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$B-$maximal operators, $B-$singular integral operators and $B-$Riesz potentials in variable exponent Lorentz spaces

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Abstract. In this paper, we prove the boundedness of $B-$maximal operator, $B-$singular integral operator and $B-$Riesz potential in the variable exponent Lorentz space $L_{p(\cdot),q(\cdot),\gamma}(\mathbb{R}^n_{k,+})$. As a consequence of the boundedness of $B-$Riesz potentials in variable exponent Lorentz spaces, we also obtain that $B-$fractional maximal operators are bounded in $L_{p(\cdot),q(\cdot),\gamma}(\mathbb{R}^n_{k,+})$.

1. Introduction

Harmonic analysis consists of significant operators such as singular integrals, maximal operators, fractional maximal operators, Riesz potentials and convolution type operators. On various function spaces, the problem of boundedness of these operators and their versions which are generated by Laplace-Bessel differential operators play a critical role in harmonic analysis and PDE's. Over the years, singular integral operators generated by Laplace-Bessel differential operators have been investigated by Aliev, Aykol, Ekincioglu, Gadjiev, Guliyev, Kaya, Kipriyanov, Klyuchantsev, Lyakhov, Safarov, Şerbetçi, Stempak, others [1–3, 7, 9–12, 15–19].

On Lorentz spaces $L_{p,q,\gamma}$, behavior of generalized $B-$potential integral operators and rough $B-$fractional integral operators have been investigated in [9, 10]. Later, in [12, 13], pointwise rearrangement estimates for generalized $B-$convolution operators and O'Neil type inequality for $B-$convolution type operators have been obtained and boundedness of generalized $B-$Riesz potentials, generalized $B-$fractional maximal function have been proved. Also, O'Neil type inequality for the Hankel convolution operator have been obtained [2]. Then, Aykol and Şerbetçi [3] have shown that fractional $B-$maximal operators defined on Lorentz spaces are bounded.

Nowadays, there is a big attention on variable exponent function spaces and several results of harmonic analysis have been discussed on these spaces, for example, Ephremidza et al. [8] have considered maximal operators, fractional integral operators and singular integral operators on variable exponent Lorentz spaces. In [8], the boundedness of such operators have been shown by assuming some decay conditions of log-type on the exponents. Several fundamental properties of variable exponent Lorentz spaces have also been obtained in [14]. The above results inspire us to investigate $B-$maximal operator, $B-$singular integral operator and $B-$Riesz potential defined on variable exponent Lorentz spaces. With this motivation we

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aim to obtain that $B-$maximal operator, $B-$singular integral operator and $B-$Riesz potential are bounded in variable exponent Lorentz spaces by using the rearrangement estimates and O'Neil type inequalities for these operators. It is worth noting that in general log-Hölder continuous conditions are well used to obtain the boundedness of these operators. But here, we are able to avoid this condition since the Hardy inequalities are applicable.

The construction of the article is as follows: The first section is devoted to introduction. In the second section, we recall some basic concepts, notations and some known results which we need throughout the paper. In the third section, we have obtained that $B-$maximal operators are bounded in variable exponent Lorentz spaces. The fourth section is devoted to boundedness of $B-$singular integral operators. Finally, in the fifth section, we present that $B-$Riesz potentials are bounded in variable exponent Lorentz spaces and as an immediate consequence of it we obtain that $B-$fractional maximal operators are bounded in these spaces.

Throughout the paper we use the letter C for a positive constant, independent of appropriate parameters and not necessary the same at each occurrence.

2. Preliminaries

We first give some basic concepts, notations and known results which are beneficial for us.

Let $x = (x', x'')$, $x' = (x_1, \dots, x_k) \in \mathbb{R}^k$, and $x'' = (x_{k+1}, \dots, x_n) \in \mathbb{R}^{n-k}$. Denote $\mathbb{R}^n_{k,+} = \{x \in \mathbb{R}^n : x_1 > 0, \dots, x_k > 0, 1 \leq k \leq n\}$, $\gamma = (\gamma_1, \dots, \gamma_k)$, $\gamma_1 > 0, \dots, \gamma_k > 0$, and $|\gamma| = \gamma_1 + \dots + \gamma_k$. We denote by $B_+(x, r)$, the open ball of radius r centered at x, namely, $B_+(x, r) = \{y \in \mathbb{R}^n_{k,+} : |x - y| < r\}$. Let $B_+(0, r) \subset \mathbb{R}^n_{k,+}$ be a measurable set, then

$$|B_+(0, r)|_\gamma = \int_{B_+(0,r)} (x')^\gamma dx = \omega(n, k, \gamma) r^Q,$$

where $\omega(n, k, \gamma) = \dfrac{\pi^{\frac{n-k}{2}}}{2^k} \displaystyle\prod_{i=1}^{k} \dfrac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\left(\frac{\gamma_i}{2}\right)}$, $Q = n + |\gamma|$.

The definition of the generalized shift operator is as follows:

$$T^y f(x) := C_{k,\gamma} \int_0^\pi \dots \int_0^\pi f\left[(x_1, y_1)_{\alpha_1}, \dots, (x_k, y_k)_{\alpha_k}, x'' - y''\right] d\gamma(\alpha),$$

where $C_{k,\gamma} = \pi^{-\frac{k}{2}} \Gamma(\frac{\gamma_i+1}{2})[\Gamma(\frac{\gamma_i}{2})]^{-1}$, $(x_i, y_i)_{\alpha_i} = (x_i^2 - 2x_i y_i \cos \alpha_i + y_i^2)^{\frac{1}{2}}$, $1 \leq i \leq k$, $1 \leq k \leq n$, and $d\gamma(\alpha) = \displaystyle\prod_{i=1}^{k} \sin^{\gamma_i - 1} \alpha_i \, d\alpha_i$ [17, 18]. Notice that the generalized shift operator is related to the Laplace-Bessel differential operator,

$$\Delta_B := \sum_{i=1}^{k} B_i + \sum_{i=k+1}^{n} \frac{\partial^2}{\partial x_i^2}, \quad B_i = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i} \frac{\partial}{\partial x_i}, \quad 1 \leq k \leq n.$$

The $B-$convolution operator associated with T^y is as follows:

$$(f \otimes g)(x) = \int_{\mathbb{R}^n_{k,+}} f(y) T^y g(x) (y')^\gamma dy.$$

For a function $f \in L^{\text{loc}}_{1,\gamma}(\mathbb{R}^n_{k,+})$, the $B-$maximal operator and $B-$fractional maximal operator are defined by, respectively,

$$M_\gamma f(x) = \sup_{r>0} |B_+(0, r)|_\gamma^{-1} \int_{B_+(0,r)} T^y |f(x)| (y')^\gamma dy,$$

$$M_\gamma^\alpha f(x) = \sup_{r>0} |B_+(0, r)|_\gamma^{\frac{\alpha}{Q}-1} \int_{B_+(0,r)} T^y |f(x)| (y')^\gamma dy, \quad 0 \leq \alpha < Q.$$

It is easy to observe that $M_\gamma^0 f = M_\gamma f$ for $\alpha = 0$ (see [11]).

$B-$Riesz potential is defined by the following:

$$I_\gamma^\alpha f(x) = \int_{\mathbb{R}^n_{k,+}} T^y |x|^{\alpha-Q} f(y)(y')^\gamma dy, \quad 0 < \alpha < Q.$$

It is well known that the inequality $M_\gamma^\alpha \leq C\, I_\gamma^\alpha$ holds.

Singular integral operator generated by a generalized shift operator ($B-$singular integral operator) is defined as

$$\begin{aligned} T_\gamma f(x) &= p.v. \int_{\mathbb{R}^n_{k,+}} \frac{\Omega(\theta)}{|y|^Q} [T^y f(x)](y')^\gamma dy \\ &= \lim_{\varepsilon\to 0} \int_{\{y\in\mathbb{R}^n_{k,+}:\,|y|>\varepsilon\}} \frac{\Omega(\theta)}{|y|^Q} [T^y f(x)](y')^\gamma dy = \lim_{\varepsilon\to 0} T_{\gamma,\varepsilon} f(x), \end{aligned} \tag{1}$$

where $\theta = y/|y|$, and the characteristic $\Omega(\theta)$ belong to some function space on the semi-sphere $S_{k,+} = \{x \in \mathbb{R}^n_{k,+} \,:\, |x| = 1\}$ and satisfying the "cancellation" condition

$$\int_{S_{k,+}} \Omega(\theta)(\theta')^\gamma d\sigma(\theta) = 0$$

($d\sigma(\theta)$ is the area element of the sphere $|\theta| = 1$). $B-$singular integral operators are the convolution type operator, where the kernel of these operator is $K(y) = \dfrac{\Omega(\theta)}{|y|^Q}$ and thus it can be written as $T_\gamma f(x) = (K \otimes f)(x)$. The existence of the limit (1) for all $x \in \mathbb{R}^n_{k,+}$ and for Schwartz test functions $f(x)$ can be proved in the standard way by considering the well-known estimate $\left|T^y f(x) - f(x)\right| \leq c(x)|y|$.

We will now introduce the variable exponent Lebesgue spaces $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$. Let $\mathcal{P}(\mathbb{R}^n_{k,+})$ be the set of all measurable functions $p(\cdot) : \mathbb{R}^n_{k,+} \to [1,\infty]$. The elements of $\mathcal{P}(\mathbb{R}^n_{k,+})$ are called variable exponent functions and also let $1 \leq p_- := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n_{k,+}} p(x) \leq p_+ := \operatorname*{ess\,sup}_{x\in\mathbb{R}^n_{k,+}} p(x) < \infty$. The space $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ is known as the set of measurable functions f such that for a variable exponent $p(\cdot) : \mathbb{R}^n_{k,+} \to [1,\infty]$,

$$\|f\|_{L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})} = \inf\left\{\lambda > 0 : \, \varrho_{p(\cdot),\gamma}\left(f/\lambda\right) \leq 1\right\},$$

where

$$\varrho_{p(\cdot),\gamma} := \int_{\mathbb{R}^n_{k,+}\setminus(\mathbb{R}^n_{k,+})_\infty} |f(x)|^{p(x)}(x')^\gamma dx + \|f\|_{L_{\infty,\gamma}(\mathbb{R}^n_{k,+})_\infty} < \infty.$$

Given $p(\cdot)$, the conjugate exponent function is as follows,

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \quad x \in \mathbb{R}^n_{k,+}.$$

Let $\mathcal{P}(0,\infty)$ be the set of all measurable functions $p(\cdot)$ on interval $(0,\infty)$.

The elements of $\mathcal{P}(0,\infty)$ are called variable exponent functions and also let

$$p_- := \operatorname*{ess\,inf}_{0<t<\infty} p(t), \qquad p_+ := \operatorname*{ess\,sup}_{0<t<\infty} p(t)$$

satisfying the conditions

$$|p(t) - p(0)| \leq \frac{C_0}{|\ln t|}, \quad 0 < t \leq \frac{1}{2}, \tag{2}$$

and

$$|p(t) - p(\infty)| \leq \frac{C_\infty}{\ln(e+t)}. \tag{3}$$

We will use the notation $\mathcal{P}_a = \{p : a < p_- \leq p_+ < \infty\}$ for $a = 0$ or $a = 1$. We denote by $\mathbb{P}(0,\infty)$, the set of all classes of measurable functions $p(\cdot) \in L_\infty(0,\infty)$ such that there exist the limits

$$p(0) = \lim_{t\to 0} p(t) \qquad \text{and} \qquad p(\infty) = \lim_{t\to\infty} p(t).$$

We denote $\mathbb{P}_a(0,\infty) = \mathbb{P}(0,\infty) \cap \mathcal{P}_a(0,\infty)$.

Definition 2.1. *[5, 8] Let $\beta(t)$ and $\nu(t)$ are measurable functions on $(0,\infty)$. The weighted Hardy operators $H^{\beta(\cdot)}_{\nu(\cdot)}$ and $\mathcal{H}^{\beta(\cdot)}_{\nu(\cdot)}$ with power weights acting on φ are defined by*

$$H^{\beta(\cdot)}_{\nu(\cdot)}\varphi(t) = t^{\beta(t)+\nu(t)-1} \int_0^t \frac{\varphi(y)}{y^{\nu(y)}} dy$$

and

$$\mathcal{H}^{\beta(\cdot)}_{\nu(\cdot)}\varphi(t) = t^{\beta(t)+\nu(t)} \int_t^\infty \frac{\varphi(y)}{y^{\nu(y)+1}} dy.$$

We need the following Hardy inequalities in variable Lebesgue spaces (see [5] and the references therein) which will be used in the proof of our main theorems.

Theorem 2.2. *[5, 8] Let $p(\cdot) \in \mathbb{P}_1(0,\infty)$, and β, $\nu \in \mathbb{P}(0,\infty)$ and*

$$0 \leq \beta(0) < \frac{1}{p(0)}, \qquad 0 \leq \beta(\infty) < \frac{1}{p(\infty)}. \tag{4}$$

Let also $q(\cdot) \in \mathbb{P}_1(0,\infty)$ such that

$$\frac{1}{q(0)} = \frac{1}{p(0)} - \beta(0), \qquad \frac{1}{q(\infty)} = \frac{1}{p(\infty)} - \beta(\infty). \tag{5}$$

Then the inequalities

$$\left\| H^{\beta(\cdot)}_{\nu(\cdot)}\varphi \right\|_{L_{q(\cdot)}(0,\infty)} \leq C \left\|\varphi\right\|_{L_{p(\cdot)}(0,\infty)}$$

and

$$\left\| \mathcal{H}^{\beta(\cdot)}_{\nu(\cdot)}\varphi \right\|_{L_{q(\cdot)}(0,\infty)} \leq C \left\|\varphi\right\|_{L_{p(\cdot)}(0,\infty)}$$

hold if and only if

$$\nu(0) < \frac{1}{p'(0)} \quad \text{and} \quad \nu(\infty) < \frac{1}{p'(\infty)}, \tag{6}$$

and

$$\nu(0) > -\frac{1}{p(0)} \quad \text{and} \quad \nu(\infty) > -\frac{1}{p(\infty)}, \tag{7}$$

respectively.

2.1. *Variable exponent Lorentz spaces*

Given a measurable function $f : \mathbb{R}^n_{k,+} \to \mathbb{R}$ and any measurable set E with $|E|_\gamma = \int_E (x')^\gamma dx$, the $\gamma-$rearrangement of f in decreasing order is defined as

$$f^*_\gamma(t) = \inf\left\{s > 0 : f_{*,\gamma}(s) \le t\right\}, \quad \forall t \in (0,\infty),$$

where $f_{*,\gamma}(s)$ denotes the $\gamma-$distribution function of f given by

$$f_{*,\gamma}(s) = \left|\left\{x \in \mathbb{R}^n_{k,+} : |f(x)| > s\right\}\right|_\gamma .$$

The average function of f^{**}_γ is defined as

$$f^{**}_\gamma(t) = \frac{1}{t}\int_0^t f^*_\gamma(s)ds,$$

for $t > 0$, and the following inequality is valid (see [6]):

$$(f+g)^{**}_\gamma(t) \le f^{**}_\gamma(t) + g^{**}_\gamma(t).$$

Some properties of $\gamma-$rearrangement of functions are given as follows (see [4, 6, 20]):

- if $0 < p < \infty$, then
$$\int_{\mathbb{R}^n_{k,+}} |f(x)|^p (x')^\gamma dx = \int_0^\infty (f^*_\gamma(t))^p dt,$$
- for any $t > 0$,
$$\sup_{|E|_\gamma = t} \int_E |f(x)|(x')^\gamma dx = \int_0^t f^*_\gamma(s)ds, \tag{8}$$
- $$\int_{\mathbb{R}^n_{k,+}} |f(x)g(x)|(x')^\gamma dx \le \int_0^\infty f^*_\gamma(t)g^*_\gamma(t)dt,$$
- it is well known that
$$(f+g)^*_\gamma(t) \le f^*_\gamma(t/2) + g^*_\gamma(t/2) \tag{9}$$
holds.

Lemma 2.3. *[3] For any measurable set $\mathcal{A} = (\mathcal{A}', \mathcal{A}_n) \subset \mathbb{R}^n_{k,+}$, $\mathcal{A}_n \subset (0,\infty)$, $\mathcal{A}' = \mathcal{A}_1 \times \cdots \times \mathcal{A}_{n-1} \subset \mathbb{R}^{n-1}$, and $y \in \mathbb{R}^n_{k,+}$, then the following equality holds*

$$\int_{\mathcal{A}} T^y g(x)(y')^\gamma dy = C_\gamma \int_{(x,0)+\widetilde{\mathcal{A}}} g\left(z', \sqrt{z_n^2 + z_{n+1}^2}\right) d\mu(z, z_{n+1}),$$

where $\widetilde{\mathcal{A}} = \mathcal{A}' \times (-m, m) \times [0, m)$, $m = \sup \mathcal{A}_n$ and $d\mu(z, z_{n+1}) = z_{n+1}^{\gamma-1} dz_1 dz_2 \cdots dz_n dz_{n+1}$.

Lemma 2.4. *[3] For any measurable set $\mathcal{A} \subset \mathbb{R}^n_{k,+}$ and for any $y \in \mathbb{R}^n_{k,+}$, the following equality holds*

$$\sup_{|\mathcal{A}|_\gamma = t} \int_{\mathcal{A}} T^y |f(x)|(y')^\gamma dy = C_\gamma \int_0^t f^*_\gamma(s)ds.$$

Definition 2.5. *The Lorentz space $L_{p,q,\gamma}(\mathbb{R}^n_{k,+})$ is the collection of all measurable functions f on $\mathbb{R}^n_{k,+}$ such that the quantity*

$$\|f\|_{p,q,\gamma} = \begin{cases} \left(\int_0^\infty \left(t^{\frac{1}{p}} f_\gamma^*(t)\right)^q \frac{dt}{t}\right)^{1/q}, & 0 < p < \infty,\ 0 < q < \infty \\ \sup\limits_{t>0} t^{\frac{1}{p}} f_\gamma^*(t), & 0 < p \le \infty,\ q = \infty \end{cases}$$

is finite.

If $0 < p \le \infty$, $q = \infty$, then $L_{p,\infty,\gamma}(\mathbb{R}^n_{k,+}) = WL_{p,\gamma}(\mathbb{R}^n_{k,+})$, where $WL_{p,\gamma}(\mathbb{R}^n_{k,+})$ is weak Lebesgue space of all measurable functions f with following norm

$$\|f\|_{WL_{p,\gamma}(\mathbb{R}^n_{k,+})} = \sup_{t>0} t^{1/p} f_\gamma^*(t) < \infty, \quad 1 \le p < \infty.$$

If $1 \le q \le p$, or $p = q = \infty$, then the functional $\|f\|_{p,q,\gamma}$ is a norm (see [4, 10, 20]). If $p = q = \infty$, then the space $L_{\infty,\infty,\gamma}(\mathbb{R}^n_{k,+})$ is denoted by $L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$.

In the case $1 < p, q < \infty$, we give a functional $\|\cdot\|^*_{p,q,\gamma}$ by

$$\|f\|^*_{p,q,\gamma} = \begin{cases} \left(\int_0^\infty \left(t^{\frac{1}{p}} f_\gamma^{**}(t)\right)^q \frac{dt}{t}\right)^{1/q}, & 0 < p < \infty,\ 0 < q < \infty \\ \sup\limits_{t>0} t^{\frac{1}{p}} f_\gamma^{**}(t), & 0 < p \le \infty,\ q = \infty \end{cases}$$

(with the usual modification if $0 < p \le \infty$, $q = \infty$) which is a norm on $L_{p,q,\gamma}(\mathbb{R}^n_{k,+})$ for $1 < p < \infty$, $1 \le q \le \infty$ or $p = q = \infty$.

We will now introduce the variable exponent Lorentz spaces $L_{p(\cdot),q(\cdot),\gamma}(\mathbb{R}^n_{k,+})$.

Definition 2.6. *Let $p(\cdot)$, $q(\cdot) \in \mathcal{P}_0(0,\infty)$ be variable exponent functions. Then the variable exponent Lorentz spaces $L_{p(\cdot),q(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ are known as the set of measurable functions f on $\mathbb{R}^n_{k,+}$ such that $t^{\frac{1}{p(t)}-\frac{1}{q(t)}} f_\gamma^*(t) \in L_{q(\cdot)}(0,\infty)$, i.e.*

$$\mathfrak{I}_{p(\cdot),q(\cdot)}(f) := \int_0^\infty t^{\frac{q(t)}{p(t)}-1} |f_\gamma^*(t)|^{q(t)} dt < \infty.$$

and the norm on these spaces is defined by

$$\|f\|_{L_{p(\cdot),q(\cdot),\gamma}(\mathbb{R}^n_{k,+})} = \inf\left\{\lambda > 0 : \mathfrak{I}_{p(\cdot),q(\cdot)}(f/\lambda) \le 1\right\} = \left\| t^{\frac{1}{p(t)}-\frac{1}{q(t)}} f_\gamma^*(t) \right\|_{L_{q(\cdot)}(0,\infty)}.$$

If we take $p(\cdot) = q(\cdot)$, then we obtain variable Lebesgue spaces $L_{p(\cdot),p(\cdot),\gamma} = L_{p(\cdot),\gamma}$. For the proof of this, the reader may refer to [14].

One can write the following norm

$$\|f\|^*_{L_{p(\cdot),q(\cdot),\gamma}(\mathbb{R}^n_{k,+})} = \left\| t^{\frac{1}{p(t)}-\frac{1}{q(t)}} f_\gamma^{**}(t) \right\|_{L_{q(\cdot)}(0,\infty)}.$$

The validity of the following can be obtained easily from the definition of Lorentz $L_{p(\cdot),q(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ spaces:

- If $p(\cdot)$, $q(\cdot) \in \mathcal{P}(0,\infty)$, then the functional $\|f\|_{L_{p(\cdot),q(\cdot),\gamma}}$ is a quasi-norm.

- Let $p(\cdot) \in \mathbb{P}_0([0,\ell])$, $q \in \mathbb{P}_1([0,\ell])$. Then $\|f\|_{L_{p(\cdot),q(\cdot),\gamma}}$ and $\|f\|^*_{L_{q(\cdot),q(\cdot),\gamma}}$ are equivalent, i.e.,

$$\|f\|_{L_{p(\cdot),q(\cdot),\gamma}} \le \|f\|^*_{L_{p(\cdot),q(\cdot),\gamma}} \le C\,\|f\|_{L_{p(\cdot),q(\cdot),\gamma}} \tag{10}$$

if and only if $p(0) > 1$ and in the case of $\ell = \infty$, also $p(\infty) > 1$, where $C > 0$ does not depend on f.

3. Boundedness of $B-$maximal operators in variable exponent Lorentz spaces

In this section, we will first give sharp rearrangement inequality for $B-$maximal operators. By using this inequality, we obtain that $B-$maximal operators are bounded in $L_{p(\cdot),q(\cdot),\gamma}(\mathbb{R}^n_{k,+})$.

Lemma 3.1. *[3] Sharp rearrangement inequality for $B-$maximal operators,*

$$(M_\gamma f)^*_\gamma(t) \leq C f^{**}_\gamma(t), \quad t > 0, \tag{11}$$

holds, where $C = C(n, \gamma)$ is a positive constant.

Theorem 3.2. *Let $p(\cdot)$, $q(\cdot) \in \mathbb{P}_1(0,\infty)$. Then $B-$maximal operator M_γ is bounded on $L_{p(\cdot),q(\cdot),\gamma}(\mathbb{R}^n_{k,+})$.*

Proof. From the inequalities (10) and (11) we obtain

$$\begin{aligned}\|M_\gamma f\|_{L_{p(\cdot),q(\cdot),\gamma}(\mathbb{R}^n_{k,+})} &= \left\| t^{\frac{1}{p(t)}-\frac{1}{q(t)}} (M_\gamma f)^*_\gamma(t) \right\|_{L_{q(\cdot)}(0,\infty)} \\ &\leq C \left\| t^{\frac{1}{p(t)}-\frac{1}{q(t)}} f^{**}_\gamma(t) \right\|_{L_{q(\cdot)}(0,\infty)} \\ &\equiv C\|f\|_{L_{p(\cdot),q(\cdot),\gamma}(\mathbb{R}^n_{k,+})}. \end{aligned} \tag{12}$$

Hence, the proof is completed. □

4. Boundedness of $B-$singular integral operators in variable exponent Lorentz spaces

In this section, we obtain that $B-$singular integral operators are bounded in variable exponent Lorentz spaces. We will first give two theorems which is used in proof of Theorem 4.3. While Theorem 4.1 states O'Neil type inequality for $B-$convolution operator, Theorem 4.2 states a pointwise rearrangement estimate of generalized $B-$potential integral. For more details, see [12].

Theorem 4.1. *[12] Let f, g be two positive measurable functions on $\mathbb{R}^n_{k,+}$. Then for all $t > 0$ the following inequality holds:*

$$(f \otimes g)^{**}_\gamma(t) \leq C_{k,\gamma}\left(f^{**}_\gamma(t) \int_0^t g^{**}_\gamma(u)du + \int_t^\infty f^*_\gamma(u) g^{**}_\gamma(u)du \right).$$

Theorem 4.2. *[12] If $K_\alpha \in WL_{Q/(Q-\alpha),\gamma}(\mathbb{R}^n_{k,+})$, $0 < \alpha < Q$, then*

$$(K_\alpha \otimes g)^*_\gamma(t) \leq (K_\alpha \otimes g)^{**}_\gamma(t) \leq C\left(t^{\alpha/Q-1} \int_0^t f^*_\gamma(s)ds + \int_t^\infty s^{\alpha/Q-1} f^*_\gamma(s)ds \right),$$

where $C = C_{k,\gamma}(Q/\alpha)^2 \|K_\alpha\|_{WL_{Q/(Q-\alpha),\gamma}(\mathbb{R}^n_{k,+})}$.

Since $T_\gamma f(x)$ is convolution type operator and from the above theorem, we can write

$$(T_\gamma f(x))^*_\gamma(t) \leq (T_\gamma f(x))^{**}_\gamma(t) \leq C\left(t^{-1} \int_0^t f^*_\gamma(s)ds + \int_t^\infty s^{-1} f^*_\gamma(s)ds \right). \tag{13}$$

Now, we are ready to present $L_{p(\cdot),q(\cdot),\gamma}(\mathbb{R}^n_{k,+})-$boundedness of $B-$singular integral operators.

Theorem 4.3. *Let $p(\cdot)$, $q(\cdot) \in \mathbb{P}_1(0,\infty)$. Then $B-$singular integral operators T_γ are bounded on $L_{p(\cdot),q(\cdot),\gamma}(\mathbb{R}^n_{k,+})$.*

Proof. Let $p(\cdot),\ q(\cdot) \in \mathbb{P}_1(0,\infty)$. Then,

$$\begin{aligned}\|T_\gamma f\|_{L_{p(\cdot),q(\cdot),\gamma}(\mathbb{R}^n_{k,+})} &= \left\| t^{\frac{1}{p(t)}-\frac{1}{q(t)}} (T_\gamma f)^*_\gamma(t) \right\|_{L_{q(\cdot)}(0,\infty)} \\ &\le C \left\| t^{\frac{1}{p(t)}-\frac{1}{q(t)}} (T_\gamma f)^{**}_\gamma(t) \right\|_{L_{q(\cdot)}(0,\infty)}.\end{aligned}$$

Then from (13) and Theorem 4.2 we get

$$\begin{aligned}\left\| t^{\frac{1}{p(t)}-\frac{1}{q(t)}} (T_\gamma f)^{**}_\gamma(t) \right\|_{L_{q(\cdot)}(0,\infty)} &\le C \left\| t^{\frac{1}{p(t)}-\frac{1}{q(t)}} \left(t^{-1}\int_0^t f^*_\gamma(s)ds + \int_t^\infty s^{-1} f^*_\gamma(s)ds \right) \right\|_{L_{q(\cdot)}(0,\infty)} \\ &\le C \left\| t^{\frac{1}{p(t)}-\frac{1}{q(t)}-1} \int_0^t f^*_\gamma(s)ds \right\|_{L_{q(\cdot)}(0,\infty)} \\ &\quad + C \left\| t^{\frac{1}{p(t)}-\frac{1}{q(t)}} \int_t^\infty s^{-1} f^*_\gamma(s)ds \right\|_{L_{q(\cdot)}(0,\infty)} \\ &= I_1 + I_2.\end{aligned}$$

From the inequality (12) we easily obtain

$$I_1 \le C \|f\|_{L_{p(\cdot),q(\cdot),\gamma}(\mathbb{R}^n_{k,+})}.$$

For I_2, taking $\beta(t) = 0$, $\nu(t) = \frac{1}{p(t)} - \frac{1}{q(t)}$ and $\varphi(t) = t^{\frac{1}{p(t)}-\frac{1}{q(t)}} f^*(t)$ in Theorem 2.2, we get

$$\begin{aligned}I_2 &= \left\| t^{\frac{1}{p(t)}-\frac{1}{q(t)}} \int_t^\infty s^{-1} f^*_\gamma(s)ds \right\|_{L_{q(\cdot)}(0,\infty)} \\ &= \left\| t^{\nu(t)} \int_t^\infty \frac{\varphi(s)}{s^{\nu(s)+1}} ds \right\|_{L_{q(\cdot)}(0,\infty)} \\ &= C \left\| \mathcal{H}^{\beta(\cdot)}_{\nu(\cdot)} \varphi \right\|_{L_{q(\cdot)}(0,\infty)} \\ &\le C \|\varphi\|_{L_{p(\cdot)}(0,\infty)} = C \|t^{\frac{1}{p(t)}-\frac{1}{q(t)}} f^*_\gamma(t)\|_{L_{q(\cdot)}(0,\infty)} = C \|f\|_{L_{p(\cdot),q(\cdot),\gamma}(\mathbb{R}^n_{k,+})}.\end{aligned}$$

This completes the proof. □

5. Boundedness of $B-$Riesz potentials in variable exponent Lorentz spaces

In this section, we will obtain the $(L_{p(\cdot),r(\cdot),\gamma}(\mathbb{R}^n_{k,+}), L_{q(\cdot),s(\cdot),\gamma}(\mathbb{R}^n_{k,+}))-$boundedness of $B-$Riesz potential. For $B-$Riesz potential, the following inequality

$$(I^\alpha_\gamma f)^*_\gamma(t) \le (I^\alpha_\gamma f)^{**}_\gamma(t) \le C_2 \left(t^{\frac{\alpha}{Q}-1} \int_0^t f^*_\gamma(s)ds + \int_t^\infty s^{\frac{\alpha}{Q}-1} f^*_\gamma(s)ds \right), \tag{14}$$

holds, where $C_2 = C_{k,\gamma}(Q/\alpha)^2 \omega(n,k,\gamma)^{(Q-\alpha)/Q}$. For more details, see [9].

Theorem 5.1. *Let* $p(\cdot),\ q(\cdot),\ r(\cdot),\ s(\cdot) \in \mathbb{P}_1(0,\infty)$, $1 < p(0), p(\infty) < \frac{Q}{\alpha}$, $\frac{1}{p(0)} - \frac{1}{q(0)} = \frac{\alpha}{Q}$ *and* $\frac{1}{p(\infty)} - \frac{1}{q(\infty)} = \frac{\alpha}{Q}$. *Then the Riesz potential* I^α_γ *is bounded from* $L_{p(\cdot),r(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ *to* $L_{q(\cdot),s(\cdot),\gamma}(\mathbb{R}^n_{k,+})$.

Proof. The proof can be obtained in similar manner as in Theorem 4.3. From (14), we get

$$\begin{aligned}
\|I_\gamma^\alpha f\|_{L_{q(\cdot),s(\cdot),\gamma}(\mathbb{R}^n_{k,+})} &= \left\| t^{\frac{1}{q(t)}-\frac{1}{s(t)}} (I_\gamma^\alpha f)^*_\gamma(t) \right\|_{L_{s(\cdot)}(0,\infty)} \\
&\le \left\| t^{\frac{1}{q(t)}-\frac{1}{s(t)}} (I_\gamma^\alpha f)^{**}_\gamma(t) \right\|_{L_{s(\cdot)}(0,\infty)} \\
&\le C \left\| t^{\frac{1}{q(t)}-\frac{1}{s(t)}} \left(t^{\frac{\alpha}{Q}-1} \int_0^t f^*_\gamma(s)ds + \int_t^\infty s^{\frac{\alpha}{Q}-1} f^*_\gamma(s)ds \right) \right\|_{L_{s(\cdot)}(0,\infty)} \\
&\le C \left\| t^{\frac{1}{q(t)}-\frac{1}{s(t)}+\frac{\alpha}{Q}-1} \int_0^t f^*_\gamma(s)ds \right\|_{L_{s(\cdot)}(0,\infty)} + C \left\| t^{\frac{1}{q(t)}-\frac{1}{s(t)}} \int_t^\infty s^{\frac{\alpha}{Q}-1} f^*_\gamma(s)ds \right\|_{L_{s(\cdot)}(0,\infty)} \\
&= J_1 + J_2.
\end{aligned}$$

We take $\nu(t) = \dfrac{1}{p(t)} - \dfrac{1}{r(t)}$ and $\varphi(t) = t^{\frac{1}{p(t)}-\frac{1}{r(t)}} f^*_\gamma(t)$. Then, we get

$$\beta(t) = \frac{1}{q(t)} - \frac{1}{s(t)} + \frac{1}{r(t)} - \frac{1}{p(t)} + \frac{\alpha}{Q}.$$

From Theorem 2.2, we obtain

$$\begin{aligned}
J_1 &= \left\| t^{\beta(t)+\nu(t)-1} \int_0^t \frac{\varphi(s)}{s^{\nu(s)}} ds \right\|_{L_{s(\cdot)}(0,\infty)} \\
&= \left\| H^{\beta(\cdot)}_{\nu(\cdot)} \varphi \right\|_{L_{s(\cdot)}(0,\infty)} \\
&\le C \|\varphi\|_{L_{r(\cdot)}(0,\infty)} = C \|t^{\frac{1}{p(t)}-\frac{1}{r(t)}} f^*_\gamma(t)\|_{L_{r(\cdot)}(0,\infty)} = C \|f\|_{L_{p(\cdot),r(\cdot),\gamma}(\mathbb{R}^n_{k,+})}.
\end{aligned}$$

We now estimate J_2. We take $\nu(t) = \dfrac{1}{p(t)} - \dfrac{1}{r(t)} - \dfrac{\alpha}{Q}$ and $\varphi(t) = t^{\frac{1}{p(t)}-\frac{1}{r(t)}} f^*_\gamma(t) \in L_{q(\cdot)}(0,\infty)$. Then, we get

$$\beta(t) = \frac{1}{q(t)} - \frac{1}{s(t)} + \frac{1}{r(t)} - \frac{1}{p(t)} + \frac{\alpha}{Q}.$$

Therefore, by using Theorem 2.2, we obtain

$$\begin{aligned}
J_2 &= \left\| t^{\beta(t)+\nu(t)} \int_t^\infty \frac{\varphi(s)}{s^{\nu(s)+1}} ds \right\|_{L_{s(\cdot)}(0,\infty)} \\
&= C \left\| \mathcal{H}^{\beta(\cdot)}_{\nu(\cdot)} \varphi \right\|_{L_{s(\cdot)}(0,\infty)} \\
&\le C \|\varphi\|_{L_{r(\cdot)}(0,\infty)} = C \|t^{\frac{1}{p(t)}-\frac{1}{r(t)}} f^*_\gamma(t)\|_{L_{r(\cdot)}(0,\infty)} = C \|f\|_{L_{p(\cdot),r(\cdot),\gamma}(\mathbb{R}^n_{k,+})}.
\end{aligned}$$

This completes the proof. □

From the inequality $M_\gamma^\alpha \le C\, I_\gamma^\alpha$ and Theorem 5.1, the following corollary is easily obtained.

Corollary 5.2. *Let $p(\cdot)$, $q(\cdot)$, $r(\cdot)$, $s(\cdot) \in \mathbb{P}_1(0,\infty)$, $1 < p(0), p(\infty) < \frac{Q}{\alpha}$, $\frac{1}{p(0)} - \frac{1}{q(0)} = \frac{\alpha}{Q}$ and $\frac{1}{p(\infty)} - \frac{1}{q(\infty)} = \frac{\alpha}{Q}$. Then the B−fractional maximal operator M_γ^α is bounded from $L_{p(\cdot),r(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q(\cdot),s(\cdot),\gamma}(\mathbb{R}^n_{k,+})$.*

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