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Geometric properties of timelike surfaces in Lorentz-Minkowski 3-space

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Abstract. In this paper, the relationships between geodesic torsions, normal curvatures and geodesic curvatures of the parameter curves intersecting at any angle on timelike surfaces in Lorentz-Minkowski 3-space are obtained by various equations. In addition, new equivalents of well-known formulas (O. Bonnet, Euler, Liouville) are found in this space. Finally, the examples of these surfaces are given.

1. Introduction

For thousands of years, Euclid had made the geometry he had established accepted all over the world with his 13-volume book called "Elements" [10]. All of Euclid's geometry was based on five axioms and five postulates. Until the 19^{th} century, the geometry consisting of these axioms and postulates was never questioned. From the 19^{th} century, some mathematicians such as Gauss, Bolyai, Riemann, Lobacevsky began to argue about the precision of these axioms and postulates. They developed opposite axioms to Euclid's axioms and thus new geometries were obtained. These geometries, which can be diversified such as hyperbolic geometry and elliptical geometry, are called non-Euclidean geometries. The interest in these geometries increased rapidly, especially with A. Einstein's demonstration that the 3-dimensional space we live in conforms to non-Euclidean geometry, not Euclidean geometry. In fact, Einstein drew the attention of many disciplines to this field with his statement that "if I had not heard of Riemann's work, I would never have been able to develop the theory of relativity". Since then, these new geometries have been used in many branches of science such as physics, mathematics and astronomy. In fact, Riemann geometry is among the most used geometries of NASA today. H.A. Lorentz and H. Minkowski joined the movement of non-Euclidean geometries in the 1950s and built the Lorentz-Minkowski space. The Lorentz-Minkowski space is an area of great interest to physicists and geometers, and its popularity is increasing day by day with its place in Einstein's theories of relativity, [13, 14]. One of the most popular areas to be studied in Euclidean or non-Euclidean spaces is the differential geometry of curves and surfaces. The basic resources in these spaces are [4, 6, 19, 20, 25–28]. Also, some studies about curves or surfaces on the Lorentz-Minkowski space are [2, 3, 5, 7, 9, 11, 12, 15–18, 21, 23, 24, 29, 30]. In our previous study, we worked on the geometry of spacelike surfaces with the same logic as in this paper, [22]. In this study, timelike surfaces, which are examined under various cases in [8] and whose basic equations are obtained, are examined more broadly. The relationships between of the geodesic torsions, the normal curvatures and the geodesic curvatures

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of the parameter curves intersecting at any angle of timelike surfaces in Lorentz-Minkowski 3-space are obtained with various equations. In addition, the new equivalents of well-known formulas (O. Bonnet, Euler, Liouville) and mean curvature are found. Besides, for all these equations obtained, special cases are examined, assuming that the parameter curves are Lorentzian orthogonal. One of these special cases refers to the resource [26], which is the guide for this paper.

2. Preliminaries

Spaces on which a metric function is defined are called metric spaces. The most familiar example of metric spaces is the Euclidean space. The metric function in Euclidean space is positive definite. Therefore, vectors, curves or surfaces in this space can be studied in a single category. But a metric function need not always be positive definite. For example, for the vectors $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3) \in \mathbb{R}^3$, the metric function defined below

$$\langle , \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R} \, , \quad \langle a, b \rangle = a_1 b_1 + a_2 b_2 - a_3 b_3$$

is not always positive definite. This metric function is called the Lorentz metric (or Lorentz inner product function). The reel vector space $\mathbb{R}^3$ with this metric function is called the Lorentz-Minkowski 3-space and is shown by $\mathbb{R}^3_1$. Here, subindex "1" is the dimensional of the largest dimensional subspace whose Lorentz metric is negative definite, and is defined as the index of the space. In the terminology, $\mathbb{R}^3_1$ is called Lorentz space also, and is known a special case of Minkowski space $\mathbb{R}^n_r$, $0 \le r \le n$. (The state $r = 0$ represents the Euclidean space.) Depending on this metric, in this space a vector a is divided into three classes: timelike ($\langle a, a \rangle < 0$), spacelike ($\langle a, a \rangle > 0$ or $a = 0$) and lightlike ($\langle a, a \rangle = 0$, $a \neq 0$), [19]. With this classification, the causal character of the vector is determined. In Lorentz-Minkowski 3-space, just like vectors, curves and surfaces are divided into three classes. Curves take on the causal character of their tangent vectors. Surfaces are expressed in terms of character of their normal vectors at each point. The surface whose normal vector at each point is timelike (spacelike) is spacelike (timelike) surface. Further, the norm (length) of the vector a is $\|a\| = \sqrt{|\langle a, a \rangle|}$. a is a unit vector, if $\|a\| = 1$. The unit vectors in $\mathbb{R}^3_1$ form the unit spheres according to their characters. The timelike vectors form the hyperbolic unit sphere $\mathbb{H}^2_0 = \left\{ a \in \mathbb{R}^3_1 \mid \langle a, a \rangle = -1 \right\}$ and the spacelike vectors form the Lorentz unit sphere $\mathbb{S}^2_1 = \left\{ a \in \mathbb{R}^3_1 \mid \langle a, a \rangle = 1 \right\}$. For $a, b \in \mathbb{R}^3_1$, if $\langle a, b \rangle = 0$, the vectors a and b are called Lorentzian ortgonal vectors. Besides, just like the inner product function, the vectorial product function in this space is defined differently from that in Euclidean space also:

$$\wedge : \quad \mathbb{R}^3_1 \times \mathbb{R}^3_1 \to \mathbb{R}^3_1 \, , \quad a \wedge b = - \begin{vmatrix} e_1 & e_2 & -e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

The operation $a \wedge b$ is called Lorentz vectorial product of a and b vectors. The causal characters of the vectors in $\mathbb{R}^3_1$ also affect the result of the vector products of these vectors with each other, and this effect causes various cases to occur. When examining the geometry of a surface, the parameter curves that create the surface can intersect perpendicularly or at any angle. The geometry of spacelike or timelike surfaces whose parameter curves intersect perpendicularly in 3- dimensional Lorentz space is given extensively in [25–27]. Akbulut studied the geometry of surfaces whose pamameter curves do not intersect perpendicularly in 3-dimensional Euclidean space, [1]. As a composition of these studies, geometry of spacelike surfaces [22] and geometry of timelike surfaces [8] whose parameter curves intersect under any angle in Lorentz 3-space are studied. While working on a spacelike surface, various cases have not occurred depending on the frames used, while various cases have been obtained on a timelike surface. Depending on these cases, the Darboux frame is shaped in various ways according to the characters of the elements that creat the frame. For example, let's take the Darboux frame $\{t, g, N\}$ on a timelike surface. Here the normal vector N is a spacelike, by definition of a timelike surface. Let's assume that vector g is also spacelike. So, this vector is defined by $g = -N \wedge t$, here the vector t is timelike. In the present case, Darboux vector of this frame is [26]

$$w = \frac{t}{T_g} + \frac{g}{R_n} - \frac{N}{R_g}. \tag{1}$$

Let the parameter curves (c_1) and (c_2) of a timelike surface be two curves that intersect at any angle. Let any curve (c) pass through the point where (c_1) and (c_2) intersect. Six different cases appear on these surface with different combinations of the characters of the elements of the Darboux frames of (c), (c_1) and (c_2), [8]. Many equations have been obtained for these six cases on the surface. There is the equation below between the Darboux vectors w, w_1 and w_2 of (c) , (c_1) and (c_2) for Case 1,

$$w = -\frac{\sinh\left(\theta-\phi\right)}{\cosh\theta}w_1 + \frac{\cosh\phi}{\cosh\theta}w_2 - \frac{d\phi}{ds}N, \tag{2}$$

for other cases, you can examine the paper [8].

3. Geometric Properties of Timelike Surfaces in Lorentz-Minkowski 3-Space

In all theorems and results given throughout the article, the data in the next sentence will be considered. Let the hyperbolic angle between the tangent vector t_1 of the parameter curve (c_1) and the tangent vector t of any curve (c) be ϕ, and the tangent vectors t_1 and t_2 of parameter curves (c_1) and (c_2) intersect under the hyperbolic angle θ on timelike surface $x = x(u, v)$. Let (c_0) be a curve perpendicular to (c), this curve will be examined in the Cases 2, 4 and 6.

Theorem 3.1. *Let radii of normal curvature be $(R_n)_1$, $(R_n)_2$ and radii of geodesic torsion be $\left(T_g\right)_1$, $\left(T_g\right)_2$ of the curves (c_1) and (c_2) on $x = x(u, v)$, respectively. For the geodesic torsion at the direction t, there are following equations:*

Case 1.

$$\frac{1}{T_g} = \frac{1}{\cosh\theta}\left[-\frac{\sinh\left(\theta-\phi\right)\sinh\phi}{\left(T_g\right)_1} + \frac{\cosh\left(\theta-\phi\right)\cosh\phi}{\left(T_g\right)_2} - \frac{\sinh\left(\theta-\phi\right)\cosh\phi}{(R_n)_1} + \frac{\sinh\left(\theta-\phi\right)\cosh\phi}{(R_n)_2}\right], \tag{3}$$

Case 2.

$$\frac{1}{\left(T_g\right)_0} = \frac{1}{\cosh\theta}\left[\frac{\cosh\left(\theta-\phi\right)\cosh\phi}{\left(T_g\right)_1} - \frac{\sinh\left(\theta-\phi\right)\sinh\phi}{\left(T_g\right)_2} + \frac{\cosh\left(\theta-\phi\right)\sinh\phi}{(R_n)_1} - \frac{\cosh\left(\theta-\phi\right)\sinh\phi}{(R_n)_2}\right], \tag{4}$$

Case 3.

$$\frac{1}{T_g} = \frac{1}{\cosh\theta}\left[\frac{\cosh\left(\theta-\phi\right)\cosh\phi}{\left(T_g\right)_1} - \frac{\sinh\left(\theta-\phi\right)\sinh\phi}{\left(T_g\right)_2} - \frac{\cosh\left(\theta-\phi\right)\sinh\phi}{(R_n)_1} + \frac{\cosh\left(\theta-\phi\right)\sinh\phi}{(R_n)_2}\right],$$

Case 4.

$$\frac{1}{\left(T_g\right)_0} = \frac{1}{\cosh\theta}\left[-\frac{\sinh\left(\theta-\phi\right)\sinh\phi}{\left(T_g\right)_1} + \frac{\cosh\left(\theta-\phi\right)\cosh\phi}{\left(T_g\right)_2} + \frac{\sinh\left(\theta-\phi\right)\cosh\phi}{(R_n)_1} - \frac{\sinh\left(\theta-\phi\right)\cosh\phi}{(R_n)_2}\right],$$

Case 5.

$$\frac{1}{T_g} = \frac{1}{\sinh\theta}\left[\frac{\sinh\left(\theta-\phi\right)\cosh\phi}{\left(T_g\right)_1} + \frac{\cosh\left(\theta-\phi\right)\sinh\phi}{\left(T_g\right)_2} + \frac{\sinh\left(\theta-\phi\right)\sinh\phi}{(R_n)_1} - \frac{\sinh\left(\theta-\phi\right)\sinh\phi}{(R_n)_2}\right],$$

Case 6.

$$\frac{1}{\left(T_g\right)_0} = \frac{1}{\sinh\theta}\left[-\frac{\cosh\left(\theta-\phi\right)\sinh\phi}{\left(T_g\right)_1} - \frac{\sinh\left(\theta-\phi\right)\cosh\phi}{\left(T_g\right)_2} - \frac{\cosh\left(\theta-\phi\right)\cosh\phi}{(R_n)_1} + \frac{\cosh\left(\theta-\phi\right)\cosh\phi}{(R_n)_2}\right].$$

Proof.

For Case 1. If we consider the expression (1), we can write the following expressions

$$w_1 = -\frac{t_1}{\left(T_g\right)_1} - \frac{t_2}{(R_n)_1} + \frac{N}{\left(R_g\right)_1}, \qquad w_2 = \frac{t_2}{\left(T_g\right)_2} + \frac{g_2}{(R_n)_2} - \frac{N}{\left(R_g\right)_2}, \tag{5}$$

[26]. If we substitute the expressions (1) and (5) in the expression (2), apply inner product with t to both sides of resulting expression, the proof is completed. If suitable vectors w, w_1 and w_2 for other cases are considered from [26], proofs of other cases are obtained by similar operations. □

Proposition 3.2. *If t_1 and t_2 are Lorentzian orthogonal vectors in Theorem 3.1, that is $\theta = 0$, then, we achive the equations below:*

Special Case 1. $$\frac{1}{T_g} = \frac{\sinh^2\phi}{\left(T_g\right)_1} + \frac{\cosh^2\phi}{\left(T_g\right)_2} + \left(\frac{1}{(R_n)_1} - \frac{1}{(R_n)_2}\right)\cosh\phi\sinh\phi,$$

Special Case 2. $$\frac{1}{\left(T_g\right)_0} = \frac{\cosh^2\phi}{\left(T_g\right)_1} + \frac{\sinh^2\phi}{\left(T_g\right)_2} + \left(\frac{1}{(R_n)_1} - \frac{1}{(R_n)_2}\right)\cosh\phi\sinh\phi,$$

Special Case 3. $$\frac{1}{T_g} = \frac{\cosh^2\phi}{\left(T_g\right)_1} + \frac{\sinh^2\phi}{\left(T_g\right)_2} + \left(-\frac{1}{(R_n)_1} + \frac{1}{(R_n)_2}\right)\cosh\phi\sinh\phi,$$

Special Case 4. $$\frac{1}{\left(T_g\right)_0} = \frac{\sinh^2\phi}{\left(T_g\right)_1} + \frac{\cosh^2\phi}{\left(T_g\right)_2} + \left(-\frac{1}{(R_n)_1} + \frac{1}{(R_n)_2}\right)\cosh\phi\sinh\phi.$$

In Case 5 and Case 6, since t_1 and t_2 are timelike vectors, they can't be Lorentzian orthogonal vectors. So, there is no Special Cases 5 and 6, [8].

Theorem 3.3 (O. Bonnet Formula). *Let radii of principal curvature of (c_1) and (c_2) on $x(u,v)$ be R_1 and R_2. If we substitute the lines of curvature for the parameter curves on the surface, we obtain following equations for the geodesic torsion of (c), respectively:*

Cases 1 − 2. $$\frac{1}{T_g} = \left(-\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{\sinh\left(\theta-\phi\right)\cosh\phi}{\cosh\theta}, \qquad \frac{1}{\left(T_g\right)_0} = \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{\cosh\left(\theta-\phi\right)\sinh\phi}{\cosh\theta}, \tag{6}$$

Cases 3 − 4. $$\frac{1}{T_g} = \left(-\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{\cosh\left(\theta-\phi\right)\sinh\phi}{\cosh\theta}, \qquad \frac{1}{\left(T_g\right)_0} = \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{\sinh\left(\theta-\phi\right)\cosh\phi}{\cosh\theta},$$

Cases 5 − 6. $$\frac{1}{T_g} = \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{\sinh\left(\theta-\phi\right)\sinh\phi}{\sinh\theta}, \qquad \frac{1}{\left(T_g\right)_0} = \left(-\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{\cosh\left(\theta-\phi\right)\cosh\phi}{\sinh\theta}.$$

Proof.

For Case 1. If we substitute the lines of curvature for the parameter curves on the surface, we have

$$\frac{1}{(T_g)_1} = \frac{1}{(T_g)_2} = 0, \qquad \frac{1}{(R_n)_1} = \frac{1}{R_1}, \qquad \frac{1}{(R_n)_2} = \frac{1}{R_2}, \tag{7}$$

[26]. Equalities in (7) are substituted in the expression (3), we complete the proof. The proof of other cases is done in a similar way. □

Proposition 3.4 (O. Bonnet Formula). *If t_1 and t_2 are Lorentzian orthogonal vectors, then for the geodesic curvature in Theorem 3.3, we obtain the equations below, respectively:*

$$\text{Special Cases } 1-2 \text{ and } 3-4. \quad \frac{1}{T_g}=\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\cosh\phi\sinh\phi \quad \text{and} \quad \frac{1}{T_g}=\left(-\frac{1}{R_1}+\frac{1}{R_2}\right)\cosh\phi\sinh\phi.$$

There is no special cases 5 - 6.

Theorem 3.5. *Let radii of normal curvature be $(R_n)_1$, $(R_n)_2$ and radii of geodesic torsion be $\left(T_g\right)_1$, $\left(T_g\right)_2$ of the curves (c_1) and (c_2) on $x(u,v)$, respectively. There are the following relationships between the geodesic torsions of (c) and (c_0) perpendicular to (c):*

$$\text{Case } 1-2. \quad \frac{1}{T_g}-\frac{1}{\left(T_g\right)_0}=-\frac{1}{\left(T_g\right)_1}+\frac{1}{\left(T_g\right)_2}+\left(-\frac{1}{(R_n)_1}+\frac{1}{(R_n)_2}\right)\frac{\sinh\theta}{\cosh\theta},$$

$$\text{Case } 3-4. \quad \frac{1}{T_g}-\frac{1}{\left(T_g\right)_0}=\frac{1}{\left(T_g\right)_1}-\frac{1}{\left(T_g\right)_2}+\left(-\frac{1}{(R_n)_1}+\frac{1}{(R_n)_2}\right)\frac{\sinh\theta}{\cosh\theta},$$

$$\text{Case } 5-6. \quad \frac{1}{T_g}-\frac{1}{\left(T_g\right)_0}=\frac{1}{\left(T_g\right)_1}+\frac{1}{\left(T_g\right)_2}+\left(\frac{1}{(R_n)_1}-\frac{1}{(R_n)_2}\right)\frac{\cosh\theta}{\sinh\theta}.$$

Proof.

Cases 1 - 2. If we subtract (4) from (3) side by side and the similar operation is applied for other cases also, the proof is completed. □

Corollary 3.6. *In Theorem 3.5, if we use curvature lines as parameter curves on $x(u,v)$, from the expression (7), for the geodesic torsions of (c) and (c_0) perpendicular to the curve (c), we get the following relations, respectively:*

$$\text{Cases } 1-2-3-4. \quad \frac{1}{T_g}-\frac{1}{\left(T_g\right)_0}=\left(-\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{\sinh\theta}{\cosh\theta},$$

$$\text{Cases } 5-6. \quad \frac{1}{T_g}-\frac{1}{\left(T_g\right)_0}=\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{\cosh\theta}{\sinh\theta}.$$

Proposition 3.7. *If t_1 and t_2 are Lorentzian orthogonal vectors in Corollary 3.6, then we have:*

$$\text{Special Cases } 1-2-3-4. \quad \frac{1}{T_g}=\frac{1}{\left(T_g\right)_0}.$$

There is no special cases 5 - 6.

Theorem 3.8. *Let radii of normal curvature be $(R_n)_1$, $(R_n)_2$ and radii of geodesic torsion be $\left(T_g\right)_1$, $\left(T_g\right)_2$ of the curves (c_1) and (c_2) on $x(u,v)$, respectively. For the normal curvature at the direction t, we get:*

$$\text{Case 1.} \quad \frac{1}{R_n}=\frac{1}{\cosh\theta}\left[\left(\frac{1}{\left(T_g\right)_1}+\frac{1}{\left(T_g\right)_2}\right)\sinh\left(\theta-\phi\right)\cosh\phi+\frac{\sinh\left(\theta-\phi\right)\sinh\phi}{(R_n)_1}+\frac{\cosh\left(\theta-\phi\right)\cosh\phi}{(R_n)_2}\right], \quad (8)$$

$$\text{Case 2.} \quad \frac{1}{(R_n)_0}=\frac{1}{\cosh\theta}\left[\left(\frac{1}{\left(T_g\right)_1}+\frac{1}{\left(T_g\right)_2}\right)\cosh\left(\theta-\phi\right)\sinh\phi+\frac{\cosh\left(\theta-\phi\right)\cosh\phi}{(R_n)_1}+\frac{\sinh\left(\theta-\phi\right)\sinh\phi}{(R_n)_2}\right], \quad (9)$$

Case 3. $$\frac{1}{R_n}=\frac{1}{\cosh\theta}\left[-\left(\frac{1}{\left(T_g\right)_1}+\frac{1}{\left(T_g\right)_2}\right)\cosh\left(\theta-\phi\right)\sinh\phi+\frac{\cosh\left(\theta-\phi\right)\cosh\phi}{(R_n)_1}+\frac{\sinh\left(\theta-\phi\right)\sinh\phi}{(R_n)_2}\right],$$

Case 4. $$\frac{1}{(R_n)_0}=\frac{1}{\cosh\theta}\left[-\left(\frac{1}{\left(T_g\right)_1}+\frac{1}{\left(T_g\right)_2}\right)\sinh\left(\theta-\phi\right)\cosh\phi+\frac{\sinh\left(\theta-\phi\right)\sinh\phi}{(R_n)_1}+\frac{\cosh\left(\theta-\phi\right)\cosh\phi}{(R_n)_2}\right],$$

Case 5. $$\frac{1}{R_n}=\frac{1}{\sinh\theta}\left[\left(\frac{1}{\left(T_g\right)_1}-\frac{1}{\left(T_g\right)_2}\right)\sinh\left(\theta-\phi\right)\sinh\phi+\frac{\sinh\left(\theta-\phi\right)\cosh\phi}{(R_n)_1}+\frac{\cosh\left(\theta-\phi\right)\sinh\phi}{(R_n)_2}\right],$$

Case 6. $$\frac{1}{(R_n)_0}=\frac{1}{\sinh\theta}\left[\left(\frac{1}{\left(T_g\right)_1}-\frac{1}{\left(T_g\right)_2}\right)\cosh\left(\theta-\phi\right)\cosh\phi+\frac{\cosh\left(\theta-\phi\right)\sinh\phi}{(R_n)_1}+\frac{\sinh\left(\theta-\phi\right)\cosh\phi}{(R_n)_2}\right].$$

Proof.

For Case 1. If the expression (1) and the expression (5) are substituted in the expression (2) and inner product with g is applied to both sides of the resulting expression, the proof is completed. If suitable vectors w, w_1 and w_2 for other cases are considered from [26], proofs of other cases are obtained by similar operations. □

Proposition 3.9. *If t_1 and t_2 are Lorentzian orthogonal vectors in Theorem 3.8, then for the normal curvature, we have equations below:*

Special Case 1. $$\frac{1}{R_n}=-\left(\frac{1}{\left(T_g\right)_1}+\frac{1}{\left(T_g\right)_2}\right)\cosh\phi\sinh\phi-\frac{\sinh^2\phi}{(R_n)_1}+\frac{\cosh^2\phi}{(R_n)_2},$$

Special Case 2. $$\frac{1}{(R_n)_0}=\left(\frac{1}{\left(T_g\right)_1}+\frac{1}{\left(T_g\right)_2}\right)\cosh\phi\sinh\phi+\frac{\cosh^2\phi}{(R_n)_1}-\frac{\sinh^2\phi}{(R_n)_2},$$

Special Case 3. $$\frac{1}{R_n}=-\left(\frac{1}{\left(T_g\right)_1}+\frac{1}{\left(T_g\right)_2}\right)\cosh\phi\sinh\phi+\frac{\cosh^2\phi}{(R_n)_1}-\frac{\sinh^2\phi}{(R_n)_2},$$

Special Case 4. $$\frac{1}{(R_n)_0}=\left(\frac{1}{\left(T_g\right)_1}+\frac{1}{\left(T_g\right)_2}\right)\cosh\phi\sinh\phi-\frac{\sinh^2\phi}{(R_n)_1}+\frac{\cosh^2\phi}{(R_n)_2}.$$

There is no special cases 5 - 6.

Theorem 3.10 (Euler Formula). *Let radii of principal curvature of (c_1) and (c_2) be R_1 and R_2, respectively. If we substitute the lines of curvature for the parameter curves on $x(u,v)$, then for the normal curvature at the direction t, we have the following equations:*

Case 1. $$\frac{1}{R_n}=\frac{1}{\cosh\theta}\left(\frac{\sinh\left(\theta-\phi\right)\sinh\phi}{R_1}+\frac{\cosh\left(\theta-\phi\right)\cosh\phi}{R_2}\right), \tag{10}$$

Case 2. $$\frac{1}{(R_n)_0}=\frac{1}{\cosh\theta}\left(\frac{\cosh\left(\theta-\phi\right)\cosh\phi}{R_1}+\frac{\sinh\left(\theta-\phi\right)\sinh\phi}{R_2}\right),$$

Case 3. $$\frac{1}{R_n}=\frac{1}{\cosh\theta}\left(\frac{\cosh\left(\theta-\phi\right)\cosh\phi}{R_1}+\frac{\sinh\left(\theta-\phi\right)\sinh\phi}{R_2}\right),$$

Case 4. $$\frac{1}{(R_n)_0} = \frac{1}{\cosh\theta}\left(\frac{\cosh\left(\theta-\phi\right)\cosh\phi}{R_1} + \frac{\sinh\left(\theta-\phi\right)\sinh\phi}{R_2}\right),$$

Case 5. $$\frac{1}{R_n} = \frac{1}{\sinh\theta}\left(\frac{\sinh\left(\theta-\phi\right)\cosh\phi}{R_1} + \frac{\cosh\left(\theta-\phi\right)\sinh\phi}{R_2}\right),$$

Case 6. $$\frac{1}{(R_n)_0} = \frac{1}{\sinh\theta}\left(\frac{\cosh\left(\theta-\phi\right)\sinh\phi}{R_1} + \frac{\sinh\left(\theta-\phi\right)\cosh\phi}{R_2}\right).$$

Proof.
For Case 1. If we substitute the lines of curvature for the parameter curves on the surface, the equations in (7) are substituted in (8), the expression (10) is gotten. The proof of other cases is done in a similar way. □

Proposition 3.11 (Euler Formula). *If t_1 and t_2 are Lorentzian orthogonal vectors in Theorem 3.10, then for the normal curvature, we obtain equations below, respectively:*

Special Cases 1 – 2. $$\frac{1}{R_n} = -\frac{\sinh^2\phi}{R_1} + \frac{\cosh^2\phi}{R_2}, \qquad \frac{1}{(R_n)_0} = \frac{\cosh^2\phi}{R_1} - \frac{\sinh^2\phi}{R_2},$$

Special Cases 3 – 4. $$\frac{1}{R_n} = \frac{\cosh^2\phi}{R_1} - \frac{\sinh^2\phi}{R_2}, \qquad \frac{1}{(R_n)_0} = -\frac{\sinh^2\phi}{R_1} + \frac{\cosh^2\phi}{R_2}.$$

There is no special cases 5 - 6.

Theorem 3.12. *Let radii of normal curvature be $(R_n)_1$, $(R_n)_2$ and radii of geodesic torsion be $\left(T_g\right)_1$, $\left(T_g\right)_2$ of the curves (c_1) and (c_2) on $x(u,v)$, respectively. Let radii of normal curvature of the curve (c) and the curve (c_0) perpendicular to the curve (c) be R_n and $(R_n)_0$, respectively. There are the following relationships between the normal curvatures:*

Cases 1 – 2. $$\frac{1}{R_n} + \frac{1}{(R_n)_0} = \frac{\sinh\theta}{\cosh\theta}\left(\frac{1}{\left(T_g\right)_1} + \frac{1}{\left(T_g\right)_2}\right) + \frac{1}{(R_n)_1} + \frac{1}{(R_n)_2},$$

Cases 3 – 4. $$\frac{1}{R_n} + \frac{1}{(R_n)_0} = -\frac{\sinh\theta}{\cosh\theta}\left(\frac{1}{\left(T_g\right)_1} + \frac{1}{\left(T_g\right)_2}\right) + \frac{1}{(R_n)_1} + \frac{1}{(R_n)_2},$$

Cases 5 – 6. $$\frac{1}{R_n} + \frac{1}{(R_n)_0} = \frac{\cosh\theta}{\sinh\theta}\left(\frac{1}{\left(T_g\right)_1} - \frac{1}{\left(T_g\right)_2}\right) + \frac{1}{(R_n)_1} + \frac{1}{(R_n)_2}.$$

Proof. If we add the expressions (8) and (9) side by side and the similar operation is applied for other cases also, the proof is completed. □

Corollary 3.13 (Mean Curvature). *The sum of the normal curvatures of (c) and (c_0) perpendicular to (c) on $x(u,v)$ is constant.*

Proof. In Theorem 3.12, If we substitute the lines of curvature for the parameter curves on the surface, from the expression (7), we get the following relation:

All Cases. $$\frac{1}{R_n} + \frac{1}{(R_n)_0} = \frac{1}{R_1} + \frac{1}{R_2} = 2H.$$

□

Theorem 3.14. *Let radii of principal curvature of* (c_1) *and* (c_2) *be* R_1 *and* R_2*, the radius of normal curvature be* R_n*, the radius of geodesic torsion of* (c) *be* T_g *and the radius of geodesic torsion of* (c_0) *perpendicular to* (c) *be* $\left(T_g\right)_0$ *on* $x(u,v)$*, respectively. Then, we get:*

$$\text{Case 1.}\quad \left(\frac{1}{R_n}-\frac{1}{R_1}\right)\left(\frac{1}{R_n}-\frac{1}{R_2}\right)=-\frac{\cosh\left(\theta-\phi\right)\sinh\phi}{\sinh\left(\theta-\phi\right)\cosh\phi}\,\frac{1}{T_g^2}\,, \tag{11}$$

$$\text{Case 2.}\quad \left(\frac{1}{(R_n)_0}-\frac{1}{R_1}\right)\left(\frac{1}{(R_n)_0}-\frac{1}{R_2}\right)=-\frac{\sinh\left(\theta-\phi\right)\cosh\phi}{\cosh\left(\theta-\phi\right)\sinh\phi}\,\frac{1}{\left(T_g\right)_0^2}\,,$$

$$\text{Case 3.}\quad \left(\frac{1}{R_n}-\frac{1}{R_1}\right)\left(\frac{1}{R_n}-\frac{1}{R_2}\right)=-\frac{\sinh\left(\theta-\phi\right)\cosh\phi}{\cosh\left(\theta-\phi\right)\sinh\phi}\,\frac{1}{T_g^2}\,,$$

$$\text{Case 4.}\quad \left(\frac{1}{(R_n)_0}-\frac{1}{R_1}\right)\left(\frac{1}{(R_n)_0}-\frac{1}{R_2}\right)=-\frac{\cosh\left(\theta-\phi\right)\sinh\phi}{\sinh\left(\theta-\phi\right)\cosh\phi}\,\frac{1}{\left(T_g\right)_0^2}\,,$$

$$\text{Case 5.}\quad \left(\frac{1}{R_n}-\frac{1}{R_1}\right)\left(\frac{1}{R_n}-\frac{1}{R_2}\right)=-\frac{\cosh\left(\theta-\phi\right)\cosh\phi}{\sinh\left(\theta-\phi\right)\sinh\phi}\,\frac{1}{T_g^2}\,,$$

$$\text{Case 6.}\quad \left(\frac{1}{(R_n)_0}-\frac{1}{R_1}\right)\left(\frac{1}{(R_n)_0}-\frac{1}{R_2}\right)=-\frac{\sinh\left(\theta-\phi\right)\sinh\phi}{\cosh\left(\theta-\phi\right)\cosh\phi}\,\frac{1}{\left(T_g\right)_0^2}\,.$$

Proof. From the equalitiy of the expressions

$$\frac{dN}{ds}=\frac{1}{\cosh\theta}\left(-\frac{\sinh\left(\theta-\phi\right)}{R_1}t_1+\frac{\cosh\phi}{R_2}t_2\right),$$

$$\frac{dN}{ds}=\frac{1}{\cosh\theta}\left[\left(-\frac{\sinh\left(\theta-\phi\right)}{R_n}+\frac{\cosh\left(\theta-\phi\right)}{T_g}\right)t_1+\left(\frac{\cosh\phi}{R_n}+\frac{\sinh\phi}{T_g}\right)t_2\right],$$

[8], we get the following equations:

$$\frac{\sinh\left(\theta-\phi\right)}{R_1}=\frac{\sinh\left(\theta-\phi\right)}{R_n}-\frac{\cosh\left(\theta-\phi\right)}{T_g},\qquad \frac{\cosh\phi}{R_2}=\frac{\cosh\phi}{R_n}+\frac{\sinh\phi}{T_g}. \tag{12}$$

From the expression (12), we get the following equalities respectively:

$$\frac{1}{R_n}-\frac{1}{R_1}=\frac{\cosh\left(\theta-\phi\right)}{\sinh\left(\theta-\phi\right)}\frac{1}{T_g},\qquad \frac{1}{R_n}-\frac{1}{R_2}=-\frac{\sinh\phi}{\cosh\phi}\frac{1}{T_g} \tag{13}$$

If we multiply two equalities in the expression (13) side by side, the proof is completed. The proof of other cases is done in a similar way. □

Proposition 3.15. *If* t_1 *and* t_2 *are Lorentzian orthogonal vectors in Theorem 3.14, that is* $\theta=0$*, then, for the normal curvature, we obtain following equations, respectively:*

Special Cases 1 - 3 - 5 and Special Cases 2 - 4 - 6.

$$\left(\frac{1}{R_n}-\frac{1}{R_1}\right)\left(\frac{1}{R_n}-\frac{1}{R_2}\right)=\frac{1}{T_g^2} \qquad \text{and} \qquad \left(\frac{1}{(R_n)_0}-\frac{1}{R_1}\right)\left(\frac{1}{(R_n)_0}-\frac{1}{R_2}\right)=\frac{1}{\left(T_g\right)_0^2}.$$

Thus, we can say that the normal curvatures take their extreme values in the principal directions, [26].

Theorem 3.16. *Let radii of principal curvature of (c_1) and (c_2) be R_1 and R_2, radius of normal curvature be R_n, radius of geodesic torsion of (c) be T_g and radius of geodesic torsion of (c_0) perpendicular to (c) be $\left(T_g\right)_0$ on $x(u,v)$, respectively. Then, we have the following equation:*

All Cases. $$\left(\frac{1}{R_n}-\frac{1}{R_1}\right)\left(\frac{1}{R_n}-\frac{1}{R_2}\right)=\frac{1}{T_g\left(T_g\right)_0}.$$

Proof.

Case 1. From (10), we get the following equations:

$$\frac{1}{R_n}-\frac{1}{R_1}=\frac{\cosh\left(\theta-\phi\right)\cosh\phi}{\cosh\theta}\left(-\frac{1}{R_1}+\frac{1}{R_2}\right), \qquad \frac{1}{R_n}-\frac{1}{R_2}=-\frac{\sinh\left(\theta-\phi\right)\sinh\phi}{\cosh\theta}\left(-\frac{1}{R_1}+\frac{1}{R_2}\right). \tag{14}$$

If we multiply the equalities in the expression (14) side by side, we obtain

$$\left(\frac{1}{R_n}-\frac{1}{R_1}\right)\left(\frac{1}{R_n}-\frac{1}{R_2}\right)=-\frac{\cosh\left(\theta-\phi\right)\cosh\phi\sinh\left(\theta-\phi\right)\sinh\phi}{\cosh^2\theta}\left(-\frac{1}{R_1}+\frac{1}{R_2}\right)^2. \tag{15}$$

Besides, if we multiply two equalities in the expression (6) side by side, we have

$$\frac{1}{T_g\left(T_g\right)_0}=-\frac{\cosh\left(\theta-\phi\right)\cosh\phi\sinh\left(\theta-\phi\right)\sinh\phi}{\cosh^2\theta}\left(-\frac{1}{R_1}+\frac{1}{R_2}\right)^2. \tag{16}$$

From the equality of (15) and (16), the proof is obtained. When we do the proof of other cases in a similar way, we see that the result is the same. □

Theorem 3.17. *Let radii of geodesic curvature be $\left(R_g\right)_1$ and $\left(R_g\right)_2$ at the direction t_1 and t_2 of (c_1) and (c_2) on $x(u,v)$, respectively. For the geodesic curvature at the direction t, we get the equalities below:*

Case 1. $$\frac{1}{R_g}=\frac{1}{\cosh\theta}\left(\frac{\sinh\left(\theta-\phi\right)}{\left(R_g\right)_1}+\frac{\cosh\phi}{\left(R_g\right)_2}\right)+\frac{d\phi}{ds}, \tag{17}$$

Case 2. $$\frac{1}{\left(R_g\right)_0}=\frac{1}{\cosh\theta}\left(\frac{\cosh\left(\theta-\phi\right)}{\left(R_g\right)_1}-\frac{\sinh\phi}{\left(R_g\right)_2}\right)-\frac{d\phi}{ds}, \tag{18}$$

Case 3. $$\frac{1}{R_g}=\frac{1}{\cosh\theta}\left(\frac{\sinh\left(\theta-\phi\right)}{\left(R_g\right)_1}+\frac{\cosh\phi}{\left(R_g\right)_2}\right)-\frac{d\phi}{ds},$$

Case 4. $$\frac{1}{\left(R_g\right)_0}=\frac{1}{\cosh\theta}\left(-\frac{\sinh\left(\theta-\phi\right)}{\left(R_g\right)_1}+\frac{\cosh\phi}{\left(R_g\right)_2}\right)+\frac{d\phi}{ds},$$

$$\textit{Case 5.}\quad \frac{1}{R_g}=\frac{1}{\sinh\theta}\left(\frac{\sinh\left(\theta-\phi\right)}{\left(R_g\right)_1}+\frac{\sinh\phi}{\left(R_g\right)_2}\right)+\frac{d\phi}{ds},$$

$$\textit{Case 6.}\quad \frac{1}{\left(R_g\right)_0}=\frac{1}{\sinh\theta}\left(\frac{\cosh\left(\theta-\phi\right)}{\left(R_g\right)_1}-\frac{\cosh\phi}{\left(R_g\right)_2}\right)-\frac{d\phi}{ds}.$$

Proof.

For Case 1. If (1) and (5) are substituted in (2) and the inner product with the vector N is applied to both sides of the resulting expression, the proof is completed. If suitable vectors w, w_1 and w_2 for other cases are considered from [26], proofs of other cases are obtained by similar operations. □

Corollary 3.18 (J. Liouville Formula). *Let radii of principal curvature of* (c_1) *and* (c_2) *be* R_1 *and* R_2 *on* $x(u,v)$*, respectively. If we substitute the lines of curvature for the parameter curves on the surface, then for geodesic curvature at the direction* t*, we have the following equations, respectively:*

$$\textit{Case 1 – 2.}\quad \frac{1}{R_g}=\frac{1}{\cosh\theta}\left(\frac{\sinh\left(\theta-\phi\right)}{R_1}+\frac{\cosh\phi}{R_2}\right)+\frac{d\phi}{ds},\qquad \frac{1}{\left(R_g\right)_0}=\frac{1}{\cosh\theta}\left(\frac{\cosh\left(\theta-\phi\right)}{R_1}-\frac{\sinh\phi}{R_2}\right)-\frac{d\phi}{ds},$$

$$\textit{Case 3 – 4.}\quad \frac{1}{R_g}=\frac{1}{\cosh\theta}\left(\frac{\cosh\left(\theta-\phi\right)}{R_1}+\frac{\sinh\phi}{R_2}\right)-\frac{d\phi}{ds},\qquad \frac{1}{\left(R_g\right)_0}=\frac{1}{\cosh\theta}\left(-\frac{\sinh\left(\theta-\phi\right)}{R_1}+\frac{\cosh\phi}{R_2}\right)+\frac{d\phi}{ds},$$

$$\textit{Case 5 – 6.}\quad \frac{1}{R_g}=\frac{1}{\sinh\theta}\left(\frac{\sinh\left(\theta-\phi\right)}{R_1}+\frac{\sinh\phi}{R_2}\right)+\frac{d\phi}{ds},\qquad \frac{1}{\left(R_g\right)_0}=\frac{1}{\sinh\theta}\left(\frac{\cosh\left(\theta-\phi\right)}{R_1}-\frac{\sinh\phi}{R_2}\right)-\frac{d\phi}{ds}.$$

Proof.

For Case 1. If we substitute the lines of curvature for the parameter curves on the surface, the equalities in (7) are substituted in (17), the proof is obtained. The proof of other cases is done in a similar way. □

Proposition 3.19 (J. Liouville Formula). *If* t_1 *and* t_2 *are Lorentzian orthogonal vectors in Theorem 3.17, then for the geodesic curvature, we obtain equations below, respectively:*

$$\textit{Special Cases 1 – 2.}\quad \frac{1}{R_g}=-\frac{\sinh\phi}{R_1}+\frac{\cosh\phi}{R_2}+\frac{d\phi}{ds},\qquad \frac{1}{\left(R_g\right)_0}=\frac{\cosh\phi}{R_1}-\frac{\sinh\phi}{R_2}-\frac{d\phi}{ds},$$

$$\textit{Special Cases 3 – 4.}\quad \frac{1}{R_g}=\frac{\cosh\phi}{R_1}+\frac{\sinh\phi}{R_2}-\frac{d\phi}{ds},\qquad \frac{1}{\left(R_g\right)_0}=\frac{\sinh\phi}{R_1}+\frac{\cosh\phi}{R_2}+\frac{d\phi}{ds}.$$

There is no special cases 5 - 6.

Theorem 3.20. *Let radii of geodesic curvature be* $\left(R_g\right)_1$, $\left(R_g\right)_2$ *of* (c_1) *and* (c_2)*, respectively. Let radii of geodesic curvature of* (c) *and* (c_0) *perpendicular to* (c) *be* R_g *and* $\left(R_g\right)_0$ *on* $x(u,v)$*, respectively. There are the following relationships between the geodesic curvatures:*

$$\textit{Cases 1 – 2.}\quad \frac{\cosh\phi}{R_g}+\frac{\sinh\phi}{\left(R_g\right)_0}=\frac{1}{\cosh\theta}\left(\frac{\sinh\theta}{\left(R_g\right)_1}+\frac{1}{\left(R_g\right)_2}\right)+\frac{d\phi}{ds}\left(\cosh\phi-\sinh\phi\right),$$

$$\textit{Cases 3 – 4.}\quad \frac{\sinh\phi}{R_g}-\frac{\cosh\phi}{\left(R_g\right)_0}=\frac{1}{\cosh\theta}\left(\frac{\sinh\theta}{\left(R_g\right)_1}-\frac{1}{\left(R_g\right)_2}\right)+\frac{d\phi}{ds}\left(\cosh\phi-\sinh\phi\right),$$

Cases 5 − 6. $$\frac{\sinh\phi}{R_g}+\frac{\cosh\phi}{\left(R_g\right)_0}=\frac{1}{\sinh\theta}\left(\frac{\cosh\theta}{\left(R_g\right)_1}-\frac{1}{\left(R_g\right)_2}\right)-\frac{d\phi}{ds}\left(\cosh\phi-\sinh\phi\right).$$

Proof.

Cases 1 - 2. If we multiply (17) by $\cosh\phi$, multiply the expression (18) by $\sinh\phi$ and add the results side by side, the proof is completed. The proof of other cases is done in a similar way. □

Corollary 3.21. *In Theorem 3.20, if we substitute the lines of curvature for the parameter curves on the surface, from (7), for the geodesic curvatures of (c) and (c_0) perpendicular to (c), we have the following relations:*

Cases 1 − 2. $$\frac{\cosh\phi}{R_g}+\frac{\sinh\phi}{\left(R_g\right)_0}=\frac{1}{\cosh\theta}\left(\frac{\sinh\theta}{R_1}+\frac{1}{R_2}\right)+\frac{d\phi}{ds}\left(\cosh\phi-\sinh\phi\right),$$

Cases 3 − 4. $$\frac{\sinh\phi}{R_g}-\frac{\cosh\phi}{\left(R_g\right)_0}=\frac{1}{\cosh\theta}\left(\frac{\sinh\theta}{R_1}-\frac{1}{R_2}\right)+\frac{d\phi}{ds}\left(\cosh\phi-\sinh\phi\right),$$

Cases 5 − 6. $$\frac{\sinh\phi}{R_g}+\frac{\cosh\phi}{\left(R_g\right)_0}=\frac{1}{\sinh\theta}\left(\frac{\cosh\theta}{R_1}-\frac{1}{R_2}\right)-\frac{d\phi}{ds}\left(\cosh\phi-\sinh\phi\right).$$

Proposition 3.22. *If t_1 and t_2 are Lorentzian orthogonal vectors in Corollary 3.21, then we obtain the follow equations:*

Special Cases 1 − 2. $$\frac{\cosh\phi}{R_g}+\frac{\sinh\phi}{\left(R_g\right)_0}=\frac{1}{R_2}+\frac{d\phi}{ds}\left(\cosh\phi-\sinh\phi\right),$$

Special Cases 3 − 4. $$\frac{\sinh\phi}{R_g}-\frac{\cosh\phi}{\left(R_g\right)_0}=-\frac{1}{R_2}+\frac{d\phi}{ds}\left(\cosh\phi-\sinh\phi\right).$$

There is no special cases 5 - 6.

Theorem 3.23. *Let radii of geodesic curvature of (c) and (c_0) perpendicular to (c) be R_g and $\left(R_g\right)_0$ on $x(u,v)$, respectively. There are the following equations for the geodesic curvature, respectively:*

$$\textit{Cases 1 − 2.}\quad \frac{1}{R_g}=\frac{1}{\cosh\theta}\left\langle t_1,\frac{dt_2}{ds}\right\rangle+\frac{d\phi}{ds},\qquad \frac{1}{\left(R_g\right)_0}=-\frac{1}{\cosh\theta}\left\langle t_1,\frac{dt_2}{ds}\right\rangle-\frac{d\phi}{ds}, \tag{19}$$

$$\textit{Cases 3 − 4.}\quad \frac{1}{R_g}=\frac{1}{\cosh\theta}\left\langle t_1,\frac{dt_2}{ds}\right\rangle-\frac{d\phi}{ds},\qquad \frac{1}{\left(R_g\right)_0}=-\frac{1}{\cosh\theta}\left\langle t_1,\frac{dt_2}{ds}\right\rangle+\frac{d\phi}{ds},$$

$$\textit{Cases 5 − 6.}\quad \frac{1}{R_g}=\frac{1}{\sinh\theta}\left\langle t_1,\frac{dt_2}{ds}\right\rangle+\frac{d\phi}{ds},\qquad \frac{1}{\left(R_g\right)_0}=-\frac{1}{\sinh\theta}\left\langle t_1,\frac{dt_2}{ds}\right\rangle-\frac{d\phi}{ds}.$$

Proof.

Case 1. If we inner product both sides of (1) by N, we get $\langle w,N\rangle=-\frac{1}{R_g}$. Also, we know $w=A-\frac{d\phi}{ds}N$, [8]. Then, from the last two equations, we have

$$\frac{1}{R_g}=-\langle A,N\rangle+\frac{d\phi}{ds}. \tag{20}$$

If the expression $\left\langle t_2,\frac{dt_1}{ds}\right\rangle=-\left\langle t_1,\frac{dt_2}{ds}\right\rangle=\langle A,N\rangle\cosh\theta$, [8], is substituted in (20), we obtain (19) . The proof of other cases is done in a similar way. □

Proposition 3.24. *If t_1 and t_2 are Lorentzian orthogonal vectors in Theorem 3.23, that is $\theta = 0$, then, we have:*

Special Cases 1 − 2. $\frac{1}{R_g} = -\frac{1}{\left(R_g\right)_0} = \left\langle t_1, \frac{dt_2}{ds} \right\rangle + \frac{d\phi}{ds}$,

Special Cases 3 − 4. $\frac{1}{R_g} = -\frac{1}{\left(R_g\right)_0} = \left\langle t_1, \frac{dt_2}{ds} \right\rangle - \frac{d\phi}{ds}$.

There is no special cases 5 - 6.

Theorem 3.25. *Let radii of geodesic curvature of (c) and (c_0) perpendicular to (c) be R_g and $\left(R_g\right)_0$ on $x(u,v)$, respectively. There are the equations below:*

Case 1.

$$\frac{1}{R_g} = \frac{\left(\sqrt{E}\right)_v - \sinh\theta\left(\sqrt{G}\right)_u}{\cosh\theta\sqrt{G}}\frac{du}{ds} + \frac{\left(\sqrt{G}\right)_u + \sinh\theta\left(\sqrt{E}\right)_v}{\cosh\theta\sqrt{E}}\frac{dv}{ds} + \frac{d}{ds}\left[\operatorname{arctanh}\left(\frac{1}{\cosh\theta}\sqrt{\frac{E}{G}}\frac{du}{dv} + \tanh\theta\right)\right],$$

Case 2.

$$\frac{1}{\left(R_g\right)_0} = \frac{-\left(\sqrt{E}\right)_v + \sinh\theta\left(\sqrt{G}\right)_u}{\cosh\theta\sqrt{G}}\frac{du}{ds} - \frac{\left(\sqrt{G}\right)_u + \sinh\theta\left(\sqrt{E}\right)_v}{\cosh\theta\sqrt{E}}\frac{dv}{ds} - \frac{d}{ds}\left[\operatorname{arccoth}\left(\frac{1}{\cosh\theta}\sqrt{\frac{E}{G}}\frac{du}{dv} + \tanh\theta\right)\right],$$

Case 3.

$$\frac{1}{R_g} = -\frac{\left(\sqrt{E}\right)_v + \sinh\theta\left(\sqrt{G}\right)_u}{\cosh\theta\sqrt{G}}\frac{du}{ds} + \frac{-\left(\sqrt{G}\right)_u + \sinh\theta\left(\sqrt{E}\right)_v}{\cosh\theta\sqrt{E}}\frac{dv}{ds} - \frac{d}{ds}\left[\operatorname{arccoth}\left(-\frac{1}{\cosh\theta}\sqrt{\frac{E}{G}}\frac{du}{dv} + \tanh\theta\right)\right],$$

Case 4.

$$\frac{1}{\left(R_g\right)_0} = \frac{\left(\sqrt{E}\right)_v + \sinh\theta\left(\sqrt{G}\right)_u}{\cosh\theta\sqrt{G}}\frac{du}{ds} + \frac{\left(\sqrt{G}\right)_u - \sinh\theta\left(\sqrt{E}\right)_v}{\cosh\theta\sqrt{E}}\frac{dv}{ds} + \frac{d}{ds}\left[\operatorname{arctanh}\left(-\frac{1}{\cosh\theta}\sqrt{\frac{E}{G}}\frac{du}{dv} + \tanh\theta\right)\right],$$

Case 5.

$$\frac{1}{R_g} = \frac{-\left(\sqrt{E}\right)_v + \cosh\theta\left(\sqrt{G}\right)_u}{\sinh\theta\sqrt{G}}\frac{du}{ds} + \frac{\left(\sqrt{G}\right)_u - \cosh\theta\left(\sqrt{E}\right)_v}{\sinh\theta\sqrt{E}}\frac{dv}{ds} + \frac{d}{ds}\left[\operatorname{arctanh}\left(\frac{1}{\sinh\theta}\sqrt{\frac{E}{G}}\frac{du}{dv} + \coth\theta\right)\right],$$

Case 6.

$$\frac{1}{\left(R_g\right)_0} = \frac{\left(\sqrt{E}\right)_v - \cosh\theta\left(\sqrt{G}\right)_u}{\sinh\theta\sqrt{G}}\frac{du}{ds} + \frac{-\left(\sqrt{G}\right)_u + \cosh\theta\left(\sqrt{E}\right)_v}{\sinh\theta\sqrt{E}}\frac{dv}{ds} - \frac{d}{ds}\left[\operatorname{arctanh}\left(\frac{1}{\sinh\theta}\sqrt{\frac{E}{G}}\frac{du}{dv} + \coth\theta\right)\right].$$

Proof.

Case 1. From the equation

$$\langle t_1, dt_2\rangle = -\langle t_2, dt_1\rangle = \frac{\left(\sqrt{E}\right)_v - \sinh\theta\left(\sqrt{G}\right)_u}{\sqrt{G}}\, du + \frac{\left(\sqrt{G}\right)_u + \sinh\theta\left(\sqrt{E}\right)_v}{\sqrt{E}}\, dv,$$

[8], we have

$$\left\langle t_1, \frac{dt_2}{ds}\right\rangle = \frac{\left(\sqrt{E}\right)_v - \sinh\theta\left(\sqrt{G}\right)_u}{\sqrt{G}}\,\frac{du}{ds} + \frac{\left(\sqrt{G}\right)_u + \sinh\theta\left(\sqrt{E}\right)_v}{\sqrt{E}}\,\frac{dv}{ds}. \tag{21}$$

Also, we know the following equalities

$$\sqrt{E}\frac{du}{ds} = -\frac{\sinh\left(\theta - \phi\right)}{\cosh\theta} \quad and \quad \sqrt{G}\frac{dv}{ds} = \frac{\cosh\phi}{\cosh\theta}, \tag{22}$$

[8]. If we substitute the expression (22) in the expression (21), we get

$$\coth\phi = -\frac{\sinh\left(\theta - \phi\right)}{\sinh\phi}\sqrt{\frac{G}{E}}\frac{dv}{du}. \tag{23}$$

If we apply some trigonometric operations in (23), we get

$$\phi = \operatorname{arctanh}\left(\frac{1}{\cosh\theta}\sqrt{\frac{E}{G}}\frac{du}{dv} + \tanh\theta\right). \tag{24}$$

If (21) and (24) are substituted in (19), the proof is obtained. The proof of other cases is done in a similar way. □

Proposition 3.26. *If t_1 and t_2 are Lorentzian orthogonal vectors in Theorem 3.25, then we have:*

Special Case 1. $$\frac{1}{R_g} = \frac{\left(\sqrt{E}\right)_v}{\sqrt{G}}\frac{du}{ds} + \frac{\left(\sqrt{G}\right)_u}{\sqrt{E}}\frac{dv}{ds} + \frac{d}{ds}\left[\operatorname{arctanh}\left(\sqrt{\frac{E}{G}}\frac{du}{dv}\right)\right],$$

Special Case 2. $$\frac{1}{\left(R_g\right)_0} = -\frac{\left(\sqrt{E}\right)_v}{\sqrt{G}}\frac{du}{ds} - \frac{\left(\sqrt{G}\right)_u}{\sqrt{E}}\frac{dv}{ds} - \frac{d}{ds}\left[\operatorname{arccoth}\left(\sqrt{\frac{E}{G}}\frac{du}{dv}\right)\right],$$

Special Case 3. $$\frac{1}{R_g} = -\frac{\left(\sqrt{E}\right)_v}{\sqrt{G}}\frac{du}{ds} - \frac{\left(\sqrt{G}\right)_u}{\sqrt{E}}\frac{dv}{ds} + \frac{d}{ds}\left[\operatorname{arccoth}\left(\sqrt{\frac{E}{G}}\frac{du}{dv}\right)\right],$$

Special Case 4. $$\frac{1}{\left(R_g\right)_0} = \frac{\left(\sqrt{E}\right)_v}{\sqrt{G}}\frac{du}{ds} + \frac{\left(\sqrt{G}\right)_u}{\sqrt{E}}\frac{dv}{ds} - \frac{d}{ds}\left[\operatorname{arctanh}\left(\sqrt{\frac{E}{G}}\frac{du}{dv}\right)\right].$$

There is no special cases 5 - 6.

4. Examples

Let's now give the examples of timelike surfaces whose parameter curves don't intersect perpendicularly.

1 : $x(v,u) = \left(-\frac{\cos(2v)\cos(2u)}{2}, v - \frac{\sin(2v)\cos(2u)}{2}, \sin(v)sin(u)\right)$, (*Figure 1*)

2 : $y(v,u) = \left(-\frac{\cosh(2v)\sinh(2u)}{2} - u, \sinh(v)\sinh(u), \frac{\cosh(2v)cosh(2u) - 1}{2}\right)$, (*Figure 2*)

3 : $z(v,u) = (\cos(u)(\cos(v) + v\sin(v)) + u\cos(v)\sin(u), \cos(u)(\sin(v) - v\cos(v)) + u\sin(v)\sin(u), 2uv)$, (*Figure* 3).

5. Conclusions

We have shown in our study [8] that six different cases occur according to the causal character of the parameter curves intersecting under any angle on a timelike surface. In this paper, relations between invariants on the timelike surface are studied, new theorems and new equivalents of well known formulas are given. This article, which will be a source for similar studies in Minkowski 3-space (or others), will also shed light on studies in different disciplines such as physics, mathematics and astronomy in this space.

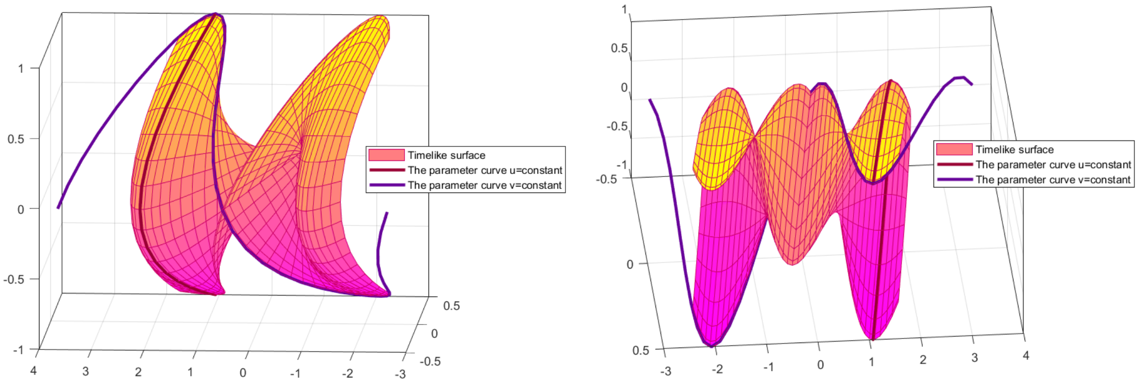

Figure 1: Timelike Surface $x(u, v)$

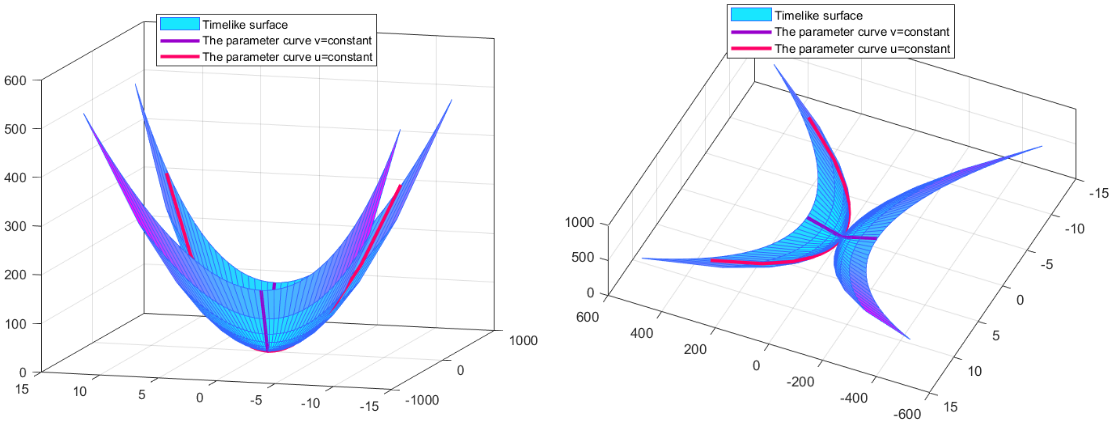

Figure 2: Timelike Surface $y(u, v)$

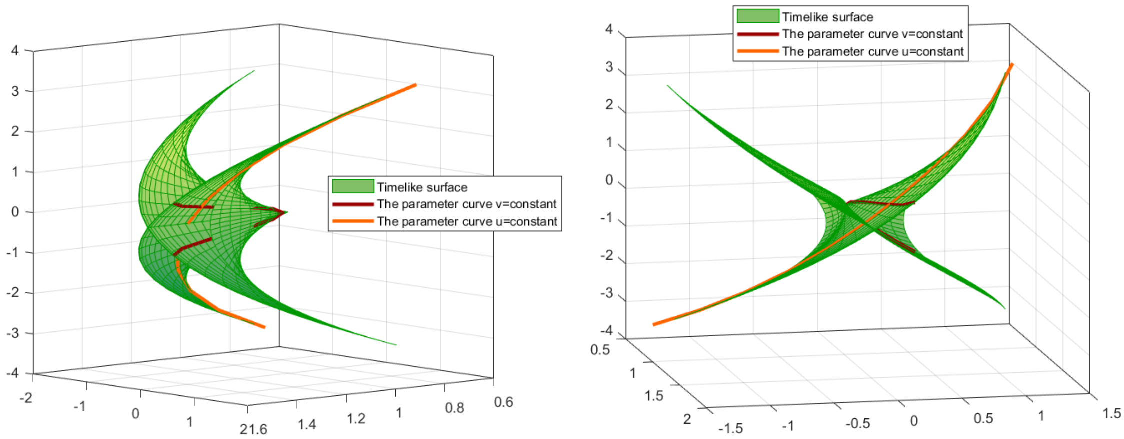

Figure 3: Timelike Surface $z(u, v)$

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References

[1] F. Akbulut, The Darboux Vectors of Curves on Any Surface, Ege University, Sciences of Faculty, İzmir, 1983.
[2] N. Alluhaibi, R. A. Abdel-Baky, Kinematic Geometry of Timelike Ruled Surfaces in Minkowski 3-Space E_1^3, Symmetry 14(4) (2022) 749.
[3] M. Babaarslan, M. Kayacik Time-like loxodromes on helicoidal surfaces in Minkowski 3-space, Filomat 31(14) (2017) 4405–4414.
[4] J. K. Beem, E. E. Paul, L. E. Kevin, Global lorentzian geometry, Routledge, 2017.
[5] F. Bulut, M. Bektaş, Special helices on equiform differential geometry of spacelike curves in Minkowski space-time, Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics 69(2) (2020) 1045–1056.
[6] Z. Chen, Y. Li, S. Sarkar, S. Dey, A. Bhattacharyya, Ricci Soliton and Certain Related Metrics on a Three-Dimensional Trans-Sasakian Manifold, Universe 8 (2022) 595.
[7] A. A. Cintra,I. I. Onnis, Enneper representation of minimal surfaces in the three-dimensional Lorentz–Minkowski space, Annali di Matematica Pura ed Applicata (1923-) 197(1) (2018) 21–39.
[8] Y. Li, S. Gür Mazlum, S. Senyurt, The Darboux Trihedrons of Timelike Surfaces in the Lorentzian 3-Space, International Journal of Geometric Methods in Modern Physics, (2022) 1–35, https://doi.org/10.1142/S0219887823500305.
[9] S. Gür, S. Şenyurt, Spacelike–timelike involute–evolute curve couple on dual Lorentzian space, J. Math. Comput. Sci. 3(4) (2013) 1054–1075.
[10] Euclid, A. S. Sertöz, B. Çelgelci, Z. Çanakçı, Ö. Kılıçkaya, D. Akca, Öklid'in elemanları (in Turkish), TÜBİTAK, (2019).
[11] Y. Li, F. Mofarreh, S. Dey, S. Roy, A. Ali, General Relativistic Space-Time with η_1-Einstein Metrics, Mathematics 10(14) (2022) 2530.
[12] Y. Li, S. H. Nazra, R. A. Abdel-Baky, Singularity Properties of Timelike Sweeping Surface in Minkowski 3-Space, Symmetry 14(10) (2022) 1996.
[13] Y. Li, A. A. Abdel-Salam, M. Khalifa Saad, Primitivoids of curves in Minkowski plane, AIMS Mathematics, 8(1) (2022) 2386-2406.
[14] Y. Li, S. Mondal, S. Dey, A. Bhattacharyya, A. Ali, A Study of Conformal η -Einstein Solitons on Trans-Sasakian 3-Manifold, Journal of Nonlinear Mathematical Physics, 2022 (2022) 1-27.
[15] Y. Li, F. Mofarreh, R. A. Abdel-Baky, Timelike Circular Surfaces and Singularities in Minkowski 3-Space, Symmetry 14(9) (2022) 1914.
[16] R. Lopez, Differential Geometry of Curves and Surfaces in Lorentz-Minkowski Space, International Electronic Journal of Geometry 7(1) (2014) 44–107.
[17] R. Lopez, Z. Milin Sipus, L. Primorac Gajcic, I. Protrka, Involutes of Pseudo-Null Curves in Lorentz–Minkowski 3-Space, Mathematics 9(11) (2021) 1256.
[18] M. A. Magid, Timelike surfaces in Lorentz 3-space with prescribed mean curvature and Gauss map, Hokkaido Mathematical Journal 20(3) (1991) 447–464.
[19] B. O'Neill, Semi-Riemannian Geometry with applications to relativity, Academic Press, London, 1983.
[20] J. G. Ratcliffe, Foundations of Hyperbolic Manifolds, Springer-Verlag, Tokyo, 1994.
[21] B. Sahiner, M. Kazaz and H. H. Ugurlu, On the Curvature Theory of Non-null Cylindrical Surfaces in Minkowski 3-Space, TWMS Journal of Applied and Engineering Mathematics 6(1) (2016) 22–29.
[22] S. Şenyurt and S. Gür, Spacelike surface geometry, International Journal of Geometric Methods in Modern Physics 14(9) (2017) 1750118.
[23] S. Şenyurt and S. Gür, Timelike–spacelike involute–evolute curve couple on dual Lorentzian space, J. Math. Comput. Sci. 2(6) (2012) 1808–1823.
[24] S. Uddin, M. S. Stanković, M. Iqbal, S. K. Yadav, M. Aslam, Slant helices in Minkowski 3-space E_1^3 with Sasai's modified frame fields, Filomat 36(1) (2022) 151–164.
[25] M. Önder, H. H. Uğurlu, Frenet frames and invariants of timelike ruled surfaces, Ain Shams Engineering Journal 4(3) (2013) 507–513.
[26] H. H. Uğurlu, On the geometry of time-like surfaces, Communications, Faculty of Sciences, University of Ankara, Al Series 46, 1997.
[27] H. H. Uğurlu and A. Çalışkan, The Geometry of Timelike and Spacelike Surface With Darboux Instantaneous Rotation Vectors, Publication of Celal Bayar University, No: 0006, 2012.
[28] V. D. I. Woestijne, Minimal surfaces of the 3-dimensional Minkowski space, Geometry and topology of submanifolds II (1990) 344–369.
[29] Y. Wang, L. Yang, P. Li, Y. Chang, Singularities of Osculating Developable Surfaces of Timelike Surfaces along Curves, Symmetry, 14 (2022) 2251.
[30] H. Yıldırım, Slant ruled surfaces and slant developable surfaces of spacelike curves in Lorentz-Minkowski 3-space, Filomat 32(14) (2018) 4875–4895.