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L-(resp. concave) down-directed convergence relation spaces and L-(resp. concave) filter convergence spaces

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Abstract. Convergence structure and relation are useful tools in interpreting many mathematical structures such as topological spaces and convex spaces. The aim of this paper is to study convergence structures in the framework of L-concave spaces by using relations. Specifically, the notion of L-down–directed relations is introduced and some simple examples are presented. Based on this, notions of L-down–directed convergence relation spaces and L-concave down–directed convergence relations are introduced. It is proved that the category of L-concave internal relation spaces can be embedded into the category of L-down–directed convergence relation spaces as a reflective subcategory. In addition, the category of L-concave down–directed convergence relation spaces is isomorphic to the category of L-concave internal relation spaces.

In order to characterize L-down–directed convergence relation space and L-concave down–directed convergence relation space, notions of L-concave filters, L-filter convergence spaces and L-concave filter convergence spaces are introduced. It is showed that the category of L-down–directed convergence relation spaces is isomorphic to the category of L-filter convergence spaces. It also showed that the category of L-concave down–directed convergence relation spaces is isomorphic to the category of L-concave filter convergence spaces and the category of L-concave spaces.

1. Introduction

In an abstract convex space, a convex structure on a nonempty set is a family of subsets containing the empty set and the underling set and is closed under arbitrary intersections and nested unions. Its theory is called the abstract convex theory which involves many mathematical structures such as lattice, graph, median algebra, metric space, poset and vector space [19].

Convex structure has been extended into fuzzy settings by many ways. Maruyama introduced L-fuzzy convex structure [6] which has being studied by many scholars [8, 12, 22, 26, 28, 34]. Also, Shi and Xiu introduced M-fuzzifying convex structures [13]. Many subsequent studies have been done [10, 20, 21, 29]. Later, Shi and Xiu introduced (L, M)-fuzzy convex structure which is a unified form of L-convex structure and M-fuzzifying convex structure [14]. It characterizations have been studied recently [11, 23, 24]. Now,

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these fuzzy forms of convex structures have being applied to many fuzzy mathematical structures such as fuzzy topology [5, 16, 20, 24, 27], fuzzy convergence [4, 7, 9, 10] and fuzzy matroid [15, 21, 29].

Relation is a useful tool to characterize fuzzy mathematical structures. In L-setting, Shi et al introduced L-topological internal relation and L-topological enclosed relation by which they characterized L-topologies [17]. Also, Liao et al introduced L-convex enclosed relation and characterized L-convex structures. Based on this, they introduced L-topological-convex enclosed relation and characterized L-topological-convex structure [5]. In (L, M)-fuzzy setting, Shi et al introduced (L, M)-fuzzy topological internal relation and (L, M)-fuzzy topological enclosed relation which are used to characterize (L, M)-fuzzy topologies [18, 25]. Wu et al introduced (L, M)-fuzzy convex enclosed relation and characterized (L, M)-fuzzy convex structures. Meanwhile, they introduced (L, M)-fuzzy topological-convex enclosed relation and obtained some characterizations of (L, M)-fuzzy topological-convex structures [24].

Convergence structures constructed by either filters or ideals are often used in interpreting topologies or convexities. To interpret fuzzy topologies, Güloğlu defined I-fuzzy convergence structure and discuss its relations with I-fuzzy topology [1]. Höhle and Šostak defined stratified L-filters and developed a direct way to constructing fuzzy convergence structures [2]. Jäger introduced stratified L-fuzzy convergence structures by using stratified L-filters and established categorical relations between stratified L-fuzzy convergence structures and stratified L-topologies [3]. Yao introduced L-fuzzifying convergence structure by L-filters and showed that L-fuzzifying convergence structures and L-fuzzifying topologies are categorically isomorphic [30]. Later, Pang further discussed categorical properties of L-fuzzifying convergence structures [9]. Also, Pang introduce (L, M)-fuzzy convergence structures by (L, M)-fuzzy filters and characterized (L, M)-fuzzy topologies [7]. To interpret fuzzy convexities, Pang introduced L-fuzzifying convex convergence structures by L-fuzzifying filters and established its relations with L-fuzzifying convexities [10]. Xiu and Pang introduced L-convex convergence structures by convex ideals and discussed its relations with L-convexities [26]. Recently, Zhang and Pang studied convergence structures via residuated lattices [31–33].

As being described above, most of discussions on fuzzy convergence structures are focused on fuzzy topological spaces. In addition, L-convergence structures in L-topological spaces or L-convex spaces are constructed by either L-filters or L-convex ideals. Then, how to interpreted L-filters in terms of relations in the framework of L-concave internal relation spaces? Further, how to construct L-convergence structures in L-concave internal relation spaces? Motivated by these problems, we present this paper. The arrangement of this paper is as follows. In Section 2, we recall some basic concepts, denotations and results related to L-concave spaces. In Section 3, we introduce notions of L-down–directed relations, L-down–directed convergence relation spaces and L-concave down–directed convergence relation spaces. We find that the category of L-concave down–directed convergence relation spaces is isomorphic to the category of L-concave internal relation spaces. In Section 4, we introduce notions of L-concave filters, L-filter convergence spaces and L-concave filter convergence spaces. We prove that the category of L-filter convergence spaces is isomorphic to the category of L-down–directed convergence relation spaces and that categories of L-concave down–directed convergence relation spaces, L-concave filter convergence spaces and L-concave spaces are all categorically isomorphic.

2. Preliminaries

In this paper, X and Y are nonempty sets. The power set of X is denoted by 2^X. L is a completely distributive lattice. The smallest (resp. largest) element in L is denoted by $\perp$ (resp. $\top$). An element $a \in L$ is called a co-prime element, if for all $b, c \in L$, $a \leq b \vee c$ implies $a \leq b$ or $a \leq c$. The set of all co-primes in $L \backslash \{\perp\}$ is denoted by $J(L)$. For any $a \in L$, there is an $L_1 \subseteq J(L)$ such that $a = \bigvee_{b \in L_1} b$. A binary relation $\prec$ on L is defined by $a \prec b$ if ad only if for each $L_1 \subseteq L$, $b \leq \bigvee L_1$ implies some $d \in L_1$ with $a \leq d$. The mapping $\beta : L \longrightarrow 2^L$, defined by $\beta(a) = \{b : b \prec a\}$, satisfies $\beta(\bigvee_{i \in I} a_i) = \bigcup_{i \in I} \beta(a_i)$ for any $\{a_i\}_{i \in I} \subseteq L$. For any $a \in L$, we denote $\beta^*(a) = \beta(a) \cap J(L)$. We have $a = \bigvee \beta(a) = \bigvee \beta^*(a)$, $\beta(a) = \bigcup_{b \in \beta^*(a)} \beta(b)$ and $\beta^*(a) = \bigcup_{b \in \beta^*(a)} \beta^*(b)$ [16].

An L-fuzzy set on X is a mapping $A : X \longrightarrow L$. The set of all L-fuzzy sets on X is denoted by L^X. The smallest (resp, largest) element in L^X is denoted by $\underline{\perp}$ (resp. $\underline{\top}$). A subset $\{A_i\}_{i \in I} \subseteq L^X$ is said to be down-directed, if for all $i, j \in I$ there is an index $k \in I$ such that $A_k \leq A_i \wedge A_j$. In this case, $\{A_i\}_{i \in I} \subseteq L^X$ and $\bigwedge_{i \in I} A_i$ are respectively denoted by $\{A_i\}_{i\in I}^{ddir} \subseteq L^X$ and $\bigwedge_{i\in I}^{ddir} A_i$. An L-fuzzy point x_λ ($\lambda \in L\backslash\{\perp\}$) is an L-fuzzy set defined by $x_\lambda(x) = \lambda$ and $x_\lambda(y) = \perp$ for any $y \in X\backslash\{x\}$. The set of all L-fuzzy points on L^X is denoted by $Pt(L^X)$. We also denote $J(L^X) = \{x_\lambda \in Pt(L^X) : \lambda \in J(L)\}$. For a mapping $f : X \longrightarrow Y$, the mapping $f_L^{\rightarrow} : L^X \longrightarrow L^Y$ is defined by $f_L^{\rightarrow}(A)(y) = \bigvee\{A(x) : f(x) = y\}$ for $A \in L^X$ and $y \in Y$, and the mapping $f_L^{\leftarrow} : L^Y \longrightarrow L^X$ is defined by $f_L^{\leftarrow}(B)(x) = B(f(x))$ for $B \in L^Y$ and $x \in X$ [16].

Definition 2.1. ([8]) A set $\mathcal{A} \subseteq L^X$ is called an L-concave structure on X and $(X, \mathcal{A})$ is called an L-concave space if

(LCA1) $\underline{\top}, \underline{\perp} \in \mathcal{A}$;

(LCA2) $\forall\{A_i\}_{i\in I} \subseteq \mathcal{A}$, $\bigvee_{i\in I} A_i \in \mathcal{A}$;

(LCA3) $\forall\{A_i\}_{i\in I}^{ddir} \in \mathcal{A}$, $\bigwedge_{i\in I}^{ddir} \in \mathcal{A}$.

Theorem 2.2. ([8]) *Let $(X, \mathcal{A})$ be an L-concave space. The L-concave hull operator $ca_{\mathcal{A}} : L^X \longrightarrow L^X$ of $\mathcal{A}$ is defined by $ca_C(A) = \bigvee\{B \in \mathcal{A} : B \leq A\}$ for any $A \in L^X$. It satisfies*

(LCAH1) $ca_{\mathcal{A}}(\underline{\top}) = \underline{\top}$;

(LCAH2) $ca_{\mathcal{A}}(A) \leq A$;

(LCAH3) $ca_{\mathcal{A}}(ca_{\mathcal{A}}(A)) = ca_{\mathcal{A}}(A)$;

(LCAH4) $ca_{\mathcal{A}}(\bigwedge_{i\in I}^{ddir} A_i) = \bigwedge ca_{\mathcal{A}}(A_i)$.

Conversely, if an operator $ca : L^X \longrightarrow L^X$ satisfies (LCAH1)–(LCAH4), then the set $\mathcal{A}_{ca} = \{A \in L^X : ca(A) = A\}$ is an L-concave hull operator satisfying $ca_{\mathcal{A}_{ca}} = ca$.

Let $(X, \mathcal{A}_X)$ and $(Y, \mathcal{A}_Y)$ be L-concave spaces. A mapping $f : X \longrightarrow Y$ is called an L-concavity preserving mapping, if $A \in \mathcal{A}_Y$ implies $f_L^{\leftarrow}(A) \in \mathcal{A}_X$ for any $A \in L^Y$. The category of L-concave spaces and L-concavity preserving mappings is denoted by L-**CAS** [8].

Definition 2.3. ([8]) A family $\mathcal{N} = \{\mathcal{N}_{x_\lambda} \subseteq L^X : x_\lambda \in J(L^X)\}$ is called an L-concave neighborhood system on L^X and the pair $(X, \mathcal{N})$ is called an L-concave neighborhood space, if for any $x_\lambda \in J(L^X)$,

(LCAN1) $\underline{\top} \in \mathcal{N}_{x_\lambda}$ and $\underline{\perp} \notin \mathcal{N}_{x_\lambda}$;

(LCAN2) $A \in \mathcal{N}_{x_\lambda}$ implies $x_\lambda \leq A$;

(LCAN3) $A \in \mathcal{N}_{x_\lambda}$ implies a set $B \in \mathcal{N}_{x_\lambda}$ such that $B \in \mathcal{N}_{y_\mu}$ for any $y_\mu \in \beta^*(B)$;

(LCAN4) $\bigwedge_{i\in I}^{ddir} A_i \in \mathcal{N}_{x_\lambda}$ if and only if $A_i \in \mathcal{N}_{x_\lambda}$ for any $i \in I$.

Let $(X, \mathcal{N}_X)$ and $(Y, \mathcal{N}_Y)$ be L-concave neighborhood spaces. A mapping $f : X \longrightarrow Y$ is called an L-concave neighborhood preserving mapping if $B \in \mathcal{N}_{f_L^{\rightarrow}(x_\lambda)}$ implies $f_L^{\leftarrow}(B) \in \mathcal{N}_{x_\lambda}$ for all $x_\lambda \in J(L^X)$ and $B \in L^Y$. The category of L-concave neighborhood spaces and L-concave neighborhood preserving mappings is denoted by L-**CANS** [8].

Theorem 2.4. ([8]) *(1) For an L-concave space $(X, \mathcal{A})$ and any $x_\lambda \in J(L^X)$, the set $\mathcal{N} = \{\mathcal{N}_{x_\lambda}^{\mathcal{A}} : x_\lambda \in J(L^X)\}$ is an L-concave neighborhood system, where $\mathcal{N}_{x_\lambda}^{\mathcal{A}} = \{B \in L^X : \exists A \in \mathcal{A},\ x_\lambda \leq A \leq B\}$.*

(2) For an L-concave neighborhood system $\mathcal{N} = \{\mathcal{N}_{x_\lambda} : x_\lambda \in J(L^X)\}$, the set $\mathcal{A}_{\mathcal{N}} = \{A \in L^X : \forall x_\lambda \in \beta^*(A),\ A \in \mathcal{N}_{x_\lambda}\}$ is an L-concave structure on X.

(3) $\mathcal{N}_{\mathcal{A}_{\mathcal{N}}} = \mathcal{N}$ and $\mathcal{A}_{\mathcal{N}_{\mathcal{A}}} = \mathcal{A}$.

*(4) The category L-**CANS** is isomorphic to the category L-**CAS**.*

Definition 2.5. ([22]) A binary relation $\preccurlyeq$ on L^X is called an L-concave internal relation and the pair $(X, \preccurlyeq)$ is called an L-concave internal relation space, if $\preccurlyeq$ satisfies

(LCIR1) $\underline{\top} \preccurlyeq \underline{\top}$;

(LCIR2) $A \preccurlyeq B$ implies $A \leq B$;

(LCIR3) $\bigvee_{i\in I} A_i \preccurlyeq B$ if and only if $A_i \preccurlyeq B$ for all $i \in I$;

(LCIR4) $A \preccurlyeq B$ implies a set $C \in L^X$ such that $A \preccurlyeq C \preccurlyeq B$;

(LCIR5) $A \preccurlyeq \bigwedge_{i\in I}^{ddir} B_i$ if and only if $A \preccurlyeq B_i$ for any $i \in I$.

Let $(X, \preccurlyeq_X)$ and $(Y, \preccurlyeq_Y)$ be L-concave internal relation spaces. A mapping $f : X \longrightarrow Y$ is called an L-concave internal relation preserving mapping, if $A \preccurlyeq_Y B$ implies $f_L^{\leftarrow}(A) \preccurlyeq_X f_L^{\leftarrow}(B)$ for all $A, B \in L^Y$. The category of L-concave internal relation spaces and L-concave internal relation preserving mappings is denoted by L-**CIRS** [22].

Theorem 2.6. ([22]) *(1) For an L-concave internal relation space $(X, \preccurlyeq)$, the operator $ca_{\preccurlyeq} : L^X \longrightarrow L^X$, defined by $ca_{\preccurlyeq}(A) = \bigvee\{B \in L^X : B \preccurlyeq A\}$ for any $A \in L^X$, is an L-concave hull operator of an L-concave structure $\mathcal{A}_{\preccurlyeq}$.*

(2) For an L-concave space $(X, \mathcal{A})$, the binary relation $\preccurlyeq_{\mathcal{A}}$, defined by $A \preccurlyeq_{\mathcal{A}} B$ if and only if $A \leq ca_{\mathcal{A}}(B)$ for any $A, B \in L^X$, is an L-concave internal relation.

(3) $\mathcal{A}_{\preccurlyeq_{\mathcal{A}}} = \mathcal{A}$ and $\preccurlyeq_{\mathcal{A}_{\preccurlyeq}} = \preccurlyeq$.

*(4) The category L-**CAS** is isomorphic to the category L-**CIRS**.*

3. *L*-down–directed relation spaces and *L*-(resp. concave) down–directed convergence relation spaces

In this section, we define the notion of L-down–directed relations and present some of examples. Based on this, we further introduce notions of L-filter convergence relation spaces and L-concave filter convergence relation spaces. Then we study their relations with L-concave internal relation spaces.

Definition 3.1. A binary operator $\preccurlyeq$ on L^X is called an L-down–directed relation and the pair $(X, \preccurlyeq)$ is called an L-down–directed relation space, if for any $A, B, C \in L^X$,

(LDDR1) $\underline{\perp} \not\preccurlyeq \underline{\perp}$ and $\underline{\top} \preccurlyeq \underline{\top}$;

(LDDR2) $A \preccurlyeq B$ if and only if $A \leq B \preccurlyeq B$;

(LDDR3) $A \preccurlyeq \bigwedge_{i \in I}^{ddir} C_i$ if and only if $A \preccurlyeq C_i$ for all $i \in I$.

In the sequel, the set of any L-down–directed relations on L^X will be denoted by $\mathcal{R}^{ddir}(L^X)$. For $\preccurlyeq_1, \preccurlyeq_2 \in \mathcal{R}^{ddir}(L^X)$, $\preccurlyeq_1$ is coarser that $\preccurlyeq_2$, denoted by $\preccurlyeq_1 \leq \preccurlyeq_2$ provided that $A \preccurlyeq_1 B$ implies $A \preccurlyeq_2 B$ for all $A, B \in L^X$.

For examples L-down–directed relations, we provide some L-down–directed relations via an L-fuzzy point $x_\lambda \in J(L^X)$ by the following proposition.

Proposition 3.2. *(1) Let $x_\lambda \in J(L^X)$. Define a binary relation $\preccurlyeq_{x_\lambda}$ on L^X by*

$$\forall A, B \in L^X, \ \ A \preccurlyeq_{x_\lambda} B \iff x_\lambda \vee A \leq B.$$

Then $\preccurlyeq_{x_\lambda}$ is an L-down–directed relation.

(2) Let $(X, \preccurlyeq)$ be an L-concave internal relation space and $x_\lambda \in J(L^X)$. Define a binary relation $\preccurlyeq_{x_\lambda}^{\preccurlyeq}$ on L^X by

$$\forall A, B \in L^X, \ \ A \preccurlyeq_{x_\lambda}^{\preccurlyeq} B \iff \exists D \in L^X \ \ s.t. \ \ x_\lambda \leq D \preccurlyeq D \leq D \vee A \leq B.$$

Then $\preccurlyeq_{x_\lambda}^{\preccurlyeq}$ is an L-down–directed relation. In addition,

(i) $\preccurlyeq_{x_\lambda}^{\preccurlyeq} \leq \preccurlyeq_{x_\lambda} = \preccurlyeq_{x_\lambda}^{\preccurlyeq_{fin}}$, where $\preccurlyeq_{fin}$, defined by $A \preccurlyeq_{fin} B$ if and only if $A \leq B$ for any $A, B \in L^X$, is the finest L-concave internal relation on L^X;

(ii) $A \preccurlyeq_{x_\lambda}^{\preccurlyeq} B$ if and only if $A \preccurlyeq_{x_\mu}^{\preccurlyeq} B$ for any $\mu \in \beta^*(\lambda)$.

(3) Let $(X, \mathcal{A})$ be an L-concave space and let $\mathcal{N}_{x_\lambda}^{\mathcal{A}}$ be the L-concave neighborhood system of $x_\lambda \in J(L^X)$. Define a binary relation $\preccurlyeq_{x_\lambda}^{\mathcal{A}}$ on L^X by

$$\forall A, B \in L^X, \ \ A \preccurlyeq_{x_\lambda}^{\mathcal{A}} B \iff \exists C \in \mathcal{N}_{x_\lambda}^{\mathcal{A}} \ \ s.t. \ \ A \leq C \leq B.$$

Then $\preccurlyeq_{x_\lambda}^{\mathcal{A}}$ is an L-down–directed relation satisfying $\preccurlyeq_{x_\lambda}^{\mathcal{A}} \leq \preccurlyeq_{x_\lambda}$.

Proof. (1) (LDDR1) is trivial. We check that (LDDR2) and (LDDR3) hold for $\preccurlyeq_{x_\lambda}$.

(LDDR2) For any $A, B \in L^X$,

$$A \preccurlyeq_{x_\lambda} B \iff x_\lambda \vee A \leq B \iff A \leq B = B \vee x_\lambda \iff A \leq B \preccurlyeq_{x_\lambda} B.$$

(LDDR3) For any $A \in L^X$ and $\{B_i\}_{i\in I}^{ddir} \subseteq L^X$,

$$A \leqslant_{x_\lambda} \bigwedge_{i\in I}^{ddir} B_i \iff x_\lambda \vee A \leq \bigwedge_{i\in I}^{ddir} B_i \iff \forall i \in I,\ x_\lambda \vee A \leq B_i \iff \forall i \in I,\ A \leqslant_{x_\lambda} B_i.$$

Therefore $\leqslant_{x_\lambda}$ is an L-down–directed relation.

(2) (LDDR1) $\perp \not\leqslant_{x_\lambda}^{\preccurlyeq} \perp$ holds trivially. Also, $\top \leqslant_{x_\lambda}^{\preccurlyeq} \top$ since $x_\lambda \leq \top \preccurlyeq \top$ by (LCIR1).

(LDDR2) For any $A, B \in L^X$,

$$\begin{aligned} A \leqslant_{x_\lambda}^{\preccurlyeq} B &\iff \exists D \in L^X \text{ s.t. } x_\lambda \leq D \preccurlyeq D \leq D \vee A \leq B \\ &\iff \exists D \in L^X \text{ s.t. } x_\lambda \leq D \preccurlyeq D \leq D \vee B \leq B \text{ and } A \leq B \\ &\iff A \leq B \leqslant_{x_\lambda}^{\preccurlyeq} B. \end{aligned}$$

(LDDR3) If $A \leq C \leqslant_{x_\lambda}^{\preccurlyeq} D \leq B$, then it is easy to check that $A \leqslant_{x_\lambda}^{\preccurlyeq} B$.

Let $A \in L^X$ and let $\{B_i\}_{i\in I}^{ddir} \subseteq L^X$. It is clear that $A \leqslant_{x_\lambda}^{\preccurlyeq} \bigwedge_{i\in I}^{ddir} B_i$ implies that $A \leqslant_{x_\lambda}^{\preccurlyeq} B_i$ for any $i \in I$.

Conversely, assume that $A \leqslant_{x_\lambda}^{\preccurlyeq} B_i$ for any $i \in I$. For any $i \in I$, there is a set $D_i \in L^X$ such that $x_\lambda \leq D_i \preccurlyeq D_i \leq D_i \vee A \leq B_i$. Further, for each $i \in I$, let

$$\psi_i = \{C_i \in L^X : x_\lambda \leq C_i \preccurlyeq C_i \leq C_i \vee A \leq B_i\}.$$

Clearly, $D_i \in \psi_i$. Let $E_i = \bigvee \psi_i$. Next, we check that $\{E_i\}_{i\in I} \subseteq L^X$ is down-directed.

Let $i, j \in I$. Since the set $\{B_i\}_{i\in I}$ is down-directed, there is an index $k \in I$ such that $B_k \leq B_i \wedge B_j$. For any $C_k \in \psi_k$, it is clear that $x_\lambda \leq C_k \preccurlyeq C_k \leq C_k \vee A \leq B_k \leq B_i \wedge B_j$. Thus $C_k \in \psi_i \cap \psi_j$ which implies that $E_k \leq E_i \wedge E_j$. So $\{E_i\}_{i\in I}$ is down-directed.

For any $i \in I$ and any $C_i \in \psi_i$, it is clear that $C_i \preccurlyeq C_i \leq E_i$. Thus $C_i \preccurlyeq E_i$ and $E_i = \bigvee_{C_i\in\psi_i} C_i \preccurlyeq E_i$ by (LCIR3). Hence $\bigwedge_{i\in I}^{ddir} E_i \preccurlyeq E_i$ for any $i \in I$. Therefore $\bigwedge_{i\in I}^{ddir} E_i \preccurlyeq \bigwedge_{i\in I}^{ddir} E_i$ by (LCIR5). Let $E = \bigwedge_{i\in I}^{ddir} E_i$. Then

$$x_\lambda \leq \bigwedge_{i\in I} D_i \leq E \preccurlyeq E \leq E \vee A \leq \bigwedge_{i\in I}^{ddir} B_i.$$

Thus $A \leqslant_{x_\lambda}^{\preccurlyeq} \bigwedge_{i\in I}^{ddir} B_i$.

Therefore $\leqslant_{x_\lambda}^{\preccurlyeq}$ is an L-down–directed relation. Next, we prove other results.

(i) For any $A, B \in L^X$,

$$\begin{aligned} A \leqslant_{x_\lambda}^{\preccurlyeq} B &\iff \exists D \in L^X \text{ s.t. } x_\lambda \leq D \preccurlyeq D \leq D \vee A \leq B \\ &\implies \exists D \in L^X \text{ s.t. } x_\lambda \vee A \leq D \vee A \leq B \\ &\implies \exists H \in L^X \text{ s.t. } x_\lambda \vee A \leq H \leq B \\ &\implies A \leqslant_{x_\lambda} B. \end{aligned}$$

Thus $\leqslant_{x_\lambda}^{\preccurlyeq} \leq \leqslant_{x_\lambda}$. Further, it is clear that $\preccurlyeq_{fin}$ is an L-concave internal relation. Thus it follows that $\leqslant_{x_\lambda}^{\preccurlyeq_{fin}} \leq \leqslant_{x_\lambda}$. To check that $\leqslant_{x_\lambda} \leq \leqslant_{x_\lambda}^{\preccurlyeq_{fin}}$, let $A, B \in L^X$ with $A \leqslant_{x_\lambda} B$. Then there is a set $C \in L^X$ such that $x_\lambda \leq C$ and $A \leq C \leq B$. Thus $C \preccurlyeq_{fin} C = C \vee A \leq B$ which implies that $A \leqslant_{x_\lambda}^{\preccurlyeq_{fin}} B$. Therefore $\leqslant_{x_\lambda} \leq \leqslant_{x_\lambda}^{\preccurlyeq_{fin}}$.

(ii) If $A \leqslant_{x_\lambda}^{\preccurlyeq} B$, then it is clear that $A \leqslant_{x_\mu}^{\preccurlyeq} B$ for any $\mu \in \beta^*(\lambda)$. Conversely, assume that $A \leqslant_{x_\mu}^{\preccurlyeq} B$ for any $\mu \in \beta^*(\lambda)$. For any $\mu \in \beta^*(\lambda)$, there is a set $D_\mu \in L^X$ such that $x_\mu \leq D_\mu \preccurlyeq D_\mu \leq D_\mu \vee A \leq B$. Let $D = \bigvee_{\mu\in\beta^*(\lambda)} D_\mu$. Then $D_\mu \preccurlyeq D$ for any $\mu \in \beta^*(\lambda)$. Thus $x_\lambda \leq D \preccurlyeq D \leq D \vee A \leq B$ by (LCIR3). Therefore $A \leqslant_{x_\lambda}^{\preccurlyeq} B$.

(3) (LDDR1) Since $\top \in \mathcal{N}_{x_\lambda}^{\mathcal{A}}$ and $\perp \notin \mathcal{N}_{x_\lambda}^{\mathcal{A}}$ by (LCAN1), it is clear that $\top \leqslant_{x_\lambda}^{\mathcal{A}} \top$ and $\perp \not\leqslant_{x_\lambda}^{\mathcal{A}} \perp$.

(LDDR2) For any $A, B \in L^X$, it follows that

$$A \leqslant_{x_\lambda}^{\mathcal{A}} B \iff \exists E \in \mathcal{N}_{x_\lambda}^{\mathcal{A}},\ A \leq E \leq B \iff B \in \mathcal{N}_{x_\lambda}^{\mathcal{A}},\ A \leq B \iff A \leq B \leqslant_{x_\lambda}^{\mathcal{A}} B.$$

(LDDR3) Let $A \in L^X$ and $\{B_i\}_{i\in I}^{ddir} \subseteq L^X$. If $A \leqslant_{x_\lambda}^{\mathcal{A}} \bigwedge_{i\in I}^{ddir} B_i$ then it is clear that $A \leqslant_{x_\lambda}^{\mathcal{A}} B_i$ for any $i \in I$. Conversely, assume that $A \leqslant_{x_\lambda}^{\mathcal{A}} B_i$ for any $i \in I$. Then, for any $i \in I$, there is a set $D_i \in \mathcal{N}_{x_\lambda}^{\mathcal{A}}$ such that

$A \le D_i \le B_i$. Let $E_i = \bigvee\{D_i \in \mathcal{N}^{\mathcal{A}}_{x_\lambda} : A \le D_i \le B_i\}$ for any $i \in I$. Then $E_i \in \mathcal{N}^{\mathcal{A}}_{x_\lambda}$ and $A \le E_i \le B_i$. In addition, it is clear that $\{E_i\}_{i\in I}$ is down-directed. Thus $\bigwedge^{ddir}_{i\in I} E_i \in \mathcal{N}^{\mathcal{A}}_{x_\lambda}$ and $A \le \bigwedge^{ddir}_{i\in I} E_i \le \bigwedge^{ddir}_{i\in I} B_i$. Hence $A \leqslant^{\mathcal{A}}_{x_\lambda} \bigwedge^{ddir}_{i\in I} B_i$.

Therefore $\leqslant^{\mathcal{A}}_{x_\lambda}$ is an L-down–directed relation. □

Lemma 3.3. *Let $(X, \leqslant)$ be an L-down–directed relation space. For any $A, B, C, D \in L^X$,*
(1) $A \le C \leqslant D \le B$ implies $A \leqslant B$;
(2) $A \leqslant B$ implies $A \vee C \leqslant B \vee C$.

Proof. (1) Since $A \le C \leqslant D$, it follows that $A \le C \le D \leqslant D$ and $A \leqslant D$ by (LDDR2). Further, since the set $\{D, B\}$ is down-directed and $A \leqslant D = D \wedge B$, we have $A \leqslant B$ by (LDDR3).

(2) Since $A \leqslant B \le B \vee C$, it is clear that $A \leqslant B \vee C$ by (1). Thus $A \le B \vee C \leqslant B \vee C$ by (LDDR2). Hence $A \vee C \le B \vee C \leqslant B \vee C$. Therefore $A \vee C \leqslant B \vee C$. □

Proposition 3.4. *Let $f : X \longrightarrow Y$ be a mapping and let $\leqslant_X \in \mathcal{R}^{ddir}(L^X)$. Then $\leqslant_{f(X)} \in \mathcal{R}^{ddir}(L^Y)$, where $\leqslant_{f(X)}$ is defined by*

$$\forall A, B \in L^Y,\ \ A \leqslant_{f(X)} B \iff A \le B \text{ and } f^{\leftarrow}_L(A) \leqslant_X f^{\leftarrow}_L(B).$$

Proof. (LDDR1) It is clear that $\top \leqslant_{f(X)} \top$ and $\bot \nleqslant_{f(X)} \bot$.

(LDDR2) For any $A, B \in L^X$,

$$\begin{aligned} A \leqslant_{f(X)} B &\iff A \le B \text{ and } f^{\leftarrow}_L(A) \leqslant_X f^{\leftarrow}_L(B) \\ &\iff A \le B \text{ and } f^{\leftarrow}_L(A) \le f^{\leftarrow}_L(B) \leqslant_X f^{\leftarrow}_L(B) \\ &\iff A \le B \leqslant_{f(X)} B. \end{aligned}$$

(LDDR3) For any $A \in L^X$ and $\{B_i\}^{ddir}_{i\in I} \subseteq L^X$,

$$\begin{aligned} A \leqslant_{f(X)} \bigwedge^{ddir}_{i\in I} B_i &\iff A \le \bigwedge^{ddir}_{i\in I} B_i \text{ and } f^{\leftarrow}_L(A) \leqslant_X f^{\leftarrow}_L(\bigwedge^{ddir}_{i\in I} B_i) \\ &\iff A \le \bigwedge^{ddir}_{i\in I} B_i \text{ and } f^{\leftarrow}_L(A) \leqslant_X \bigwedge^{ddir}_{i\in I} f^{\leftarrow}_L(B_i) \\ &\iff \forall i \in I,\ A \le B_i \text{ and } f^{\leftarrow}_L(A) \leqslant_X f^{\leftarrow}_L(B_i) \\ &\iff \forall i \in I,\ A \leqslant_{f(X)} B_i. \end{aligned}$$

Therefore $\leqslant_{f(X)}$ is an L-down–directed relation. □

Based on examples of L-down–directed relations presented in Proposition 3.2, we introduce the notion of L-down–directed convergence relation spaces and study its relation with L-concave internal spaces.

Definition 3.5. A relation $\sqsubseteq$ on $J(L^X) \times \mathcal{R}^{ddir}(L^X)$ is called an L-pre-down-directed convergence relation on L^X if for any $x_\lambda \in J(L^X)$ and $\leqslant_1, \leqslant_2 \in \mathcal{R}^{ddir}(L^X)$,
(LDDCR1) $x_\lambda \sqsubseteq \leqslant_{x_\lambda}$;
(LDDCR2) $x_\lambda \sqsubseteq \leqslant_1$ and $\leqslant_1 \le \leqslant_2$ imply $x_\lambda \sqsubseteq \leqslant_2$.

Lemma 3.6. *Let $\sqsubseteq$ be an L-pre-down-directed convergence relation on L^X and $x_\lambda \in J(L^X)$. Define a binary relation $\leqslant^{\sqsubseteq}_{x_\lambda}$ on L^X by*

$$\forall A, B \in L^X,\ \ A \leqslant^{\sqsubseteq}_{x_\lambda} B \iff \forall \leqslant \in \mathcal{R}^{ddir}_{\sqsubseteq}(x_\lambda),\ A \leqslant B,$$

where $\mathcal{R}^{ddir}_{\sqsubseteq}(x_\lambda) = \{\leqslant \in \mathcal{R}^{ddir}(L^X) : x_\lambda \sqsubseteq \leqslant\}$. Then $\leqslant^{\sqsubseteq}_{x_\lambda}$ is an L-down–directed relation.

Proof. (LDDR1) Since $x_\lambda \sqsubseteq \leqslant_{x_\lambda}$ by (LDDCR1) and $\perp \not\leqslant_{x_\lambda} \perp$, it is clear that $\perp \not\leqslant^{\sqsubseteq}_{x_\lambda} \perp$. For any $\leqslant \in \mathcal{R}^{ddir}(L^X)$, it is clear that $\top \leqslant \top$ by (LDDR1). Thus $\top \leqslant^{\sqsubseteq}_{x_\lambda} \top$ holds trivially.

(LDDR2) For any $A, B \in L^X$,

$$\begin{aligned} A \leqslant^{\sqsubseteq}_{x_\lambda} B &\iff \forall \leqslant \in \mathcal{R}^{ddir}_{\sqsubseteq}(x_\lambda),\ A \leqslant B \\ &\iff \forall \leqslant \in \mathcal{R}^{ddir}_{\sqsubseteq}(x_\lambda),\ A \leq B \leqslant B \\ &\iff A \leq B \leqslant^{\sqsubseteq}_{x_\lambda} B. \end{aligned}$$

(LDDR3) For any $A \in L^X$ and $\{B_i\}^{ddir}_{i\in I} \subseteq L^X$,

$$\begin{aligned} A \leqslant^{\sqsubseteq}_{x_\lambda} \bigwedge_{i\in I}^{ddir} B_i &\iff \forall \leqslant \in \mathcal{R}^{ddir}_{\sqsubseteq}(x_\lambda),\ A \leqslant \bigwedge_{i\in I}^{ddir} B_i \\ &\iff \forall \leqslant \in \mathcal{R}^{ddir}_{\sqsubseteq}(x_\lambda), \forall i \in I,\ A \leqslant B_i \\ &\iff \forall i \in I,\ A \leqslant^{\sqsubseteq}_{x_\lambda} B_i. \end{aligned}$$

Therefore $\leqslant^{\sqsubseteq}_{x_\lambda}$ is an L-down–directed relation. □

Definition 3.7. An L-pre-down-directed convergence relation $\sqsubseteq$ on L^X is called an L-down–directed convergence relation and the pair $(X, \sqsubseteq)$ is called an L-down–directed convergence relation space if

(LDDCR3) $\forall A, B \in L^X$, $A \leqslant^{\sqsubseteq}_{x_\lambda} B$ if and only if $A \leqslant^{\sqsubseteq}_{x_\mu} B$ for any $\mu \in \beta^*(\lambda)$.

Let $(X, \sqsubseteq_X)$ and $(Y, \sqsubseteq_Y)$ be L-down–directed convergence relation spaces. A mapping $f : X \longrightarrow Y$ is called an L-down–directed convergence relation preserving mapping if $x_\lambda \sqsubseteq_X \leqslant_X$ implies $f^{\rightarrow}_L(x_\lambda) \sqsubseteq_Y \leqslant_{f(X)}$ for any $x_\lambda \in J(L^X)$ and $\leqslant_X \in \mathcal{R}^{ddir}(L^X)$. The category of L-down–directed convergence relation spaces and L-down–directed convergence relation preserving mappings is denoted by L-**DDCRS**.

Next, we study relations between L-down–directed convergence spaces and L-concave internal spaces.

Theorem 3.8. *Let $(X, \leqslant)$ be an L-concave internal relation space. Define a relation $\sqsubseteq_\leqslant$ on $J(L^X) \times \mathcal{R}^{ddir}(L^X)$ by*

$$\forall x_\lambda \in J(L^X), \forall \leqslant \in \mathcal{R}^{ddir}(L^X),\ x_\lambda \sqsubseteq_\leqslant \leqslant \iff \leqslant^{\leqslant}_{x_\lambda} \leq \leqslant .$$

Then $\sqsubseteq_\leqslant$ is an L-down–directed convergence relation satisfying $\leqslant^{\sqsubseteq_\leqslant}_{x_\lambda} = \leqslant^{\leqslant}_{x_\lambda}$.

Proof. (LDDCR1) For any $x_\lambda \in J(L^X)$, it follows from Proposition 3.2(2) that $\leqslant^{\leqslant}_{x_\lambda} \leq \leqslant_{x_\lambda}$. Thus $x_\lambda \sqsubseteq_\leqslant \leqslant_{x_\lambda}$.

(LDDCR2) If $x_\lambda \sqsubseteq_\leqslant \leqslant_1$ and $\leqslant_1 \leq \leqslant_2$, then $\leqslant^{\leqslant}_{x_\lambda} \leq \leqslant_1 \leq \leqslant_2$. Thus $\leqslant^{\leqslant}_{x_\lambda} \leq \leqslant_2$. This shows that $x_\lambda \sqsubseteq_\leqslant \leqslant_2$.

(LDDCR3) For any $A, B \in L^X$ and $x_\lambda \in J(L^X)$,

$$A \leqslant^{\sqsubseteq_\leqslant}_{x_\lambda} B \iff \forall x_\lambda \sqsubseteq_\leqslant \leqslant,\ A \leqslant B \iff \forall \leqslant^{\leqslant}_{x_\lambda} \leq \leqslant,\ A \leqslant B \iff A \leqslant^{\leqslant}_{x_\lambda} B.$$

Thus $\leqslant^{\sqsubseteq_\leqslant}_{x_\lambda} = \leqslant^{\leqslant}_{x_\lambda}$. Further, It follows from Proposition 3.2(2) that

$$A \leqslant^{\sqsubseteq_\leqslant}_{x_\lambda} B \iff A \leqslant^{\leqslant}_{x_\lambda} B \iff \forall \mu \in \beta^*(\lambda),\ A \leqslant^{\leqslant}_{x_\mu} B \iff \forall \mu \in \beta^*(\lambda),\ A \leqslant^{\sqsubseteq_\leqslant}_{x_\mu} B.$$

This implies that (LDDCR3) holds.

Therefore $\sqsubseteq_\leqslant$ is an L-down–directed convergence relation satisfying $\leqslant^{\sqsubseteq_\leqslant}_{x_\lambda} = \leqslant^{\leqslant}_{x_\lambda}$. □

Theorem 3.9. *Let $(X, \leqslant_X)$ and $(Y, \leqslant_Y)$ be L-concave internal relation spaces. If $f : X \longrightarrow Y$ is an L-concave internal relation preserving mapping with respect to $(X, \leqslant_X)$ and $(Y, \leqslant_Y)$, then $f : (X, \sqsubseteq_{\leqslant_X}) \longrightarrow (Y, \sqsubseteq_{\leqslant_Y})$ is an L-down–directed convergence relation preserving mapping.*

Proof. For any $x_\lambda \in J(L^X)$ and $\preccurlyeq_X \in \mathcal{R}^{ddir}(L^X)$ with $x_\lambda \sqsubseteq_{\preccurlyeq_X} \preccurlyeq_X$, it follows that $\preccurlyeq_{x_\lambda}^{\preccurlyeq_X} \leq \preccurlyeq_X$. In order to prove $f_L^{\rightarrow}(x_\lambda) \sqsubseteq_{\preccurlyeq_Y} \preccurlyeq_{f(X)}$, it is sufficient to prove that $\preccurlyeq^{\preccurlyeq_Y}_{f_L^{\rightarrow}(x_\lambda)} \leq \preccurlyeq_{f(X)}$. Indeed, for any $A, B \in L^Y$,

$$\begin{aligned} A \preccurlyeq^{\preccurlyeq_Y}_{f_L^{\rightarrow}(x_\lambda)} B &\iff \exists D \in L^Y \text{ s.t. } f_L^{\rightarrow}(x_\lambda) \leq D \preccurlyeq_Y D \leq D \vee A \leq B \\ &\Longrightarrow A \leq B \text{ and } \exists D \in L^Y \text{ s.t. } x_\lambda \leq f_L^{\leftarrow}(D) \preccurlyeq_X f_L^{\leftarrow}(D) \leq f_L^{\leftarrow}(D) \vee f_L^{\leftarrow}(A) \leq f_L^{\leftarrow}(B) \\ &\Longrightarrow A \leq B \text{ and } f_L^{\leftarrow}(A) \preccurlyeq^{\preccurlyeq_X}_{x_\lambda} f_L^{\leftarrow}(B) \\ &\Longrightarrow A \leq B \text{ and } f_L^{\leftarrow}(A) \preccurlyeq_X f_L^{\leftarrow}(B) \\ &\Longrightarrow A \preccurlyeq_{f(X)} B. \end{aligned}$$

Thus $\preccurlyeq^{\preccurlyeq_Y}_{f_L^{\rightarrow}(x_\lambda)} \leq \preccurlyeq_{f(X)}$. This implies that $f_L^{\rightarrow}(x_\lambda) \sqsubseteq_{\preccurlyeq_Y} \preccurlyeq_{f(X)}$. Therefore f is an L-down–directed convergence relation preserving mapping. □

Theorem 3.10. *Let $(X, \sqsubseteq)$ be an L-down–directed convergence relation space. Define a binary relation $\preccurlyeq_\sqsubseteq$ on L^X by*

$$\forall A, B \in L^X,\ A \preccurlyeq_\sqsubseteq B \iff \exists A \leq D \leq B \text{ s.t. } \forall x_\lambda \in \beta^*(D), \forall \preccurlyeq \in \mathcal{R}^{ddir}_\sqsubseteq(x_\lambda),\ A \preccurlyeq D.$$

Then $\preccurlyeq_\sqsubseteq$ is an L-concave internal relation.

Proof. By Lemma 3.3(1), it is easy to see that $A \leq C \preccurlyeq_\sqsubseteq D \leq B$ implies $A \preccurlyeq_\sqsubseteq B$ for any $A, B, C, D \in L^X$. In addition, (LCIR1) and (LCIR2) hold trivially for $\preccurlyeq_\sqsubseteq$. We next verify (LCIR3)–(LCIR5) hold for $\preccurlyeq_\sqsubseteq$.

(LCIR3) If $\bigvee_{i\in I} A_i \preccurlyeq_\sqsubseteq B$ then it is clear that $A_i \preccurlyeq_\sqsubseteq B$ for any $i \in I$. Conversely, assume that $A_i \preccurlyeq_\sqsubseteq B$ for any $i \in I$. By $A_i \preccurlyeq_\sqsubseteq B$, there is a $D_i \in L^X$ such that $A_i \leq D_i \leq B$ and $A_i \preccurlyeq D_i$ for any $x_\lambda \in \beta^*(D_i)$ and $\preccurlyeq \in \mathcal{R}^{ddir}_\sqsubseteq(x_\lambda)$. Let $D = \bigvee_{i\in I} D_i$. Then $\bigvee_{i\in I} A_i \leq D \leq B$. For all $y_\mu \in \beta^*(D)$ and $\preccurlyeq \in \mathcal{R}^{ddir}_\sqsubseteq(y_\mu)$, there is an index $j \in I$ such that $y_\mu \in \beta^*(D_j)$. Thus $A_j \preccurlyeq D_j$ by $A_i \preccurlyeq_\sqsubseteq B$. Hence Lemma 3.3(2) implies

$$\bigvee_{i\in I} A_i \leq D = D \vee A_j \preccurlyeq D \vee D_j = D$$

As a result, $\bigvee_{i\in I} A_j \preccurlyeq D$ by Lemma 3.3(1). Therefore $\bigvee_{i\in I} A_j \preccurlyeq_\sqsubseteq B$.

(LCIR4) Let $A \preccurlyeq_\sqsubseteq B$. We need to find a $D \in L^X$ such that $A \preccurlyeq_\sqsubseteq D \preccurlyeq_\sqsubseteq B$. By $A \preccurlyeq_\sqsubseteq B$, there is $D \in L^X$ such that $A \leq D \leq B$ and $A \preccurlyeq D$ for any $x_\lambda \in \beta^*(D)$ and $\preccurlyeq \in \mathcal{R}^{ddir}_\sqsubseteq(x_\lambda)$.

We say that $D \preccurlyeq_\sqsubseteq B$. Indeed, it is clear that $D \leq D \leq B$. In addition, for all $y_\mu \in \beta^*(D)$ and $\preccurlyeq \in \mathcal{R}^{ddir}_\sqsubseteq(y_\mu)$, it follows that $A \preccurlyeq D$. Thus Lemma 3.3(2) yields that $D = D \vee A \preccurlyeq D \vee D = D$. This shows that $D \preccurlyeq_\sqsubseteq B$.

We also say that $A \preccurlyeq_\sqsubseteq D$. Indeed, it is clear that $A \leq D \leq D$. In addition, for all $z_\eta \in \beta^*(D)$ and $\preccurlyeq \in \mathcal{R}^{ddir}_\sqsubseteq(z_\eta)$, it is clear that $A \preccurlyeq D$ by $A \preccurlyeq_\sqsubseteq B$. Thus $A \preccurlyeq_\sqsubseteq D$. Therefore $A \preccurlyeq_\sqsubseteq D \preccurlyeq_\sqsubseteq B$ as desired.

(LCIR5) If $A \preccurlyeq_\sqsubseteq \bigwedge^{ddir}_{i\in I} B_i$, then it is clear that $A \preccurlyeq_\sqsubseteq B_i$ for any $i \in I$. Conversely, assume that $A \preccurlyeq_\sqsubseteq B_i$ for any $i \in I$. Thus, for any $i \in I$, there is a set $D_i \in L^X$ such that $A \leq D_i \leq B_i$ and $A \preccurlyeq D_i$ for any $x_\lambda \in \beta^*(D_i)$ and any $\preccurlyeq \in \mathcal{R}^{ddir}_\sqsubseteq(x_\lambda)$. Let $E_i = \bigvee \varphi_i$, where

$$\varphi_i = \{D_i \in L^X : A \leq D_i \leq B_i \text{ s.t. } \forall x_\lambda \in \beta^*(D_i), \forall \preccurlyeq \in \mathcal{R}^{ddir}_\sqsubseteq(x_\lambda),\ A \preccurlyeq D_i\}.$$

Form Lemma 3.3(1), it is easy to check that $E_i \in \varphi_i$. Since $\{B_i\}_{i\in I}$ is down-directed, $\{E_i\}_{i\in I}$ is also down-directed. Thus $A \leq \bigwedge^{ddir}_{i\in I} E_i \leq \bigwedge^{ddir}_{i\in I} B_i$. In addition, for any $x_\lambda \in \beta^*(\bigwedge^{ddir}_{i\in I} E_i)$, it is clear that $x_\lambda \in \beta^*(E_i)$ for any $i \in I$. For any $\preccurlyeq \in \mathcal{R}^{ddir}_\sqsubseteq(x_\lambda)$, we have $A \preccurlyeq D_i \leq E_i$ by $A \preccurlyeq_\sqsubseteq B_i$. Thus $A \preccurlyeq E_i$ for any $i \in I$. Hence $A \preccurlyeq \bigwedge^{ddir}_{i\in I} E_i$ by (LDDR3). Therefore $A \preccurlyeq_\sqsubseteq \bigwedge^{ddir}_{i\in I} B_i$. □

Theorem 3.11. *Let $(X, \sqsubseteq_X)$ and $(Y, \sqsubseteq_Y)$ be L-down–directed convergence relation spaces. If $f : X \longrightarrow Y$ is an L-down–directed convergence relation preserving mapping with respect to $(X, \sqsubseteq_X)$ and $(Y, \sqsubseteq_Y)$, then $f : (X, \preccurlyeq_{\sqsubseteq_X}) \longrightarrow (Y, \preccurlyeq_{\sqsubseteq_Y})$ is an L-concave internal relation preserving mapping.*

Proof. If $A \preccurlyeq_{\sqsubseteq_Y} B$ then there is $D \in L^X$ such that $A \leq D \leq B$ and $A \preccurlyeq_Y D$ for all $y_\mu \in \beta^*(D)$ and $\preccurlyeq_Y \in \mathcal{R}^{ddir}_{\sqsubseteq_Y}(y_\mu)$. Thus $f_L^{\leftarrow}(A) \leq f_L^{\leftarrow}(D) \leq f_L^{\leftarrow}(B)$. In order to prove that $f_L^{\leftarrow}(A) \preccurlyeq_{\sqsubseteq_X} f_L^{\leftarrow}(B)$, let $x_\lambda \in \beta^*(f_L^{\leftarrow}(D))$ and $\preccurlyeq_X \in \mathcal{R}^{ddir}_{\sqsubseteq_X}(x_\lambda)$. Then $f_L^{\rightarrow}(x_\lambda) \in \beta^*(D)$ and $\preccurlyeq_{f(X)} \in \mathcal{R}^{ddir}_{\sqsubseteq_Y}(f_L^{\rightarrow}(x_\lambda))$. Thus $A \preccurlyeq_{f(X)} D$ which implies that $f_L^{\leftarrow}(A) \preccurlyeq_X f_L^{\leftarrow}(D)$. This shows that $f_L^{\leftarrow}(A) \preccurlyeq_{\sqsubseteq_X} f_L^{\leftarrow}(B)$.

Therefore f is an L-concave internal relation preserving mapping. □

Lemma 3.12. *Let $(X, \sqsubseteq)$ be an L-down–directed convergence relation space. Then $\preccurlyeq_{x_\lambda}^{\preccurlyeq_\sqsubseteq} \leq \preccurlyeq_{x_\lambda}^{\sqsubseteq} \leq \preccurlyeq_{x_\lambda}$ for any $x_\lambda \in J(L^X)$.*

Proof. Let $x_\lambda \in J(L^X)$ and $A, B \in L^X$. If $A \preccurlyeq_{x_\lambda}^{\preccurlyeq_\sqsubseteq} B$ then there is a $D \in L^X$ such that $x_\lambda \leq D \preccurlyeq_\sqsubseteq D \leq D \vee A \leq B$. So

$$\begin{aligned}
D \preccurlyeq_\sqsubseteq D \quad &\Longleftrightarrow \quad \forall y_\eta \in \beta^*(D), \forall \preccurlyeq_1 \in \mathcal{R}^{ddir}_{\sqsubseteq}(y_\eta),\ D \preccurlyeq_1 D \\
&\Longrightarrow \quad \forall \mu \in \beta^*(\lambda), \forall \preccurlyeq_1 \in \mathcal{R}^{ddir}_{\sqsubseteq}(x_\mu),\ D \preccurlyeq_1 D \\
&\Longleftrightarrow \quad \forall \mu \in \beta^*(\lambda),\ D \preccurlyeq_{x_\mu}^{\sqsubseteq} D \\
&\overset{\text{(LDDCR3)}}{\Longleftrightarrow} \quad D \preccurlyeq_{x_\lambda}^{\sqsubseteq} D \\
&\Longrightarrow \quad A \leq B \preccurlyeq_{x_\lambda}^{\sqsubseteq} B \quad \text{(by Lemma 3.3(2))} \\
&\overset{\text{(LDDR2)}}{\Longleftrightarrow} \quad A \preccurlyeq_{x_\lambda}^{\sqsubseteq} B.
\end{aligned}$$

This shows that $\preccurlyeq_{x_\lambda}^{\preccurlyeq_\sqsubseteq} \leq \preccurlyeq_{x_\lambda}^{\sqsubseteq}$. Further, since $x_\lambda \sqsubseteq \preccurlyeq_{x_\lambda}$ by (LDDCR1), it follows that

$$A \preccurlyeq_{x_\lambda}^{\sqsubseteq} B \iff \forall x_\lambda \sqsubseteq \preccurlyeq,\ A \preccurlyeq B \implies A \preccurlyeq_{x_\lambda} B.$$

This implies that $\preccurlyeq_{x_\lambda}^{\sqsubseteq} \leq \preccurlyeq_{x_\lambda}$. So $\preccurlyeq_{x_\lambda}^{\preccurlyeq_\sqsubseteq} \leq \preccurlyeq_{x_\lambda}^{\sqsubseteq} \leq \preccurlyeq_{x_\lambda}$. □

Theorem 3.13. *Let $(X, \sqsubseteq)$ be an L-down–directed convergence relation space. Then $\sqsubseteq \leq \sqsubseteq_{\preccurlyeq_\sqsubseteq}$.*

Proof. Let $x_\lambda \in J(L^X)$ and let $\preccurlyeq \in \mathcal{R}^{ddir}(L^X)$ with $x_\lambda \sqsubseteq \preccurlyeq$. It follows from Lemma 3.12 that $\preccurlyeq_{x_\lambda}^{\preccurlyeq_\sqsubseteq} \leq \preccurlyeq_{x_\lambda}^{\sqsubseteq} \leq \preccurlyeq$. Thus $\preccurlyeq_{x_\lambda}^{\preccurlyeq_\sqsubseteq} \leq \preccurlyeq$ followed by $x_\lambda \sqsubseteq_{\preccurlyeq_\sqsubseteq} \preccurlyeq$. Therefore $\sqsubseteq \leq \sqsubseteq_{\preccurlyeq_\sqsubseteq}$. □

Theorem 3.14. $\preccurlyeq_{\sqsubseteq_\preccurlyeq} = \preccurlyeq$ *for any L-concave internal relation space $(X, \preccurlyeq)$.*

Proof. Let $A, B \in L^X$ with $A \preccurlyeq_{\sqsubseteq_\preccurlyeq} B$. It follows from Theorem 3.8 that $\preccurlyeq_{x_\lambda}^{\sqsubseteq_\preccurlyeq} = \preccurlyeq_{x_\lambda}^{\preccurlyeq}$. Thus

$$\begin{aligned}
A \preccurlyeq_{\sqsubseteq_\preccurlyeq} B \quad &\Longleftrightarrow \quad \exists A \leq D \leq B \ \text{ s.t. } \ \forall x_\lambda \in \beta^*(D), \forall \preccurlyeq \in \mathcal{R}^{ddir}_{\sqsubseteq_\preccurlyeq}(x_\lambda),\ A \preccurlyeq D \\
&\Longleftrightarrow \quad \exists A \leq D \leq B \ \text{ s.t. } \ \forall x_\lambda \in \beta^*(D),\ A \preccurlyeq_{x_\lambda}^{\sqsubseteq_\preccurlyeq} D \\
&\Longleftrightarrow \quad \exists A \leq D \leq B, \ \text{ s.t. } \ \forall x_\lambda \in \beta^*(D),\ A \preccurlyeq_{x_\lambda}^{\preccurlyeq} D.
\end{aligned}$$

There is a set $D \in L^X$ such that $A \leq D \leq B$ and $A \preccurlyeq_{x_\lambda}^{\preccurlyeq} D$ for any $x_\lambda \in \beta^*(D)$. By $A \preccurlyeq_{x_\lambda}^{\preccurlyeq} D$, there is a set $E_{x_\lambda} \in L^X$ such that

$$x_\lambda \leq E_{x_\lambda} \preccurlyeq E_{x_\lambda} \leq E_{x_\lambda} \vee A \leq D \leq B.$$

Let $E = \bigvee_{x_\lambda \in \beta^*(D)} E_{x_\lambda}$. Then $D = \bigvee_{x_\lambda \in \beta^*(D)} x_\lambda \leq \bigvee_{x_\lambda \in \beta^*(D)} E_{x_\lambda} = E$. In addition, $E \preccurlyeq E$ by (LCIR3) of $\preccurlyeq$. Hence $A \leq D \leq E \preccurlyeq E \leq B$ which implies that $A \preccurlyeq B$. Therefore $\preccurlyeq_{\sqsubseteq_\preccurlyeq} \leq \preccurlyeq$.

Conversely, let $A \preccurlyeq B$. By (LCIR4), there is a set $C \in L^X$ such that $A \preccurlyeq C \preccurlyeq B$. Put

$$D = \bigvee \{C \in L^X : A \preccurlyeq C \preccurlyeq B\}.$$

Then $A \preccurlyeq D \preccurlyeq B$. Further, by $D \preccurlyeq B$ and (LCIR4), there is an $E \in L^X$ such that $D \preccurlyeq E \preccurlyeq B$. Thus $A \preccurlyeq E \preccurlyeq B$ which implies $D \preccurlyeq E \leq D$. Hence $D \preccurlyeq D$.

In order to prove that $A \preccurlyeq_{\sqsubseteq_\preccurlyeq} B$, let $x_\lambda \in \beta^*(D)$ and $\preccurlyeq \in \mathcal{R}^{ddir}_{\sqsubseteq_\preccurlyeq}(x_\lambda)$. Then $x_\lambda \sqsubseteq_\preccurlyeq \preccurlyeq$ implies $\preccurlyeq_{x_\lambda}^{\preccurlyeq} \leq \preccurlyeq$. Further, since $x_\lambda \leq D \preccurlyeq D = D \vee A = D$, it follows that $A \preccurlyeq_{x_\lambda}^{\preccurlyeq} D$. Thus $A \preccurlyeq D$ followed by $A \preccurlyeq_{\sqsubseteq_\preccurlyeq} B$. Therefore $\preccurlyeq \leq \preccurlyeq_{\sqsubseteq_\preccurlyeq}$.

In conclusion, we proved that $\preccurlyeq_{\sqsubseteq_\preccurlyeq} = \preccurlyeq$, as desired. □

Based on Theorems 3.8 and 3.9, we obtain a functor $\mathbb{F} : L$-**CIRS** $\longrightarrow L$-**DDCRS** defined by:

$$\mathbb{F}((X, \preccurlyeq)) = (X, \sqsubseteq_{\preccurlyeq}) \quad \text{and} \quad \mathbb{F}(f) = f.$$

Similarly, based on Theorems 3.10 and 3.11, we obtain an functor $\mathbb{G} : L$-**DDCRS** $\longrightarrow L$-**CIRS** defined by:

$$\mathbb{G}((X, \sqsubseteq)) = (X, \preccurlyeq_{\sqsubseteq}) \quad \text{and} \quad \mathbb{G}(f) = f.$$

Based on Theorems 3.8–3.14, we have the following conclusions.

Corollary 3.15. $(\mathbb{F}, \mathbb{G})$ *is a Galois connection, where* $\mathbb{G}$ *is a left inverse of* $\mathbb{F}$.

Corollary 3.16. *The category L-***CAS** *can be embedded in the category L-***DDCRS** *as a reflective subcategory.*

Now, we have established the connection between L-down–directed convergence relations and L-concave internal relations. Then, is there any L-down–directed convergence relation with special properties which can enhance this connection? In order to discuss this, we present the notion of L-concave down–directed relations as follows.

Definition 3.17. An L-down–directed convergence relation $\sqsubseteq$ on L^X is called an L-concave down–directed convergence relation and the pair $(X, \sqsubseteq)$ is called an L-concave down–directed convergence relation space, if $\sqsubseteq$ satisfies

(LCDDCR1) $x_\lambda \sqsubseteq \preccurlyeq^{\sqsubseteq}_{x_\lambda}$;

(LCDDCR2) $A \preccurlyeq^{\sqsubseteq}_{x_\lambda} B$ if and only if $\exists D \in L^X$ s.t. $\forall y_\mu \in \beta^*(D),\ x_\lambda \leq D \preccurlyeq^{\sqsubseteq}_{y_\mu} D \leq D \vee A \leq B$.

The category of L-concave down–directed convergence relation spaces and L-down–directed convergence relation preserving mappings is denoted by L-**CDDCRS**. Next, we discuss relationships between L-**CIRS** and L-**CDDCRS**.

Theorem 3.18. *Let* $(X, \preccurlyeq)$ *be an L-concave internal relation space. Then* $\sqsubseteq_{\preccurlyeq}$ *is an L-concave down–directed convergence relation.*

Proof. By Theorem 3.8, it is sufficient to prove that $\sqsubseteq_{\preccurlyeq}$ satisfies (LCDDCR1) and (LCDDCR2).

(LCDDFCR1). For any $x_\lambda \in J(L^X)$, $\preccurlyeq^{\preccurlyeq}_{x_\lambda} \leq \preccurlyeq^{\preccurlyeq}_{x_\lambda}$ implies $x_\lambda \sqsubseteq_{\preccurlyeq} \preccurlyeq^{\preccurlyeq}_{x_\lambda}$. Thus $x_\lambda \sqsubseteq_{\preccurlyeq} \preccurlyeq^{\sqsubseteq_{\preccurlyeq}}_{x_\lambda}$ by Theorem 3.8.

(LCDDCR2). For any $D \in L^X$ and $y_\mu \in \beta^*(D)$, it is clear that $D \preccurlyeq D$ implies $D \preccurlyeq^{\preccurlyeq}_{y_\mu} D$. Theorem 3.8 implies

$$\begin{aligned}
A \preccurlyeq^{\sqsubseteq_{\preccurlyeq}}_{x_\lambda} B \quad &\Longleftrightarrow \quad A \preccurlyeq^{\preccurlyeq}_{x_\lambda} B \\
&\Longleftrightarrow \quad \exists D \in L^X \text{ s.t. } x_\lambda \leq D \preccurlyeq D \leq D \vee A \leq B \\
&\Longrightarrow \quad \exists D \in L^X \text{ s.t. } \forall y_\mu \in \beta^*(D),\ x_\lambda \leq D \preccurlyeq^{\preccurlyeq}_{y_\mu} D \leq D \vee A \leq B \\
&\Longleftrightarrow \quad \exists D \in L^X \text{ s.t. } \forall y_\mu \in \beta^*(D),\ x_\lambda \leq D \preccurlyeq^{\sqsubseteq_{\preccurlyeq}}_{y_\mu} D \leq D \vee A \leq B \\
&\Longrightarrow \quad \exists D \in L^X \text{ s.t. } \forall \mu \in \beta^*(\lambda),\ x_\lambda \leq D \preccurlyeq^{\sqsubseteq_{\preccurlyeq}}_{x_\mu} D \leq D \vee A \leq B \\
&\overset{\text{(LDDCR3)}}{\Longrightarrow} \quad \exists D \in L^X \text{ s.t. } x_\lambda \leq D \preccurlyeq^{\sqsubseteq_{\preccurlyeq}}_{x_\lambda} D \leq D \vee A \leq B \\
&\Longleftrightarrow \quad \exists D \in L^X \text{ s.t. } x_\lambda \leq D \preccurlyeq^{\preccurlyeq}_{x_\lambda} D \leq D \vee A \leq B \quad \text{(by Theorem 3.8)} \\
&\Longrightarrow \quad A \leq B \preccurlyeq^{\preccurlyeq}_{x_\lambda} B \quad \text{(by Lemma 3.3(2))} \\
&\Longrightarrow \quad A \leq B \preccurlyeq^{\sqsubseteq_{\preccurlyeq}}_{x_\lambda} B \quad \text{(by Theorem 3.8)} \\
&\Longleftrightarrow \quad A \preccurlyeq^{\sqsubseteq_{\preccurlyeq}}_{x_\lambda} B.
\end{aligned}$$

Thus (LCDDCR2) holds for $\sqsubseteq_{\preccurlyeq}$.

Therefore $\sqsubseteq_{\preccurlyeq}$ is an L-concave down–directed convergence relation. $\square$

Lemma 3.19. *Let* $(X, \sqsubseteq)$ *be an L-concave down–directed convergence relation space. Then* $\preccurlyeq^{\preccurlyeq_{\sqsubseteq}}_{x_\lambda} = \preccurlyeq^{\sqsubseteq}_{x_\lambda}$ *for* $x_\lambda \in J(L^X)$.

Proof. For any $A, B \in L^X$, (LCDDCR2) yields that

$$\begin{aligned} A \leqslant_{x_\lambda}^{\leqslant_{\sqsubseteq}} B &\Longleftrightarrow \exists D \in L^X \text{ s.t. } x_\lambda \leq D \leqslant_{\sqsubseteq} D \geq D \wedge B \geq A \\ &\Longleftrightarrow \exists D \in L^X \text{ s.t. } \forall y_\mu \in \beta^*(D), \forall \leqslant \in \mathcal{R}_{\sqsubseteq}^{ddir}(y_\mu),\ x_\lambda \leq D \leqslant D \leq D \vee A \leq B \\ &\Longleftrightarrow \exists D \in L^X,\ \forall y_\mu \in \beta^*(D),\ x_\lambda \leq D \leqslant_{y_\mu}^{\sqsubseteq} D \leq D \vee A \leq B \\ &\Longleftrightarrow A \leqslant_{x_\lambda}^{\sqsubseteq} B. \end{aligned}$$

Therefore $\leqslant_{x_\lambda}^{\leqslant_{\sqsubseteq}} = \leqslant_{x_\lambda}^{\sqsubseteq}$. □

Theorem 3.20. $\sqsubseteq_{\leqslant_{\sqsubseteq}} = \sqsubseteq$ *for any L-concave down–directed convergence relation space* $(X, \sqsubseteq)$*.*

Proof. $\sqsubseteq \leq \sqsubseteq_{\leqslant_{\sqsubseteq}}$ by Theorem 3.13. In order to prove that $\sqsubseteq_{\leqslant_{\sqsubseteq}} \leq \sqsubseteq$, let $x_\lambda \in J(L^X)$ and $\leqslant \in \mathcal{R}^{ddir}(L^X)$ with $x_\lambda \sqsubseteq_{\leqslant_{\sqsubseteq}} \leqslant$. Then $\leqslant_{x_\lambda}^{\leqslant_{\sqsubseteq}} \leq \leqslant$. Thus $x_\lambda \sqsubseteq \leqslant_{x_\lambda}^{\sqsubseteq} = \leqslant_{x_\lambda}^{\leqslant_{\sqsubseteq}} \leq \leqslant$ by (LCDDCR1) and Lemma 3.19. Hence $x_\lambda \sqsubseteq \leqslant$ by (LDDCR2). Therefore $\sqsubseteq_{\leqslant_{\sqsubseteq}} \leq \sqsubseteq$. □

Based on Theorems 3.8- 3.11, 3.13, 3.18 and 3.20, we have the following conclusion.

Theorem 3.21. *The category L-***CIRS** *is isomorphic to the category L-***CDDCRS***.*

Remark 3.22. Based on Theorems 2.6 and 3.21, relationships between L-**CDDCRS** and L-**CAS** are as follows.

(1) Let $(X, \sqsubseteq)$ be an L-concave down–directed convergence space. The set

$$\mathcal{A}_{\sqsubseteq} = \{A \in L^X : \forall x_\lambda \in \beta^*(A),\ A \leqslant_{x_\lambda}^{\sqsubseteq} A\}$$

is an L-concave structure on L^X.

(2) Let $(X, \mathcal{A})$ be an L-concave space. Define a mapping $\sqsubseteq_{\mathcal{A}}$ on $J(L^X) \times \mathcal{R}^{ddir}(L^X)$ by

$$\forall x_\lambda \in J(L^X), \forall \leqslant \in \mathcal{R}^{ddir}(L^X),\ \ x_\lambda \sqsubseteq_{\mathcal{A}} \Longleftrightarrow \leqslant_{x_\lambda}^{\mathcal{A}} \leq \leqslant .$$

Then $\sqsubseteq_{\mathcal{A}}$ is an L-concave down–directed convergence relation.

(3) $\sqsubseteq_{\mathcal{A}_{\sqsubseteq}} = \sqsubseteq$ and $\mathcal{A}_{\sqsubseteq_{\mathcal{A}}} = \mathcal{A}$.

(4) The category L-**CDDCRS** is isomorphic to the category L-**CAS**.

4. *L*-concave filters and *L*-(resp. concave) filter convergence spaces

In [26], Xiu et al. presented the notion of L-convex ideals by which they introduced L-convex convergence spaces and discussed its relationships with L-convex space. Then, is it possible to introduce L-concave filter or L-concave filter convergence space? Further, how about their relationships with L-down–directed relation and L-concave down–directed convergence relation spaces? In order to solve these problems, we define L-concave filter and discuss its relationships with L-concave down–directed convergence relations.

Definition 4.1. A set $\mathcal{F} \subseteq L^X$ is called an L-concave filter on L^X and the pair $(X, \mathcal{F})$ is called an L-concave filter space, if

(LCF1) $\underline{\perp} \notin \mathcal{F}$ and $\underline{\top} \in \mathcal{F}$;

(LCF2) $A \in \mathcal{F}$ and $A \leq B$ imply $B \in \mathcal{F}$;

(LCF3) $\{A_i\}_{i \in I}^{ddir} \subseteq \mathcal{F}$ implies $\bigwedge_{i \in I}^{ddir} A_i \in \mathcal{F}$.

The set of any L-concave filters on L^X is denoted by $\mathcal{F}_c(L^X)$.

Example 4.2. (1) For any $x_\lambda \in J(L^X)$, the set $\mathcal{F}_{x_\lambda} = \{F \in L^X : x_\lambda \leq F\}$ is an L-concave filter on L^X.

(2) For any $x_\lambda \in J(L^X)$ and any L-concave space $(X, \mathcal{A})$, the set $\mathcal{N}_{x_\lambda}^{\mathcal{A}}$ is an L-concave filter on L^X.

Theorem 4.3. *Let* $(X, \leqslant)$ *be an L-down–directed relation space. Then* $\mathcal{F}_{\leqslant} = \{B \in L^X : B \leqslant B\}$ *is an L-concave filter.*

Proof. (LCF1) It directly follows from (LDDR1) that $\perp \notin \mathcal{F}_{\preccurlyeq}$ and $\top \in \mathcal{F}_{\preccurlyeq}$.

(LCF2) If $B \in \mathcal{F}_{\preccurlyeq}$ and $B \leq C$, then $B \preccurlyeq B \leq C$. Thus $B \preccurlyeq C$ by Lemma 3.3(1). Hence $C \preccurlyeq C$ by (LDDR2). Therefore $C \in \mathcal{F}_{\preccurlyeq}$.

(LCF3) Let $\{B_i\}_{i\in I}^{ddir} \subseteq \mathcal{F}_{\preccurlyeq}$. Then $B_i \preccurlyeq B_i$ for any $i \in I$. Thus $\bigwedge_{i\in I}^{ddir} B_i \preccurlyeq B_i$ for any $i \in I$. Hence $\bigwedge_{i\in I}^{ddir} B_i \preccurlyeq \bigwedge_{i\in I}^{ddir} B_i$ by (LDDR3). Therefore $\bigwedge_{i\in I}^{ddir} B_i \in \mathcal{F}_{\preccurlyeq}$. □

Theorem 4.4. *Let $(X, \mathcal{F})$ be an L-concave filter spaces. Define a binary relation $\preccurlyeq_{\mathcal{F}}$ by*

$$\forall A, B \in L^X, \ \ A \preccurlyeq_{\mathcal{F}} B \iff A \leq B \in \mathcal{F}.$$

Then $\preccurlyeq_{\mathcal{F}}$ is an L-down–directed relation on L^X.

Proof. (LDDR1) By (LCF1), it is clear that $\top \preccurlyeq_{\mathcal{F}} \top$ and $\perp \not\preccurlyeq_{\mathcal{F}} \perp$.

(LDDR2) For $A, B \in L^X$, (LCF2) implies that $A \preccurlyeq_{\mathcal{F}} B$ if and only if $A \leq B \in \mathcal{F}$ if and only if $A \leq B \preccurlyeq_{\mathcal{F}} B$.

(LDDR3) Let $A \in L^X$ and $\{B_i\}_{i\in I}^{ddir} \subseteq L^X$. If $A \preccurlyeq_{\mathcal{F}} \bigwedge_{i\in I}^{ddir} B_i$ then it is clear that $A \preccurlyeq_{\mathcal{F}} B_i$ for any $i \in I$. Conversely, assume that $A \preccurlyeq_{\mathcal{F}} B_i$ for any $i \in I$. For each $i \in I$, it is clear that $A \leq B_i \in \mathcal{F}$. Since $\{B_i\}_{i\in I}^{ddir} \in \mathcal{F}$, it follows from (LCF3) that $A \leq \bigwedge_{i\in I}^{ddir} B_i \in \mathcal{F}$. Therefore $A \preccurlyeq_{\mathcal{F}} \bigwedge_{i\in I}^{ddir} B_i$. □

Theorem 4.5. *$\mathcal{F}_c(L^X)$ and $\mathcal{R}^{ddir}(L^X)$ are one-to-one correspondent.*

Proof. Let $(X, \preccurlyeq)$ be an L-down–directed relation space. For any $A, B \in L^X$, (LDDR2) implies that

$$A \preccurlyeq B \iff A \leq B \preccurlyeq B \iff A \leq B \in \mathcal{F}_{\preccurlyeq} \iff A \leq B \preccurlyeq_{\mathcal{F}_{\preccurlyeq}} B \iff A \preccurlyeq_{\mathcal{F}_{\preccurlyeq}} B.$$

This shows that $\preccurlyeq = \preccurlyeq_{\mathcal{F}_{\preccurlyeq}}$.

Let $(X, \mathcal{F})$ be an L-concave filter space. For any $B \in L^X$, it is clear that

$$B \in \mathcal{F}_{\preccurlyeq_{\mathcal{F}}} \iff B \preccurlyeq_{\mathcal{F}} B \iff B \in \mathcal{F}.$$

Therefore $\mathcal{F}_{\preccurlyeq_{\mathcal{F}}} = \mathcal{F}$. □

Lemma 4.6. *(1) If $(X, \preccurlyeq)$ be an L-concave internal relation space then $\mathcal{F}_{x_\lambda} = \mathcal{F}_{\preccurlyeq_{x_\lambda}}$ and $\preccurlyeq_{x_\lambda} = \preccurlyeq_{\mathcal{F}_{x_\lambda}}$ for any $x_\lambda \in J(L^X)$.*

(2) If $(X, \mathcal{A})$ is an L-concave space then $\mathcal{F}_{\preccurlyeq_{x_\lambda}^{\preccurlyeq}} = \mathcal{N}_{x_\lambda}^{\mathcal{A}_{\preccurlyeq}}$ and $\preccurlyeq_{x_\lambda}^{\preccurlyeq_{\mathcal{A}}} = \preccurlyeq_{x_\lambda}^{\mathcal{A}}$ for any $x_\lambda \in J(L^X)$.

(3) If $\{\mathcal{F}_i\}_{i\in I} \subseteq \mathcal{F}_c(L^X)$ then $\bigcap_{i\in I} \mathcal{F}_i$ is also an L-concave filter.

(4) If $(X, \mathcal{F}_X)$ is an L-concave filter space and if $f : X \longrightarrow Y$ is a mapping, then $\mathcal{F}_{f(X)} = \{G \in L^Y : f_L^{\leftarrow}(G) \in \mathcal{F}_X\}$ is an L-concave filter on L^Y.

Proof. (1) For any $F \in L^X$,

$$F \in \mathcal{F}_{x_\lambda} \iff x_\lambda \leq F \iff x_\lambda \vee F \leq F \iff F \preccurlyeq_{x_\lambda} F \iff F \in \mathcal{F}_{\preccurlyeq_{x_\lambda}}.$$

Thus $\mathcal{F}_{x_\lambda} = \mathcal{F}_{\preccurlyeq_{x_\lambda}}$. Also, for any $A, B \in L^X$,

$$A \preccurlyeq_{x_\lambda} B \iff A \leq B \in \mathcal{F}_{x_\lambda} \iff A \leq B \preccurlyeq_{\mathcal{F}_{x_\lambda}} B \iff A \preccurlyeq_{\mathcal{F}_{x_\lambda}} B.$$

Therefore $\preccurlyeq_{x_\lambda} = \preccurlyeq_{\mathcal{F}_{x_\lambda}}$.

(2) For any $D \in L^X$, it is clear that $D \preccurlyeq D$ if and only if $D \in \mathcal{A}_{\preccurlyeq}$. For any $F \in L^X$,

$$F \in \mathcal{F}_{\preccurlyeq_{x_\lambda}^{\preccurlyeq}} \iff F \preccurlyeq_{x_\lambda}^{\preccurlyeq} F \iff x_\lambda \leq F \preccurlyeq F \iff x_\lambda \leq F \in \mathcal{A}_{\preccurlyeq} \iff F \in \mathcal{N}_{x_\lambda}^{\mathcal{A}_{\preccurlyeq}}.$$

Thus $\mathcal{F}_{\preccurlyeq_{x_\lambda}^{\preccurlyeq}} = \mathcal{N}_{x_\lambda}^{\mathcal{A}_{\preccurlyeq}}$. Also, for any $D \in L^X$, it is clear that $D \in \mathcal{A}$ if and only if $D \preccurlyeq_{\mathcal{A}} D$. For any $A, B \in L^X$,

$$\begin{aligned} A \preccurlyeq_{x_\lambda}^{\preccurlyeq_{\mathcal{A}}} B &\iff \exists D \in L^X \text{ s.t. } x_\lambda \leq D \preccurlyeq_{\mathcal{A}} D \leq D \vee A \leq B \\ &\iff \exists D \in \mathcal{A} \text{ s.t. } x_\lambda \leq D \leq D \vee A \leq B \\ &\iff A \leq B \in \mathcal{N}_{x_\lambda}^{\mathcal{A}} \\ &\iff A \preccurlyeq_{x_\lambda}^{\mathcal{A}} B. \end{aligned}$$

Therefore $\leqslant_{x_\lambda}^{\leqslant_{\mathcal{A}}}=\leqslant_{x_\lambda}^{\mathcal{A}}$.

(3) Its proof is direct.

(4) We verify that $\mathcal{F}_{f(X)}$ satisfies (LCF1)–(LCF3).

(LCF1) It is clear that $\underline{\perp} \notin \mathcal{F}_{f(X)}$ and $\underline{\top} \in \mathcal{F}_{f(X)}$ since $f_L^{\leftarrow}(\underline{\perp}) = \underline{\perp} \notin \mathcal{F}_X$ and $f_L^{\leftarrow}(\underline{\top}) = \underline{\top} \in \mathcal{F}_X$.

(LCF2) If $A \in \mathcal{F}_{f(X)}$ and $A \leq B \in L^Y$ then $f_L^{\leftarrow}(A) \in \mathcal{F}_X$. Thus $f_L^{\leftarrow}(A) \leq f_L^{\leftarrow}(B) \in \mathcal{F}_X$ which implies $B \in \mathcal{F}_{f(X)}$.

(LCF3) If $\{A_i\}_{i\in I}^{ddir} \subseteq \mathcal{F}_{f(X)}$ then $f_L^{\leftarrow}(\bigwedge_{i\in I}^{ddir} A_i) = \bigwedge_{i\in I}^{ddir} f_L^{\leftarrow}(A_i) \in \mathcal{F}_X$. Thus $\bigwedge_{i\in I}^{ddir} A_i \in \mathcal{F}_{f(X)}$. □

By Theorem 4.5, there is a one-to-one correspondence between L-concave filters and L-down–directed relations. Next, we introduce L-filter convergence and L-concave filter convergence and discuss their relationships with L-down–directed relations and L-concave down–directed convergence relations. For this, we present the following lemma.

Definition 4.7. A mapping $\lim : \mathcal{F}_c(L^X) \longrightarrow 2^{J(L^X)}$ is called an L-filter convergence structure and the pair $(X, \lim)$ is called an L-filter convergence space if lim satisfies

(LFC1) $\forall x_\lambda \in J(L^X)$, $x_\lambda \in \lim(\mathcal{F}_{x_\lambda})$;

(LFC2) $\forall \mathcal{F}_1, \mathcal{F}_2 \in \mathcal{F}(L^X)$, $\mathcal{F}_1 \subseteq \mathcal{F}_2$ implies $\lim(\mathcal{F}_1) \subseteq \lim(\mathcal{F}_2)$;

(LFC3) $\mathcal{F}_{x_\lambda}^{\lim} = \bigcap_{\mu\in\beta^*(\lambda)} \mathcal{F}_{x_\mu}^{\lim}$, where $\mathcal{F}_{x_\lambda}^{\lim} = \bigcap_{x_\lambda\in\lim(\mathcal{F})} \mathcal{F}$.

Let $(X, \lim_X)$ and $(Y, \lim_Y)$ be L-filter convergence spaces. A mapping $f : X \longrightarrow Y$ is called an L-filter convergence preserving mapping, if $x_\lambda \in \lim_X(\mathcal{F})$ implies $f_L^{\rightarrow}(x_\lambda) \in \lim_Y(\mathcal{F}_{f(X)})$ for any $x_\lambda \in J(L^X)$ and $\mathcal{F}_X \in \mathcal{F}_c(L^X)$. The category of any L-filter convergence spaces and L-filter convergence preserving mappings is denoted by L-**FCS**.

Lemma 4.8. *Let $(X, \sqsubseteq)$ be an L-down–directed convergence relation space. Define $\lim_{\sqsubseteq} : \mathcal{F}_c(L^X) \longrightarrow J(L^X)$ by*

$$\forall \mathcal{F} \in \mathcal{F}_c(L^X),\ \ \lim_{\sqsubseteq}(\mathcal{F}) = \{x_\lambda \in J(L^X) : x_\lambda \sqsubseteq \leqslant_{\mathcal{F}}\}.$$

Then $\mathcal{F}_{x_\lambda}^{\lim_{\sqsubseteq}} = \{A \in L^X : A \leqslant_{x_\lambda}^{\sqsubseteq} A\}$ for any $x_\lambda \in J(L^X)$.

Proof. Let $A \in L^X$. Theorem 4.5 implies that

$$\begin{aligned}
A \in \mathcal{F}_{x_\lambda}^{\lim_{\sqsubseteq}} &\iff \forall \mathcal{F} \in \mathcal{F}_c(L^X),\ \ x_\lambda \in \lim_{\sqsubseteq}(\mathcal{F}) \ \text{ implies } \ A \in \mathcal{F} \\
&\iff \forall \mathcal{F} \in \mathcal{F}_c(L^X),\ \ x_\lambda \sqsubseteq \leqslant_{\mathcal{F}} \ \text{ implies } \ A \leqslant_{\mathcal{F}} A \\
&\iff \forall \leqslant \in \mathcal{R}^{ddir}(L^X),\ \ x_\lambda \sqsubseteq \leqslant_{\mathcal{F}_{\leqslant}} \ \text{ implies } A \in \mathcal{F}_{\leqslant} \\
&\iff \forall \leqslant \in \mathcal{R}^{ddir}(L^X),\ \ x_\lambda \sqsubseteq \leqslant \ \text{ implies } \ A \leqslant A \\
&\iff \forall \leqslant \in \mathcal{R}_{\sqsubseteq}^{ddir}(x_\lambda),\ \ A \leqslant A \\
&\iff A \leqslant_{x_\lambda}^{\sqsubseteq} A.
\end{aligned}$$

This shows that $\mathcal{F}_{x_\lambda}^{\lim_{\sqsubseteq}} = \{A \in L^X : A \leqslant_{x_\lambda}^{\sqsubseteq} A\}$. □

Theorem 4.9. *Let $(X, \sqsubseteq)$ be an L-down–directed convergence relation space. Define $\lim_{\sqsubseteq} : \mathcal{F}_c(L^X) \longrightarrow J(L^X)$ by*

$$\forall \mathcal{F} \in \mathcal{F}_c(L^X),\ \ \lim_{\sqsubseteq}(\mathcal{F}) = \{x_\lambda \in J(L^X) : x_\lambda \sqsubseteq \leqslant_{\mathcal{F}}\}.$$

Then $\lim_{\sqsubseteq}$ is an L-filter convergence structure on L^X.

Proof. (LFC1) For any $x_\lambda \in J(L^X)$, (LDDCR1) yields that $x_\lambda \sqsubseteq \leqslant_{x_\lambda}$. Since $\leqslant_{\mathcal{F}_{x_\lambda}} = \leqslant_{x_\lambda}$ by Lemma 4.6(1), it is clear that $x_\lambda \sqsubseteq \leqslant_{\mathcal{F}_{x_\lambda}}$. Thus $x_\lambda \in \lim_{\sqsubseteq}(\mathcal{F}_{x_\lambda})$.

(LFC2) If $\mathcal{F}_1 \subseteq \mathcal{F}_2$ and $x_\lambda \in \lim_{\sqsubseteq}(\mathcal{F}_1)$, then $\leqslant_{\mathcal{F}_1} \leq \leqslant_{\mathcal{F}_2}$ and $x_\lambda \sqsubseteq \leqslant_{\mathcal{F}_1}$. Thus $x_\lambda \sqsubseteq \leqslant_{\mathcal{F}_2}$ by (LDDR2). Hence $x_\lambda \in \lim_{\sqsubseteq}(\mathcal{F}_2)$. Therefore $\lim_{\sqsubseteq}(\mathcal{F}_1) \subseteq \lim_{\sqsubseteq}(\mathcal{F}_2)$.

(LFC3) Let $x_\lambda \in J(L^X)$ and $A \in L^X$. Then Lemma 4.8 and (LDDCR3) imply that

$$\begin{aligned} A \in \mathcal{F}_{x_\lambda}^{\lim_\sqsubseteq} &\iff A \leqslant_{x_\lambda}^{\sqsubseteq} A \\ &\iff \forall \mu \in \beta^*(\lambda),\ A \leqslant_{x_\mu}^{\sqsubseteq} F \\ &\iff \forall \mu \in \beta^*(\lambda),\ A \in \mathcal{F}_{x_\mu}^{\lim_\sqsubseteq} \\ &\iff F \in \bigcap_{\mu \in \beta^*(\lambda)} \mathcal{F}_{x_\mu}^{\lim_\sqsubseteq}. \end{aligned}$$

Hence $\mathcal{F}_{x_\lambda}^{\lim_\sqsubseteq} = \bigcap_{\mu \in \beta^*(\lambda)} \mathcal{F}_{x_\mu}^{\lim_\sqsubseteq}$.

Therefore $\lim_\sqsubseteq$ is an L-filter convergence structure. □

Theorem 4.10. *Let $(X, \sqsubseteq_X)$ and $(Y, \sqsubseteq_Y)$ be L-down–directed convergence relation spaces. If $f : X \longrightarrow Y$ is an L-down–directed convergence relation preserving mapping with respect to $(X, \sqsubseteq_X)$ and $(Y, \sqsubseteq_Y)$, then $f : (X, \lim_{\sqsubseteq_X}) \longrightarrow (Y, \lim_{\sqsubseteq_Y})$ is an L-filter convergence preserving mapping.*

Proof. Let $\mathcal{F} \in \mathcal{F}(L^X)$ and let $A, B \in L^Y$. Then

$$\begin{aligned} A(\leqslant_{\mathcal{F}})_{f(X)}B &\iff A \le B \text{ and } f_L^{\leftarrow}(A) \leqslant_{\mathcal{F}} f_L^{\leftarrow}(B) \\ &\iff A \le B \text{ and } f_L^{\leftarrow}(A) \le f_L^{\leftarrow}(B) \in \mathcal{F} \\ &\iff A \le B \in \mathcal{F}_{f(X)} \\ &\iff A \le B \leqslant_{\mathcal{F}_{f(X)}} B \\ &\iff A \leqslant_{\mathcal{F}_{f(X)}} B. \end{aligned}$$

Thus $(\leqslant_{\mathcal{F}})_{f(X)} = \leqslant_{\mathcal{F}_{f(X)}}$. For any $x_\lambda \in J(L^X)$, it follows that

$$\begin{aligned} x_\lambda \in \lim_{\sqsubseteq_X}(\mathcal{F}) &\iff x_\lambda \sqsubseteq_X \leqslant_{\mathcal{F}} \\ &\implies f_L^{\rightarrow}(x_\lambda) \sqsubseteq_Y (\leqslant_{\mathcal{F}})_{f(X)} \\ &\iff f_L^{\rightarrow}(x_\lambda) \sqsubseteq_Y \leqslant_{\mathcal{F}_{f(X)}} \\ &\iff f_L^{\rightarrow}(x_\lambda) \in \lim_{\sqsubseteq_Y}(\mathcal{F}_{f(X)}). \end{aligned}$$

Therefore f is an L-down–directed convergence relation preserving mapping. □

Theorem 4.11. *Let $(X, \lim)$ be an L-filter convergence space. Define a relation $\sqsubseteq_{\lim}$ on $J(L^X) \times \mathcal{R}^{ddir}(L^X)$ by*

$$\forall x_\lambda \in J(L^X), \forall \leqslant \in \mathcal{R}^{ddir}(L^X),\ x_\lambda \sqsubseteq_{\lim} \leqslant \iff x_\lambda \in \lim(\mathcal{F}_\leqslant).$$

Then $\sqsubseteq_{\lim}$ is an L-down–directed convergence relation satisfying $\leqslant_{x_\lambda}^{\sqsubseteq_{\lim}} = \leqslant_{\mathcal{F}_{x_\lambda}^{\lim}}$.

Proof. (LDDCR1) $\mathcal{F}_{\leqslant_{x_\lambda}} = \mathcal{F}_{x_\lambda}$ by Lemma 4.6(1). Since $x_\lambda \in \lim(\mathcal{F}_{x_\lambda})$ by (LFC1), it follows that $x_\lambda \sqsubseteq_{\lim} \leqslant_{x_\lambda}$.

(LDDCR2) Let $\leqslant_1, \leqslant_2 \in \mathcal{R}^{ddir}(L^X)$ with $\leqslant_1 \le \leqslant_2$ and $x_\lambda \sqsubseteq_{\lim} \leqslant_1$. Then $\mathcal{F}_{\leqslant_1} \subseteq \mathcal{F}_{\leqslant_2}$ and $x_\lambda \in \lim(\mathcal{F}_{\leqslant_1}) \subseteq \lim(\mathcal{F}_{\leqslant_2})$. Thus $x_\lambda \sqsubseteq_{\lim} \leqslant_2$.

(LDDCR3). Let $x_\lambda \in J(L^X)$. We check that $\leqslant_{x_\lambda}^{\sqsubseteq_{\lim}} = \leqslant_{\mathcal{F}_{x_\lambda}^{\lim}}$. Indeed, for any $A, B \in L^X$, Theorem 4.5 yields that

$$\begin{aligned} A \leqslant_{x_\lambda}^{\sqsubseteq_{\lim}} B &\iff \forall \leqslant \in \mathcal{R}_{\sqsubseteq_{\lim}}^{ddir}(x_\lambda),\ A \leqslant B \\ &\iff \forall \leqslant \in \mathcal{R}^{ddir}(L^X),\ x_\lambda \in \lim(\mathcal{F}_\leqslant) \text{ implies } A \leqslant B \\ &\iff \forall \leqslant \in \mathcal{R}^{ddir}(L^X),\ x_\lambda \in \lim(\mathcal{F}_\leqslant) \text{ implies } A \le B \in \mathcal{F}_\leqslant \\ &\iff \forall \mathcal{F} \in \mathcal{F}(L^X),\ x_\lambda \in \lim(\mathcal{F}_{\leqslant_{\mathcal{F}}}) \text{ implies } A \le B \in \mathcal{F}_{\leqslant_{\mathcal{F}}} \\ &\iff \forall \mathcal{F} \in \mathcal{F}(L^X),\ x_\lambda \in \lim(\mathcal{F}) \text{ implies } A \le B \in \mathcal{F} \\ &\iff A \le B \in \mathcal{F}_{x_\lambda}^{\lim} \\ &\iff A \leqslant_{\mathcal{F}_{x_\lambda}^{\lim}} B. \end{aligned}$$

Thus $\leqslant_{x_\lambda}^{\sqsubseteq_{\lim}}=\leqslant_{\mathcal{F}_{x_\lambda}^{\lim}}$. For any $A, B \in L^X$, (LFC3) implies that

$$\begin{aligned} A \leqslant_{x_\lambda}^{\sqsubseteq_{\lim}} B &\iff A \leqslant_{\mathcal{F}_{x_\lambda}^{\lim}} B \\ &\iff A \leqslant_{\bigcap_{\mu\in\beta^*(\lambda)}\mathcal{F}_{x_\mu}^{\lim}} B \\ &\iff \forall \mu \in \beta^*(\lambda),\ A \leqslant_{\mathcal{F}_{x_\mu}^{\lim}} B \\ &\iff \forall \mu \in \beta^*(\lambda),\ A \leqslant_{x_\mu}^{\sqsubseteq_{\lim}} B. \end{aligned}$$

Hence (LDDCR3) holds for $\sqsubseteq_{\lim}$.

Therefore $\sqsubseteq_{\lim}$ is an L-filter convergence relation satisfying $\leqslant_{x_\lambda}^{\sqsubseteq_{\lim}}=\leqslant_{\mathcal{F}_{x_\lambda}^{\lim}}$. □

Theorem 4.12. *Let $(X, \lim_X)$ and $(Y, \lim_Y)$ be L-filter convergence spaces with respect to $(X, \lim_X)$ and $(Y, \lim_Y)$. If $f : X \longrightarrow Y$ is an L-filter convergence preserving mapping, then $f : (X, \sqsubseteq_{\lim_X}) \to (Y, \sqsubseteq_{\lim_Y})$ is an L-filter convergence preserving mapping.*

Proof. Let $x_\lambda \in J(L^X)$ and $\leqslant\in \mathcal{R}^{ddir}(L^X)$ with $x_\lambda \sqsubseteq_{\lim_X}\leqslant$. Thus $x_\lambda \in \lim_X(\mathcal{F}_\leqslant)$. Hence $f_L^{\rightarrow}(x_\lambda) \in \lim_Y((\mathcal{F}_\leqslant)_{f(X)})$. So $f_L^{\rightarrow}(x_\lambda) \in \lim_Y(\mathcal{F}_{\leqslant_{f(X)}})$ which implies that $f_L^{\rightarrow}(x_\lambda) \sqsubseteq_{\lim_Y}\leqslant_{f(X)}$. Therefore f is an L-filter convergence relation preserving mapping. □

Theorem 4.13. *Let $(X, \sqsubseteq)$ be an L-filter convergence relation space. Then $\sqsubseteq_{\lim_\sqsubseteq}=\sqsubseteq$.*

Proof. Let $x_\lambda \in J(L^X)$ and $\leqslant\in \mathcal{R}^{ddir}(L^X)$. By Theorem 4.5, it follows that

$$x_\lambda \sqsubseteq_{\lim_\sqsubseteq}\leqslant \iff x_\lambda \in \lim_\sqsubseteq(\mathcal{F}_\leqslant) \iff x_\lambda \sqsubseteq\leqslant_{\mathcal{F}_\leqslant} \iff x_\lambda \sqsubseteq\leqslant .$$

This shows that $\sqsubseteq_{\lim_\sqsubseteq}=\sqsubseteq$. □

Theorem 4.14. *Let $(X, \lim)$ be an L-filter convergence space. Then* $\lim_{\sqsubseteq_{\lim}} = \lim$.

Proof. Let $x_\lambda \in J(L^X)$ and $\mathcal{F} \in \mathcal{F}_c(L^X)$. It follows from Theorem 4.5 that

$$x_\lambda \in \lim_{\sqsubseteq_{\lim}}(\mathcal{F}) \iff x_\lambda \sqsubseteq_{\lim}\leqslant_\mathcal{F} \iff x_\lambda \in \lim(\mathcal{F}_{\leqslant_\mathcal{F}}) \iff x_\lambda \in \lim(\mathcal{F}).$$

Thus $\lim_{\sqsubseteq_{\lim}} = \lim$. □

Based on Theorems 4.9 and 4.10, we obtain a functor $\mathbb{T} : L$-**DDCRS** $\longrightarrow L$-**FCS** by

$$\mathbb{T}((X, \sqsubseteq)) = (X, \lim_\sqsubseteq) \quad \text{and} \quad \mathbb{T}(f) = f.$$

Based on Theorems 4.9–4.14, $\mathbb{T}$ is an isomorphic functor. Thus we have the following result.

Theorem 4.15. *The category L-**DDCRS** is isomorphic to the category L-**FCS**.*

Definition 4.16. An L-filter convergence structure $\lim : \mathcal{F}_c(L^X) \longrightarrow 2^{J(L^X)}$ is called an L-concave filter convergence structure and the pair $(X, \lim)$ is called an L-concave filter convergence space if

(LCFC1) $x_\lambda \in \lim(\mathcal{F}_{x_\lambda}^{\lim})$;

(LCFC2) $A \in \mathcal{F}_{x_\lambda}^{\lim}$ if and only if there is a set $B \in L^X$ such that $x_\lambda \leq B \leq A$ and $B \in \mathcal{F}_{x_\mu}^{\lim}$ for any $y_\mu \in \beta^*(B)$.

The category of any L-concave filter convergence spaces and L-filter convergence preserving mappings is denoted by L-**CFCS**.

Theorem 4.17. *If $(X, \sqsubseteq)$ is an L-concave down–directed convergence relation space then $(X, \lim_\sqsubseteq)$ is an L-concave filter convergence space.*

Proof. Based on Theorem 4.9, $\lim_{\sqsubseteq}$ is an L-filter convergence . It is sufficient to check that (LCFC1) and (LCFC2) hold for $\lim_{\sqsubseteq}$.

(LCFC1) Let $x_\lambda \in J(L^X)$. Then $\preccurlyeq^{\sqsubseteq}_{x_\lambda} = \preccurlyeq^{\sqsubseteq_{\lim_{\sqsubseteq}}}_{x_\lambda} = \preccurlyeq_{\mathcal{F}^{\lim_{\sqsubseteq}}_{x_\lambda}}$ by Theorems 4.13 and 4.11. Since $x_\lambda \sqsubseteq \preccurlyeq^{\sqsubseteq}_{x_\lambda}$ by (LCDDCR1), it follows that $x_\lambda \sqsubseteq \preccurlyeq_{\mathcal{F}^{\lim_{\sqsubseteq}}_{x_\lambda}}$. Thus $x_\lambda \in \lim_{\sqsubseteq}(\mathcal{F}^{\lim_{\sqsubseteq}}_{x_\lambda})$.

(LCFC2) Let $A \in L^X$. Then Lemma 4.8 and (LCDDCR2) yield that

$$\begin{aligned} A \in \mathcal{F}^{\lim_{\sqsubseteq}}_{x_\lambda} &\iff A \preccurlyeq^{\sqsubseteq}_{x_\lambda} A \\ &\iff \exists D \in L^X \text{ s.t. } \forall y_\mu \in \beta^*(D),\ x_\lambda \leq D \preccurlyeq^{\sqsubseteq}_{y_\mu} D \leq D \vee A \leq A \\ &\iff \exists x_\lambda \leq D \leq A \text{ s.t. } \forall y_\mu \in \beta^*(D),\ D \in \mathcal{F}^{\lim_{\sqsubseteq}}_{y_\mu}. \end{aligned}$$

This shows that (LCFC2) holds for $\lim_{\sqsubseteq}$. $\square$

Theorem 4.18. *If* $(X, \lim)$ *is an L-concave filter convergence space then* $(X, \sqsubseteq_{\lim})$ *is an L-concave down–directed convergence relation space.*

Proof. Based on Theorem 4.11, $\sqsubseteq_{\lim}$ is an L-down–directed convergence relation. Thus it is sufficient to prove that (LCDDCR1) and (LCDDFCR2) hold for $\sqsubseteq_{\lim}$.

(LCDDCR1) Let $x_\lambda \in J(L^X)$. Then $\preccurlyeq^{\sqsubseteq_{\lim}}_{x_\lambda} = \preccurlyeq_{\mathcal{F}^{\lim}_{x_\lambda}}$ by Theorem 4.11. Since $x_\lambda \in \lim(\mathcal{F}^{\lim}_{x_\lambda})$ by (LCFC1), it follows that $x_\lambda \sqsubseteq_{\lim} \preccurlyeq_{\mathcal{F}^{\lim}_{x_\lambda}}$. This shows that $x_\lambda \sqsubseteq_{\lim} \preccurlyeq^{\sqsubseteq_{\lim}}_{x_\lambda}$.

(LCDDCR2) Let $A, B \in L^X$. It follows from Theorem 4.11 and (LDDR2) that

$$A \preccurlyeq^{\sqsubseteq_{\lim}}_{x_\lambda} B \iff A \preccurlyeq_{\mathcal{F}^{\lim}_{x_\lambda}} B \iff A \leq B \preccurlyeq_{\mathcal{F}^{\lim}_{x_\lambda}} B \iff A \leq B \in \mathcal{F}^{\lim}_{x_\lambda}.$$

Further, it follows from (LCFC2) and Theorem 4.11 that

$$\begin{aligned} A \preccurlyeq^{\sqsubseteq_{\lim}}_{x_\lambda} B &\iff A \leq B \in \mathcal{F}^{\lim}_{x_\lambda} \\ &\iff \exists x_\lambda \leq D \vee A \leq B \text{ s.t. } \forall y_\mu \in \beta^*(D),\ D \in \mathcal{F}^{\lim}_{y_\mu} \\ &\iff \exists x_\lambda \leq D \vee A \leq B \text{ s.t. } \forall y_\mu \in \beta^*(D),\ D \preccurlyeq_{\mathcal{F}^{\lim}_{y_\mu}} D \\ &\iff \exists x_\lambda \leq D \vee A \leq B \text{ s.t. } \forall y_\mu \in \beta^*(D),\ D \preccurlyeq^{\sqsubseteq_{\lim}}_{y_\mu} D \\ &\iff \exists D \in L^X \text{ s.t. } \forall y_\mu \in \beta^*(D),\ x_\lambda \leq D \preccurlyeq^{\sqsubseteq_{\lim}}_{y_\mu} D \leq D \vee A \leq B. \end{aligned}$$

Thus (LCFCR2) hold for $\sqsubseteq_{\lim}$. $\square$

Based on Theorems 4.15, 4.17 and 4.18, we have the following conclusion.

Theorem 4.19. *The category L-***CFCRS** *is isomorphic to the category L-***CFCS***.*

Remark 4.20. Based on Theorems 3.20 and 4.19, relationships between L-**CFCS** and L-**CIRS** are present as follows.

(1) Let $(X, \lim)$ be an L-concave filter convergence space. Define a binary relation $\preccurlyeq_{\lim}$ on L^X by

$$\forall A, B \in L^X,\ A \preccurlyeq_{\lim} B \iff \exists A \leq D \leq B,\ \forall x_\lambda \in \beta^*(D),\ D \in \mathcal{F}^{\lim}_{x_\lambda}.$$

Then $\preccurlyeq_{\lim}$ is an L-concave internal relation on L^X.

(2) Let $(X, \preccurlyeq)$ be an L-concave internal relation space. Define a mapping $\lim_{\preccurlyeq}$ on $\mathcal{F}(L^X) \times 2^{J(L^X)}$ by

$$\forall \mathcal{F} \in \mathcal{F}(L^X),\ \lim_{\preccurlyeq}(\mathcal{F}) = \{x_\lambda \in J(L^X) : \preccurlyeq^{\preccurlyeq}_{x_\lambda} \leq \preccurlyeq_{\mathcal{F}}\}.$$

Then $\lim_{\preccurlyeq}$ is an L-concave filter convergence structure on L^X.

(3) $\preccurlyeq_{\lim_{\preccurlyeq}} = \preccurlyeq$ and $\lim_{\preccurlyeq_{\lim}} = \lim$.

(4) The category L-**CFCS** is isomorphic to the category L-**CIRS**.

Remark 4.21. Based on Remark 4.20 and Theorem 2.6, relationships between L-**CFCS** and L-**CAS** are as follows.

(1) Let $(X, \lim)$ be an L-concave filter convergence space. Then the set

$$\mathcal{A}_{\lim} = \{A \in L^X : \forall x_\lambda \in \beta^*(A),\ A \in \mathcal{F}^{\lim}_{x_\lambda}\}$$

is an L-concave structure on L^X.

(2) Let $(X, \mathcal{A})$ be an L-concave space. Define a mapping $\lim_{\mathcal{A}} : \mathcal{F}(L^X) \longrightarrow 2^{J(L^X)}$ by

$$\forall \mathcal{F} \in \mathcal{F}(L^X),\ \lim_{\mathcal{A}}(\mathcal{F}) = \{x_\lambda \in J(L^X) : \mathcal{N}^{\mathcal{A}}_{x_\lambda} \subseteq \mathcal{F}\}.$$

Then $\lim_{\mathcal{A}}$ is an L-concave filter convergence structure on L^X.

(3) $\mathcal{A}_{\lim_{\mathcal{A}}} = \mathcal{A}$ and $\lim_{\mathcal{A}_{\lim}} = \lim$.

(4) The category L-**CFCS** is isomorphic to the category L-**CAS**.

Conclusions

In this paper, we introduced L-down–directed relations, L-down–directed convergence relations and L-concave down–directed convergence relations. We proved that the category of L-concave internal relation spaces can be embedded into the category of L-down–directed convergence relation spaces as a reflective subcategory, and that the category of L-concave down–directed convergence relation spaces is isomorphic to the category of L-concave internal relation spaces. We further introduced L-concave filters, L-filter convergence spaces and L-concave filter convergence spaces. We prove that that L-filter convergence space and L-down–directed convergence relation spaces are isomorphic. In addition, we also proved that L-concave down–directed convergence relation spaces, L-concave filter convergence spaces, L-concave internal relations and L-concave space are all categorically isomorphic.

In [26], Xiu et al introduced notions of L-convex ideals and L-convergence structures. Indeed, if L is a complete lattice with an inverse involution, then L-concave filter is a dual concept of L-convex ideal. Similarly, L-concave filter convergence structure is a dual concept of L-convergence structure. However, L-concave filter convergence structure can adapt to a more general environment where the complete lattice L has no inverse involution.

In recent years, fuzzy relations have been applied to many mathematical structures such as L-topological spaces, (L, M)-fuzzy topological spaces, L-concave space, L-convex spaces, (L, M)-fuzzy convex spaces and M-fuzzifying convex spaces [5, 17, 18, 22, 24]. Thus fuzzy relations may provide some alternative ways to study convergence structures in these spaces.

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References

[1] M. Güloğlu, D. Coker, Convergence in I-fuzzy topological spaces, Fuzzy Sets Syst. 151 (2005) 615–623.
[2] U. Höhle, A.P. Šostak, Axiomatic foundations of fixed-basis fuzzy topology, in: U. Höhle, S.E. Rodabaugh (Eds), Mathematics of Fuzzy Sets: Logic, Topolgoy, and Measure Theory, in: Handbook Series, vol. 3, Kluwer Academic Publishers, Boston, Dordrecht, London, 1999.
[3] G. Jäger, Lattice-valued convergence spaces and regularity, Fuzzy Sets Syst. 159 (2008) 2248–2502.
[4] L.Q. Li, Q. Jin, K. Hu, Lattice-valued convergence associated with CNS spaces, Fuzzy Sets Syst. 370 (2019) 91–98.
[5] C.Y. Liao, X.Y. Wu, L-topological-convex spaces generated by L-convex bases, Open Math. 17 (2019) 1547-1566.
[6] Y. Maruyama, Lattice-valued fuzzy convex geometry, RIMS Kokyroku 1641 (2009) 22–37.
[7] B. Pang, On (L, M)-fuzzy convergence spaces, Fuzzy Sets Syst. 238 (2014) 46–70.
[8] B. Pang, F.G. Shi, Subcategories of the category of L-convex spaces, Fuzzy Sets Syst. 313 (2017) 61–74.
[9] B. Pang, F.G. Shi, Convenient properties of stratified L-convergence tower spaces, Filomat, 33 (2019) 4811-4825.
[10] B. Pang, Convergence structures in M-fuzzifying convex spaces, Quaes. Math. 43 (2020) 1541–1561.

[11] B. Pang, Hull operators and interval operators in (L, M)-fuzzy convex spaces, Fuzzy Sets Syst. 405 (2021) 106–127.
[12] B. Pang, Fuzzy convexities via overlap functions, IEEE Trans. Fuzzy Syst. (2022). DOI: 10.1109/TFUZZ.2022.3194354.
[13] F.G. Shi and Z.Y. Xiu, A new approach to the fuzzification of convex structures, J. Appl. Math. 2014 (2014) 1-12.
[14] F.G. Shi, Z. Y. Xiu, (L, M)-fuzzy convex structures, J. Nonlinear Sci. Appl. 10 (2017) 3655–3669.
[15] F.G. Shi. (L, M)-fuzzy matroids. Fuzzy Sets Syst. 160 (2009) 2387–2400.
[16] F.G. Shi, L-fuzzy interiors and L-fuzzy closures, Fuzzy Sets Syst. 160 (2009) 1218–1232.
[17] Y. Shi, F.G. Shi, Characterizations of L-topologies, J. Intell. Fuzzy Syst. 34 (2018) 613–623.
[18] Y. Shi, F.G. Shi, (L, M)-fuzzy internal relations and (L, M)-fuzzy enclosed relations, J. Intell. Fuzzy Syst. 36 (2019) 5153–5165.
[19] M.L.J. van de Vel, Theory of convex structures, Noth-Holland, Amsterdam, 1993.
[20] K. Wang, F.G. Shi, M-fuzzifying topological convex spaces, Iran. J. Fuzzy Syst. 15 (2018) 159–174.
[21] X.Y. Wu, B. Davvaz, S.Z. Bai, On M-fuzzifying convex matroids and M-fuzzifying independent structures, J. Intell. Fuzzy Syst. 33 (2017) 269–280.
[22] X.Y. Wu, F.G. Shi. L-concave bases and L-topological-concave spaces. J. Intell. Fuzzy Syst. 35 (2018) 4731–4743.
[23] X.Y. Wu, E.Q. Li, Category and subcategories of (L, M)-fuzzy convex spaces, Iran. J. Fuzzy Syst. 15 (2019) 129–146.
[24] X.Y. Wu, C.Y. Liao, (L, M)-fuzzy topological-convex spaces, Filomat, 33 (2019) 6435–6451.
[25] X.Y. Wu, Y. Shi, (L, M)-fuzzy topological derived internal relations and (L, M)-fuzzy topological derived enclosed relations, Iran. J. Fuzzy Syst. 19 (2022) 89–106.
[26] Z.Y. Xiu, Q.H. Li and B. Pang, Fuzzy convergence structures in the framework of L-convex spaces, Iran. J. Fuzzy Syst. 17 (2020) 139–150.
[27] Z.Y. Xiu and Q.H. Li, Degrees of L-continuity for mappings between L-topological spaces, Mathematics 7 (2019) 1013–1028.
[28] H. Yang, B. Pang, Fuzzy points based betweenness relations in L-convex spaces, Filomat 35 (2021) 3521–3532.
[29] S.J. Yang, F.G. Shi, M-fuzzifying independent spaces, J. Intell. Fuzzy Syst. 34 (2018) 11–21.
[30] W. Yao, On many-valued stratified L-fuzzy convergence spaces, Fuzzy Sets Syst. 159 (2008) 2503–2519.
[31] L. Zhang, B. Pang, The category of residuated lattice valued filter spaces, Quaes. Math. (2021). https://doi.org/10.2989/16073606.2021.1973140.
[32] L. Zhang, B. Pang, Monoidal closedness of the category of $\top$-semiuniform convergence spaces, Fuzzy Sets Syst. 51 (2022) 1348–1370.
[33] L. Zhang, B. Pang, A new approach to lattice-valued convergence groups via $\top$-filters, Fuzzy Sets Syst. (2022). https://doi.org/10.1016/j.fss.2022.06.026.
[34] F.F. Zhao, B. Pang, Equivalence among L-closure (interior) operators, L-closure (interior) systems and L-enclosed (internal) relations, Filomat 36 (2022) 979–1003.