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Bernstein-Nikolskii-Markov-type inequalities for algebraic polynomials in a weighted Lebesgue space

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Abstract. In this paper, we study Bernstein, Markov and Nikol'skii type inequalities for arbitrary algebraic polynomials with respect to a weighted Lebesgue space, where the contour and weight functions have some singularities on a given contour.

1. Introduction

Let $\mathbb{C}$ be a complex plane, $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$; $G \subset \mathbb{C}$ be a bounded Jordan region, with $0 \in G$ and the boundary $L := \partial G$ be a closed Jordan curve, $\Omega := \overline{\mathbb{C}} \setminus \overline{G} = extL$. Let $\wp_n$ denotes the class of arbitrary algebraic polynomials $P_n(z)$ of degree at most $n \in \mathbb{N}$.

Let $0 < p \leq \infty$; $h(z)$, $z \in \mathbb{C}$, be a some weight function. For a rectifiable Jordan curve L, we denote:

$$\begin{aligned} \|P_n\|_{\mathcal{L}_p} &:= \|P_n\|_{\mathcal{L}_p(h,L)} := \left(\int_L h(z) \left|P_n(z)\right|^p |dz| \right)^{1/p}, \ 0 < p < \infty, \\ \|P_n\|_{\mathcal{L}_\infty} &:= \|P_n\|_{\mathcal{L}_\infty(1,L)} := \max_{z \in L} |P_n(z)|, \ p = \infty. \end{aligned}$$

Clearly, $\|\cdot\|_{\mathcal{L}_p}$ is a quasinorm (i.e. a norm for $1 \leq p \leq \infty$ and a $p-$norm for $0 < p < 1$).

Denoted by $w = \Phi(z)$, the univalent conformal mapping of Ω onto $\Delta := \{w : |w| > 1\}$ with normalization $\Phi(\infty) = \infty$, $\lim_{z\to\infty} \frac{\Phi(z)}{z} > 0$ and $\Psi := \Phi^{-1}$. For $t \geq 1$, we set:

$$L_t := \{z : \ |\Phi(z)| = t\}, \ L_1 \equiv L, \ G_t := intL_t, \ \Omega_t := extL_t.$$

Let $\left\{z_j\right\}_{j=1}^{l}$ be a fixed system of distinct points on curve L which is located in the positive direction. For some fixed R_0, $1 < R_0 < \infty$, and $z \in G_{R_0}$, consider generalized Jacobi weight function $h(z)$ which is defined as follows:

$$h(z) := h_0(z) \prod_{j=1}^{l} \left|z - z_j\right|^{\gamma_j}, \tag{1}$$

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where $\gamma_j > -1$ for all $j = 1, 2, ..., l$, and h_0 is uniformly separated from zero in G_{R_0}, i.e. there exists a constant $c_0 := c_0(G_{R_0}) > 0$, such that for all $z \in G_{R_0}$ $h_0(z) \geq c_0 > 0$.

In this work, we study the following type estimates

$$\|P_n^{(m)}\|_X \leq \lambda_n \|P_n\|_Y \tag{2}$$

for some spaces X and Y, where $\lambda_n > 0$, $\lambda_n \to \infty$, $n \to \infty$, is a constant, depending on the geometrical properties of the curve L, the weight function h and spaces X, Y. In the literature, these inequalities are often called Bernstein-type for $X = Y = \mathcal{L}_\infty$; Markov-type for $X = Y = \mathcal{L}_p$, $p > 0$, and Nikolskii-type for $m = 0, X = \mathcal{L}_q, Y = \mathcal{L}_p$, $0 < p < q < \infty$, inequalities in Lebesgue space for all polynomials $P_n \in \wp_n$ and any $m = 0, 1, 2, ...$

One of the first results analogous to (2), in the case $m = 0$, $X = \mathcal{L}_\infty$, $Y = \mathcal{L}_p$; $h(z) \equiv 1$, $L = \{z : |z| = 1\}$ and $0 < p < \infty$ was found by Jackson [32] as follows:

$$\|P_n\|_{\mathcal{L}_\infty} \leq 2n^{\frac{1}{p}} \left(\int_0^{2\pi} \left|P_n(e^{it})\right|^p dt \right)^{\frac{1}{p}}.$$

For some $m \geq 0$, X, Y, $h(z)$, L and $0 < p < \infty$, estimates of (2)- type was investigated by Szegö and Zigmund [46], Suetin [47], [48], Mamedhanov [14, 15], Nikol'skii [17, pp. 122–133], Dzyadyk [29], Andrashko [18], Nevai, Totik [38], Pritsker [42], Ditzian, Pritsker [28], Ditzian, Tikhonov [27], Andrievskii [20], [21] (see also the references cited therein) and others.

The last few years, analogous estimates of (2) for some $m \geq 0$, X, Y, $h(z)$, L and $0 < p \leq \infty$, were obtained in [3–16, 22, 43] and others.

In this work, we continue to study the estimation of (2)-type for quasidisks and for weight function $h(z)$ defined as in (1) for various regions in the complex plane.

2. Definitions and main results

Throughout this paper, $c, c_0, c_1, c_2, ...$ are positive and $\varepsilon_0, \varepsilon_1, \varepsilon_2, ...$ are sufficiently small positive constants (generally, different in different relations), which depends on G in general and, on parameters inessential for the argument; otherwise, such dependence will be explicitly stated.

For any $k \geq 0$ and $l > k$, notation $i = \overline{k, l}$ means $i = k, k+1, ..., l$.

Let $z = \psi(w)$ be the univalent conformal mapping of $B := \{w : |w| < 1\}$ onto the G normalized by $\psi(0) = 0$, $\psi'(0) > 0$. According to [40, pp.286-294], we say a bounded Jordan region G is called κ *-quasidisk*, $0 \leq \kappa < 1$, if any conformal mapping ψ can be extended to a K -quasiconformal, $K = \frac{1+\kappa}{1-\kappa}$, the homeomorphism of the plane $\overline{\mathbb{C}}$ on plane $\overline{\mathbb{C}}$. In that case, the curve $L := \partial G$ is called a κ *-quasicircle*. The region G (curve L) is called a *quasidisk (quasicircle)*, if it is κ*-quasidisk* (κ *-quasicircle)* for some $0 \leq \kappa < 1$.

We denote the class of κ *-quasicircle* by $Q(\kappa)$, $0 \leq \kappa < 1$, and write $L \in Q$, if $L \in Q(\kappa)$ for some $0 \leq \kappa < 1$. It is well-known that the quasicircle may not even be locally rectifiable [33, p.104].

Since, the object of study is the $\mathcal{L}_p(h, L)$, it is natural to give the following definition.

Definition 2.1. We say that $L \in \widetilde{Q}(\kappa)$, $0 \leq \kappa < 1$, if $L \in Q(\kappa)$ and L is rectifiable.

Let z_1, z_2 be arbitrary points on L and $l(z_1, z_2)$ denotes the subarc of L of shorter diameter with endpoints z_1 and z_2. The curve L is a quasicircle if and only if (three point property)

$$L(z; z_1, z_2) := \sup_{z_1,\, z_2 \in L,\, z \in l(z_1, z_2)} \frac{|z_1 - z| + |z - z_2|}{|z_1 - z_2|} < \infty.$$

Lesley [34, p.341] said that the curve L is "c−quasiconformal", if there exists the constant $c > 0$, independent from points z_1, z_2 and z such that $L(z; z_1, z_2) \leq c$. The Jordan curve L is called asymptotically conformal

[26], [41], if $L(z;z_1,z_2) \to 1$, as $|z_1 - z_2| \to 0$. According to the geometric criteria of quasiconformality of the curves ([17, p.81], [41, p.107]), every asymptotically conformal curve is a quasicircle. Every smooth curve is asymptotically conformal but corners are not allowed. The asymptotically conformal curves can be non-rectifiable.

Let S be rectifiable Jordan curve or arc and let $z = z(s)$, $s \in [0, |S|]$, $|S| := mes\ S$, be the natural parametrization of S. A Jordan curve or arc is called smooth, if S has a continuous tangent $\theta(z) := \theta(z(s))$ at every point $z(s)$. The class of such curves or arcs is denoted by C_θ.

Following [40, p.163], we say that a bounded Jordan curve L is $\lambda-$quasismooth (in the sense of Lavrentiev) curve, if for every pair $z_1,\ z_2 \in L$, there exists a constant $\lambda := \lambda(L) \geq 1$, such that

$$|l(z_1, z_2)| \leq \lambda\, |z_1 - z_2|\,,\ z_1,\ z_2 \in L,$$

where $|l(z_1, z_2)|$ is the linear measure (length) of $l(z_1, z_2)$. The region G is called a $\lambda-$quasismooth region, if $L = \partial G$ is a $\lambda-$quasismooth curve.

Following [40, p.48], we say that a Jordan curve S called Dini-smooth, if it has a parametrization $z = z(s)$, $0 \leq s \leq |S| := mes\ S$, such that $z'(s) \neq 0$, $0 \leq s \leq |S|$ and $\left|z'(s_2) - z'(s_1)\right| < g(s_2 - s_1)$, $s_1 < s_2$, where g is an increasing function for which

$$\int_0^1 \frac{g(x)}{x} dx < \infty.$$

A Jordan region G has a piecewise Dini-smooth boundary, if $L := \partial G$ consists of the union of finite Dini-smooth arcs L_j, $j = \overline{1,l}$, such that they have exterior (with respect to $\overline{G}$) angles $\lambda_j\pi$, $0 < \lambda_j < 2$, at the corner points $\left\{z_j\right\}$, $j = \overline{1,m}$, where two arcs meet.

According to the "three-point" criterion [33, p.100], every piecewise $C_\theta-$curve, Dini-smooth curve (without cusps), $\lambda-$quasismooth (in the sense of Lavrentiev) curve is quasiconformal. But, we know that calculating the quasi-conformity coefficient κ for all such curves is not an easy task. Therefore, we now give a more general definition of a class of curves with a different functional characteristic.

Definition 2.2. We say that $L = \partial G \in Q_\alpha$ if L is a quasicircle and $\Phi \in H^\alpha(\overline{\Omega})$ for some $0 < \alpha \leq 1$ (i.e., $|\Phi(z') - \Phi(z'')| \leq c\, |z' - z''|^\alpha$, for any pair $z', z' \in \overline{\Omega}$ and c constant, independent from z', z'').

Since the objects will be integrals along the curve, we must also require from the curve their rectifiability.

Definition 2.3. We say that $L \in \widetilde{Q}_\alpha, 0 < \alpha \leq 1$, *if* $L \in Q_\alpha$ and L is rectifiable.

We note that the class Q_α is sufficiently large. A detailed account on it and the related topics are contained in [34], [41], [49] (see also the references cited therein). This can be seen from the following:

Remark 2.4. a) If L is a piecewise Dini-smooth curve and largest exterior (interior) angle on L has opening $\vartheta\pi\ (\nu\pi)$, $0 < \vartheta \leq 1\ (1 \leq \nu < 2)$, then $L \in \widetilde{Q}_\vartheta \left(L \in \widetilde{Q}_{\frac{1}{2-\nu}}\right)$[34], [41, p.52]. If L is a smooth curve having continuous tangent line, then $G \in \widetilde{Q}_\alpha$ for all $0 < \alpha < 1$.

b) If G is "L-shaped" region, then $G \in \widetilde{Q}_\alpha$ for $\alpha = \frac{2}{3}$.
c) If L is quasismooth, then $G \in \widetilde{Q}_\alpha$ for $\alpha = \frac{1}{2}(1 - \frac{1}{\pi} \arcsin \frac{1}{c})^{-1}$ [49], [50].
d) If L is "c-quasiconformal", then $G \in Q_\alpha$ for $\alpha = \frac{\pi}{2(\pi - \arcsin \frac{1}{c})}$ [34].
e) If L is an asymptotically conformal curve, then $G \in Q_\alpha$ for all $0 < \alpha < 1$ [34].
Now, we start to formulate the new results.
Throughout in the text, we denote:

$$\gamma := \max\left\{0; \gamma_j,\ j = \overline{1,l}\right\}. \tag{3}$$

Theorem 2.5. *Let $0 < p \leq \infty$; $L \in \widetilde{Q}(\kappa)$ for some $0 \leq \kappa < 1$ and $h(z)$ be defined by (1). Then for any $P_n \in \wp_n$, $n \in \mathbb{N}$, and every $m = 0, 1, 2, \ldots$*

$$\|P_n^{(m)}\|_\infty \leq c_1 n^{\left(\frac{1+\gamma}{p}+m\right)(1+k)} \|P_n\|_p, \tag{4}$$

where γ is defined as in (3).

Corollary 2.6. *Let $L \in \widetilde{Q}(\kappa)$ for some $0 \leq \kappa < 1$. Then for any $P_n \in \wp_n$, $n \in \mathbb{N}$, and every $m = 1, 2, \ldots$*

$$\|P_n^{(m)}\|_\infty \leq c_1 n^{m(1+k)} \|P_n\|_\infty. \tag{5}$$

Theorem 2.7. *Let $1 < p \leq \infty$; $L \in \widetilde{Q}(\kappa)$ for some $0 \leq \kappa < 1$ and $h(z)$ be defined by (1). Then for any $P_n \in \wp_n$, $n \in \mathbb{N}$, and every $m = 0, 1, 2, \ldots$*

$$\left|P_n^{(m)}(z_j)\right| \leq c_2 n^{\left(\frac{1+\gamma_j}{p}+m\right)(1+k)} \|P_n\|_p.$$

Theorem 2.8. *Let $p > 1$; $L \in \widetilde{Q}(\kappa)$ for some $0 \leq k < 1$ and $h(z)$ be defined by (1); $R = 1 + \frac{c}{n}$. Then for arbitrary $P_n \in \wp_n$ and any $m = 0, 1, 2, \ldots$*

$$\|P_n^{(m)}\|_{\mathcal{L}_p(h, L_R)} \leq c_3 n^{m(1+k)} \|P_n\|_p. \tag{6}$$

Theorem 2.9. *Let $L \in \widetilde{Q}(\kappa)$ for some $0 \leq \kappa < 1$ and $h(z)$ be defined by (1). Then for arbitrary $P_n \in \wp_n$, $0 < p \leq q < \infty$ and any $m = 0, 1, 2, \ldots$*

$$\|P_n^{(m)}\|_q \leq c_4 n^{\left(\frac{1}{p}-\frac{1}{q}\right)(1+\gamma)(1+k)} \|P_n^{(m)}\|_p, \tag{7}$$

where γ define as in (3).

Now, we can state the corresponding results for the class of regions $G \in Q_\alpha$.

Theorem 2.10. *Let $0 < p \leq \infty$; $L \in \widetilde{Q}_\alpha$ for some $0 < \alpha \leq 1$ and $h(z)$ be defined by (1).Then for any $P_n \in \wp_n$, $n \in \mathbb{N}$, and every $m = 0, 1, 2, \ldots$*

$$\|P_n^{(m)}\|_\infty \leq c_5 \|P_n\|_p \begin{cases} n^{\delta\left(\frac{1+\gamma}{p}+m\right)}, & \alpha < \frac{1}{2}, \\ n^{\frac{1}{\alpha}\left(\frac{1+\gamma}{p}+m\right)}, & \alpha \geq \frac{1}{2}. \end{cases} \tag{8}$$

where γ is defined as in (3) and $\delta = \delta(G)$, $1 \leq \delta \leq 2$.

Corollary 2.11. *$L \in \widetilde{Q}_\alpha$ for some $0 < \alpha \leq 1$. Then for any $P_n \in \wp_n$, $n \in \mathbb{N}$, and every $m = 1, 2, \ldots$*

$$\|P_n^{(m)}\|_\infty \leq c_5 \|P_n\|_\infty \begin{cases} n^{\delta m}, & \alpha < \frac{1}{2}, \\ n^{\frac{m}{\alpha}}, & \alpha \geq \frac{1}{2}. \end{cases} \tag{9}$$

Theorem 2.12. *Let $1 < p \leq \infty$; $L \in \widetilde{Q}_\alpha$ for some $0 < \alpha \leq 1$ and $h(z)$ be defined by (1). Then for any $P_n \in \wp_n$, $n \in \mathbb{N}$, and every $m = 0, 1, 2, \ldots$*

$$\left|P_n^{(m)}(z_j)\right| \leq c_6 \|P_n\|_p \begin{cases} n^{\delta\left(\frac{1+\gamma_j}{p}+m\right)}, & \alpha < \frac{1}{2}, \\ n^{\frac{1}{\alpha}\left(\frac{1+\gamma_j}{p}+m\right)}, & \alpha \geq \frac{1}{2}. \end{cases} \tag{10}$$

Theorem 2.13. *Let $p > 1$; $L \in Q_\alpha$ for some $0 < \alpha \leq 1$ and $h(z)$ be defined by (1); $R = 1 + \frac{c}{n}$. Then for arbitrary $P_n \in \wp_n$ and any $m = 0, 1, 2, \ldots$*

$$\|P_n^{(m)}\|_{\mathcal{L}_p(h, L_R)} \leq c_7 \|P_n\|_p \begin{cases} n^{\delta m}, & \alpha < \frac{1}{2}; \\ n^{\frac{m}{\alpha}}, & \alpha \geq \frac{1}{2}. \end{cases} \tag{11}$$

Theorem 2.14. *Let $L \in \widetilde{Q}_\alpha$ for some $0 < \alpha \leq 1$ and $h(z)$ be defined by (1). Then for arbitrary $P_n \in \wp_n$, $0 < p \leq q < \infty$ and any $m = 0, 1, 2, \dots,$ we have:*

$$\|P_n^{(m)}\|_q \leq c_8 \|P_n^{(m)}\|_p \begin{cases} n^{\delta\left(\frac{1}{p}-\frac{1}{q}\right)(1+\gamma)}, & \alpha < \frac{1}{2}, \\ n^{\frac{1}{\alpha}\left(\frac{1}{p}-\frac{1}{q}\right)(1+\gamma)}, & \alpha \geq \frac{1}{2}, \end{cases} \tag{12}$$

where γ is defined as in (3).

Remark 2.15. For some regions and the weight function $h(z)$, similar statements to Theorems 2.5-2.14 were obtained earlier. In particular, for the $m = 0$, $h(z) \equiv 1$, $p > 0$ in [32, $\kappa = 0$], [6, $0 \leq \kappa < 1$]; for $m = 0$, same $h(z)$, $p > 0$, $L-$piecewise smooth curve and for $h(z) \equiv 1$ and rectifiable curve with corners L in [18]; for $h(z) \equiv 1$ and quasicircle L in [36]; for doubling weight function $h(z)$, quasismooth curve L and $R = 1$ in [21]; for $m \geq 1$, same $h(z)$, $p > 1$ in [15] and for $m = 0$, same $h(z)$, $p > 1$ in [8]; for $m = 0$, $h(z) \equiv 1$, $0 < p \leq q < \infty$ in [35, $\kappa = 1$], [42, $\kappa = 1; 0 \leq \kappa < 1$] and others.

Theorems 2.10–2.14 are analogues of Theorems 2.5–2.9 for a wider class of regions. Taking into account Remark 2.4 , we can write analogues of these theorems for other more simple regions.

Remark 2.16. The given estimates in Theorems 2.5-2.14 are exact.

3. Some auxiliary results

For $a > 0$ and $b > 0$, we use the notations "$a \preceq b$" (order inequality), if $a \leq cb$ and "$a \asymp b$" are equivalent to $c_1 a \leq b \leq c_2 a$ for some constants c, c_1, c_2 (independent of a and b) respectively.

The following definitions of the K-quasiconformal curves are well-known (see, for example, [17], [33, p.97] and [44]):

Definition 3.1. The Jordan arc (or curve) L is called $K-$quasiconformal ($K \geq 1$), if there is a $K-$quasiconformal mapping f of the region $D \supset L$ such that $f(L)$ is a line segment (or circle).

Let $F(L)$ denotes the set of all sense preserving plane homeomorphisms f of the region $D \supset L$ such that $f(L)$ is a line segment (or circle) and lets define

$$K_L := \inf \{K(f) : f \in F(L)\},$$

where $K(f)$ is the maximal dilatation of a such mapping f. L is a quasiconformal curve, if $K_L < \infty$, and L is a $K-$quasiconformal curve, if $K_L \leq K$.

Remark 3.2. It is well-known that, if we are not interested with the coefficients of quasiconformality of the curve, then the definitions of "quasicircle" and "quasiconformal curve" are identical. However, if we are also interested with the coefficients of quasiconformality of the given curve, then we will consider that if the curve L is $K-$quasiconformal, then it is $\kappa-$quasicircle with $\kappa = \frac{K^2-1}{K^2+1}$.

By the following Remark 3.2 for simplicity, we will use both terms, depending on the situation.

For $z \in \mathbb{C}$ and $M \subset \mathbb{C}$, we set

$$d(z, M) = dist(z, M) := \inf \{|z - \zeta| : \zeta \in M\}.$$

Let $\varphi := \psi^{-1}$, i.e. $w = \varphi(z)$ be the univalent conformal mapping of G onto the B normalized by $\varphi(0) = 0$, $\varphi'(0) > 0$. For $t \geq 1$, we set: $L_t := \left\{z : \left|\varphi(z)\right| = t\right\}$, $L_1 \equiv L$, $G_t := intL_t$, $\Omega_t := extL_t$.

Lemma 3.3. ([1]) *Let L be a $K-$quasiconformal curve, $z_1 \in L$, $z_2, z_3 \in \Omega \cap \{z : |z - z_1| \leq d(z_1, L_{r_0})\}$; $w_j = \Phi(z_j)$, $j = 1, 2, 3$. Then* *a) The statements* $|z_1 - z_2| \preceq |z_1 - z_3|$ *and* $|w_1 - w_2| \preceq |w_1 - w_3|$ *are equivalent. and similarly so are* $|z_1 - z_2| \asymp |z_1 - z_3|$ *and* $|w_1 - w_2| \asymp |w_1 - w_3|$.

b) If $|z_1 - z_2| \preceq |z_1 - z_3|$, *then*

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{\epsilon} \preceq \left|\frac{z_1 - z_3}{z_1 - z_2}\right| \preceq \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{c},$$

where $\epsilon < 1,\ c > 1,\ 0 < r_0 < 1$, *are constants, depending on* G.

Lemma 3.4. *Let* $G \in Q(\kappa)$ *for some* $0 \le \kappa < 1$. *Then*

$$|\Psi(w_1) - \Psi(w_2)| \succeq |w_1 - w_2|^{1+\kappa},$$

for all $w_1, w_2 \in \overline{\Delta}$.

This fact follows from [40, p.287, Lemma 9.9] and the estimation for the Ψ' (see, for example, [19, Th.2.8]):

$$|\Psi'(\tau)| \asymp \frac{d(\Psi(\tau), L)}{|\tau| - 1}. \tag{13}$$

Lemma 3.5. *Let* $G \in Q_\alpha$, *we have*

$$d(t, L_R) \succeq (R-1)^{\mu} \succeq n^{-\mu},$$

where

$$\mu = \begin{cases} \frac{1}{\alpha}, & \alpha \ge \frac{1}{2}, \\ \delta, & \alpha < \frac{1}{2}, \end{cases} \tag{14}$$

and $\delta = \delta(\alpha, G)$, $1 \le \delta \le 2$, *is a certain number.*

This fact follows easily from [34] and [19].

Let $\{z_j\}_{j=1}^{l}$ be a fixed the system of the points on L and the weight function $h(z)$ defined as (1).

Lemma 3.6. ([4]) *Let* L *be a rectifiable Jordan curve,* $h(z)$ *defined as in (1). Then for arbitrary* $P_n(z) \in \wp_n$, *any* $R > 1$ *and* $n \in \mathbb{N}$, *we have:*

$$\|P_n\|_{\mathcal{L}_p(h, L_R)} \preceq R^{n + \frac{1+\gamma}{p}} \|P_n\|_{\mathcal{L}_p(h, L)},\ p > 0, \tag{15}$$

where γ *is defined as in (3).*

Remark 3.7. In case of $h(z) \equiv 1$, the estimation (15) has been proved in [31].

4. Proof of theorems

Proof. [Proof of Theorems 2.5 and 2.10] First of all, we give two theorem that we will use in this case, and after than we give estimate for $\left|P_n^{(m)}(z)\right|, z \in \overline{G}$, for each $m \ge 1$.

Theorem A [6, Cor.2.3]*Let* $p > 0$; $L \in \widetilde{Q}(\kappa)$, $0 \le \kappa < 1$ *and* $h(z)$ *be defined by (1). Then for any* $P_n \in \wp_n$, $n \in N$, *there exists a constant* $c_9 = c_9(L, p, \gamma) > 0$ *such that the following is fulfilled:*

$$\|P_n\|_\infty \le c_9 n^{\frac{(1+\gamma)(1+k)}{p}} \|P_n\|_p, \tag{16}$$

where $\gamma := \max\left\{0; \gamma_j,\ j = \overline{1, l}\right\}$.

Theorem B [8, Th.2.4] *Let $p > 0$; $L \in \widetilde{Q}_\alpha$, $0 < \alpha \le 1$, and $h(z)$ be defined by (1). Then, there exists a constant $c_{10} = c_{10}(L, p, \gamma) > 0$ such that the following is fulfilled:*

$$\|P_n\|_\infty \le c_{10} \|P_n\|_p \begin{cases} n^{\frac{(1+\gamma)\delta}{p}}, & \alpha < \frac{1}{2}, \\ n^{\frac{1+\gamma}{\alpha p}}, & \alpha \ge \frac{1}{2}, \end{cases} \tag{17}$$

where $\delta = \delta(G)$, $1 \le \delta \le 2$, is a certain number.

Now, we will give estimate for $\left|P_n^{(m)}(z)\right|$, $z \in \overline{G}$, for each $m \ge 0$ and for the regions of the classes $\widetilde{Q}(\kappa)$ and $\widetilde{Q}_\alpha$, respectively. Let $z \in L$ is an arbitrary fixed point; $B(z, d(z, L_R)) := \{\xi : |\xi - z| < d(z, L_R)\}$. By the Cauchy integral formulas for mth derivatives, we have:

$$P_n^{(m)}(z) = \frac{m!}{2\pi i} \int\limits_{\partial B(z,d(z,L_R))} \frac{P_n(t)}{(t-z)^{m+1}} dt, \ m = 0, 1, 2, ...$$

Then, applying well-known Bernstein-Walsh inequality [51, p.101], we obtain:

$$\|P_n\|_{C(\overline{G}_R)} \le |\Phi(z)|^n \|P_n\|_{C(\overline{G})}, \ \forall P_n \in \wp_n; \tag{18}$$

$$\begin{aligned} \left|P_n^{(m)}(z)\right| &\le \frac{m!}{2\pi} \max_{z \in \partial B(z,d(z,L_R))} |P_n(t)| \cdot \int\limits_{\partial B(z,d(z,L_R))} \frac{|dt|}{|t-z|^{m+1}} \\ &\le \frac{m!}{2\pi} \max_{t \in \overline{G}_R} |P_n(t)| \cdot \frac{2\pi d(z, L_R)}{d^{m+1}(z, L_R)} \le \frac{\max_{t \in \overline{G}} |P_n(t)|}{d^m(z, L_R)}. \end{aligned}$$

If $p = \infty$, we get:

$$\left|P_n^{(m)}(z)\right| \le \frac{1}{d^m(z, L_R)} \|P_n\|_\infty, \forall z \in L.$$

If $0 < p < \infty$, applying Theorems A and B and using the Lemmas 3.4, 3.5, for all $\forall z \in L$, we get:

$$\begin{aligned} \left|P_n^{(m)}(z)\right| &\le n^{\frac{\gamma+1}{p}(1+\kappa)} \|P_n\|_p \cdot n^{\frac{m}{\alpha}} \le n^{\left(\frac{\gamma+1}{p}+m\right)\frac{1}{\alpha}} \|P_n\|_p; \\ \left|P_n^{(m)}(z)\right| &\le n^{\frac{\gamma+1}{p\alpha}} \|P_n\|_p \cdot n^{m(1+\kappa)} \le n^{\left(\frac{\gamma+1}{p}+m\right)(1+\kappa)} \|P_n\|_p. \end{aligned}$$

Since $z \in L$ is arbitrary, we complete the proof of Theorems 2.5, Corollary 2.6 and Theorem 2.10, Corollary 2.11. □

Proof. [Proof of Theorems 2.7 and 2.12] Suppose that $L \in \widetilde{Q}_\alpha$ for some $\frac{1}{2} \le \alpha \le 1$, $i = \overline{1, l}$, be given and $h(z)$ defined as in (1). The Cauchy integral representation for the $P_n(z)$ in G_R, $R = 1 + \frac{\varepsilon_0}{n}$, gives:

$$P_n^{(m)}\left(z_j\right) = \frac{1}{2\pi i} \int\limits_{L_R} \frac{P_n(\zeta) \, d\zeta}{\left(\zeta - z_j\right)^{m+1}}, \ z \in G_R, \ m = 0, 1, 2, ...$$

Multiplying the numerator and the determinator of the integrand by $h^{1/p}(\zeta)$, according to the Hölder inequality, we obtain:

$$\begin{aligned} \left|P_n^{(m)}\left(z_j\right)\right| &\le \left(\int\limits_{L_R} h(\zeta) |P_n(\zeta)|^p |d\zeta| \right)^{1/p} \times \left(\int\limits_{L_R} \frac{|d\zeta|}{\prod_{j=1}^{l} \left|\zeta - z_j\right|^{(q-1)\gamma_j} \left|\zeta - z_j\right|^{q(m+1)}} \right)^{1/q} \\ &=: J_{n,1} \times J_{n,2}. \end{aligned} \tag{19}$$

According Lemma 3.6 for any points $\left\{z_j\right\}_{j=1}^{l} \in L$, we have:

$$\left|P_n\left(z_j\right)\right| \leq \|P_n\|_{\mathcal{L}_p} \cdot (J_{n,2})^{1/q}. \tag{20}$$

To estimate the integral $J_{n,2}$, we introduce $w_j := \Phi(z_j)$, $\varphi_j := \arg w_j$, $j = \overline{1,l}$. Without loss of generality, we will assume that $\varphi_l < 2\pi$; $L^j := L \cap \overline{\Omega}^j$, $L_R^j := L_R \cap \overline{\Omega}^j$, $j = \overline{1,l}$, where $\Omega^j := \Psi(\Delta_j^{'})$;

$$\begin{aligned}
\Delta_1^{'} \quad &: \quad = \left\{t = Re^{i\theta} : R > 1,\ \frac{\varphi_m + \varphi_1}{2} \ \leq \theta < \frac{\varphi_1 + \varphi_2}{2}\right\},\\
\Delta_m^{'} \quad &: \quad = \left\{t = Re^{i\theta} : R > 1,\ \frac{\varphi_{m-1} + \varphi_m}{2} \ \leq \theta < \frac{\varphi_m + \varphi_1}{2}\right\},
\end{aligned}$$

and for $j = \overline{2,l}$

$$\Delta_j^{'} := \left\{t = Re^{i\theta} : R > 1,\ \frac{\varphi_{j-1} + \varphi_j}{2} \ \leq \theta < \frac{\varphi_j + \varphi_0}{2}\right\},$$

where $\varphi_0 = 2\pi - \varphi_l$.Then, we get:

$$(J_{n,2})^q = \sum_{i=1}^{l} \int\limits_{L_R^i} \frac{|d\zeta|}{\prod\limits_{j=1}^{l} \left|\zeta - z_j\right|^{(q-1)\gamma_j} \left|\zeta - z_j\right|^{q(m+1)}} \asymp \sum_{i=1}^{l} \int\limits_{L_R^i} \frac{|d\zeta|}{\left|\zeta - z_j\right|^{(q-1)\gamma_j + q(m+1)}} =: \sum_{i=1}^{l} J_{n,2}^i, \tag{21}$$

since the points $\left\{z_j\right\}_{j=1}^{l} \in L$ are distinct. For simplicity of further calculations, we will estimate only for z_1. Let for the $\delta_1, 0 < \delta_1 < \delta_0 < \frac{1}{2} diam\ \overline{G}$, denote:

$$l_{R,1}^1 := L_R^1 \cap \Omega(z_1, \delta_1), l_{R,2}^1 := L_R^1 \backslash l_{R,1}^1,\ F_{R,i}^1 := \Phi(l_{R,i}^1),\ i = 1, 2.$$

We get:

$$J_{n,2}^1 := \int\limits_{L_R^1} \frac{|d\zeta|}{|\zeta - z_1|^{(q-1)\gamma_1 + q(m+1)}} = \int\limits_{l_{R,1}^1} \frac{|d\zeta|}{|\zeta - z_1|^{(q-1)\gamma_1 + q(m+1)}} + \int\limits_{l_{R,2}^1} \frac{|d\zeta|}{|\zeta - z_1|^{(q-1)\gamma_1 + q(m+1)}}. \tag{22}$$

Applying the Lemma's 3.4 and 3.5, we have:

$$\begin{aligned}
&\int\limits_{l_{R,1}^1} \frac{|d\zeta|}{|\zeta - z_1|^{(q-1)\gamma_1 + q(m+1)}} = \int\limits_{\Phi(l_{R,1}^1)} \frac{d(\Psi(\tau), L)\,|d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{(q-1)\gamma_1 + q(m+1)}\,(|\tau| - 1)} \\
\preceq \quad & n \int\limits_{\Phi(l_{R,1}^1)} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{(q-1)\gamma_1 + q(m+1) - 1}} \preceq n \int\limits_{\Phi(l_{R,1}^1)} \frac{|d\tau|}{|\tau - w_1|^{\left[(q-1)\gamma_1 + q(m+1) - 1\right](1+\kappa)}} \\
\preceq \quad & n^{\left[(q-1)\gamma_1 + q(m+1) - 1\right](1+\kappa)};
\end{aligned} \tag{23}$$

$$\int\limits_{l_{R,2}^1} \frac{|d\zeta|}{|\zeta - z_1|^{(q-1)\gamma_1 + q(m+1)}} \leq (\delta_1)^{(q-1)\gamma_1 + q(m+1)} mes l_{R,1}^1 \preceq 1, \tag{24}$$

for $L \in \widetilde{Q}(\kappa)$, and

$$\int\limits_{l_{R,1}^1} \frac{|d\zeta|}{|\zeta - z_1|^{(q-1)\gamma_1 + q(m+1)}} \preceq n \int\limits_{\Phi(l_{R,1}^1)} \frac{|d\tau|}{|\tau - w_1|^{\frac{(q-1)\gamma_1 + q(m+1) - 1}{\alpha}}} \preceq n^{\frac{(q-1)\gamma_1 + q(m+1) - 1}{\alpha}},$$

for $L \in \widetilde{Q}_\alpha$. Then, from (22), we get:

$$J_{n,2}^1 \preceq n^{\left(\frac{\gamma_1+1}{p}+m\right)\frac{1}{\alpha}}. \tag{25}$$

Combining the relations (20)-(25), we obtain:

$$|P_n(z_1)| \preceq n^{\left(\frac{\gamma_1+1}{p}+m\right)(1+\kappa)} \|P_n\|_{\mathcal{L}_p},$$

for $L \in \widetilde{Q}(\kappa)$;

$$|P_n(z_1)| \preceq n^{\left(\frac{\gamma_1+1}{p}+m\right)\frac{1}{\alpha}} \|P_n\|_{\mathcal{L}_p},$$

for $L \in \widetilde{Q}_\alpha$, and, we complete the proof of Theorems 2.7 and 2.12. □

Proof. [Proof of Theorems 2.8 and 2.13] For $z \in \mathbb{C}$ and $p > 0$ consider the function:

$$g^{\frac{1}{p}}(z) := \prod_{j=1}^{l} \left(z - z_j\right)^{\frac{\gamma_j}{p}}, \gamma_j > -1, \ j = \overline{1, l}.$$

Since $\left\{z_j\right\}_{j=1}^{l} \in L$, then, the function $g^{\frac{1}{p}}(z)$ analytic in Ω (we take an arbitrary continuous branch of the $g^{\frac{1}{p}}(z)$ and we maintain the same designation for this branch). We have:

$$\left[g^{\frac{1}{p}}(z)\right]' = \left(\prod_{j=1}^{l} \left(z - z_j\right)^{\frac{\gamma_j}{p}}\right)' = \frac{g^{\frac{1}{p}}(z)}{p} \sum_{j=1}^{l} \frac{\gamma_j}{z - z_j};$$

$$\left[g^{\frac{1}{p}}(z)P_n(z)\right]' = \left[g^{\frac{1}{p}}(z)\right]' P_n(z) + g^{\frac{1}{p}}(z)P_n'(z)$$

and so:

$$g^{\frac{1}{p}}(z)P_n'(z) = -\left[g^{\frac{1}{p}}(z)\right]' P_n(z) + \left[g^{\frac{1}{p}}(z)P_n(z)\right]' = -\frac{g^{\frac{1}{p}}(z)P_n(z)}{p} \sum_{j=1}^{l} \frac{\gamma_j}{z - z_j} + \left[g^{\frac{1}{p}}(z)P_n(z)\right]'.$$

Therefore,

$$h(z)\left|P_n'(z)\right|^p \preceq h(z)\left|P_n(z)\right|^p \left(\sum_{j=1}^{l} \frac{1}{\left|z - z_j\right|}\right)^p + \left|\left[g^{\frac{1}{p}}(z)P_n(z)\right]'\right|^p,$$

since $h(z) = \left|g(z)\right|$.

Integrating over the L_R and using Lemma 3.6, we get:

$$\|P_n'\|_{\mathcal{L}_p(h,L_R)} \preceq \sup_{z \in L_R} \frac{1}{\left|z - z_j\right|} \|P_n\|_{\mathcal{L}_p(h,L_R)} + \left\{\int_{L_R} \left|\left[g^{\frac{1}{p}}(z)P_n(z)\right]'\right|^p |dz|\right\}^{\frac{1}{p}} \tag{26}$$

$$\begin{aligned}
&\preceq \frac{1}{d(z_j, L_R)} \|P_n\|_{\mathcal{L}_p(h,L_R)} + \left\{\int_{L_R} \left|\left[g^{\frac{1}{p}}(z)P_n(z)\right]'\right|^p |dz|\right\}^{\frac{1}{p}} \\
&\preceq \frac{1}{d(z_j, L_R)} \|P_n\|_{\mathcal{L}_p(h,L)} + \left\{\int_{L_R} \left|\left[g^{\frac{1}{p}}(z)P_n(z)\right]'\right|^p |dz|\right\}^{\frac{1}{p}} \\
&\preceq \frac{1}{d(z_j, L_R)} \|P_n\|_{\mathcal{L}_p(h,L)} + J(n) \preceq \frac{1}{d(z_j, L_R)} \|P_n\|_{\mathcal{L}_p(h,L)} + J(n),
\end{aligned}$$

where $J(n)$ defined as follows:

$$J(n) := \left\{ \int\limits_{L_R} \left| \left[g^{\frac{1}{p}}(z) P_n(z) \right]' \right|^p |dz| \right\}^{\frac{1}{p}}. \tag{27}$$

Therefore, it is necessary to calculate the integral $J(n)$. Let

$$B_j(z) := \frac{\Phi(z) - \Phi(z_j)}{1 - \overline{\Phi(z_j)}\Phi(z)}; \; B(z) := \prod_{j=1}^{l} B_j(z)$$

denote a Blaschke function [51, Ch.10] with respect to the $\{z_j\}_{j=1}^{l}$. Clearly, $B(z_j) = 0$, $|B(z)| \equiv 1$ for all $z \in L$ and $|B(z)| < 1$ for all $z \in \Omega$.

For $p > 0$, let us set:

$$\begin{aligned} G(z) \; &: = \prod_{j=1}^{l} G_j(z) := \prod_{j=1}^{l} \left(\frac{z - z_j}{B_j(z)\Phi(z)} \right)^{\frac{\gamma_j}{p}}, \gamma_j > -1, \; j = \overline{1,l}; \\ H_{n,p}(z) \; &: = G(z) \frac{P_n(z)}{\Phi^{n+1}(z)} = \left(\frac{g(z)}{B(z)\Phi(z)} \right)^{\frac{1}{p}} \frac{P_n(z)}{\Phi^{n+1}(z)}, \; z \in \Omega. \end{aligned}$$

The function $G(z)$ is analytic in Ω continuous on $\overline{\Omega}$ (we take an arbitrary continuous branch of the $G(z)$ and we maintain the same designation for this branch). The function $H_{n,p}$ is analytic in Ω, continuous on $\overline{\Omega}$. Then,

$$\begin{aligned} H'_{n,p}(z) \; &: = \left[\frac{g^{\frac{1}{p}}(z) P_n(z)}{B^{\frac{1}{p}}(z) \Phi^{n+1+\frac{1}{p}}(z)} \right]' \\ &= \left[g^{\frac{1}{p}}(z) P_n(z) \right]' \left[\frac{1}{B^{\frac{1}{p}}(z) \Phi^{n+1+\frac{1}{p}}(z)} \right] + \left[g^{\frac{1}{p}}(z) P_n(z) \right] \left[\frac{1}{B^{\frac{1}{p}}(z) \Phi^{n+1+\frac{1}{p}}(z)} \right]', \; z \in \Omega. \end{aligned}$$

Therefore

$$\left[g^{\frac{1}{p}}(z) P_n(z) \right]' = \left[B^{\frac{1}{p}}(z) \Phi^{n+1+\frac{1}{p}}(z) \right] \left\{ \left[\frac{g^{\frac{1}{p}}(z) P_n(z)}{B^{\frac{1}{p}}(z) \Phi^{n+1+\frac{1}{p}}(z)} \right]' - \left[g^{\frac{1}{p}}(z) P_n(z) \right] \left[\frac{1}{B^{\frac{1}{p}}(z) \Phi^{n+1+\frac{1}{p}}(z)} \right]' \right\}$$

and

$$\begin{aligned} \left| \left[g^{\frac{1}{p}}(z) P_n(z) \right]' \right| &= \left| \left[B^{\frac{1}{p}}(z) \Phi^{n+1+\frac{1}{p}}(z) \right] \right| \left\{ \left| \left[\frac{g^{\frac{1}{p}}(z) P_n(z)}{B^{\frac{1}{p}}(z) \Phi^{n+1+\frac{1}{p}}(z)} \right]' \right| + \left| \left[g^{\frac{1}{p}}(z) P_n(z) \right] \right| \left| \left[\frac{1}{B^{\frac{1}{p}}(z) \Phi^{n+1+\frac{1}{p}}(z)} \right]' \right| \right\} \\ &\leq \left| \left[\frac{g^{\frac{1}{p}}(z) P_n(z)}{B^{\frac{1}{p}}(z) \Phi^{n+1+\frac{1}{p}}(z)} \right]' \right| + \left| \left[g^{\frac{1}{p}}(z) P_n(z) \right] \right| \left| \left[\frac{1}{B^{\frac{1}{p}}(z) \Phi^{n+1+\frac{1}{p}}(z)} \right]' \right| \\ &= : \left| A'_n(z) \right| + \left| \left[g^{\frac{1}{p}}(z) P_n(z) \right] \right| \left| B'_n(z) \right|, \; z \in L_R, \end{aligned}$$

where

$$A_n(z) := \frac{g^{\frac{1}{p}}(z) P_n(z)}{B^{\frac{1}{p}}(z) \Phi^{n+1+\frac{1}{p}}(z)}; \; B_n(z) := \frac{1}{B^{\frac{1}{p}}(z) \Phi^{n+1+\frac{1}{p}}(z)},$$

and, so,

$$\left|\left[g^{\frac{1}{p}}(z)P_n(z)\right]'\right|^p \leq \left|A_n'(z)\right|^p + \left|\left[g^{\frac{1}{p}}(z)P_n(z)\right]\right|^p \left|B_n'(z)\right|^p, \ z \in L_R. \tag{28}$$

The function $A_n(z)$ is analytic in Ω and is continuous on $\overline{\Omega}$, Cauchy integral representations for $A_n(z)$ at the all points $z \in \Omega$, we have:

$$A_n'(z) = -\frac{1}{2\pi i}\int\limits_L \frac{A_n(\zeta)}{(\zeta - z)^2}d\zeta, \ z \in \Omega.$$

For $z \in L_R$, we get:

$$\begin{aligned}\left|g^{\frac{1}{p}}(z)P_n(z)\right| &\leq \frac{1}{2\pi}\left|B^{\frac{1}{p}}(z)\Phi^{n+1+\frac{1}{p}}(z)\right|\int\limits_L \left|\frac{g^{\frac{1}{p}}(\zeta)P_n(\zeta)}{B^{\frac{1}{p}}(\zeta)\Phi^{n+1+\frac{1}{p}}(\zeta)}\right|\frac{|d\zeta|}{|\zeta - z|^2} \\ &\leq \left(\int\limits_L h(\zeta)\left|P_n(\zeta)\right|^p |d\zeta|\right)^{\frac{1}{p}}\left(\int\limits_L \frac{|d\zeta|}{|\zeta - z|^{2q}}\right)^{\frac{1}{q}} = \|P_n\|^p_{\mathcal{L}_p(h,L)}\left(\int\limits_L \frac{|d\zeta|}{|(\zeta - z|^{2q}}\right)^{\frac{1}{q}}, \ \frac{1}{p}+\frac{1}{q} = 1.\end{aligned}$$

$$\left|g^{\frac{1}{p}}(z)P_n(z)\right|^p \leq \|P_n\|^p_{\mathcal{L}_p(h,L)}\left(\int\limits_L \frac{|d\zeta|}{|\zeta - z|^{2q}}\right)^{\frac{p}{q}}.$$

Integrating the last inequality over the curve L_R, we have:

$$\begin{aligned}\int\limits_{L_R}\left|g^{\frac{1}{p}}(z)P_n(z)\right|^p |dz| &\leq \|P_n\|^p_{\mathcal{L}_p(h,L)}\int\limits_{L_R}\left(\int\limits_L \frac{|d\zeta|}{|\zeta - z|^{2q}}\right)^{\frac{p}{q}}|dz| \\ &\leq \|P_n\|^p_{\mathcal{L}_p(h,L)}\int\limits_{L_R}\frac{|dz|}{\inf_{\zeta\in L}|\zeta - z|^{(2q-1)\frac{p}{q}}} \leq \|P_n\|^p_{\mathcal{L}_p(h,L)}\frac{1}{\inf_{\zeta\in L}|\zeta - z|^{(2q-1)\frac{p}{q}-1}}.\end{aligned}$$

So,

$$\|P_n\|_{\mathcal{L}_p(h,L_R)} \leq \|P_n\|_{\mathcal{L}_p(h,L)}\frac{1}{\inf_{\zeta\in L} d^{\left[(2q-1)\frac{p}{q}-1\right]\frac{1}{p}}(\zeta, L_R)} = \|P_n\|_{\mathcal{L}_p(h,L)}\frac{1}{d(L, L_R)}. \tag{29}$$

Let us estimate $\left|B_n'(z)\right|$. Cauchy integral representations for $B_n'(z)$, $z \in \Omega$, gives:

$$B_n'(z) = -\frac{1}{2\pi i}\int\limits_L \frac{1}{B^{\frac{1}{p}}(\zeta)\Phi^{n+1+\frac{1}{p}}(\zeta)}\frac{d\zeta}{(\zeta - z)^2}, \ z \in \Omega.$$

So, for $z \in L_R$, we have:

$$\left|B_n'(z)\right| \leq \int\limits_L \frac{1}{\left|B^{\frac{1}{p}}(\zeta)\Phi^{n+1+\frac{1}{p}}(\zeta)\right|}\frac{|d\zeta|}{|\zeta - z|^2} \leq \int\limits_L \frac{|d\zeta|}{|\zeta - z|^2} \leq \frac{1}{d(L, L_R)}. \tag{30}$$

Now, integrating the (28) over the curve L_R, according Lemma 3.6, (29) and (30), we obtain:

$$\begin{aligned}\int\limits_{L_R}\left|\left[g^{\frac{1}{p}}(z)P_n(z)\right]'\right|^p |dz| &\leq \int\limits_{L_R}\left|A_n'(z)\right|^p |dz| + \int\limits_{L_R}\left|g^{\frac{1}{p}}(z)P_n(z)\right|^p \left|B_n'(z)\right|^p |dz| \\ &\leq \|P_n\|^p_{\mathcal{L}_p(h,L)}\frac{1}{d^p(L, L_R)} + \frac{1}{d^p(L, L_R)}\|P_n\|^p_{\mathcal{L}_p(h,L_R)} \\ &\leq \|P_n\|^p_{\mathcal{L}_p(h,L)}\frac{1}{d^p(L, L_R)};\end{aligned} \tag{31}$$

$$\left(\int\limits_{L_R}\left|\left[g^{\frac{1}{p}}(z)P_n(z)\right]'\right|^p|dz|\right)^{\frac{1}{p}}\preceq\frac{1}{d(L,L_R)}\|P_n\|_{\mathcal{L}_p(h,L)}.$$

Therefore, from (26), we get:

$$\|P_n'\|_{\mathcal{L}_p(h,L_R)}\preceq\frac{1}{d(z_j,L_R)}\|P_n\|_{\mathcal{L}_p(h,L)}+\frac{1}{d(L,L_R)}\|P_n\|_{\mathcal{L}_p(h,L)}=\|P_n\|_{\mathcal{L}_p(h,L)}\left[\frac{1}{d(z_j,L_R)}+\frac{1}{d(L,L_R)}\right].$$

Combining with Lemmas 3.4 and 3.5, respectively, we complete the proofs of Theorems 2.8 and 2.13.

Therefore, we proved that (6) and (11) hold for $m=1$. We verify that they also hold for each $m\geq 2$. Suppose that it hold for some $m=s\geq 2$, as following:

$$\|P_n^{(s)}\|_{\mathcal{L}_p(h,L_R)}\preceq\frac{1}{[d(L,L_R)]^s}\|P_n\|_p,\ z\in\overline{G}. \tag{32}$$

We show that it also holds for $m=s+1$. Then, after re-applying of the estimate (32), we have:

$$\begin{aligned}\|P_n^{(s+1)}\|_{\mathcal{L}_p(h,L_R)} &= \|\left[P_n^{(s)}\right]'\|_{\mathcal{L}_p(h,L_R)}\preceq\frac{1}{d(L,L_R)}\|P_n^{(s)}\|_p\\ &= \frac{1}{d(L,L_R)}\|\left[P_n^{(s-1)}\right]'\|_p\preceq\frac{1}{[d(L,L_R)]^2}\|P_n^{(s-1)}\|_p\\ &\preceq \ldots\preceq\frac{1}{[d(L,L_R)]^{s+1}}\|P_n\|_p.\end{aligned}$$

Thus, by the method of induction, we can claim that what estimate is true for any $m=1,2,\ldots$:

$$\|P_n^{(m)}\|_{\mathcal{L}_p(h,L_R)}\preceq\frac{1}{[d(L,L_R)]^m}\|P_n\|_p,\ z\in\overline{G}.$$

Now, if G be a $k-$quasidisk, then according to Lemma 3.4, we have:

$$\|P_n^{(m)}\|_{\mathcal{L}_p(h,L_R)}\preceq n^{m(1+k)}\|P_n\|_p,$$

and if $G\in Q_\alpha$, then according to Lemma 3.5, we have:

$$\|P_n^{(m)}\|_{\mathcal{L}_p(h,L_R)}\preceq n^{\mu(1+k)}\|P_n\|_p,$$

and we complete the proofs. □

Proof. [Proofs of Theorems 2.9 and 2.14] A simple calculation gives:

$$\|P_n^{(m)}\|_q=\left(\int\limits_L h(z)\left|P_n^{(m)}(z)\right|^q|dz|\right)^{1/q}=\left(\int\limits_L\left|P_n^{(m)}(z)\right|^{q-p}h(z)\left|P_n^{(m)}(z)\right|^p|dz|\right)^{1/q} \tag{33}$$

$$\leq\max_{z\in L}\left|P_n^{(m)}(z)\right|^{1-\frac{p}{q}}\left(\int\limits_L h(z)\left|P_n^{(m)}(z)\right|^p|dz|\right)^{1/q}=\left\|P_n^{(m)}\right\|_\infty^{1-\frac{p}{q}}\|P_n^{(m)}\|_p^{\frac{p}{q}}.$$

Let $T_N(z):=P_n^{(m)}(z)$, $\deg T_N=N\leq n-m$. Using Theorem 2.5 and Theorem 2.10, for $m=0$, $0\leq\kappa<1$, we have:

$$\|T_N\|_\infty\leq c_1N^{\frac{1+\gamma}{p}(1+k)}\|T_N\|_p,$$

and

$$\|T_N\|_\infty \le c_1 N^{\frac{1+\gamma}{p}\mu}\|T_N\|_p,$$

where μ defined as in (14). Consequently,

$$\begin{aligned} \|P_n^{(m)}\|_\infty &\le (n-m)^{\frac{1+\gamma}{p}(1+k)}\|P_n^{(m)}\|_p \le n^{\frac{1+\gamma}{p}(1+k)}\|P_n^{(m)}\|_p; \\ \|P_n^{(m)}\|_\infty &\le (n-m)^{\frac{1+\gamma}{p}\mu}\|P_n^{(m)}\|_p \le n^{\frac{1+\gamma}{p}\mu}\|P_n^{(m)}\|_p, \end{aligned} \tag{34}$$

respectively. Now, combining (33) and (34), we get:

$$\begin{aligned} \|P_n^{(m)}\|_q &\le n^{\frac{1+\gamma}{p}(1-\frac{p}{q})(1+k)}\|P_n^{(m)}\|_p^{(1-\frac{p}{q})}\|P_n^{(m)}\|_p^{\frac{p}{q}} \le n^{(\frac{1}{p}-\frac{1}{q})(1+\gamma)(1+k)}\|P_n^{(m)}\|_p; \\ \|P_n^{(m)}\|_q &\le n^{\frac{1+\gamma}{p}(1-\frac{p}{q})\mu}\|P_n^{(m)}\|_p^{(1-\frac{p}{q})}\|P_n^{(m)}\|_p^{\frac{p}{q}} \le n^{(\frac{1}{p}-\frac{1}{q})(1+\gamma)\mu}\|P_n^{(m)}\|_p. \end{aligned}$$

Thus, we completed the proofs. □

Proof. [*Proof of Remark 2.16*] Sharpness of the inequalities (4) and (8) can be argued as follows. These inequalities can be interpreted as a combination of the well-known sharp Markov inequalities $\left\|P_n^{(m)}\right\|_\infty \le n^m \|P_n\|_\infty$, $m \ge 1$, with inequalities (16) and (17), respectively. And the sharpness of the last inequalities can be verified to the following examples: For the polynomial $T_n(z) = 1 + z + ... + z^n$, $h^*(z) = h_0(z)$, $h^{**}(z) = |z-1|^\gamma$, $\gamma > 0$, $L := \{z : |z| = 1\}$ and any $n \in \mathbb{N}$ there exist $c_3 = c_3(h^*, p) > 0$, $c_3' = c_3'(h^{**}, p) > 0$ such that:

$$\begin{aligned} a)\ \|T\|_\infty &\ge c_3 n^{\frac{1}{p}}\|T\|_{\mathcal{L}_p(h^*,\ L)},\ \ p > 1; \\ b)\ \|T\|_\infty &\ge c_3' n^{\frac{\gamma+1}{p}}\|T\|_{\mathcal{L}_p(h^{**},\ L)},\ \ p > \gamma + 1. \end{aligned}$$

Really, if $L := \{z : |z| = 1\}$,then, $L \in \widetilde{Q}(0)$ and $L \in \widetilde{Q}_1$.
a) $h^*(z) \equiv 1$; b) $h^{**}(z) = |z-1|^\gamma$, $\gamma > 0$.
Obviously,

$$|T(z)| \le \sum_{j=0}^{n-1}\left|z^j\right| = n,\ \ |z| = 1;\ \ |T(1)| = n.$$

So,

$$\|T\|_{\mathcal{L}_\infty} = n.$$

On the other hand, according to [46, p. 236], we have:

$$\|T\|_{\mathcal{L}_p(h^*,L)} \asymp n^{1-\frac{1}{p}},\ p > 1,$$

and

$$\|T\|_{\mathcal{L}_p(h^{**},L)} \asymp n^{1-\frac{\gamma+1}{p}},\ \ \ p > \gamma + 1.$$

Therefore,

$$\begin{aligned} a)\ \|T\|_{\mathcal{L}_\infty} &= n \asymp n^{\frac{1}{p}}\|T\|_{\mathcal{L}_p(h^*,L)},\ p > 1; \\ b)\ \|T\|_{\mathcal{L}_\infty} &= n = n \cdot n^{1-\frac{\gamma+1}{p}} \cdot n^{\frac{\gamma+1}{p}-1} \asymp n^{\frac{\gamma+1}{p}}\|T\|_{\mathcal{L}_p(h^{**},L)},\ p > \gamma + 1. \end{aligned}$$

□

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