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New results on fractional relaxation integro differential equations with impulsive conditions

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Abstract. The aim of this paper is to study the existence and uniqueness of solutions for nonlinear fractional relaxation impulsive integro-differential equations with boundary conditions. Some results are established by using the Banach contraction mapping principle and the Schauder fixed point theorem. An example is provided which illustrates the theoretical results.

1. Introduction

Fractional differential equations have many applications in different problems and phenomenons in science and engineering, see [1]-[19], [21]-[23]. Recently, fractional differential equations have been proved to be useful tools in the modelling of many phenomena in various fields of engineering, physics and economics. It finds an extensive use in fluid dynamic traffic models, nonlinear earthquake oscillations, and many other physical phenomena including seepage flow in porous media. Actually, fractional differential equations are studied as an alternative model to integer differential equations. Since the turn of the century, some authors have used impulsive differential systems to describe the model, particularly in describing dynamics of populations subject to abrupt changes as well as other phenomena like harvesting, diseases, and so forth. Impulsive differential equations have played an important role in modelling phenomena.

In [10], Chidouh, Guezane-Lakoud and Bebbouchi studied the existence and uniqueness of positive solutions of the following nonlinear fractional relaxation differential equation

$$\begin{cases} {}^{LC}\mathfrak{D}^{\gamma}\mathfrak{u}(\iota) + \alpha\mathfrak{u}(\iota) = f(\iota, \mathfrak{u}(\iota)), \quad 0 < \iota \leq 1, \\ \mathfrak{u}(0) = \mathfrak{u}_0 > 0, \end{cases}$$

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where ${}^{LC}\mathfrak{D}^{\gamma}$ is fractional derivative of Liouville-Caputo, $0 < \gamma \le 1$. By using the method of the upper and lower solutions and the Schauder and Banach fixed point theorems, the existence and uniqueness of solutions have been established.

In [11], Guezane Lakoud, Khaldi and Kilicman discussed the existence of solutions for the following nonlinear differential equation with boundary conditions

$$\begin{cases} {}^{LC}\mathfrak{D}^{\gamma}_{1^-}\mathfrak{D}^{\mu}_{0^+}\mathfrak{u}(\iota) = f(\iota,\mathfrak{u}(\iota)), \quad \iota \in (0,1), \\ \mathfrak{u}(0) = \mathfrak{u}'(0) = \mathfrak{u}(1) = 0, \end{cases}$$

where ${}^{LC}\mathfrak{D}^{\gamma}_{1^-}$ and $\mathfrak{D}^{\mu}_{0^+}$ are correct Caputo and Liouville to the left fractional derivatives of Riemann-Liouville respectively, $0 < \gamma \le 1$, $1 < \mu \le 2$. By employing the Krasnoselskii fixed point theorem, the authors produced results for existence.

In [2], Abdo, Wahash and Panchat investigated the existence and uniqueness of positive solutions of the following nonlinear fractional differential equation with integral boundary conditions

$$\begin{cases} {}^{LC}\mathfrak{D}^{\gamma}\mathfrak{u}(\iota) = f(\iota,\mathfrak{u}(\iota)), \quad 0 < \iota \le T, \\ \mathfrak{u}(0) = a\int_0^T \mathfrak{u}(\xi)d\xi + b, \end{cases}$$

where $1 < \gamma < 1$. The existence and uniqueness of solutions have been demonstrated using the method of the upper and lower solutions and the Schauder and Banach fixed point theorems.

Motivated and inspired by the works mentioned above, by applying the Banach and Schauder fixed point theorems, we investigate the existence and uniqueness of solutions to the following fractional relaxation impulsive integro-differential equation of the form

$$\begin{cases} \mathfrak{D}^{\mu} \quad {}^{LC}\mathfrak{D}^{\gamma}\mathfrak{u}(\iota) + \alpha\mathfrak{u}(\iota) = f(\iota,\mathfrak{u}(\iota), I^{\vartheta}\mathfrak{u}(\iota)), \quad \iota \ne \iota_z, \quad \iota \in (0,T), \quad \alpha \in \mathbb{R}, \\ \Delta\mathfrak{u}(\iota_z) = G_z(\mathfrak{u}(\iota_z^-)), \quad z = 1,2,...,m, \\ {}^{LC}\mathfrak{D}^{\gamma}\mathfrak{u}(0) = {}^{LC}\mathfrak{D}^{\gamma}\mathfrak{u}(T) = 0, \quad \mathfrak{u}(0) = a\int_0^T \mathfrak{u}(\xi)d\xi + b, \quad a,b \in \mathbb{R}, \end{cases} \tag{1}$$

where $\mathfrak{D}^{\mu}$ and ${}^{LC}\mathfrak{D}^{\gamma}$ are the fractional derivative of Riemann-Liouville and Liouville-Caputo fractional derivative of orders μ and γ respectively, $1 < \mu < 2$, $0 < \gamma < 1$, I^{ϑ} is fractional integral order $\vartheta \in (0,1)$ by Riemann-Liouville, and $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a nonlinear continuous function. $\Delta\mathfrak{u}(\iota_z) = \mathfrak{u}(\iota_z^+) - \mathfrak{u}(\iota_z^-)$ denotes the jump of $\mathfrak{u}$ at $\iota = \iota_z$, $\mathfrak{u}(\iota_z^+)$ and $\mathfrak{u}(\iota_z^-)$ respresent the right and left limits of $\mathfrak{u}(\iota)$ at $\iota = \iota_z$ respectively, $z = 1,2,...,m$.

The remaining part of the paper is divided into four sections. Section 2 presents notations, fractional calculus definitions, and fixed point theorems. In Section 3, results about the existence and uniqueness of nonlinear fractional relaxation impulsive integro-differential equations are obtained. Section 4 provides an example.

2. Preliminaries

In this section, we mention some definitions, notations and results of the fractional calculus. Consider the Banach space

$$\mathcal{PC}(J,X) = \{\mathfrak{u} : J \to X : \mathfrak{u} \in C(\iota_z, \iota_{z+1}], X\}, \quad z = 0,1,2,...,m$$

and there exist $\mathfrak{u}(\iota_z^-)$ and $\mathfrak{u}(\iota_z^+)$, $z = 0,1,2,...,m$ with $\mathfrak{u}(\iota_z^-) = \mathfrak{u}(\iota_z)$ with the norm

$$\|\mathfrak{u}\|_{\mathcal{PC}} := \sup\{\|\mathfrak{u}(\iota)\| : \iota \in J\}.$$

Now we're giving out some fractional calculus results and properties.

Definition 2.1. ([15]) *The fractional integral of a function $\mathcal{K} : J \to \mathbb{R}$ of order $\gamma > 0$ is defined by*

$$I^{\gamma}\mathcal{K}(\iota) = \frac{1}{\Lambda(\gamma)}\int_0^{\iota}(\iota - \xi)^{\gamma-1}\mathcal{K}(\xi)d\xi,$$

provided the integral exists.

Definition 2.2. ([15]) *The Liouville-Caputo fractional derivative of a function $\mathcal{K} : J \to \mathbb{R}$ of order $\gamma > 0$ is defined by*

$$^{LC}\mathfrak{D}^{\gamma}\mathcal{K}(\iota) = \mathfrak{D}^{\gamma}\left[\mathcal{K}(\iota) - \sum_{z=0}^{m_1-1}\frac{\mathcal{K}^{(z)}(0)}{z!}\iota^{z}\right],$$

where

$$m_1 = [\gamma] + 1 \quad for \quad \gamma \notin \mathbb{N}_0, \quad m_1 = \gamma \quad for \quad \gamma \in \mathbb{N}_0, \tag{2}$$

and $\mathfrak{D}^{\gamma}_{0^+}$ is a fractional derivative in Riemann-Liouville sense of order γ given by

$$\mathfrak{D}^{\gamma}\mathcal{K}(\iota) = \mathfrak{D}^{m_1}I^{m_1-\gamma}\mathcal{K}(\iota) = \frac{1}{\Lambda(n-\gamma)}\frac{d^{m_1}}{d\iota^{m_1}}\int_0^{\iota}(\iota-\xi)^{m_1-\gamma-1}\mathcal{K}(\xi)d\xi.$$

The Liouville-Caputo fractional derivative $^{LC}\mathfrak{D}^{\gamma}_{0^+}$ exists for $\mathfrak{u}$ belonging to $AC^{m_1}(J)$. In this case, it is defined by

$$^{LC}\mathfrak{D}^{\gamma}\mathcal{K}(\iota) = I^{m_1-\gamma}\mathfrak{u}^{(m_1)}(\iota) = \frac{1}{\Lambda(n-\gamma)}\int_0^{\iota}(\iota-\xi)^{m_1-\gamma-1}\mathcal{K}^{(m)_1}(\xi)d\xi.$$

Remark that when $\gamma = m_1$, we get $^{LC}\mathfrak{D}^{\gamma}\mathcal{K}(\iota) = \mathcal{K}^{(m_1)}(\iota)$.

Lemma 2.3. ([15]) *Let $\gamma > 0$ and m be the given by* (2). *If $\mathcal{K} \in AC^{m}(J, \mathbb{R})$, then*

$$(I^{\gamma LC}\mathfrak{D}^{\gamma}\mathcal{K})(\iota) = \mathcal{K}(\iota) - \sum_{z=0}^{m-1}\frac{\mathcal{K}^{(z)}(0)}{z!}\iota^{z},$$

where $\mathcal{K}^{(z)}$ is the usual derivative of $\mathcal{K}$ of order z.

Lemma 2.4. ([15]) *For $\gamma > 0$ and m be given by* (2), *then the Liouville-Caputo fractional differential equation $^{LC}\mathfrak{D}^{\gamma}\mathcal{K}(t) = 0$ has a general solution*

$$\mathcal{K}(\iota) = a_0 + a_1\iota + a_2\iota^2 + ... + a_{m-1}\iota^{m-1},$$

where $a_i \in \mathbb{R}$, $i = 0, 1, 2, ..., m-1$. Further, the Riemann-Liouville fractional differential equation $\mathfrak{D}^{\gamma}\mathcal{K}(\iota) = 0$ has a general solution

$$\mathcal{K}(\iota) = a_1\iota^{\gamma-1} + a_2\iota^{\gamma-2} + a_3\iota^{\gamma-3} + ... + a_m\iota^{\gamma-m}, \quad a_i \in \mathbb{R}, \quad i = 1, 2, ..., m.$$

Lemma 2.5. ([15]) *For any $\gamma, \mu \in [0, \infty)$ and $\epsilon > -1$, we have*

$$\frac{1}{\Lambda(\gamma)}\int_0^{\iota}(\iota-\xi)^{\mu-1}\xi^{\gamma-1}d\xi = \frac{\Lambda(\mu)}{\Lambda(\gamma+\mu)}\iota^{\gamma+\mu-1}.$$

Lemma 2.6. ([20]) *(Banach fixed point theorem) Let Υ be a nonempty closed convex subset of a Banach space $(S, \|\cdot\|)$, then any contraction mapping Φ of Υ into itself has a unique fixed point.*

Lemma 2.7. ([20]) *(Schauder fixed point theorem) Let Υ be a nonempty bounded closed convex subset of a Banach space S and $\Phi : \Upsilon \to \Upsilon$ be a continuous compact operator. Then has a fixed point in Υ.*

We require the following lemma in order to get our results.

Lemma 2.8. *For any $\mathcal{K} \in C(J)$, the following problem*

$$\begin{cases} \mathfrak{D}^{\mu} \quad {}^{LC}\mathfrak{D}^{\gamma}\mathfrak{u}(\iota) + \alpha\mathfrak{u}(\iota) = \mathcal{K}(\iota), \quad \iota \neq \iota_z, \quad \iota \in [0, T], \quad \alpha \in \mathbb{R}, \\ \Delta\mathfrak{u}(\iota_z) = G_z(\mathfrak{u}(\iota_z^-)), \quad z = 1, 2, ..., m, \\ {}^{LC}\mathfrak{D}^{\gamma}\mathfrak{u}(0) = {}^{LC}\mathfrak{D}^{\gamma}\mathfrak{u}(T) = 0, \quad \mathfrak{u}(0) = a\int_0^T \mathfrak{u}(\xi)d\xi + b, \quad a, b \in \mathbb{R}, \end{cases} \tag{3}$$

is equivalent to the integral equation

$$\mathfrak{u}(\iota) = I^{\gamma+\mu}\mathcal{K}(\iota) - \alpha I^{\gamma+\mu}\mathfrak{u}(\iota) - \frac{\iota^{\mu+\gamma-1}}{T^{\mu-1}\Lambda(\mu+\gamma)}(I^{\mu}\mathcal{K}(T) - \alpha I^{\mu}\mathfrak{u}(T)) + a\int_0^T \mathfrak{u}(\xi)d\xi + b + \sum_{z=1}^{m} G_z(\mathfrak{u}(\iota_z)) \quad (4)$$

$$= \begin{cases} \frac{1}{\Lambda(\gamma+\mu)}\left(\int_0^{\iota}(\iota-\xi)^{\gamma+\mu-1}\mathcal{K}(\xi)d\xi - \alpha\int_0^{\iota}(\iota-\xi)^{\gamma+\mu-1}\mathfrak{u}(\xi)d\xi\right) \\ -\frac{\iota^{\mu+\gamma-1}}{T^{\mu-1}\Lambda(\mu+\gamma)}\left(\int_0^T(T-\xi)^{\mu-1}\mathcal{K}(\xi)d\xi - \alpha\int_0^T(T-\xi)^{\mu-1}\mathfrak{u}(\xi)d\xi\right) \\ +a\int_0^T \mathfrak{u}(\xi)d\xi + b \text{ if } \iota\in[0,\iota_1] \\ \frac{1}{\Lambda(\gamma+\mu)}\left(\int_{\iota_1}^{\iota_2}(\iota_2-\xi)^{\gamma+\mu-1}\mathcal{K}(\xi)d\xi - \alpha\int_{\iota_1}^{\iota_2}(\iota_2-\xi)^{\gamma+\mu-1}\mathfrak{u}(\xi)d\xi\right) \\ +\frac{1}{\Lambda(\gamma+\mu)}\left(\int_{\iota_1}^{\iota}(\iota-\xi)^{\gamma+\mu-1}\mathcal{K}(\xi)d\xi - \alpha\int_{\iota_1}^{\iota}(\iota-\xi)^{\gamma+\mu-1}\mathfrak{u}(\xi)d\xi\right) \\ -\frac{\iota^{\mu+\gamma-1}}{T^{\mu-1}\Lambda(\mu+\gamma)}\left(\int_0^T(T-\xi)^{\mu-1}\mathcal{K}(\xi)d\xi - \alpha\int_0^T(T-\xi)^{\mu-1}\mathfrak{u}(\xi)d\xi\right) \\ +a\int_0^T \mathfrak{u}(\xi)d\xi + b + G_1(\mathfrak{u}(\iota_1^-)) \text{ if } \iota\in(\iota_1,\iota_2]. \\ \vdots \\ \frac{1}{\Lambda(\gamma+\mu)}\sum_{i=1}^{z}\left(\int_{\iota_{i-1}}^{\iota_i}(\iota_i-\xi)^{\gamma+\mu-1}\mathcal{K}(\xi)d\xi - \alpha\int_{\iota_{i-1}}^{\iota_i}(\iota_i-\xi)^{\gamma+\mu-1}\mathfrak{u}(\xi)d\xi\right) \\ +\frac{1}{\Lambda(\gamma+\mu)}\left(\int_{\iota_z}^{\iota}(\iota-\xi)^{\gamma+\mu-1}\mathcal{K}(\xi)d\xi - \alpha\int_{\iota_z}^{\iota}(\iota-\xi)^{\gamma+\mu-1}\mathfrak{u}(\xi)d\xi\right) \\ -\frac{\iota^{\mu+\gamma-1}}{T^{\mu-1}\Lambda(\mu+\gamma)}\left(\int_0^T(T-\xi)^{\mu-1}\mathcal{K}(\xi)d\xi - \alpha\int_0^T(T-\xi)^{\mu-1}\mathfrak{u}(\xi)d\xi\right) \\ +a\int_0^T \mathfrak{u}(\xi)d\xi + b + \sum_{z=1}^{m} G_z(\mathfrak{u}(\iota_z^-)) \text{ if } \iota\in(\iota_z,\iota_{z+1}]. \end{cases}$$

Proof. Taking the integral operator I^{μ} to the first equation of (3), and from Lemma 2.4, we get

$$^{LC}\mathfrak{D}^{\gamma}\mathfrak{u}(\iota) = I^{\mu}\mathcal{K}(\iota) - \alpha I^{\mu}\mathfrak{u}(\iota) + a_1\iota^{\mu-1} + a_2\iota^{\mu-2}. \quad (5)$$

According to the conditions $^{LC}\mathfrak{D}^{\gamma}\mathfrak{u}(0) = {}^{LC}\mathfrak{D}^{\gamma}\mathfrak{u}(T) = 0$, it yields

$$a_1 = \frac{1}{T^{\mu-1}}(\alpha I^{\mu}\mathfrak{u}(T) - I^{\mu}\mathcal{K}(T)), \quad a_2 = 0.$$

Replacing a_1 and a_2 by their values in (5), we find

$$^{LC}\mathfrak{D}^{\gamma}\mathfrak{u}(\iota) = I^{\mu}\mathcal{K}(\iota) - \alpha I^{\mu}\mathfrak{u}(\iota) + \frac{\iota^{\mu-1}}{T^{\mu-1}}(\alpha I^{\mu}\mathfrak{u}(T) - I^{\mu}\mathcal{K}(T)).$$

If we take the integral operator I^{γ} again to the above equation and use Lemma 2.4 and Lemma 2.5, we observe

$$\mathfrak{u}(\iota) = I^{\gamma+\mu}\mathcal{K}(\iota) - \alpha I^{\gamma+\mu}\mathfrak{u}(\iota) - \frac{\Lambda(\mu)\iota^{\mu+\gamma-1}}{T^{\mu-1}\Lambda(\mu+\gamma)}(I^{\mu}\mathcal{K}(T) - \alpha I^{\mu}\mathfrak{u}(T)) + a_3. \quad (6)$$

Using the integral condition, we find

$$a_3 = a\int_0^T \mathfrak{u}(\xi)d\xi + b.$$

As a result, we obtain the integral equation (4) by substituting the value of a_3 into (6). The reverse is followed by a direct calculation which completes the proof. □

3. Main results

In the following, we employ some fixed point theorems to prove existence and uniqueness results for the problem (1). The following hypotheses are required in order to get our results.

(H1) There exist constants $l_1, l_2 > 0$ such that

$$|f(\iota, \mathfrak{u}_1, \mathfrak{v}_1) - f(\iota, \mathfrak{u}_2, \mathfrak{v}_2)| \leq l_1|\mathfrak{u}_1 - \mathfrak{u}_2| + l_2|\mathfrak{v}_1 - \mathfrak{v}_2|,$$

for any $\iota \in J$ and each $\mathfrak{u}_i, \mathfrak{v}_i \in \mathbb{R}$, $i = 1, 2$.

(H2) There exists a function $\Psi \in L^1(J, \mathbb{R}^+)$ such that

$$|f(\iota, \mathfrak{u}, \mathfrak{v})| \leq \Psi(\iota), \quad \forall(\iota, \mathfrak{u}, \mathfrak{v}) \in J \times \mathbb{R} \times \mathbb{R}.$$

(H3) There exists $\rho > 0$ that says

$$|G_z(\mathfrak{u}) - G_z(\mathfrak{v})| \leq \rho|\mathfrak{u} - \mathfrak{v}|, \quad \text{for all} \quad \mathfrak{u}, \mathfrak{v} \in X \quad \textit{with} \quad z = 1, 2, ..., m.$$

Existence and uniqueness results via Banach fixed point theorem

Theorem 3.1. *Let (H1) holds. If*

$$\theta = \left(\frac{(m+1)T^{\gamma+\mu}}{\Lambda(\gamma+\mu+1)} + \frac{T^{2\mu+\gamma-1}}{\mu T^{\mu-1}\Lambda(\mu+\gamma)}\right)\left(l_1 + l_2\frac{T^\eta}{\Lambda(\eta+1)} + |\alpha|\right) + |a|T + m\rho < 1, \tag{7}$$

then the problem (1) *has at least one solution.*

Proof. Our aim is to use Banach fixed point theorem. For this reason, we define an operator $\Phi : C \to C$ as follows:

$$\begin{aligned}
(\Phi\mathfrak{u})(\iota) &= \frac{1}{\Lambda(\gamma+\mu)}\sum_{0<\iota_z<\iota}\left(\int_{\iota_{z-1}}^{\iota_z}(\iota_z-\xi)^{\gamma+\mu-1}\mathcal{K}(\xi)d\xi - \alpha\int_{\iota_{z-1}}^{\iota_z}(\iota_z-\xi)^{\gamma+\mu-1}\mathfrak{u}(\xi)d\xi\right)\\
&\quad+\frac{1}{\Lambda(\gamma+\mu)}\left(\int_{\iota_m}^{\iota}(\iota-\xi)^{\gamma+\mu-1}\mathcal{K}(\xi)d\xi - \alpha\int_{\iota_m}^{\iota}(\iota-\xi)^{\gamma+\mu-1}\mathfrak{u}(\xi)d\xi\right)\\
&\quad-\frac{\iota^{\mu+\gamma-1}}{T^{\mu-1}\Lambda(\mu+\gamma)}\left(\int_0^T(T-\xi)^{\mu-1}\mathcal{K}(\xi)d\xi - \alpha\int_0^T(T-\xi)^{\mu-1}\mathfrak{u}(\xi)d\xi\right)\\
&\quad+a\int_0^T\mathfrak{u}(\xi)d\xi + b + \sum_{0<\iota_z<\iota}G_z(\mathfrak{u}(\iota_z^-)).
\end{aligned} \tag{8}$$

So we transform the problem (1) into a fixed point problem. Obviously, the fixed points of operator Φ are solutions of problem (1). By (H1) and (H3) , for each $\mathfrak{u}, \mathfrak{v} \in C$ and $\iota \in J$, we get

$|(\Phi\mathfrak{u})(\iota) - (\Phi\mathfrak{v})(\iota)|$

$$\begin{aligned}
&\leq \frac{1}{\Lambda(\gamma+\mu)}\sum_{0<\iota_z<\iota}\left(\int_{\iota_{z-1}}^{\iota_z}(\iota_z-\xi)^{\gamma+\mu-1}|f(\xi,\mathfrak{u}(\xi),I^\eta\mathfrak{u}(\xi)) - f(\xi,\mathfrak{v}(\xi),I^\eta\mathfrak{v}(\xi))|d\xi\right)\\
&\quad+\frac{1}{\Lambda(\gamma+\mu)}\int_{\iota_m}^{\iota}(\iota-\xi)^{\gamma+\mu-1}|f(\xi,\mathfrak{u}(\xi),I^\eta\mathfrak{u}(\xi)) - f(\xi,\mathfrak{v}(\xi),I^\eta\mathfrak{v}(\xi))|d\xi\\
&\quad+\frac{|\alpha|}{\Lambda(\gamma+\mu)}\sum_{0<\iota_z<\iota}\int_{\iota_{z-1}}^{\iota_z}(\iota_z-\xi)^{\gamma+\mu-1}|\mathfrak{u}(\xi)-\mathfrak{v}(\xi)|d\xi
\end{aligned}$$

$$
\begin{aligned}
&+\frac{|\alpha|}{\Lambda(\gamma+\mu)}\int_{\iota_m}^{\iota}(\iota-\xi)^{\gamma+\mu-1}|\mathfrak{u}(\xi)-\mathfrak{v}(\xi)|d\xi \\
&+\frac{\iota^{\mu+\gamma-1}}{T^{\mu-1}\Lambda(\mu+\gamma)}\bigg(\int_0^T(T-\xi)^{\mu-1}|f(\xi,\mathfrak{u}(\xi),I^{\eta}\mathfrak{u}(\xi)) \\
&-f(\xi,\mathfrak{v}(\xi),I^{\eta}\mathfrak{v}(\xi))|d\xi+|\alpha|\int_0^T(T-\xi)^{\mu-1}|\mathfrak{u}(\xi)-\mathfrak{v}(\xi)|d\xi\bigg) \\
&+|a|\int_0^T|\mathfrak{u}(\xi)-\mathfrak{v}(\xi)|d\xi+\sum_{z=1}^{m}|G_z(\mathfrak{u}(\iota_z^-))-G_z(\mathfrak{v}(\iota_z^-))| \\
\leq\ &\left[\left(\frac{(m+1)T^{\gamma+\mu}}{\Lambda(\gamma+\mu+1)}+\frac{T^{2\mu+\gamma-1}}{\mu T^{\mu-1}\Lambda(\mu+\gamma)}\right)\times\left(l_1+l_2\frac{T^{\eta}}{\Lambda(\eta+1)}+|\alpha|\right)+|a|T+m\rho\right]\|\mathfrak{u}-\mathfrak{v}\|.
\end{aligned}
$$

Thus we obtain

$$\|\Phi\mathfrak{u}-\Phi\mathfrak{v}\|\leq\theta\|\mathfrak{u}-\mathfrak{v}\|.$$

From (7), we conclude that Φ is a contraction. Banach fixed point theorem states that Φ has a unique fixed point, which is the unique solution of the problem (1) on J. This completes the proof. □

Existence results via Schauder's fixed point theorem

For the sake convenience, we put

$$\kappa_1=\frac{\Psi^*(m+1)T^{\gamma+\mu}}{\Lambda(\gamma+\mu+1)}+\frac{\Psi^*T^{\gamma+2\mu-1}}{\mu T^{\mu-1}\Lambda(\gamma+\mu)}+|b|,$$

where $\Psi^*=\sup\{\Psi(\iota):\iota\in J\}$.

Theorem 3.2. *Let us assume that the (H1) and (H2) are satisfied. If*

$$\omega=|\alpha|\left(\frac{(m+1)T^{\gamma+\mu}}{\Lambda(\gamma+\mu+1)}+\frac{T^{\gamma+2\mu-1}}{\mu T^{\mu-1}\Lambda(\gamma+\mu)}\right)+|a|T+m\rho<1,$$

then the problem (1) *has at least one solution on J.*

Proof. We consider the nonempty closed bounded convex subset

$$\Upsilon=\{\mathfrak{u}\in C:\|\mathfrak{u}\|\leq M\}$$

of C, where M is chosen such

$$M\geq\frac{\kappa_1}{1-\omega}.$$

Notice that the continuity of the operator Φ follows from the continuity of the function f. Now, applying the Arzela-Ascoli theorem, we need to show that the operator Φ is compact. Therefore, we will show that

$\Phi(\Upsilon) \subset \Upsilon$ and $\Phi(\Upsilon)$ is uniformly bounded and equicontinuous set. For $\mathfrak{u} \in \Upsilon$, we have

$$\begin{aligned}
|(\Phi\mathfrak{u})(\iota)| \leq\ & \frac{1}{\Lambda(\gamma+\mu)} \sum_{0<\iota_z<\iota} \int_{\iota_{z-1}}^{\iota_z} (\iota_z-\xi)^{\gamma+\mu-1} |f(\xi,\mathfrak{u}(\xi), I^\eta \mathfrak{u}(\xi))| d\xi \\
&+ \frac{1}{\Lambda(\gamma+\mu)} \int_{\iota_z}^{\iota} (\iota-\xi)^{\gamma+\mu-1} |f(\xi,\mathfrak{u}(\xi), I^\eta \mathfrak{u}(\xi))| d\xi \\
&+ \frac{|\alpha|}{\Lambda(\gamma+\mu)} \sum_{0<\iota_z<\iota} \int_{\iota_{z-1}}^{\iota_z} (\iota_z-\xi)^{\gamma+\mu-1} |\mathfrak{u}(\xi)| d\xi + \frac{|\alpha|}{\Lambda(\gamma+\mu)} \int_{\iota_z}^{\iota} (\iota-\xi)^{\gamma+\mu-1} |\mathfrak{u}(\xi)| d\xi \\
&+ \frac{\iota^{\mu+\gamma-1}}{T^{\mu-1}\Lambda(\mu+\gamma)} \left(\int_0^T (T-\xi)^{\mu-1} |f(\xi,\mathfrak{u}(\xi), I^\eta \mathfrak{u}(\xi))| d\xi + |\alpha| \int_0^T (T-\xi)^{\mu-1} |\mathfrak{u}(\xi)| d\xi \right) \\
&+ |a| \int_0^T |\mathfrak{u}(\xi)| d\xi + |b| + \sum_{z=1}^{m} |G_z(\mathfrak{u}(\iota_z^-))| \\
\leq\ & \frac{\Psi^*(m+1)T^{\gamma+\mu}}{\Lambda(\gamma+\mu+1)} + |\alpha| M \left(\frac{(m+1)T^{\gamma+\mu}}{\Lambda(\gamma+\mu+1)} + \frac{T^{2\mu+\gamma-1}}{\mu T^{\mu-1}\Lambda(\mu+\gamma)} \right) + \frac{\Psi^* T^{\gamma+2\mu-1}}{\mu T^{\mu-1}\Lambda(\gamma+\mu)} + |a|TM + |b| + m\rho \\
\leq\ & M.
\end{aligned}$$

Then $\|\Phi\mathfrak{u}\| \leq M$, which means that $\Phi(\Upsilon) \subset \Upsilon$ and the set $\Phi(\Upsilon)$ is uniformly bounded. Next, we will prove that $\Phi(\Upsilon)$ is equicontinuous set. For $\iota_1, \iota_2 \in J$ such that $\iota_{z-1} < \iota_z$ and for $\mathfrak{u} \in \Upsilon$, we get
$|(\Phi\mathfrak{u})(\iota_z) - (\Phi\mathfrak{u})(\iota_{z-1})|$

$$\begin{aligned}
\leq\ & \frac{1}{\Lambda(\gamma+\mu)} \sum_{0<\iota_z<\iota} \int_0^{\iota_{z-1}} \left((\iota_z-\xi)^{\gamma+\mu-1} - (\iota_{z-1}-\xi)^{\gamma+\mu-1} \right) |f(\xi,\mathfrak{u}(\xi), I^\eta \mathfrak{u}(\xi))| d\xi \\
&+ \frac{1}{\Lambda(\gamma+\mu)} \sum_{0<\iota_z<\iota} \int_{\iota_{z-1}}^{\iota_z} (\iota_z-\xi)^{\gamma+\mu-1} |f(\xi,\mathfrak{u}(\xi), I^\eta \mathfrak{u}(\xi))| d\xi \\
&+ \frac{1}{\Lambda(\gamma+\mu)} \int_0^{\iota_{z-1}} \left((\iota_z-\xi)^{\gamma+\mu-1} - (\iota_{z-1}-\xi)^{\gamma+\mu-1} \right) |f(\xi,\mathfrak{u}(\xi), I^\eta \mathfrak{u}(\xi))| d\xi \\
&+ \frac{1}{\Lambda(\gamma+\mu)} \int_{\iota_{z-1}}^{\iota_z} (\iota_z-\xi)^{\gamma+\mu-1} |f(\xi,\mathfrak{u}(\xi), I^\eta \mathfrak{u}(\xi))| d\xi \\
&+ \frac{|\alpha|}{\Lambda(\gamma+\mu)} \sum_{0<\iota_z<\iota} \left(\int_0^{\iota_{z-1}} (\iota_z-\xi)^{\gamma+\mu-1} - (\iota_{z-1}-\xi)^{\gamma+\mu-1} |\mathfrak{u}(\xi)| d\xi + \int_{\iota_{z-1}}^{\iota_z} (\iota_z-\xi)^{\gamma+\mu-1} |\mathfrak{u}(\xi)| d\xi \right) \\
&+ \frac{|\alpha|}{\Lambda(\gamma+\mu)} \left(\int_0^{\iota_{z-1}} (\iota_z-\xi)^{\gamma+\mu-1} - (\iota_{z-1}-\xi)^{\gamma+\mu-1} |\mathfrak{u}(\xi)| d\xi + \int_{\iota_{z-1}}^{\iota_z} (\iota_z-\xi)^{\gamma+\mu-1} |\mathfrak{u}(\xi)| d\xi \right) \\
&+ \frac{\iota_2^{\mu+\gamma-1} - \iota_1^{\mu+\gamma-1}}{T^{\mu-1}\Lambda(\mu+\gamma)} \left(\int_0^T (T-\xi)^{\mu-1} |f(\xi,\mathfrak{u}(\xi), I^\eta \mathfrak{u}(\xi))| d\xi + |\alpha| \int_0^T (T-\xi)^{\mu-1} |\mathfrak{u}(\xi)| d\xi \right) \\
&+ \sum_{z=1}^{m} |G_z(\mathfrak{u}(\iota_2^-)) - G_z(\mathfrak{v}(\iota_1^1))| \\
\leq\ & \frac{\Psi^*(m+1)}{\Lambda(\gamma+\mu+1)} \left(\iota_2^{\gamma+\mu} - \iota_1^{\gamma+\mu} \right) + \frac{\left(\iota_2^{\mu+\gamma-1} - \iota_1^{\mu+\gamma-1} \right)}{T^{\mu-1}\Lambda(\mu+\gamma)} \left(\frac{\Psi^* T^\mu}{\mu} + \frac{|\alpha| T^\mu M}{\mu} \right) + \rho \|(\mathfrak{u}(\iota_z)) - (\mathfrak{u}(\iota_{z-1}))\|.
\end{aligned}$$

As $\iota_{z-1} \to \iota_z$, we can observe that the above inequality's right-hand side tends to zero and that the convergence is independent of the $\mathfrak{u}$ in Υ parameters, which means $\Phi(\Upsilon)$ is equicontinuous. According to the Arzela-Ascoli theorem, Φ is compact. Therefore, using the Schauder fixed point theorem, we prove that Φ has at least one fixed point $\mathfrak{u} \in \Upsilon$ which is a solution of the problem (1) on J. □

4. An example

Consider the following fractional relaxation impulsive integro-differential equation

$$\begin{cases} \mathfrak{D}^{\frac{3}{2}} \quad {}^{LC}\mathfrak{D}^{\frac{1}{2}}\mathfrak{u}(\iota) + \frac{1}{4}\mathfrak{u}(\iota) = f\left(\iota, \mathfrak{u}(\iota), I^{\frac{1}{3}}\mathfrak{u}(\iota)\right), \quad \iota \neq \iota_z \quad \iota \in (0,1), \\ \Delta\mathfrak{u}(\iota_z) = G_z(\mathfrak{u}(\iota_z^-)), \quad z = 1,2,...,m, \\ {}^{LC}\mathfrak{D}^{\frac{1}{2}}\mathfrak{u}(0) = {}^{LC}\ \mathfrak{D}^{\frac{1}{2}}\mathfrak{u}(1) = 0, \quad \mathfrak{u}(0) = \frac{1}{10}\int_0^1 \mathfrak{u}(\xi)d\xi + 2. \end{cases} \tag{9}$$

Here $\gamma = \frac{1}{2}$, $\mu = \frac{3}{2}$, $\eta = \frac{1}{3}$, $\alpha = \frac{1}{4}$, $a = \frac{1}{10}$, and $b = 2$. Set

$$f\left(\iota, \mathfrak{u}(\iota), I^{\frac{1}{3}}\mathfrak{u}(\iota)\right) = \frac{sin(\iota)}{\exp(\iota^2)+7}\left(\frac{|\mathfrak{u}(\iota)|}{|\mathfrak{u}(\iota)|+1} + \frac{|I^{\frac{1}{3}\mathfrak{u}(\iota)}|}{1+|I^{\frac{1}{3}\mathfrak{u}(\iota)}|}\right).$$

For $\mathfrak{u}_i, \mathfrak{v}_i \in \mathbb{R}$, $i = 1,2$, we have

$$\begin{aligned} |f(\iota, \mathfrak{u}_1, \mathfrak{u}_2) - f(\iota, \mathfrak{v}_1, \mathfrak{v}_2)| &= \left|\frac{sin(\iota)}{\exp(\iota^2)+7}\left(\left(\frac{|\mathfrak{u}_1|}{|\mathfrak{u}_1|+1} - \frac{|\mathfrak{v}_1|}{|\mathfrak{v}_1|+1}\right) + \left(\frac{|\mathfrak{u}_2|}{|\mathfrak{u}_2|+1} - \frac{|\mathfrak{v}_2|}{|\mathfrak{v}_2|+1}\right)\right)\right| \\ &\leq \frac{1}{\exp(\iota^2)+7}\left(\frac{|\mathfrak{u}_1 - \mathfrak{v}_1|}{(1+|\mathfrak{u}_1|)(1+|\mathfrak{v}_1|)} + \frac{|\mathfrak{u}_2 - \mathfrak{v}_2|}{(1+|\mathfrak{u}_2|)(1+|\mathfrak{v}_2|)}\right) \\ &\leq \frac{1}{8}(|\mathfrak{u}_1 - \mathfrak{v}_1| + |\mathfrak{u}_2 - \mathfrak{v}_2|). \end{aligned}$$

Thus the assumption (H1) is satisfied with $l_1 = l_2 = \frac{1}{8}$, $\rho = \frac{1}{2}$ and $m = 1$. We shall verify that condition (7) is satisfied. Indeed

$$\begin{aligned} \theta &= \left(\frac{(m+1)T^{\gamma+\mu}}{\Lambda(\gamma+\mu+1)} + \frac{T^{2\mu+\gamma-1}}{\mu T^{\mu-1}\Lambda(\mu+\gamma)}\right)\left(l_1 + l_2\frac{T^\eta}{\Lambda(\eta+1)} + |\alpha|\right) + |a|T + m\rho \\ &= \left(\frac{1}{\Lambda(3)} + \frac{2}{3\Lambda(2)}\right)\left(\frac{1}{8} + \frac{1}{8}\frac{1}{\Lambda(\frac{1}{3}+1)} + \frac{1}{4}\right) + \frac{1}{10} \\ &\simeq 0.85 < 1. \end{aligned}$$

Consequently, the problem (9) has a unique solution on $[0,1]$ according to the Theorem 3.1. Also we have

$$f(\iota, \mathfrak{u}, \mathfrak{v}) \leq \frac{2}{\exp(\iota^2)+7}, \quad \forall(\iota, \mathfrak{u}, \mathfrak{v}) \in J \times \mathbb{R} \times \mathbb{R}.$$

Hence condition (H2) holds with $\Psi(\iota) = \frac{2}{\exp(\iota^2)+7}$, it follows from Theorem 3.2 that the problem (9) has at least one solution on $[0,1]$.

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