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An integral transform via the bounded linear operators on abstract Wiener space

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Abstract. In this paper, we obtain some results of a more rigorous mathematical structure that can guarantee the orthogonality of an orthogonal set even when results and formula on abstract Wiener integrals or some transforms using bounded linear operators. We then establish the existence of an integral transform on abstract Wiener space. Finally, we obtain some fundamental formulas with respect to the integral transform involving the Cameron-Storvick type theorem.

1. Introduction

Let H be a real separable infinite-dimensional Hilbert space. Let $\langle\cdot,\cdot\rangle_H$ be an inner product on H with the norm $|\cdot|_H = \sqrt{\langle\cdot,\cdot\rangle_H}$. Let $\|\cdot\|_0$ be a measurable norm on H with respect to the Gaussian cylinder measure ν_0 on H. Let B denote the completion of H with respect to $\|\cdot\|_0$. Let i be the natural injection from H to B. The adjoint operator i^* of i is one to one and maps B^* continuously onto a dense subset H^*, where B^* and H^* are topological duals of B and H, respectively. By identifying H^* with H and B^* with i^*B^*, we have a triple $B^* \subset H^* \approx H \subset B$ with $\langle x, y\rangle = (x, y)$ for all x in H and y in B^*, where $(\cdot,\cdot)$ denotes the natural dual pairing between B and B^*. By some results of Gross [11], $\nu_0 \circ i^{-1}$ has a unique countably additive extension ν to the Borel σ-algebra $\mathcal{B}(B)$ of B. The triple (B, H, ν) is called an abstract Wiener space. For more details, see [4, 11–13, 16, 20, 21].

Let $\{\alpha_j\}_{j=1}^{\infty}$ be a complete orthonormal set in H with α_j's are in B^*. For each $h \in H$ and for $x \in B$, we define a stochastic inner product $(h, x)^\sim$ by

$$(h,x)^\sim = \begin{cases} \lim\limits_{n\to\infty} \sum_{j=1}^{n} \langle h, \alpha_j\rangle_H (x, \alpha_j), & \text{if the limit exists} \\ 0, & \text{otherwise} \end{cases}.$$

Then for $h(\neq 0)$ in H, the stochastic inner product $(h, x)^\sim$ exists for all $x \in B$, $(h,\cdot)^\sim$ is a Gaussian random variable on B with mean zero and variance $|h|_H^2$, and is essentially independent of the choice of the complete orthonormal set. If both h and x are in H, then $(h, x)^\sim = \langle h, x\rangle$. Furthermore, $(h, \lambda x)^\sim = (\lambda h, x)^\sim = \lambda(h, x)^\sim$ for all $\lambda \in \mathbb{R}$, $h \in H$ and $x \in B$. One can see that if $\{h_1, \ldots, h_n\}$ is an orthonormal set in H, then the random variables $(h_j, x)^\sim$'s are independent and for $h \in B^* \subset H$, $(h, x)^\sim = (h, x)$, see [10, 13, 16].

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Lee defined an integral transform

$$\mathcal{F}_{\gamma,\beta}(F)(y) = \int_B F(\gamma x + \beta y)d\nu(x)$$

of analytic functionals on abstract Wiener space [20]. One can see that many transforms : the Fourier-Wiener transform [1], the modified Fourier-Wiener transform [2], the Fourier-Feynman transform [3] and the Gauss transform are special cases of Lee's integral transform $\mathcal{F}_{\gamma,\beta}$. Later, many mathematicians have studied integral transforms in conjunction with related topics for functionals in various classes. Recently, the authors obtained basic formulas for integral transforms and convolution products of functionals in several classes, see [4–7, 9, 17, 18].

In [10], there are many research results and formulas for the integral transform via the bounded linear operators with related topics. Consider a cylinder functional of the form

$$F(x) = f((g_1, x), \dots, (g_n, x)) \tag{1}$$

where f is an appropriate function on $\mathbb{R}^n$. According to the results in [4, 10, 17, 20], when we calculate the abstract Wiener integrals or transforms of functional of the form (1), the orthogonality of a set $\{g_1, \dots, g_n\}$ is very important. Using the orthogonality of the set $\{g_1, \dots, g_n\}$, we can use the change of variable formulas (3) from the abstract Wiener integrals into the Lebesgue integrals. For this reason, the papers on abstract Wiener integrals or some transforms using bounded linear operators have only been done on one-dimensional functionals. Studies on multi-dimensional functionals have not been conducted yet.

In this paper, we give an idea to solve these mathematical difficulties, and use it to obtain research results for the integral transform. We then establish some formulas and results with respect to the integral transform.

2. Definitions and notations

In this section, we give some definitions and notations to understand this paper.

We first give an integration formula for the abstract Weiner integrals used in this paper. Let $\{g_1, \dots, g_n\}$ be an orthogonal set in H with g_j in B^* for $j = 1, \dots, n$ and $F : B \to \mathbb{C}$ a functional defined by the formula

$$F(x) = f((g_1, x), \dots, (g_n, x)) \tag{2}$$

where $f : \mathbb{R}^n \to \mathbb{C}$ is a Lebesgue measurable function. Then

$$\begin{aligned} &\int_B f((g_1, x), \dots, (g_n, x))d\nu(x) \\ &\qquad \doteq \Big(\prod_{j=1}^n \frac{1}{2\pi|g_j|_H^2}\Big)^{\frac{1}{2}} \int_{\mathbb{R}^n} f(\vec{u}) \exp\Big\{-\sum_{j=1}^n \frac{u_j^2}{2|g_j|_H^2}\Big\} d\vec{u} \end{aligned} \tag{3}$$

in the sense that if either side of (3) exists, both sides exist and equality holds, where $\vec{u} = (u_1, \dots, u_n) \in \mathbb{R}^n$ and $d\vec{u} = du_1 \cdots du_n$.

We are ready to state the definition of the integral transform, the convolution product and the first variation via the bounded linear operators.

Definition 2.1. *Let F, F_1 and F_2 be measurable functionals on B. Let T and S be bounded linear operators from B to B. Then the integral transform $\mathcal{T}_{T,S}(F)$ of F is defined by the formula*

$$\mathcal{T}_{T,S}(F)(y) = \int_B F(Tx + Sy)d\nu(x), \qquad y \in B, \tag{4}$$

and the convolution product $(F_1 * F_2)_T$ of F_1 and F_2 is defined by the formula

$$(F_1 * F_1)_T(y) = \int_B F_1\Big(\frac{y+Tx}{\sqrt{2}}\Big)F_2\Big(\frac{y-Tx}{\sqrt{2}}\Big)d\nu(x), \qquad y \in B \tag{5}$$

if they exist. Furthermore, the first variation δF of F is defined by the formula

$$\delta F(x|w) = \frac{\partial}{\partial k}F(x+kw)\Big|_{k=0}, \qquad x, w \in B \tag{6}$$

if it exists.

We shall introduce a class of functionals on B. Let $\mathcal{A}$ be the class of functionals F of the form

$$F(x) = f((g_1, x), \dots, (g_n, x)), \tag{7}$$

where $f : \mathbb{R}^n \to \mathbb{C}$ is Lebesgue measurable and

$$|f(\vec{u})| \le M_f \exp\Big\{N_f \sum_{j=1}^{n} |u_j|\Big\} \tag{8}$$

for some real numbers $M_f > 0$ and $N_f \ge 0$.

3. A class of some angle preserving operators

In this section, we will explain why these research results and formulas are necessary and important. In order to do this, we need a class of operators.

First, let $T : B^* \to B^*$ be a bounded linear operator. Although the set $\{g_1, g_2\}$ is an orthogonal set in B^*, the set $\{Tg_1, Tg_2\}$ might not an orthogonal set in B^*. Like this assertion, we can find a mathematical problem. The orthogonality of a set is very important to obtain some abstract Wiener integrals of a cylinder functional

$$F(x) = f((g_1, x), \dots, (g_n, x))$$

where $f : \mathbb{R}^n \to \mathbb{C}$ is an appropriate function and $\{g_1, \dots, g_n\} \subset B^*$. If the set $\{g_1, \dots, g_n\}$ is orthogonal, then the existence of a following abstract Wiener integral

$$\int_B F(x)d\nu(x)$$

exists under some conditions for f. On the other hands, the following abstract Wiener integral

$$\int_B f((Tg_1, x), \dots, (Tg_n, x))d\nu(x)$$

might not exist or it may be difficult to show its existence unless the orthogonality of the set $\{Tg_1, \dots, Tg_n\}$ is given. Therefore, the problem of how to construct the orthogonality of the set $\{Tg_1, \dots, Tg_n\}$ naturally arises under giving an orthogonal set $\{g_1, \dots, g_n\} \subset B^*$.

Lemma 3.1. *Let $T : B^* \to B^*$ be a bounded linear operator and λ a positive real number. Then the following statements are equivalents:*

(1) $\langle Tg_1, Tg_2\rangle_H = \lambda^2\langle g_1, g_2\rangle_H$ for all g_1 and g_2 in B^*.

(2) $|Tg|_H = \lambda|g|_H$ for all g in B^*.

Proof. Suppose that $\langle Tg_1, Tg_2\rangle_H = \lambda^2\langle g_1, g_2\rangle_H$ for all g_1 and g_2 in B^*. Then we see that

$$|Tg|_H^2 = \langle Tg, Tg\rangle_H = \lambda^2\langle g, g\rangle_H = \lambda^2|g|_H^2.$$

Since $\lambda > 0$, we can conclude that $|Tg|_H = \lambda|g|_H$ for all g in B^*. Conversely, let $|Tg|_H = \lambda|g|_H$ for all g in B^*. Then for all g_1 and g_2 in B^*, we have

$$\begin{aligned}
\langle g_1, g_2\rangle_H &= \frac{1}{4}\Big[|g_1 + g_2|_H - |g_1 - g_2|_H\Big]\\
&= \frac{1}{4\lambda^2}\Big[|Tg_1 + Tg_2|_H - |Tg_1 - Tg_2|_H\Big]\\
&= \frac{1}{\lambda^2}\Big[\frac{1}{4}|Tg_1 + Tg_2|_H - \frac{1}{4}|Tg_1 - Tg_2|_H\Big]\\
&= \frac{1}{\lambda^2}\langle Tg_1, Tg_2\rangle_H,
\end{aligned}$$

which completes the first argument in Lemma 3.1 as desired. □

In our next theorem, we are going to suggest a method to solve the previously presented mathematical problem.

Theorem 3.2. *Let T be as in Lemma 3.1 above. Suppose that there is a $\lambda > 0$ such that*

$$\langle Tg_1, Tg_2\rangle_H = \lambda^2\langle g_1, g_2\rangle_H \tag{9}$$

for all g_1 and g_2 in B^*. Then T preserves angles between non-zero elements in B^*.

Proof. Let g_1 and g_2 are given with $g_1, g_2 \neq 0$. Let θ be the angle between g_1 and g_2, and let $\overline{\theta}$ be the angle between Tg_1 and Tg_2. Then using Lemma 3.1, we have

$$\cos\overline{\theta} = \frac{\langle Tg_1, Tg_2\rangle_H}{|Tg_1|_H|Tg_2|_H} = \frac{\lambda^2\langle g_1, g_2\rangle_H}{\lambda^2|g_1|_H|g_2|_H} = \frac{\langle g_1, g_2\rangle_H}{|g_1|_H|g_2|_H} = \cos\theta.$$

Thus, the angles, being in the $[0, \pi]$, are equal. Hence the proof of Theorem 3.2 is established. □

We next give some examples of the preserving angles operators on B^*.

Example 3.3. *For a fixed $w \in B^*$, let T_1, T_2 and T_w be bounded linear operators from B^* to B^* defined by the formulas*

$$T_1(g) = \alpha g$$
$$T_2(g) = |g|_H^2 g$$

and

$$T_w(g) = (\sin\theta + 2)g$$

where $\alpha > 0$ and θ is the angle between g and w. Then T_1, T_2 and T_w satisfy the condition (9). *In fact, we see that*

$$|T_2g|_H^2 = \langle T_2g, T_2g\rangle_H = |g|_H^4\langle g, g\rangle_H = |g|_H^4|g|_H^2,$$

and so $|T_2g|_H = |g|_H^2|g|_H$. In this case, $\lambda = |g|_H^2 > 0$. Furthermore,

$$|T_wg|_H^2 = \langle T_wg, T_wg\rangle_H = (\sin\theta + 2)^2|g|_H^2,$$

and so $|T_wg|_H = (\sin\theta + 2)|g|_H$. In this case, $\lambda = (\sin\theta + 2) > 0$. Hence using Lemma 3.1 and Theorem 3.2, T_2 and T_w preserve angles.

Let X and Y be Banach spaces and let $\mathcal{L}(X:Y)$ be set of all bounded operators from X to Y. Let $\mathcal{L}_0(B^*:B^*)$ be the space of all operators in $\mathcal{L}(B^*:B^*)$ which satisfy the condition (9) above, namely,

$$\mathcal{L}_0(B^*:B^*) = \{T \in \mathcal{L}(B^*:B^*) | \, T \text{ satisfies the condition (9)}\}.$$

Then we have the following assertions.

(1) Using some facts and results of the adjoint operator, give a operator $A \in \mathcal{L}(B:B)$, there is a bounded linear operator $A^*: B^* \to B^*$ so that for all $g \in B^*$ and $x \in B$,

$$(A^* g, x) = (g, Ax).$$

In fact, it is valid by in the sense of Riesz representation theorem, see [14].

(2) Since $(B^*)^* = B$, for $T \in \mathcal{L}_0(B^*:B^*)$, we have

$$T^*: B \to B$$

and $T^* \in \mathcal{L}(B:B)$.

(3) Let

$$\mathcal{L}_{AP}(B:B) = \{A \in \mathcal{L}(B:B) | A = T^*, T \in \mathcal{L}_0(B^*:B^*)\}.$$

Then $\mathcal{L}_{AP}(B:B) \subset \mathcal{L}(B:B)$ and for each $A \in \mathcal{L}_{AP}(B:B)$, A^* preserves angles between non-zero elements in B^*.

In order to express simply, we shall introduce some notations. Let F be an element of $\mathcal{A}$ and T an element of $\mathcal{L}_{AP}(B:B)$. For a given orthogonal set $\mathcal{G} \equiv \{g_1, \ldots, g_n\} \subset B^*$, let

$$T^*\mathcal{G} = \{T^* g_1, \ldots, T^* g_n\}.$$

Then the set $T^*\mathcal{G}$ is an orthogonal set in H with $T^* g_j \in B^*$ for all $j = 1, \ldots, n$ because $T \in \mathcal{L}_{AP}(B:B)$. For $x \in B$, let

$$F(x) = f((g_1, x), \ldots, (g_n, x)) \equiv f((\mathcal{G}, x)).$$

Then one can see that

$$f((T^*\mathcal{G}, x)) = f((T^* g_1, x), \ldots, (T^* g_n, x)). \tag{10}$$

4. Existence theorems

In this section, we establish the existence of the integral transform, the convolution product and the first variation for functionals in $\mathcal{A}$.

In the first theorem in this section, we give the existence of the integral transform $\mathcal{T}_{T,S}(F)$ of $F \in \mathcal{A}$.

Theorem 4.1. *Let F be an element of $\mathcal{A}$, and let T and S be are elements of $\mathcal{L}_{AP}(B:B)$. Then the integral transform $\mathcal{T}_{T,S}(F)$ of F exists, belongs to $\mathcal{A}$ and is given by the formula*

$$\mathcal{T}_{T,S}(F)(y) = \Gamma_1((S^*\mathcal{G}, y)) \tag{11}$$

for $y \in B$, where

$$\Gamma_1(\vec{v}) = \Big(\prod_{j=1}^{n} \frac{1}{2\pi |T^* g_j|_H^2}\Big)^{\frac{1}{2}} \int_{\mathbb{R}^n} f(\vec{u} + \vec{v}) \exp\Big\{-\sum_{j=1}^{n} \frac{u_j^2}{2|T^* g_j|_H^2}\Big\} d\vec{u}.$$

Proof. Using equations (4), (10) and (3), it follows that for $y \in B$

$$\begin{aligned}\mathcal{T}_{T,S}(F)(y) &= \int_B f((T^*\mathcal{G}, x) + (S^*\mathcal{G}, y))d\nu(x) \\ &= \Big(\prod_{j=1}^{n} \frac{1}{2\pi|T^*g_j|_H^2}\Big)^{\frac{1}{2}} \int_{\mathbb{R}^n} f(\vec{u} + (S^*\mathcal{G}, y)) \exp\Big\{-\sum_{j=1}^{n} \frac{u_j^2}{2|T^*g_j|_H^2}\Big\}d\vec{u} \\ &= \Gamma_1((S^*\mathcal{G}, y)).\end{aligned}$$

Furthermore, using equation (8), we have

$$\begin{aligned}|\Gamma_1(\vec{v})| &\le \Big(\prod_{j=1}^{n} \frac{M_f^2}{2\pi|T^*g_j|_H^2}\Big)^{\frac{1}{2}} \int_{\mathbb{R}^n} \exp\Big\{N_f \sum_{j=1}^{n}(|u_j| + |v_j|) - \sum_{j=1}^{n} \frac{u_j^2}{2|T^*g_j|_H^2}\Big\}d\vec{u} \\ &= M_{\Gamma_1} \exp\Big\{N_{\Gamma_1} \sum_{j=1}^{n} |v_j|\Big\},\end{aligned}$$

where

$$M_{\Gamma_1} = \Big(\prod_{j=1}^{n} \frac{M_f^2}{2\pi|T^*g_j|_H^2}\Big)^{\frac{1}{2}} \int_{\mathbb{R}^n} \exp\Big\{N_f \sum_{j=1}^{n} |u_j| - \sum_{j=1}^{n} \frac{u_j^2}{2|T^*g_j|_H^2}\Big\}d\vec{u} < \infty$$

and $N_{\Gamma_1} = N_f$. This means that $\mathcal{T}_{T,S}(F) \in \mathcal{A}$ and so the proof is completed as desired. □

In Theorem 4.2 below, we establish the existence of the convolution product of functionals in $\mathcal{A}$.

Theorem 4.2. *Let $T \in \mathcal{L}_{AP}(B : B)$, and let F_1 and F_2 be elements of $\mathcal{A}$. Then the convolution product $(F_1 * F_2)_T$ of F_1 and F_2 exists, belongs to $\mathcal{A}$ and is given by the formula*

$$(F_1 * F_2)_T(y) = \Gamma_2((\mathcal{G}, y)) \tag{12}$$

for $y \in B$, where

$$\begin{aligned}\Gamma_2(\vec{v}) = \Big(\prod_{j=1}^{n} \frac{1}{2\pi|T^*g_j|_H^2}\Big)^{\frac{1}{2}} \int_{\mathbb{R}^n} f_1\Big(\frac{1}{\sqrt{2}}\vec{v} + \frac{1}{\sqrt{2}}\vec{u}\Big)f_2\Big(\frac{1}{\sqrt{2}}\vec{v} - \frac{1}{\sqrt{2}}\vec{u}\Big) \\ \times \exp\Big\{-\sum_{j=1}^{n} \frac{u_j^2}{2|T^*g_j|_H^2}\Big\}d\vec{u}.\end{aligned}$$

Proof. Equations (5), (10) and (3) yield the following calculation

$$\begin{aligned}&(F_1 * F_2)_T(y) \\ &= \int_B f_1\Big(\frac{1}{\sqrt{2}}(\mathcal{G}, y) + \frac{1}{\sqrt{2}}(T^*\mathcal{G}, x)\Big)f_2\Big(\frac{1}{\sqrt{2}}(\mathcal{G}, y) - \frac{1}{\sqrt{2}}(T^*\mathcal{G}, x)\Big)d\nu(x) \\ &= \Big(\prod_{j=1}^{n} \frac{1}{2\pi|T^*g_j|_H^2}\Big)^{\frac{1}{2}} \int_{\mathbb{R}^n} f_1\Big(\frac{1}{\sqrt{2}}(\mathcal{G}, y) + \frac{1}{\sqrt{2}}\vec{u}\Big)f_2\Big(\frac{1}{\sqrt{2}}(\mathcal{G}, y) - \frac{1}{\sqrt{2}}\vec{u}\Big) \\ &\qquad \times \exp\Big\{-\sum_{j=1}^{n} \frac{u_j^2}{2|T^*g_j|_H^2}\Big\}d\vec{u} \\ &= \Gamma_2((\mathcal{G}, y))\end{aligned}$$

for $y \in B$. Thus, equation (12) is established. Furthermore, using equation (8), we have

$$
\begin{aligned}
&|\Gamma_2(\vec{v})| \\
&\le \Big(\prod_{j=1}^{n} \frac{1}{2\pi|T^*g_j|_H^2}\Big)^{\frac{1}{2}} \int_{\mathbb{R}^n} \Big|f_1\Big(\frac{1}{\sqrt{2}}\vec{v}+\frac{1}{\sqrt{2}}\vec{u}\Big)\Big|\Big|f_2\Big(\frac{1}{\sqrt{2}}\vec{v}-\frac{1}{\sqrt{2}}\vec{u}\Big)\Big| \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times \exp\Big\{-\sum_{j=1}^{n}\frac{u_j^2}{2|T^*g_j|_H^2}\Big\}d\vec{u} \\
&\le \Big(\prod_{j=1}^{n} \frac{M_{f_1}^2M_{f_2}^2}{2\pi|T^*g_j|_H^2}\Big)^{\frac{1}{2}} \int_{\mathbb{R}^n} \exp\Big\{\frac{N_{f_1}+N_{f_2}}{\sqrt{2}}\sum_{j=1}^{n}(|u_j|+|v_j|)-\sum_{j=1}^{n}\frac{u_j^2}{2|T^*g_j|_H^2}\Big\}d\vec{u} \\
&= M_{\Gamma_2}\exp\Big\{N_{\Gamma_2}\sum_{j=1}^{n}|v_j|\Big\},
\end{aligned}
$$

where

$$M_{\Gamma_2} = \Big(\prod_{j=1}^{n} \frac{M_{f_1}^2M_{f_2}^2}{2\pi|T^*g_j|_H^2}\Big)^{\frac{1}{2}} \int_{\mathbb{R}^n} \exp\Big\{\frac{N_{f_1}+N_{f_2}}{\sqrt{2}}\sum_{j=1}^{n}|u_j|-\sum_{j=1}^{n}\frac{u_j^2}{2|T^*g_j|_H^2}\Big\}d\vec{u} < \infty$$

and $N_{\Gamma_2} = \frac{N_{f_1}+N_{f_2}}{\sqrt{2}}$, and hence $(F_1 * F_2)_T$ is in $\mathcal{A}$. Thus we obtain the desired results. □

In the last theorem of this section, we give the existence of the first variation exists.

Theorem 4.3. *Let F be an element of $\mathcal{A}$ with f is differentiable and its all partial derivatives satisfy the condition* (8) *and let $w \in H$. Then the first variation δF of F exists, belongs to $\mathcal{A}$ and is given by the formula*

$$\delta F(x|w) = \Gamma_3((\mathcal{G}, x)) \tag{13}$$

for $y \in B$, where

$$\Gamma_3(\vec{v}) = \sum_{j=1}^{n}\langle g_j, w\rangle_H \frac{\partial f}{\partial v_j}(\vec{v})$$

and $\frac{\partial f}{\partial v_j}$ is the j-th partial derivative of f for $j = 1, 2, \dots, n$.

Proof. Using equations (6) and (10), we have

$$\delta F(x|w) = \frac{\partial}{\partial k} f((\mathcal{G}, x) + k(\mathcal{G}, w))\Big|_{k=0} = \sum_{j=1}^{n}\langle g_j, w\rangle_H \frac{\partial f}{\partial v_j}((\mathcal{G}, x)).$$

In fact, using equation (8), we obtain that

$$
\begin{aligned}
|\Gamma_3(\vec{v})| &\le \sum_{j=1}^{n}|\langle g_j, w\rangle_H|\Big|\frac{\partial f}{\partial v_j}(\vec{v})\Big| = \sum_{j=1}^{n} M_{\frac{\partial f}{\partial v_j}}|g_j|_H|w|_H\exp\Big\{N_{\frac{\partial f}{\partial v_j}}\sum_{k=1}^{n}|v_k|\Big\} \\
&\le M_{\Gamma_3}\exp\Big\{N_{\Gamma_3}\sum_{k=1}^{n}|v_k|\Big\},
\end{aligned}
$$

where $M_{\Gamma_3} = |w|_H M_0$, $N_{\Gamma_3} = nN_0$, $M_0 = \max\Big\{M_{\frac{\partial f}{\partial v_1}}|g_1|_H, \dots, M_{\frac{\partial f}{\partial v_n}}|g_n|_H\Big\}$ and $N_0 = \max\Big\{N_{\frac{\partial f}{\partial v_1}}, \dots, N_{\frac{\partial f}{\partial v_n}}\Big\}$. Hence we have the desired results. □

5. Relationships

In this section, we give various relationships among the integral transform, the convolution product and the first variation of functionals in $\mathcal{A}$.

As the first relationship, we establish the convolution theorem for the integral transform $\mathcal{T}_{T,S}$.

Theorem 5.1. (Convolution Theorem) *Let F_1 and F_2 be as in Theorem 4.2 above, let $T, S \in \mathcal{L}_{AP}(B:B)$. Then we have*

$$\mathcal{T}_{T,S}(F_1 * F_2)_T(y) = \mathcal{T}_{T,S}(F_1)(y/\sqrt{2})\mathcal{T}_{T,S}(F_2)(y/\sqrt{2}) \tag{14}$$

for $y \in B$.

Proof. In Theorems 4.1 and 4.2, we established the existence of both sides of equation (14). We shall establish the equality of equation (14). Using equations (4), (5), (10) and (3), we have

$$\begin{aligned}
&\mathcal{T}_{T,S}(F_1 * F_2)_T(y)\\
&= \int_B \int_B f_1\Big(\frac{1}{\sqrt{2}}(T^*\mathcal{G}, x) + \frac{1}{\sqrt{2}}(T^*\mathcal{G}, z) + \frac{1}{\sqrt{2}}(S^*\mathcal{G}, y)\Big)\\
&\qquad\qquad \times f_2\Big(\frac{1}{\sqrt{2}}(T^*\mathcal{G}, x) - \frac{1}{\sqrt{2}}(T^*\mathcal{G}, z) + \frac{1}{\sqrt{2}}(S^*\mathcal{G}, y)\Big)d\nu(z)d\nu(x)\\
&= \Big(\prod_{j=1}^n \frac{1}{2\pi|T^*g_j|_H^2}\Big)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f_1\Big(\frac{1}{\sqrt{2}}\vec{u} + \frac{1}{\sqrt{2}}\vec{v} + \frac{1}{\sqrt{2}}(S^*\mathcal{G}, y)\Big)\\
&\qquad\qquad \times f_2\Big(\frac{1}{\sqrt{2}}\vec{u} - \frac{1}{\sqrt{2}}\vec{v} + \frac{1}{\sqrt{2}}(S^*\mathcal{G}, y)\Big)\exp\Big\{-\sum_{j=1}^n \frac{u_j^2 + v_j^2}{2|T^*g_j|_H^2}\Big\}d\vec{u}d\vec{v}.
\end{aligned}$$

Substituting $s_j = \frac{u_j+v_j}{\sqrt{2}}$ and $r_j = \frac{u_j-v_j}{\sqrt{2}}$ for each $j = 1, 2, \dots, n$. Then using equation (3) again, we have

$$\begin{aligned}
&\mathcal{T}_{T,S}(F_1 * F_2)_T(y)\\
&= \Big(\prod_{j=1}^n \frac{1}{2\pi|T^*g_j|_H^2}\Big)\int_{\mathbb{R}^n} f_1\Big(s_j + \frac{1}{\sqrt{2}}(S^*\mathcal{G}, y)\Big)\exp\Big\{-\sum_{j=1}^n \frac{s_j^2}{2|T^*g_j|_H^2}\Big\}d\vec{s}\\
&\qquad\qquad \times \int_{\mathbb{R}^n} f_2\Big(r_j + \frac{1}{\sqrt{2}}(S^*\mathcal{G}, y)\Big)\exp\Big\{-\sum_{j=1}^n \frac{r_j^2}{2|T^*g_j|_H^2}\Big\}d\vec{r}\\
&= \int_B f_1\Big((T^*\mathcal{G}, w_1) + \frac{1}{\sqrt{2}}(S^*\mathcal{G}, y)\Big)d\nu(w_1) \times \int_B f_2\Big((T^*\mathcal{G}, w_2) + \frac{1}{\sqrt{2}}(S^*\mathcal{G}, y)\Big)d\nu(w_2)\\
&= \mathcal{T}_{T,S}(F_1)(y/\sqrt{2})\mathcal{T}_{T,S}(F_2)(y/\sqrt{2}),
\end{aligned}$$

which establishes Theorem 5.1 as desired. □

Theorem 5.2 tells us that the integral transform and the first variation are commutable.

Theorem 5.2. *Let $T, S \in \mathcal{L}_{AP}(B:B)$ and let F be as in Theorem 4.3. Let $w \in H$. Then*

$$\mathcal{T}_{T,S}(\delta F(\cdot|Sw))(y) = \delta\mathcal{T}_{T,S}(F)(y|w) \tag{15}$$

for $y \in B$. Furthermore, the both sides of equation (15) *is given by the formula*

$$\sum_{j=1}^n \int_B \langle g_j, x\rangle_H \frac{\partial f}{\partial v_j}((T^*\mathcal{G}, x) + (S^*\mathcal{G}, y))d\nu(x)$$

where $\frac{\partial f}{\partial v_j}$ is the j-th partial derivative of f for $j = 1, 2, \dots, n$.

Proof. From Theorems 4.1 and 4.3, we see that $\mathcal{T}_{T,S}(\delta F(\cdot|Sw))(y)$ and $\delta\mathcal{T}_{T,S}(F)(y|w)$ exist, and they are elements of $\mathcal{A}$. We left show that the equality in equation (15) holds. First, using equations (4), (10) and (6), we have

$$\begin{aligned}
&\mathcal{T}_{T,S}(\delta F(\cdot|Sw))(y)\\
&= \int_B \frac{\partial}{\partial k} f((T^*\mathcal{G},x)+(S^*\mathcal{G},y)+k(S^*\mathcal{G},w))\Big|_{k=0} d\nu(x)\\
&= \sum_{j=1}^n \int_B \langle g_j, x\rangle_H \frac{\partial f}{\partial v_j}((T^*\mathcal{G},x)+(S^*\mathcal{G},y))d\nu(x)
\end{aligned}$$

for $y \in B$. Next, using equations (6), (10) and (4) again, we have

$$\begin{aligned}
&\delta\mathcal{T}_{T,S}(F)(y|w)\\
&= \frac{\partial}{\partial k}\int_B f((T^*\mathcal{G},x)+(S^*\mathcal{G},y)+k(S^*\mathcal{G},w))d\nu(x)\Big|_{k=0}\\
&= \sum_{j=1}^n \int_B \langle g_j, x\rangle_H \frac{\partial f}{\partial v_j}((T^*\mathcal{G},x)+(S^*\mathcal{G},y))d\nu(x)
\end{aligned}$$

for $y \in B$. Hence we have the desired result. □

In Theorem 5.3, we establish a relationship between the convolution product and the first variation.

Theorem 5.3. *Let F_1 and F_2 be as in Theorem 4.2 with f_1 and f_2 are differentiable and its derivatives satisfies the condition (8). Let $T \in \mathcal{L}_{AP}(B:B)$ and let $w \in H$. Then we have*

$$\delta(F_1 * F_2)_T(y|w) = (\delta F_1(\cdot|w/\sqrt{2}) * F_2)_T(y) + (F_1 * \delta F_2(\cdot|w/\sqrt{2}))_T(y) \tag{16}$$

for $y \in B$.

Proof. We obtained the existence of both side of equation (16) by Theorems 4.2 and 4.3. So, we shall show that the equality holds. Using equations (5), (10) and (6), we have

$$\begin{aligned}
&\delta(F_1 * F_2)_T(y|w)\\
&= \frac{\partial}{\partial k}\Big[\int_B f_1\Big(\frac{1}{\sqrt{2}}(\mathcal{G},y)+\frac{k}{\sqrt{2}}(\mathcal{G},k)+\frac{1}{\sqrt{2}}(\mathcal{G},x)\Big)\\
&\qquad\qquad \times f_2\Big(\frac{1}{\sqrt{2}}(\mathcal{G},y)+\frac{k}{\sqrt{2}}(\mathcal{G},k)-\frac{1}{\sqrt{2}}(\mathcal{G},x)\Big)d\nu(x)\Big]\Big|_{k=0}\\
&= \sum_{j=1}^n \int_B \langle g_j, w\rangle_H \frac{\partial f_1}{\partial v_j}\Big(\frac{1}{\sqrt{2}}(\mathcal{G},y)+\frac{1}{\sqrt{2}}(\mathcal{G},x)\Big)d\nu(x)\\
&\qquad\qquad + \sum_{j=1}^n \int_B \langle g_j, w\rangle_H \frac{\partial f_2}{\partial v_j}\Big(\frac{1}{\sqrt{2}}(\mathcal{G},y)-\frac{1}{\sqrt{2}}(\mathcal{G},x)\Big)d\nu(x)\\
&= (\delta F_1(\cdot|w/\sqrt{2}) * F_2)_T(y) + (F_1 * \delta F_2(\cdot|w/\sqrt{2}))_T(y),
\end{aligned}$$

which completes the proof of Theorem 5.3 as desired. □

We next give some more relationships among the integral transform $\mathcal{T}_{T,S}$, the convolution product and the first variation of functionals in $\mathcal{A}$. Here is the list of some relationships.

Let F_1 and F_2 be satisfy all conditions in previous sections. Then we have the following assertions :

(i) Using equations (16) and (15), we have

$$\begin{aligned}
&\delta(\mathcal{T}_{T,S}(F_1) * \mathcal{T}_{T,S}(F_2))_T(y|w)\\
&= (\delta\mathcal{T}_{T,S}(F_1(\cdot|\frac{w}{\sqrt{2}})) * \mathcal{T}_{T,S}(F_2))_T(y) + (\mathcal{T}_{T,S}(F_1) * \delta\mathcal{T}_{T,S}((\cdot|\frac{w}{\sqrt{2}}))_T(y)\\
&= (\mathcal{T}_{T,S}(\delta F_1(\cdot|\frac{Sw}{\sqrt{2}})) * \mathcal{T}_{T,S}(F_2))_T(y) + (\mathcal{T}_{T,S}(F_1) * \mathcal{T}_{T,S}(\delta F_2(\cdot|\frac{Sw}{\sqrt{2}}))_T(y)
\end{aligned}$$

for $y \in B$ and $w \in H$.

(ii) Using equations (14) and (15), we have

$$\begin{aligned}
&\mathcal{T}_{T,S}(\delta F_1(\cdot|Sw) * \delta F_2(\cdot|Sw))_T(y)\\
&= \mathcal{T}_{T,S}(\delta F_1(\cdot|Sw))(\frac{y}{\sqrt{2}})\mathcal{T}_{T,S}(\delta F_2(\cdot|Sw))(\frac{y}{\sqrt{2}})\\
&= \delta\mathcal{T}_{T,S}(F_1)(\frac{y}{\sqrt{2}}|w)\delta\mathcal{T}_{T,S}(F_2)(\frac{y}{\sqrt{2}}|w)
\end{aligned}$$

for $y \in B$ and $w \in H$.

(iii) Using equations (16) and (14), we have

$$\begin{aligned}
&\delta\mathcal{T}_{T,S}(F_1 * F_2)_T(y|w)\\
&= \mathcal{T}_{T,S}(\delta(F_1 * F_2)_T(\cdot|Sw))(y)\\
&= \mathcal{T}_{T,S}(\delta F_1(\cdot|\frac{Sw}{\sqrt{2}}) * F_2)_T(y) + \mathcal{T}_{T,S}(F_1 * \delta F_2(\cdot|\frac{Sw}{\sqrt{2}}))_T(y)
\end{aligned}$$

for $y \in B$ and $w \in H$.

(iv) Using equations (15), (14) and (16), we have

$$\begin{aligned}
&\mathcal{T}_{T,S}(\delta(F_1 * F_2)_T(\cdot|Sw))(y)\\
&= \delta\mathcal{T}_{T,S}((F_1 * F_2)_T)(y|w)\\
&= \delta(\mathcal{T}_{T,S}(F_1)(\frac{\cdot}{\sqrt{2}})\mathcal{T}_{T,S}(F_2)(\frac{\cdot}{\sqrt{2}}))(y|w)\\
&= \delta\mathcal{T}_{T,S}(F_1)(\frac{y}{\sqrt{2}}|w)\mathcal{T}_{T,S}(F_2)(\frac{y}{\sqrt{2}}) + \mathcal{T}_{T,S}(F_1)(\frac{y}{\sqrt{2}})\delta\mathcal{T}_{T,S}(F_2)(\frac{y}{\sqrt{2}}|w)
\end{aligned}$$

for $y \in B$ and $w \in H$. Furthermore, using equations (16), (14) and (15), we can obtain an another relationship as below

$$\begin{aligned}
&\mathcal{T}_{T,S}(\delta(F_1 * F_2)_T(\cdot|Sw))(y)\\
&= \mathcal{T}_{T,S}((\delta F_1(\cdot|\frac{Sw}{\sqrt{2}}) * F_2)_T)(y) + \mathcal{T}_{T,S}((F_1 * \delta F_2(\cdot|\frac{Sw}{\sqrt{2}})_T)(y)\\
&= \mathcal{T}_{T,S}(\delta F_1(\cdot|\frac{Sw}{\sqrt{2}}))(\frac{y}{\sqrt{2}})\mathcal{T}_{T,S}(F_2)(\frac{y}{\sqrt{2}}) + \mathcal{T}_{T,S}(F_1)(\frac{y}{\sqrt{2}})\mathcal{T}_{T,S}(\delta F_2(\cdot|\frac{Sw}{\sqrt{2}}))(\frac{y}{\sqrt{2}})
\end{aligned}$$

for $y \in B$ and $w \in H$.

6. The Cameron-Storvick type theorem

In this section, we establish the Cameron-Storvick type theorem with respect to the integral transform $\mathcal{T}_{T,S}$.

Before do this, we need the following Lemma 6.1 below. The following lemma was established in [15] and used in [19].

Lemma 6.1. (Translation theorem) *Let x_0 be an element of H. If F is ν-integrable on B, then*

$$\int_B F(x+x_0)d\nu(x) = \exp\left\{-\frac{1}{2}|x_0|_H^2\right\}\int_B F(x)\exp\{(x_0,x)\}d\nu(x). \tag{17}$$

The Cameron-Storvick theorem is that the abstract Wiener integrals involving the first variation can be expressed by the ordinary forms without concept the first variation. Numerous constructions and theories regarding the Cameron-Storvick theorem have been studied and applied in [4, 15, 19, 20].

In Theorem 6.2, we establish the Cameron-Storvick type theorem for the integral transform $\mathcal{T}_{T,S}$.

Theorem 6.2. *Let $T \in \mathcal{L}_{AP}(B:B)$ with $T^*T = I$ and let $S \in \mathcal{L}_{AP}(B:B)$ with $\mathcal{R}(S) \subset H$, where $\mathcal{R}(S)$ is the range of S. Let F be as in Theorem 5.2 and let $w \in H$. Then we have*

$$\mathcal{T}_{T,S}(\delta F(\cdot|Sw))(y) = \mathcal{T}_{T,S}((Sw,\cdot)F(\cdot))(y) - (Sw,Sy)\mathcal{T}_{T,S}(F)(y) \tag{18}$$

for $y \in B$.

Proof. The existence of equation (18) is obtained from Theorem 5.2. We left to show that the equality in equation (18) holds. Using equations (11) and (15), we have

$$\mathcal{T}_{T,S}(\delta F(\cdot|Sw))(y) = \frac{\partial}{\partial k}\left[\int_B F(Tx+Sy+kSw)d\nu(x)\right]\Big|_{k=0}$$

for $y \in B$ and $w \in H$. Since T and S in $\mathcal{L}_{AP}(B:B)$, $T^*Sw \in B^*$ Now, let $F_y(x) = F(x+y)$ and let $(F)^T(x) = F(Tx)$. Then using equations (17), (4) and some algebraic calculations by replacing $(F_{Sy})^T$ with F and replacing T^*Sw with w, we have

$$\begin{aligned}
&\mathcal{T}_{T,S}(\delta F(\cdot|Sw))(y)\\
&= \frac{\partial}{\partial k}\left[\exp\left\{-\frac{k^2}{2}|T^*Sw|_H^2\right\}\int_B F(Tx+Sy)\exp\{k(T^*Sw,x)\}d\nu(x)\right]\Big|_{k=0}\\
&= \int_B F(Tx+Sy)(T^*Sw,x)d\nu(x)\\
&= \int_B F(Tx+Sy)(Sw,Tx+Sy)d\nu(x) - \int_B F(Tx+Sy)(Sw,Sy)d\nu(x)\\
&= \mathcal{F}_{S,R}((Sw,\cdot)F(\cdot))(y) - (Sw,Sy)\mathcal{T}_{T,S}(F)(y)
\end{aligned}$$

for $y \in B$ and $w \in H$. Hence we have the desire results. □

We will explain the usefulness of the Cameron-Storvick type theorem with an example. Equation (18) tells us that

$$\mathcal{T}_{T,S}((w,\cdot)F(\cdot))(y) = \mathcal{T}_{T,S}(\delta F(\cdot|Sw))(y) + (Sw,Sy)\mathcal{T}_{T,S}(F)(y). \tag{19}$$

In fact, it is not easy to calculate the integral transform involving polynomial weight. That is to say, a calculation of the following abstract Wiener integral

$$\int_B (S^*u_1,x)\exp\{(T^*u_2,x)\}d\nu(x)$$

is not easy unless S^*u_1 and T^*u_2 are orthogonal. From equation (19), we note that the integral transform of functionals with polynomial weight can be calculated very easily from the integral transform of functionals. For example, let

$$F(x) = \sum_{j=1}^n (g_j, x).$$

Then one can see that $F \in \mathcal{A}$ and so using equations (11) and (15), we have

$$\mathcal{T}_{T,S}(F)(y) = \sum_{j=1}^{n} (S^* g_j, y)$$

and

$$\mathcal{T}_{T,S}(\delta F(\cdot|Sw))(y) = \sum_{j=1}^{n} (S^* g_j, y) + \sum_{j=1}^{n} (S^* g_j, w).$$

Hence we can conclude that

$$\begin{aligned}\mathcal{T}_{T,S}((w,\cdot)F(\cdot))(y) &= \sum_{j=1}^{n} (S^* g_j, y) + \sum_{j=1}^{n} (S^* g_j, w) + (Sw, Sy) \sum_{j=1}^{n} (S^* g_j, y) \\ &= \sum_{j=1}^{n} (S^* g_j, y)[1 + (Sw, Sy)] + \sum_{j=1}^{n} (S^* g_j, w).\end{aligned}$$

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