quad \text{for} \quad X \in \mathfrak{m}, \quad \text{where} \quad j := \begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \in K\ .$$

As j is in the center of K, the orthogonal linear map J is $\mathrm{Ad}(K)$-invariant, and thus defines an $\mathrm{Ad}(G)$-invariant Hermitian structure on Q^{m*}. By identifying the multiplication by the unit complex number i with the application of the linear map J, the tangent spaces of Q^{m*} thus become m-dimensional complex linear spaces, and we will adopt this point of view in the sequel.

As for the complex quadric (again compare [10], [11], and [20] with [5] and [30]), there is another important structure on the tangent bundle of the complex quadric besides the Riemannian metric and the complex structure, namely an S^1-bundle $\mathfrak{A}$ of real structures. The situation here differs from that of the complex quadric in that for Q^{m*}, the real structures in $\mathfrak{A}$ cannot be interpreted as the shape operator of a complex hypersurface in a complex space form, but as the following considerations will show, $\mathfrak{A}$ still plays a fundamental role in the description of the geometry of Q^{m*}.

Let

$$a_0 := \begin{pmatrix} 1 & & & & & \\ & -1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}.$$

Note that we have $a_0 \notin K$, but only $a_0 \in O_2\, SO_m$. However, $\mathrm{Ad}(a_0)$ still leaves $\mathfrak{m}$ invariant, and therefore defines an $\mathbb{R}$-linear map A_0 on the tangent space $\mathfrak{m} \cong T_{p_0}Q^{m*}$. A_0 turns out to be an involutive orthogonal map with $A_0 \circ J = -J \circ A_0$ (i.e. A_0 is anti-linear with respect to the complex structure of $T_{p_0}Q^{m*}$), and hence a real structure on $T_{p_0}Q^{m*}$. But A_0 commutes with $\mathrm{Ad}(g)$ not for all $g \in K$, but only for $g \in SO_m \subset K$. More specifically, for $g = (g_1, g_2) \in K$ with $g_1 \in SO_2$ and $g_2 \in SO_m$, say $g_1 = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$ with $t \in \mathrm{Ric}$ (so that $\mathrm{Ad}(g_1)$ corresponds to multiplication with the complex number $\mu := e^{it}$), we have

$$A_0 \circ \mathrm{Ad}(g) = \mu^{-2} \cdot \mathrm{Ad}(g) \circ A_0\ .$$

This equation shows that the object which is $\mathrm{Ad}(K)$-invariant and therefore geometrically relevant is not the real structure A_0 by itself, but rather the "circle of real structures"

$$\mathfrak{A}_{p_0} := \{\lambda\, A_0 | \lambda \in S^1\}\ .$$

$\mathfrak{A}_{p_0}$ is $\mathrm{Ad}(K)$-invariant, and therefore generates an $\mathrm{Ad}(G)$-invariant S^1-subbundle $\mathfrak{A}$ of the endomorphism bundle $\mathrm{End}(TQ^{m*})$, consisting of real structures on the tangent spaces of Q^{m*}. For any $A \in \mathfrak{A}$, the tangent line to the fibre of $\mathfrak{A}$ through A is spanned by JA.

For any $p \in Q^{m*}$ and $A \in \mathfrak{A}_p$, the real structure A induces a splitting

$$T_pQ^{m*} = V(A) \oplus JV(A)$$

into two orthogonal, maximal totally real subspaces of the tangent space T_pQ^{m*}. Here $V(A)$ resp. $JV(A)$ are the $(+1)$-eigenspace resp. the (-1)-eigenspace of A. For every unit vector $Z \in T_pQ^{m*}$ there exist $t \in [0, \frac{\pi}{4}]$, $A \in \mathfrak{A}_p$ and orthonormal vectors $X, Y \in V(A)$ so that

$$Z = \cos(t) \cdot X + \sin(t) \cdot JY$$

holds; see [21, Proposition 3]. Here t is uniquely determined by Z. The vector Z is singular, i.e. contained in more than one Cartan subalgebra of $\mathfrak{m}$, if and only if either $t = 0$ or $t = \frac{\pi}{4}$ holds. The vectors with $t = 0$ are called $\mathfrak{A}$-*principal*, whereas the vectors with $t = \frac{\pi}{4}$ are called $\mathfrak{A}$-*isotropic*. If Z is regular, i.e. $0 < t < \frac{\pi}{4}$ holds, then also A and X, Y are uniquely determined by Z.

As for the complex quadric, the Riemannian curvature tensor $\bar{R}$ of Q^{m*} can be fully described in terms of the "fundamental geometric structures" g, J and $\mathfrak{A}$. In fact, under the correspondence $T_{p_0}Q^{m*} \cong \mathfrak{m}$, the curvature $\bar{R}(X, Y)Z$ corresponds to $-[[X, Y], Z]$ for $X, Y, Z \in \mathfrak{m}$, see [8, Chapter XI, Theorem 3.2(1)]. By evaluating the latter expression explicitly, one can show that one has

$$\begin{aligned} \bar{R}(X, Y)Z = &- g(Y, Z)X + g(X, Z)Y \\ &- g(JY, Z)JX + g(JX, Z)JY + 2g(JX, Y)JZ \\ &- g(AY, Z)AX + g(AX, Z)AY \\ &- g(JAY, Z)JAX + g(JAX, Z)JAY \end{aligned} \tag{2.1}$$

for arbitrary $A \in \mathfrak{A}_{p_0}$. Therefore the curvature of Q^{m*} is the negative of that of the complex quadric Q^m, compare [21, Theorem 1]. This confirms that the symmetric space Q^{m*} which we have constructed here is indeed the non-compact dual of the complex quadric.

It has been shown by Nomizu [15, Theorem 15.3] that there exists one and only one torsion-free covariant derivative $\bar{\nabla}$ on Q^{m*} so that the symmetric involutions $s_p : Q^{m*} \to Q^{m*}$ at $p \in Q^{m*}$ are all affine. Here $\bar{\nabla}$ denotes the *canonical covariant derivative* of Q^{m*}. Concerned with the derivative $\bar{\nabla}$, the action of any member of G on Q^{m*} is also affine. Moreover, $\bar{\nabla}$ is the Levi-Civita connection corresponding to the Riemannian metric g, and therefore g is parallel with respect to $\bar{\nabla}$. Moreover, the complex hyperbolic quadric Q^{m*} becomes a Kähler manifold in this way, i.e. the complex structure J is also parallel. Since the S^1-subbundle $\mathfrak{A}$ of the endomorphism bundle $\mathrm{End}(TQ^{m*})$ is $\mathrm{Ad}(G)$-invariant, it is also parallel with respect to the same covariant derivative $\bar{\nabla}$ induced by $\bar{\nabla}$ on $\mathrm{End}(TQ^{m*})$. Because the tangent line of the fiber of $\mathfrak{A}$ through some $A_p \in \mathfrak{A}$ is spanned by JA_p, this means precisely that for any section A of $\mathfrak{A}$ there exists a real-valued 1-form $q : TQ^{m*} \to \mathcal{R}$ so that

$$\bar{\nabla}_v A = q(v) \cdot JA_p \quad \text{holds for } p \in Q^{m*}, v \in T_pQ^{m*}. \tag{2.2}$$

3. The maximal $\mathfrak{A}$-invariant subbundle $\mathcal{Q}$ of TM

Let M be a real hypersurface in complex hyperbolic quadric Q^{m*} and denote by (ϕ, ξ, η, g) the induced almost contact metric structure on M and by ∇ the induced Riemannian connection on M. Note that $\xi = -JN$, where N is a (local) unit normal vector field of M. The vector field ξ is known as the Reeb vector field of M. If the integral curves of ξ are geodesics in M, the hypersurface M is called a Hopf hypersurface. The integral curves of ξ are geodesics in M if and only if ξ is a principal curvature vector of M everywhere. The tangent bundle TM of M splits orthogonally into $TM = \mathcal{C} \oplus \mathcal{F}$, where $\mathcal{C} = \ker(\eta)$ is the maximal complex subbundle of TM and $\mathcal{F} = \mathbb{R}\xi$. The structure tensor field ϕ restricted to $\mathcal{C}$ coincides with the complex structure J restricted to $\mathcal{C}$, and we have $\phi\xi = 0$. We denote by νM the normal bundle of M.

We first introduce some notations. For a fixed real structure $A \in \mathfrak{A}_{[z]}$ and $X \in T_{[z]}M$ we decompose AX into its tangential and normal component, that is,

$$AX = BX + \rho(X)N$$

where BX is the tangential component of AX and

$$\rho(X) = g(AX, N) = g(X, AN) = g(X, AJ\xi) = g(JX, A\xi).$$

Since $JX = \phi X + \eta(X)N$ and $A\xi = B\xi + \rho(\xi)N$ we also have

$$\rho(X) = g(\phi X, B\xi) + \eta(X)\rho(\xi) = \eta(B\phi X) + \eta(X)\rho(\xi).$$

We also define

$$\delta = g(N, AN) = g(JN, JAN) = -g(JN, AJN) = -g(\xi, A\xi).$$

Now at each point $z \in M$ let us consider a maximal $\mathfrak{A}$-invariant subspace $\mathcal{Q}_z$ of T_zM, $z{\in}M$, defined by

$$\mathcal{Q}_z = \{X \in \mathcal{C}_z \mid AX \in T_zM \text{ for all } A \in \mathfrak{A}_z\}$$

of T_zM, $z{\in}M$. Thus for a case where the unit normal vector field N is $\mathfrak{A}$-isotropic it can be easily checked that the orthogonal complement $\mathcal{Q}_z^{\perp} = \mathcal{C}_z \ominus \mathcal{Q}_z$, $z{\in}M$, of the distribution $\mathcal{Q}$ in the complex subbundle $\mathcal{C}$, becomes $\mathcal{Q}_z^{\perp} = \text{Span}[A\xi, AN]$. Here it can be easily checked that the vector fields $A\xi$ and AN belong to the tangent space T_zM, $z{\in}M$ if the unit normal vector field N becomes $\mathfrak{A}$-isotropic. Then by using the same method for real hypersurfaces in complex hyperbolic quadric Q^{m*} as in Suh [31] we get the following

Lemma 3.1. *Let M be a real hypersurface in complex hyperbolic quadric Q^{m*}. Then the following statements are equivalent:*

(i) *The normal vector $N_{[z]}$ of M is $\mathfrak{A}$-principal,*
(ii) $\mathcal{Q}_{[z]} = \mathcal{C}_{[z]}$,
(iii) *There exists a real structure $A \in \mathfrak{A}_{[z]}$ such that $AN_{[z]} \in \mathbb{C}\nu_{[z]}M$.*

We now assume that M is a Hopf hypersurface. Then the Reeb vector field ξ becomes

$$S\xi = \alpha\xi$$

for the smooth Reeb function $\alpha = g(S\xi, \xi)$ on M. The transformed vector field JX by the Kähler structure J on Q^{m*} for any vector field X on M in Q^{m*} is given by

$$JX = \phi X + \eta(X)N$$

for a unit normal vector field N to M. Then the *equation of Codazzi* is given by

$$\begin{aligned} &g((\nabla_X S)Y - (\nabla_Y S)X, Z) \\ &\quad = -\eta(X)g(\phi Y, Z) + \eta(Y)g(\phi X, Z) + 2\eta(Z)g(\phi X, Y) - g(X, AN)g(AY, Z) \\ &\qquad + g(Y, AN)g(AX, Z) - g(X, A\xi)g(JAY, Z) + g(Y, A\xi)g(JAX, Z). \end{aligned}$$

From this, if we put $Z = \xi$, it follows that

$$\begin{aligned} &g((\nabla_X S)Y - (\nabla_Y S)X, \xi) \\ &\quad = 2g(\phi X, Y) - g(X, AN)g(Y, A\xi) + g(Y, AN)g(X, A\xi) \\ &\qquad + g(X, A\xi)g(JY, A\xi) - g(Y, A\xi)g(JX, A\xi). \end{aligned}$$

On the other hand, by differentiating $S\xi = \alpha\xi$ we have

$$\begin{aligned} &g((\nabla_X S)Y - (\nabla_Y S)X, \xi) \\ &\quad = g((\nabla_X S)\xi, Y) - g((\nabla_Y S)\xi, X) \\ &\quad = (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g((S\phi + \phi S)X, Y) - 2g(S\phi SX, Y). \end{aligned}$$

If we compare the previous two equations and putting $X = \xi$, we have the following

$$Y\alpha = (\xi\alpha)\eta(Y) - 2g(\xi, AN)g(Y, A\xi) + 2g(Y, AN)g(\xi, A\xi). \tag{3.1}$$

Substituting (3.1) into the obtained equation implies

$$\begin{aligned} & g((\nabla_X S)Y - (\nabla_Y S)X, \xi) \\ & \quad = 2g(\xi, AN)g(X, A\xi)\eta(Y) - 2g(X, AN)g(\xi, A\xi)\eta(Y) \\ & \qquad - 2g(\xi, AN)g(Y, A\xi)\eta(X) + 2g(Y, AN)g(\xi, A\xi)\eta(X) \\ & \qquad + \alpha g((\phi S + S\phi)X, Y) - 2g(S\phi SX, Y). \end{aligned}$$

Summing up all the facts above, we have the following

$$\begin{aligned} 0 = & \, 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) + 2g(\phi X, Y) \\ & - g(X, AN)g(Y, A\xi) + g(Y, AN)g(X, A\xi) \\ & + g(X, A\xi)g(JY, A\xi) - g(Y, A\xi)g(JX, A\xi) \\ & - 2g(\xi, AN)g(X, A\xi)\eta(Y) + 2g(X, AN)g(\xi, A\xi)\eta(Y) \\ & + 2g(\xi, AN)g(Y, A\xi)\eta(X) - 2g(Y, AN)g(\xi, A\xi)\eta(X). \end{aligned}$$

At each point $z \in M$ we can choose $A \in \mathfrak{A}_z$ such that

$$N = \cos(t)Z_1 + \sin(t)JZ_2$$

for some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $0 \leq t \leq \frac{\pi}{4}$ (see Proposition 3 in [21]). Note that t is a function on M. First of all, since $\xi = -JN$, we have

$$\begin{aligned} \xi &= \sin(t)Z_2 - \cos(t)JZ_1, \\ AN &= \cos(t)Z_1 - \sin(t)JZ_2, \\ A\xi &= \sin(t)Z_2 + \cos(t)JZ_1. \end{aligned} \tag{3.2}$$

On the other hand, we have $JA\xi = -AJ\xi = -AN$, and inserting this formula into the previous equation implies

Lemma 3.2. *Let M be a Hopf hypersurface in Q^{m*} with (local) unit normal vector field N. For each point $z \in M$ we choose $A \in \mathfrak{A}_z$ such that $N_z = \cos(t)Z_1 + \sin(t)JZ_2$ holds for some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $0 \leq t \leq \frac{\pi}{4}$. Then*

$$\begin{aligned} 0 = & \, 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) + 2g(\phi X, Y) - 2g(X, AN)g(Y, A\xi) \\ & + 2g(Y, AN)g(X, A\xi) - 2g(\xi, A\xi)\{g(Y, AN)\eta(X) - g(X, AN)\eta(Y)\} \end{aligned}$$

holds for all vector fields X and Y on M.

By the equation of Gauss, the curvature tensor $R(X, Y)Z$ for a real hypersurface M in Q^{m*} induced from the curvature tensor $\bar{R}$ of Q^{m*} can be described in terms of the complex structure J and the complex conjugations $A \in \mathfrak{A}$ as follows: for any tangent vector fields X, Y and Z on M in Q^{m*}

$$\begin{aligned} R(X, Y)Z = & -g(Y, Z)X + g(X, Z)Y - g(JY, Z)(JX)^T + g(JX, Z)(JY)^T \\ & + 2g(JX, Y)(JZ)^T - g(AY, Z)(AX)^T + g(AX, Z)(AY)^T \\ & - g(JAY, Z)(JAX)^T + g(JAX, Z)(JAY)^T \\ & + g(SY, Z)SX - g(SX, Z)SY, \end{aligned} \tag{3.3}$$

where $(\cdots)^T$ denotes the tangential component of the vector $(\cdots)$ in Q^{m*}.

Let $\{e_1, e_2, \cdots, e_{2m-1}, e_{2m} := N\}$ be a basis of the tangent vector space T_zQ^{m*} of Q^{m*} at $z \in Q^{m*}$. By the definition of the Ricci operator of M in Q^{m*}, it is given by $\mathrm{Ric}(X) = \Sigma_{i=1}^{2m-1} R(X, e_i)e_i$. So from (3.3) it follows that

$$\begin{aligned}\mathrm{Ric}(X) = &-(2m-1)X + 3\eta(X)\xi + g(AN,N)(AX)^T - g(AX,N)(AN)^T \\ &+ g(JAN,N)(JAX)^T - g(JAX,N)(JAN)^T \\ &+ (\mathrm{Tr}\, S)SX - S^2X,\end{aligned} \tag{3.4}$$

where we have used some basic formulas induced from contracting of the curvature tensor in (3.3)(see also Berndt and Suh [1]).

In this paper, we consider the notion of *cyclic Ricci semi-symmetric*

$$\mathfrak{S}_{X,Y,Z}(R(X,Y)Ric)(Z) = 0,$$

which is weaker than Ricci semi-symmetric $R(X,Y)Ric(Z) = 0$ or Ricci parallel, $\nabla_X Ric = 0$, where the curvature tensor $R(X,Y)Z$ is defined by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

for any vector fields X, Y and Z on M in the complex hyperbolic quadric Q^{m*}. Then by the Ricci formula and the first Bianchi identity on Riemannan manifolds the cyclic Ricci semi-symmetric $\mathfrak{S}_{X,Y,Z}(R(X,Y)Ric)(Z) = 0$ implies

$$\begin{aligned}\mathfrak{S}_{X,Y,Z}R(X,Y)Ric(Z) =&R(X,Y)\mathrm{Ric}(Z) + R(Y,Z)\mathrm{Ric}(X) + R(Z,X)\mathrm{Ric}(Y) \\ =&0\end{aligned} \tag{3.5}$$

for any tangent vector fields X, Y and Z on M, where $\mathfrak{S}_{X,Y,Z}$ denotes the cyclic sum of the vector fields X, Y and Z. Hereafter, unless otherwise stated the equation (3.5) is said to be *cyclic Ricci semi-symmetric* or otherwise *cyclic semi-parallel Ricci tensor* of M in Q^{m*}.

On the other hand, for a real structure A of Q^{m*} we decompose AX into its tangential and normal components given by $AX = BX + g(AX,N)N$. From this and the anti-commuting property between the complex structure J and real structure A, we get

$$AN = AJ\xi = -JA\xi = -\phi A\xi - g(A\xi,\xi)N. \tag{3.6}$$

In addition, from the expression of the vector fields $A\xi$ and N in (3.2) we obtain that $g(A\xi, N) = 0$, which means that the unit vector field $A\xi$ is tangent to M.

Now let us use the equation of Gauss which is given by

$$\bar{\nabla}_X Y = \nabla_X Y + g(SX,Y)N,$$

where S denotes the shape operator of M in Q^{m*}. Then we get the following

$$\begin{aligned}\nabla_X(A\xi) =&\bar{\nabla}_X(A\xi) - g(SX,A\xi)N \\ =&(\bar{\nabla}_X A)\xi + A(\bar{\nabla}_X\xi) - g(SX,A\xi)N \\ =&q(X)JA\xi + A(\nabla_X\xi + g(SX,\xi)N) - g(SX,A\xi)N \\ =&q(X)JA\xi + A\phi SX + g(SX,\xi)AN - g(SX,A\xi)N \\ =&q(X)\phi A\xi + q(X)g(A\xi,\xi)N + B\phi SX + g(\phi SX,AN)N \\ &\quad - g(SX,\xi)\phi A\xi - g(SX,\xi)g(A\xi,\xi)N - g(SX,A\xi)N \\ =&q(X)\phi A\xi + q(X)g(A\xi,\xi)N + B\phi SX \\ &- g(A\xi,SX)N + g(A\xi,\xi)g(SX,\xi)N \\ &- g(SX,\xi)\phi A\xi - g(SX,\xi)g(A\xi,\xi)N - g(SX,A\xi)N,\end{aligned}$$

where we have used the formulas $(\bar{\nabla}_X A)Y = q(X)JAY$ and (3.6). From this, by comparing the tangential and normal parts of both sides, we can assert the following:

Lemma 3.3. *Let M be a real hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 3$. Then we obtain*

$$\nabla_X(A\xi) = q(X)\phi A\xi + B\phi SX - g(SX, \xi)\phi A\xi \tag{3.7}$$

and

$$q(X)g(A\xi, \xi) = 2g(SX, A\xi) \tag{3.8}$$

for any tangent vector field X on M.

4. Proof of Main Theorem 1 with $\mathfrak{A}$-principal unit normal vector field

Now in this section we consider only an $\mathfrak{A}$-principal unit normal vector field N for a real hypersurface M in Q^{m*} with cyclic semi-symmetric Ricci tensor. As in section 2, we denote by $\bar{\nabla}$ the canonical covariant derivative of Q^{m*}, and by $\bar{\nabla}^{\mathrm{End}}$ the induced covariant derivative on the endomorphism bundle $\mathrm{End}(TQ^{m*})$.

Lemma 4.1. *Let M be a Hopf hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 3$, such that the normal vector field N is $\mathfrak{A}$-principal everywhere. Let A be the section of the S^1-bundle $\mathfrak{A}$ so that $AN = N$ holds. Then we have the following:*

(i) The Reeb curvature function α is constant.

(ii) If $X \in \mathcal{C}$ is a principal vector of M with principal curvature λ, then $\alpha = \pm 2$, $\lambda = \pm 1$ for $\alpha = 2\lambda$ or ϕX is a principal curvature vector with principal curvature $\mu = \frac{\alpha\lambda - 2}{2\lambda - \alpha}$ for $\alpha \neq 2\lambda$.

(iii) $\bar{\nabla}^{\mathrm{End}}_X A = 0$ for any $X \in \mathcal{C}$.

(iv) $ASX = SX$ for any $X \in \mathcal{C}$.

(v) The shape operator commutes with the complex conjugation, that is, $AS = SA$.

(vi) $q(\xi) = 2\alpha$.

Proof. In Suh, Pérez and Woo [32] and Lemma 3.2, we know the following for any $X \in T_\lambda$

$$(2\lambda - \alpha)S\phi X = (\alpha\lambda - 2)\phi X$$

when the unit normal vector field N is $\mathfrak{A}$-principal. From this, together with (3.1) and the method in [32], it can be verified that (i) the Reeb function α is constant and (ii) holds on M.

Now let us prove (iii) and (iv). In order to do this we consider the real valued 1-form $q : TQ^{m*} \to \mathbb{R}$ on the complex hyperbolic quadric Q^{m*}. Then by (2.2) we know that

$$\bar{\nabla}^{\mathrm{End}}_X A = q(X) \cdot JA \quad \text{holds for every } X \in TQ^{m*}.$$

Now let us differentiate the equation $g(AN, JN) = 0$ along any $X \in T_xM$, $x \in M$. Thereby we obtain

$$\begin{aligned} 0 =& g((\bar{\nabla}^{\mathrm{End}}_X A)N + A\bar{\nabla}_X N, JN) + g(AN, (\bar{\nabla}^{\mathrm{End}}_X J)N + J\bar{\nabla}_X N) \\ =& q(X) - g(ASX, JN) - g(\xi, SX) \end{aligned}$$

for the second equality it was used that $\bar{\nabla}^{\mathrm{End}} J = 0$ holds, because Q^{m*} is Kählerian. This gives us for the 1-form q

$$q(X) = -g(ASX, \xi) + g(\xi, SX) = g(S\xi, X) + g(\xi, SX) = 2\alpha\eta(X), \tag{4.1}$$

where we have used that $A\xi = -AJN = JAN = JN = -\xi$, because of $N \in V(A)$. It follows from Equation (4.1) that $q(X) = 0$ holds for any $X \in \mathcal{C}$, whence (iii) follows.

Second, we differentiate the formula $AJN = -JAN = -JN$ along the distribution $\mathcal{C}$. By applying equation (4.1) and again $\bar{\nabla}^{\mathrm{End}} J = 0$, we obtain for $X \in \mathcal{C}$

$$q(X)JAJN - AJSX = JSX.$$

Because of (4.1) for any $X \in \mathcal{C}$, we have $q(X) = 0$, and therefore $-AJSX = JSX$, which implies $ASX = SX$, completing the proof of (iv).

From the $\mathfrak{A}$-principal, let us differentiate $AN = N$. Then it follows that

$$\begin{aligned} -SX =& \bar{\nabla}_X N = \bar{\nabla}_X (AN) \\ =& (\bar{\nabla}_X A)N + A\bar{\nabla}_X N = q(X)JAN - ASX \\ =& -2\alpha\eta(X)\xi - ASX, \end{aligned}$$

where in the final equality we have used the formula (4.2) and $\mathfrak{A}$-principal. Then it can be arranged again as follows:

$$ASX = SX - 2\alpha\eta(X)\xi$$

for any $X \in T_xM$, $x \in M$. Then by the symmetric property of the above equation it becomes $g(ASX, Y) = g(ASY, X)$ for any $X, Y \in T_xM$, $x \in M$. That is, (v) holds on M.

Since $A\xi \in T_xM$, $x \in M$, for M in Q^{m*}, by the equation of Gauss and (4.1) we know that

$$\begin{aligned} \nabla_X(A\xi) =& \bar{\nabla}_X(A\xi) - \sigma(X, A\xi) \\ =& q(X)JA\xi + A(\nabla_X \xi) + g(SX, \xi)AN - g(SX, A\xi)N \end{aligned}$$

for any $X \in TM$. From this, by taking the inner product with the unit normal N and using $A\xi = -\xi$ for an $\mathfrak{A}$-principal, we have

$$q(X) = 2\alpha\eta(X),$$

which implies $q(\xi) = 2\alpha$, which gives a complete proof of (vi). □

On the other hand, from (3.8) and using $AN = N$, $A\xi = -\xi$ and $AX = BX$ for an $\mathfrak{A}$-principal unit normal vector field, we have

$$\mathrm{Ric}(X) = -(2m-1)X + 2\eta(X)\xi + AX + hSX - S^2X,$$

where the mean curvature $h = \mathrm{Tr}\, S$ is defined by the trace of the shape operator S of M in Q^{m*}. Now, let us use the assumption of cyclic semi-symmetric Ricci tensor, that is, $\mathfrak{S}_{X,Y,Z}(R(X,Y)\mathrm{Ric})Z = 0$. This is equivalent to

$$\mathfrak{S}_{X,Y,Z} R(X,Y)\mathrm{Ric}(Z) = 0 \tag{4.2}$$

for any vector fields X, Y and $Z \in T_zM$, $z \in M$, where $\mathfrak{S}_{X,Y,Z}$ denotes the cyclic sum of the vector fields X, Y and Z. Then in order to find some geometric structures of the cyclic semi-symmetric Ricci tensor (4.2), we want to consider some formulas as follows:

By virtue of the expression of the curvature tensor $R(X,Y)Z$ in (3.3), we calculate the term $R(X,Y)\xi$ as follows:

$$\begin{aligned} R(X,Y)\xi =& -\eta(Y)X + \eta(X)Y - \eta(AY)(AX)^T + \eta(AX)(AY)^T \\ &+ g(AY,N)(JAX)^T - g(AX,N)(JAY)^T \\ &+ \alpha\{\eta(Y)SX - \eta(X)SY\} \\ =& -\eta(Y)X + \eta(X)Y + \eta(Y)(AX)^T - \eta(X)(AY)^T \\ &+ \alpha\{\eta(Y)SX - \eta(X)SY\}. \end{aligned} \tag{4.3}$$

On the other hand, it is known that a pseudo-Einstein real hypersurface M in Q^{m*} is defined by

$$\mathrm{Ric}(X) = aX + b\eta(X)\xi, \tag{4.4}$$

where a and b are constant on M.

Now let us check that whether this kind of pseudo-Einstein real hypersurface satisfies the cyclic Ricci semi-symmetric (4.2) or not. In order to do this, let us substitute (4.4) into (4.2) and use (4.3) in the obtained equation. Then it follows that

$$\begin{aligned}
&\mathfrak{S}_{X,Y,Z}\eta(Z)R(X,Y)\xi\\
&\quad=\eta(Z)\Big\{-\eta(Y)X+\eta(X)Y-\eta(AY)(AX)^T+\eta(AX)(AY)^T\\
&\qquad+g(AY,N)(JAX)^T-g(AX,N)(JAY)^T+\alpha\{\eta(Y)SX-\eta(X)SY\}\Big\}\\
&\qquad+\eta(X)\Big\{-\eta(Z)Y+\eta(Y)Z-\eta(AZ)(AY)^T+\eta(AY)(AZ)^T\\
&\qquad+g(AZ,N)(JAY)^T-g(AY,N)(JAZ)^T+\alpha\{\eta(Z)SY-\eta(Y)SZ\}\Big\}\\
&\qquad+\eta(Y)\Big\{-\eta(X)Z+\eta(Z)X-\eta(AX)(AZ)^T+\eta(AZ)(AX)^T\\
&\qquad+g(AX,N)(JAZ)^T-g(AZ,N)(JAX)^T+\alpha\{\eta(X)SZ-\eta(Z)SX\}\Big\}.
\end{aligned}\tag{4.5}$$

Since the unit normal vector field N is $\mathfrak{A}$-principal, we know that $AN = N$ and $A\xi = -\xi$. So (4.5) implies the following

$$\begin{aligned}
&\mathfrak{S}_{X,Y,Z}\eta(Z)R(X,Y)\xi = \eta(Z)\Big\{\eta(Y)(AX)^T-\eta(X)(AY)^T\Big\}\\
&+\eta(X)\Big\{\eta(Z)(AY)^T-\eta(Y)(AZ)^T\Big\}+\eta(Y)\Big\{\eta(X)(AZ)^T-\eta(Z)(AX)^T\Big\}\\
&=0.
\end{aligned}\tag{4.6}$$

From this we can assert the following

Theorem 4.2. *Let M be a pseudo-Einstein real hypersurface in the complex hyperbolic quadric Q^{m*}, $m\geq 3$. If the unit normal vector field N is $\mathfrak{A}$-principal, then M satisfies the cyclic Ricci semi-symmetric, that is,*

$$\mathfrak{S}_{X,Y,Z}R(X,Y)Ric(Z) = 0, \tag{4.7}$$

where $\mathfrak{S}_{X,Y,Z}$ denotes the cyclic sum of the vector fields X, Y and Z on M.

As a converse problem, we can assert the following theorem

Theorem 4.3. *Let M be a Hopf real hypersurface in the complex hyperbolic quadric Q^{m*}, $m\geq 3$, with $\mathfrak{A}$-principal unit singular normal vector field. Then there does not exist a real hypersurface in Q^{m*} satisfying the cyclic Ricci semi-symmetric. In particular, there do not exist a Hopf pseudo-Einstein real hypersurface in Q^{m*}.*

Proof. In order to prove this theorem, we need Lemma 4.1 and the following proposition.

By virtue of Lemma 4.1, some geometric properties of Hopf hypersurfaces in Q^m are being investigated when the unit normal vector field N is $\mathfrak{A}$-principal. Among them, as a new characterization of contact hypersurfaces in the complex hyperbolic quadric Q^{m*}, we proved the following results:

Proposition 4.4. *Let M be a Hopf real hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 3$. Then M has an $\mathfrak{A}$-principal singular normal vector field N if and only if M is locally congruent to one of the following:*

(i) the tube of radius r around the Hermitian symmetric space $Q^{*(m-1)}$ which is imbedded in Q^{m*} as a totally geodesic complex hypersurface,

(ii) a horosphere in Q^{m*} whose center at infinity is the equivalence class of an $\mathfrak{A}$-principal geodesic in Q^{m*},

(iii) the tube of radius r around the m-dimensional real hyperbolic space $\mathbb{R}H^m$ which is embedded in Q^{m*} as a real space form of Q^{m*}.

Proof. From items (iv) and (v) of Lemma 4.1 we see that $SX = 0$ for any $X \in \mathcal{C} \cap JV(A)$, then by Lemma 3.2 it holds that $S\phi X = \frac{2}{\alpha}\phi X$. Thus, the expression of the shape operator S of M can be given by

$$S = \mathrm{diag}(\alpha, \underbrace{0, 0, \ldots, 0}_{(m-1)}, \underbrace{\frac{2}{\alpha}, \frac{2}{\alpha}, \ldots, \frac{2}{\alpha}}_{(m-1)})$$

It follows that the shape operator S satisfies $S\phi + \phi S = \frac{2}{\alpha}\phi$, and M is a contact real hypersurface. Then, Proposition 4.4 follows directly from Theorem C. □

Let M be a Hopf real hypersurface in the complex hyperbolic quadric Q^{m*} with $\mathfrak{A}$-principal singular normal vector field. Then from (4.1) the Ricci operator becomes

$$\mathrm{Ric}(X) = -(2m-1)X + 2\eta(X)\xi + AX + HX, \tag{4.8}$$

where HX is given by $HX = hSX - S^2X$ for any vector field X on M. Then the Ricci cyclic semi-symmetric (4.7) and the first Bianchi identity implies that

$$\begin{aligned}
0 =& \mathfrak{S}_{X,Y,Z} R(X,Y)\mathrm{Ric}(Z) \\
=& \mathfrak{S}_{X,Y,Z} R(X,Y)\{-(2m-1)Z + 2\eta(Z)\xi + AZ + HZ\} \\
=& -(2m-1)\mathfrak{S}_{X,Y,Z} R(X,Y)Z + 2\mathfrak{S}_{X,Y,Z}\eta(Z)R(X,Y)\xi \\
&+ \mathfrak{S}_{X,Y,Z} R(X,Y)AZ + \mathfrak{S}_{X,Y,Z} R(X,Y)HZ \\
=& \mathfrak{S}_{X,Y,Z} R(X,Y)AZ + \mathfrak{S}_{X,Y,Z} R(X,Y)HZ.
\end{aligned}$$

This implies

$$\mathfrak{S}_{X,Y,Z} R(X,Y)HZ = -\mathfrak{S}_{X,Y,Z} R(X,Y)AZ, \tag{4.9}$$

where the curvature tensor of the left side in (4.9) is given by

$$\begin{aligned}
R(X,Y)HZ =& -g(Y,HZ)X + g(X,HZ)Y - g(JY,HZ)(JX)^T \\
&+ g(JX,HZ)(JY)^T + 2g(JX,Y)(JHZ)^T - g(AY,HZ)(AX)^T \\
&+ g(AX,HZ)(AY)^T - g(JAY,HZ)(JAX)^T \\
&+ g(JAX,HZ)(JAY)^T \\
&+ g(SY,HZ)SX - g(SX,HZ)SY.
\end{aligned} \tag{4.10}$$

Now let us consider $X, Y \in T_\lambda \cap JV(A)$ in Lemma 4.1 and Proposition 4.4 with $\lambda = 0$ and $\mu = \frac{2}{\alpha}$, because M is contact in Proposition 4.4. Then it follows that $SX = SY = 0$, $AX = -X$ and $AY = -Y$ for the principal curvature $\lambda = 0$. Then $HX = hSX - S^2X = 0$, and $HY = hSY - S^2Y = 0$. Moreover, for $Z \in T_\mu \cap V(A)$ in Lemma 4.1 and Proposition 4.4 it implies that

$$SZ = \mu Z, \quad \mu = \frac{2}{\alpha}, \quad \text{and} \quad HZ = hSZ - S^2Z = (h\mu - \mu^2)Z.$$

Accordingly, from (4.9) and the 1st Bianchi identity, together with $HX = 0$ and $HY = 0$ mentioned above, it satisfies that

$$\begin{aligned}\mathfrak{S}_{X,Y,Z}R(X,Y)HZ =&R(X,Y)HZ + R(Y,Z)HX + R(Z,X)HY\\ =&(h\mu - \mu^2)R(X,Y)Z.\end{aligned} \tag{4.11}$$

On the other hand, from the right side of (4.9), together with $AX = -X$, $AY = -Y$, and $AZ = Z$, it follows that

$$\begin{aligned}\mathfrak{S}_{X,Y,Z}R(X,Y)AZ =&R(X,Y)AZ + R(Y,Z)AX + R(Z,X)AY\\ =&R(X,Y)Z - R(Y,Z)X - R(Z,X)Y\\ =&R(X,Y)Z + R(X,Y)Z\\ =&2R(X,Y)Z\end{aligned} \tag{4.12}$$

Then (4.9), (4.11) and (4.12) imply

$$(h\mu - \mu^2 + 2)R(X,Y)Z = 0.$$

But here the curvature tensor $R(X,Y)Z$ never vanishing for $X, Y \in T_\lambda \cap JV(A)$ and $Z \in T_\mu \cap V(A)$.

In fact, by virtue of Theorem C, in Proposition 4.4 we know that M is contact and the principal curvature μ is given by $\mu = \frac{2}{\alpha}$ and $AX = -X$, and $AY = -Y$ and $AZ = Z$. Moreover, $SX = SY = 0$ and $SZ = \mu Z$. If the curvature tensor $R(X,Y)Z = 0$ for $X, Y \in T_\lambda \cap JV(A)$, $X \perp Y$ and $Z \in T_\mu \cap V(A)$, then it follows that

$$\begin{aligned}0 = R(X,Y)Z =& -g(JY,Z)(JX)^T + g(JX,Z)(JY)^T\\ &+ 2g(JX,Y)(JZ)^T - g(JAY,Z)(JAX)^T + g(JAX,Z)(JAY)^T\\ =& -(JX)^T + g(X,Y)(JY)^T - g(JY,Z)(JX)^T + g(JX,Z)(JY)^T\\ =& -2(JX)^T,\end{aligned}$$

where we have taken $Z = JY$, $g(X,Y) = 0$, $AX = -X$, $AY = -Y$ and $AZ = Z$. So it gives a contradiction.

So we should have $h\mu - \mu^2 + 2 = 0$. From this, we want to calculate the principle curvatures of cyclic Ricci semi-symmetric real hypersurfaces in Q^{m*} as follows:

$$\begin{aligned}0 =&h\mu - \mu^2 + 2\\ =&\{\alpha + (m-1)(\frac{2}{\alpha})\}(\frac{2}{\alpha}) - \frac{4}{\alpha^2} + 2\\ =&(m-2)\frac{4}{\alpha^2} + 4.\end{aligned} \tag{4.13}$$

Then it follows that $\alpha^2 + m - 2 = 0$. This gives a contradiction. So we get a complete proof of our Main Theorem 1 in the introduction. □

5. Proof of Main Theorem 2 with $\mathfrak{A}$-isotropic unit normal vector field

In section 4, we proved that there does not exist a Hopf real hypersurface with *cyclic semi-symmetric Ricci tensor* in the complex hyperbolic quadric Q^{m*} with $\mathfrak{A}$-principal unit normal vector field. Motivated by the result of section 4, in this section we give a complete proof of our Main Theorem 2 for real hypersurfaces with *cyclic semi-symmetric Ricci tensor* when M has an $\mathfrak{A}$-isotropic unit normal vector field. Then by putting $Z = \xi$ in the assumption of cyclic Ricci semi-symmetric it is given by

$$R(X,Y)\mathrm{Ric}(\xi) + R(Y,\xi)\mathrm{Ric}(X) + R(\xi,X)\mathrm{Ric}(Y) = 0. \tag{5.1}$$

Since we assumed that the unit normal N is $\mathfrak{A}$-isotropic, by the definition in section 3 we know that $t = \frac{\pi}{4}$. Then by the expression of the $\mathfrak{A}$-isotropic unit normal vector field, the equation (3.2) gives $N = \frac{1}{\sqrt{2}} Z_1 + \frac{1}{\sqrt{2}} JZ_2$. This implies that

$$g(\xi, A\xi) = 0,\ g(\xi, AN) = 0,\ g(AN, N) = 0,\ g(A\xi, N) = 0,$$

and

$$g(JAN, \xi) = -g(AN, N) = 0.$$

Then the vector fields AN and $A\xi$ become tangent vector fields to M in Q^{m*}.

When M is $\mathfrak{A}$-isotropic, the Ricci operator becomes

$$\begin{aligned} \mathrm{Ric}(X) = &-(2m-1)X + 3\eta(X)\xi - g(AX, N)(AN)^T - g(JAX, N)(JAN)^T \\ &+ hSX - S^2X. \end{aligned} \tag{5.2}$$

Then by putting $X = \xi$ in (5.2) and M being Hopf, we have

$$\mathrm{Ric}(\xi) = \kappa\xi, \quad \kappa = -2m + 4 + h\alpha - \alpha^2. \tag{5.3}$$

Since AN is a tangent vector field for an $\mathfrak{A}$-isotropic normal vector field, we know that

$$\nabla_Y(AN) = \{(\bar{\nabla}_Y A)N + A\bar{\nabla}_Y N\}^T = \{q(Y)JAN - ASY\}^T,$$

and

$$\nabla_Y(A\xi) = -q(Y)AN + B\phi SY + g(SY, \xi)AN,$$

where we have used (3.2) and (3.6), and $(\cdots)^T$ denotes the tangential component of the vector $(\cdots)$ in Q^m.

Now we assert an important lemma which gives a key role in the proof of our Main Theorem 2 as follows:

Lemma 5.1. *Let M be a Hopf real hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 3$, with $\mathfrak{A}$-isotropic unit normal vector field N. Then we have*

$$SA\xi = 0 \quad \textit{and} \quad SAN = -S\phi A\xi = 0.$$

Proof. Let us denote by $\mathcal{Q}^\perp = \mathrm{Span}\{A\xi, AN\}$, where $\mathcal{Q}$ is the maximal $\mathfrak{A}$-invariant subspace in the complex subbundle of $\mathcal{C}$. By differentiating $g(AN, N) = 0$ and using $(\bar{\nabla}_X A)Y = q(X)JAY$ and the equation of Weingarten, we know that

$$\begin{aligned} 0 &= g(\bar{\nabla}_X(AN), N) + g(AN, \bar{\nabla}_X N) \\ &= g(q(X)JAN - ASX, N) - g(AN, SX) \\ &= -2g(ASX, N). \end{aligned}$$

Then $SAN = 0$. From (3.2), we obtain $AN = -\phi A\xi$. So, it implies that $S\phi A\xi = 0$. Moreover, by differentiating $g(A\xi, N) = 0$ and using $g(AN, N) = 0$, we have:

$$\begin{aligned} 0 &= g(\bar{\nabla}_X(A\xi), N) + g(A\xi, \bar{\nabla}_X N) \\ &= g(q(X)JA\xi + A(\phi SX + g(SX, \xi)N), N) - g(SA\xi, X) \\ &= -2g(SA\xi, X) \end{aligned}$$

for any $X \in T_zM$, $z \in M$, where in the third equality we have used $\phi AN = JAN = -AJN = A\xi$. Then it follows that

$$SA\xi = 0.$$

It completes the proof of our assertion. □

Then by (3.3) and (3.5) the third term of the cyclic Ricci semi-symmetric (5.1) becomes

$$\begin{aligned}R(\xi, X)\mathrm{Ric}(Y) = &- g(X, \mathrm{Ric}(Y))\xi + g(\xi, \mathrm{Ric}(Y))X - g(AX, \mathrm{Ric}(Y))(A\xi)^T \\ &+ g(A\xi, \mathrm{Ric}(Y))(AX)^T - g(JAX, \mathrm{Ric}(Y))(JA\xi)^T \\ &+ g(JA\xi, \mathrm{Ric}(Y))(JAX)^T \\ &+ g(SX, \mathrm{Ric}(Y))S\xi - g(S\xi, \mathrm{Ric}(Y))SX,\end{aligned} \tag{5.4}$$

where we have used $J\xi = N$ which becomes a unit normal vector field to M in Q^m. Then we can calculate the above terms one by one as follows:

$$\begin{aligned}g(X,\mathrm{Ric}(Y))\xi &= g(\mathrm{Ric}(X), Y)\xi \\ &=\Big\{ - (2m-1)g(X, Y) + 3\eta(X)\eta(Y) - g(\phi A\xi, X)g(\phi A\xi, Y) \\ &\quad - g(A\xi, X)g(A\xi, Y) + hg(SX, Y) - g(S^2X, Y)\Big\}\xi,\end{aligned}$$

and

$$g(\xi, \mathrm{Ric}(Y))X = g(\mathrm{Ric}(\xi), Y)X = \kappa\eta(Y)X,$$

where we have used (5.2) and (5.3), and put $\kappa = -2(m-2) + (\alpha h - \alpha^2)$. The other six terms in (5.4) can be computed as follows:

$$\begin{aligned}-g(AX, \mathrm{Ric}(Y))(A\xi)^T =&\Big\{(2m-1)g(AX, Y) - 3\eta(Y)\eta(AX) + g(A\xi, Y)\eta(X) \\ &- hg(SY, AX) + g(S^2Y, AX)\Big\}(A\xi)^T,\end{aligned}$$

where we have used $\phi A\xi = -AN$ and $g(A\xi, AX) = \eta(X)$ in the $\mathfrak{A}$-isotropic unit normal vector field. Moreover, by virtue of $\mathfrak{A}$-isotropic $g(A\xi, \xi) = \eta(A\xi) = 0$ and $SA\xi = 0$ in Lemma 5.1, it follows that

$$g(A\xi, \mathrm{Ric}(Y))(AX)^T = 2mg(A\xi, Y)(AX)^T,$$

$$\begin{aligned}-g(JAX, \mathrm{Ric}(Y))(JA\xi)^T =&\Big\{(2m-1)g(JAX, Y) - 3\eta(Y)\eta(JAX) \\ &- g(\phi A\xi, Y)g(AN, JAX) - hg(SY, JAX) \\ &+ g(S^2Y, JAX)\Big\}(JA\xi)^T,\end{aligned}$$

where the term $g(A\xi, Y)g(A\xi, JAX) = g(A\xi, Y)g(N, X) = 0$ is used. Then by Lemma 5.1 the remained terms are calculated as follows:

$$\begin{aligned}g(JA\xi, \mathrm{Ric}(Y))(JAX)^T =&\Big\{ - (2m-1)g(JA\xi, Y) + 3\eta(Y)\eta(JA\xi) \\ &- g(\phi A\xi, Y)g(\phi A\xi, JA\xi) - g(A\xi, Y)g(A\xi, JA\xi) \\ &+ hg(SY, JA\xi) + hg(SY, JA\xi) - g(S^2Y, JA\xi)\Big\}(AX)^T \\ =& - 2mg(AN, Y)(JAX)^T,\end{aligned}$$

where we have used $g(JA\xi, Y) = g(\phi A\xi, Y) = -g(AN, Y)$ in the $\mathfrak{A}$-isotropic unit normal vector field.

$$\begin{aligned}g(SX, \mathrm{Ric}(Y))S\xi =&\Big\{ - (2m-1)g(Y, SX) + 3\eta(Y)\eta(SX) \\ &- g(\phi A\xi, Y)g(\phi A\xi, SX) \\ &- g(A\xi, Y)g(A\xi, SX) + hg(SY, SX) - g(S^2Y, SX)\Big\}S\xi \\ =&\alpha\Big\{ - (2m-1)g(SX, Y) + 3\alpha\eta(X)\eta(Y) \\ &+ hg(S^2X, Y) - g(S^3X, Y)\Big\}\xi,\end{aligned}$$

where in the second equality we have used $SA\xi = 0$ in Lemma 5.1. Now the final term in (5.4) is given by

$$\begin{aligned}-g(S\xi, \mathrm{Ric}(Y))SX =& -\alpha g(\mathrm{Ric}(Y), \xi)SX\\ =&\alpha\Big\{(2m-1)\eta(Y) - 3\eta(Y)\\ &+ g(\phi A\xi, Y)g(\phi A\xi, \xi) + g(A\xi, Y)g(A\xi, \xi)\\ &- hg(SY, \xi) + g(S^2Y, \xi)\Big\}SX\\ =&\alpha\{2(m-2) - h\alpha + \alpha^2\}\eta(Y)SX\end{aligned}$$

Substituting all the formulas above into (5.4) and using (5.2) and (5.3), we have the following

$$\begin{aligned}0 =&\kappa R(X, Y)\xi + R(\xi, X)\mathrm{Ric}(Y) - R(\xi, Y)\mathrm{Ric}(X)\\ =&(4 + h\alpha - \alpha^2)\Big\{ - g(A\xi, Y)(AX)^T + g(A\xi, X)(AY)^T\\ &+ g(AN, Y)(JAX)^T - g(AN, X)(JAY)^T\Big\}\\ &- \Big\{4(\eta(Y)\eta(AX) - \eta(X)\eta(AY)) + h\{g(SY, AX) - g(SX, AY)\}\\ &- (g(S^2Y, AX) - g(S^2X, AY))\Big\}(A\xi)^T\\ &- \Big\{ - (2m-1)(g(JAX, Y) - g(JAY, X)) + 3\eta(Y)\eta(JAX) - 3\eta(X)\eta(JAY)\\ &- g(\phi A\xi, Y)\eta(X) + g(\phi A\xi, X)\eta(Y) + h\{g(SY, JAX) - g(SX, JAY)\}\\ &- (g(S^2Y, JAX) - g(S^2X, JAY))\Big\}(JA\xi)^T.\end{aligned} \tag{5.5}$$

Now by Lemma 5.1, let us consider the distribution $\mathcal{Q}$ in section 3. At each point $z \in M$, we define the maximal $\mathfrak{A}$-invariant subspace of T_zM as follows:

$$\mathcal{Q}_z = \{X \in \mathcal{C}_z \mid AX \in T_zM \text{ for all } A \in \mathfrak{A}_z\}.$$

By putting $X \in \mathcal{Q} \cap V(A)$ in (5.5), then $Y = \phi X \in \mathcal{Q} \cap JV(A)$. Moreover, by Lemma 3.4, if we put $SX = \lambda X$, then $\phi X = \mu\phi X$, where $\mu = \frac{\lambda\alpha-2}{2\lambda-\alpha}$. From this it follows that

$$(\mu - \lambda)\{h - (\mu + \lambda)\} = 0. \tag{5.6}$$

This implies $\mu = \lambda$ or $h = \mu + \lambda$. Then we consider the following two cases:

Case 1. $\lambda = \mu$

In this case $\lambda = \frac{\alpha\lambda-2}{2\lambda-\alpha}$ implies $\lambda^2 - \alpha\lambda + 1 = 0$. This means that the principal curvatures are given by $\coth r$, $\tanh r$, and $\alpha = 2\coth 2r = \coth r + \tanh r$, and the shape operator S and the structure tensor ϕ commutes with each other, that is, $S\phi = \phi S$. So by virtue of Theorem B in the introduction due to Suh [30], M is locally congruent to a tube of radius r over the complex hyperbolic space $\mathbb{C}H^k$ in Q^{2k^*} or a horosphere whose center at infinity is $\mathfrak{A}$-isotropic singular.

Case 2. $h = \lambda + \mu$

In this subcase we know that the trace h becomes $h = -\frac{\alpha}{m-3}$, because

$$h = \lambda + \mu = \alpha + (m-2)(\lambda + \mu) = \alpha + (m-2)h.$$

Since $h = \lambda + \mu = \lambda + \frac{\alpha\lambda-2}{2\lambda-\alpha}$, it implies the following

$$2\lambda^2 - 2h\lambda + \alpha h - 2 = 0. \tag{5.7}$$

From this, together with $h = -\frac{\alpha}{m-3}$, it follows that the principal curvatures λ and μ satisfies the quadratic equation

$$2(m-3)x^2 + 2\alpha x - \alpha^2 - 2(m-3) = 0.$$

In this case the shape operator is given by

$$S = \begin{bmatrix} \alpha & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \lambda & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \mu \end{bmatrix}$$

where the two principal curvatures λ and μ are respectively given by

$$\lambda = \frac{-\alpha + \sqrt{(2m-5)\alpha^2 + 4(m-3)^2}}{2(m-3)}$$

and

$$\mu = \frac{-\alpha - \sqrt{(2m-5)\alpha^2 + 4(m-3)^2}}{2(m-3)}.$$

Here, the corresponding principal curvature spaces are $\xi \in T_\alpha$, $A\xi, AN \in T_{\beta=\gamma}$, T_λ and T_μ with multiplicities 1, 2, $m-2$ and $m-2$ respectively.

When $m = 3$, then from $h = \alpha + (m-2)h$ it follows that the Reeb function $\alpha = 0$. Moreover, by virtue of (5.7), principal curvatures λ and μ satisfy the quadratic equation

$$x^2 - hx - 1 = 0.$$

So we have that the principal curvatures are given by

$$\alpha = 0, \quad \beta = \gamma = 0, \quad \lambda = \frac{h + \sqrt{h^2+4}}{2}, \quad \text{and} \quad \mu = \frac{h - \sqrt{h^2+4}}{2}$$

with multiplicities 1, 2, 1 and 1 respectively. This gives a complete proof of our Main Theorem 2.

References

[1] J. Berndt and Y.J. Suh, Real hypersurfaces in Hermitian Symmetric Spaces, Volume **5** in the series Advances in Analysis and Geometry, Walter de Gruyter GmbH, Berlin/Boston, 2022.

[2] D.E. Blair, Almost contact manifolds with Killing structure tensors, Pacific J. Math. **39**(1971), 285-292.

[3] S.K. Chaubey, Y.J. Suh and U.C. De, Characterizations of the Lorentzian manifolds admitting a type of semi-symmetric connection, Analysis and Math. Physics, **10**(2020), 61(15 pages).

[4] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, Graduate Studies in Math., Amer. Math. Soc., 34, 2001.

[5] S. Klein and Y.J. Suh, Contact real hypersurfaces in the complex hyperbolic quadric, Ann. Mat. Pura Appl. **198** (2019), 1481-1494.

[6] M. Kimura, Real hypersurfaces of a complex projective space, Bull. Aust. Math. Soc. **33**(1986), 383-387.

[7] A.W. Knapp, *Lie Groups Beyond an Introduction*, Progress in Math., Birkhäuser, 2002.

[8] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vol. II, A Wiley-Interscience Publ., Wiley Classics Library Ed., 1996.

[9] J-H. Kwon and H. Nakagawa, Real hypersurfaces with cyclic-parallel Ricci tensor of a complex projective space, Hokkaido Math. J. **17**(1988), 355-371.

[10] H. Lee and Y.J. Suh, Commuting Jacobi operators on real hypersurfaces of Type B in the complex quadric, Math. Phys., Analysis and Geom. **23** (2020), 44.

[11] H. Lee and Y.J. Suh, A new result on parallel Ricci tensor for real hypersurfaces in the complex quadric, to appear in Proc. Royal Soc. Edinburgh Sect. A **151**(2021).

[12] A. Martinez and J.D. Pérez, Real hypersurfaces in quaternionic projective space, Ann. Math. Pura Appl. **145** (1968), 355–384.

[13] S. Montiel and A. Romero, On some real hypersurfaces in a complex hyperbolic space, Geom. Dedicata **212**(1991), 245-261.

[14] S. Montiel and A. Romero, Complex Einstein hypersurfaces of indefinite complex space forms, *Proc. Camb. Phil. Soc.* **94**(1983), 495-508.

[15] K. Nomizu, Invariant affine connections on homogeneous spaces, Amer. J. Math. **76** (1954), 33-65.

[16] M. Okumura, On some real hypersurfaces of a complex projective space, Trans. Amer. Math. Soc. **212**(1975), 355-364.

[17] J.D. Pérez, Cyclic-parallel real hypersurfaces of quaternionic projective space, Tsukuba J. Math. **17**(1993), 189-191.

[18] J.D. Pérez and Y.J. Suh, Real hypersurfaces of quaternionic projective space satisfying $\nabla_{U_i} R = 0$, Differential Geom. Appl. **7**(1997), 211-217.

[19] J.D. Pérez and Y.J. Suh, The Ricci tensor of real hypersurfaces in complex two-plane Grassmannians, J. Korean Math. Soc. **44**(2007), 211-235.

[20] J.D. Pérez and Y.J. Suh, Derivatives of the shape operator of real hypersurfaces in the complex quadric, Results Math. **73**(2018), 73:126.

[21] H. Reckziegel, On the geometry of the complex quadric, in: Geometry and Topology of Submanifolds VIII (Brussels/Nordfjordeid 1995), World Sci. Publ., River Edge, NJ, 1995, pp. 302-315.

[22] A. Romero, Some examples of indefinite complete complex Einstein hypersurfaces not locally symmetric, Proc. Amer. Math. Soc. **98** (1986), 283—286.

[23] A. Romero, On a certain class of complex Einstein hypersurfaces in indefinite complex space forms, Math. Z. **192** (1986), 627—635.

[24] B. Smyth, Differential geometry of complex hypersurfaces, Ann. Math. **85** (1967), 246-266.

[25] B. Smyth, Homogeneous complex hypersurfaces, J. Math. Soc. Japan **20** (1968), 643-647.

[26] Y.J. Suh, Real hypersurfaces in complex two-plane Grassmannians with parallel Ricci tensor, Proc. Roy. Soc. Edinburgh Sect. A **142**(2012), 1309-1324.

[27] Y.J. Suh, Hypersurfaces with isometric Reeb flow in complex hyperbolic two-plane Grassmannians, Adv. in Appl. Math., **50** (2013), 645—659.

[28] Y.J. Suh, Real hypersurfaces of type B in complex two-plane Grassmannians, Monatsh. Math. **147** (2006), 337-355.

[29] Y.J. Suh, Pseudo-Einstein real hypersurfaces in the complex quadric, Math. Nachr. **290** (2017), 1884-1904.

[30] Y.J. Suh, Real hypersurfaces in the complex hyperbolic quadrics with isometric Reeb flow, Comm. in Contemp. Math. **20** (2018), 1750031(20 pages).

[31] Y.J. Suh, Ricci soliton and pseudo-Einstein real hypersurfaces in the complex hyperbolic quadric, J. of Geom. and Phys. **162**(2021), 103888(16 pages).

[32] Y.J. Suh, J.D. Pérez and C. Woo, Real hypersurfaces in the complex hyperbolic quadric with parallel structure Jacobi operator, Publ. Math. Debrecen. **94**(2019), 75-107.

[33] C. Woo, G.J. Kim and Y.J. Suh, Cyclic Ricci semi-symmetric real hypersurfaces in the complex quadric, Rocky Mt. J. Math. (to appear).

[34] K. Yano, On harmonic and Killing vectors, Ann. of Math. **55**(1952), 38-45.