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On Gallai's path decomposition conjecture

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Abstract. Gallai conjectured that every connected graph on n vertices can be decomposed into at most $\frac{n+1}{2}$ paths. Let G be a connected graph on n vertices. The E-subgraph of G, denoted by F, is the subgraph induced by the vertices of even degree in G. The maximum degree of G is denoted by $\triangle(G)$. In 2020, Botler and Sambinelli verified Gallai's Conjecture for graphs whose E-subgraphs F satisfy $\triangle(F) \leq 3$. If the E-subgraph of G has at most one vertex with degree greater than 3, Fan, Hou and Zhou verified Gallai's Conjecture for G. In this paper, it is proved that if there are two adjacent vertices $x, y \in V(F)$ such that $d_F(v) \leq 3$ for every vertex $v \in V(F)\backslash\{x, y\}$, then G has a path-decomposition $\mathcal{D}_1$ such that $|\mathcal{D}_1| \leq \frac{n+1}{2}$ and $\mathcal{D}_1(x) \geq 2$, and a path-decomposition $\mathcal{D}_2$ such that $|\mathcal{D}_2| \leq \frac{n+1}{2}$ and $\mathcal{D}_2(y) \geq 2$.

1. Introduction

All graphs considered in this paper are finite and simple. A *decomposition* of a graph is a set of subgraphs that partition its edge set. If all these subgraphs are isomorphic to path, then it is called a path-decomposition. Let $\mathcal{D}$ be a path-decomposition of a graph G. The number of elements of $\mathcal{D}$ is denoted by $|\mathcal{D}|$. For a vertex $v \in V(G)$, the number of paths in $\mathcal{D}$ with v as an end vertex is denoted by $\mathcal{D}(v)$. Gallai [6] proposed the following conjecture.

Conjecture 1.1. *(Gallai's conjecture [6]) Let G be a connected graph on n vertices. Then G has a path-decomposition $\mathcal{D}$ such that $|\mathcal{D}| \leq \frac{n+1}{2}$.*

The first breakthrough in the study of Gallai's conjecture is Lovász [6] made.

Theorem 1.1. *(Lovász [6]) Let G be a graph on n vertices. If G has at most one vertex of even degree, then G has a path-decomposition $\mathcal{D}$ such that $|\mathcal{D}| \leq \frac{n}{2}$.*

Given a graph G, the sets of vertices and edges of G are denoted by $V(G)$ and $E(G)$, respectively. A *cut vertex* of G is a vertex whose removal increases the number of components of G. The *even subgraph* of G (E-subgraph, for short), denoted by $EV(G)$, is the subgraph of G induced by its even degree vertices. The maximum degree of a graph G is denoted by $\triangle(G)$. A *block* in a graph G is a maximal 2-connected subgraph of G. We use S_{k_1,k_2} to denote a double-star with center vertices x and y, where the degree of x is k_1 and the degree of y is k_2.

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By Theorem 1.1, Gallai's conjecture is true if the E-subgraph of G has at most one vertex. The conjecture was verified by Favaron and Kouider [5] for Eulerian graphs with degrees 2 and 4, by Botler and Jiménez [1] for $2k$-regular ($k \geq 3$) graphs of girths at least $2k-2$ that have a pair of disjoint perfect matchings. Pyber [7] verified Gallai's conjecture for graphs whose E-subgraphs are forests. Each block of a forest is a single edge. If each block of the E-subgraph of G has maximum degree at most 3 and contains no triangles, Fan [3] verified Gallai's conjecture is true. If the maximum degree of the E-subgraph of G less than or equal to 3, Botler and Sambinelli [2] verified that G has a path-decomposition $\mathcal{D}_1$ such that $|\mathcal{D}_1| \leq \frac{|V(G)|}{2}$, or a path-decomposition $\mathcal{D}_2$ such that $|\mathcal{D}_2| \leq \frac{|V(G)|+1}{2}$. From this result, we can get the following theorem.

Theorem 1.2. *(Theorem 13, [2]) Let G be a connected graph on n vertices and F be the E-subgraph of G. If $\triangle(F) \leq 3$, then G has a path-decomposition $\mathcal{D}$ such that $|\mathcal{D}| \leq \frac{n+1}{2}$.*

Fan, Hou and Zhou [4] generalized the result above.

Theorem 1.3. *(Theorem 5, [4]) Let G be a connected graph on n vertices and F be the E-subgraph of G. If there is a vertex $x \in V(F)$ such that $d_F(v) \leq 3$ for every vertex $v \in V(F)\backslash\{x\}$, then G has a path-decomposition $\mathcal{D}$ such that $|\mathcal{D}| \leq \frac{n+1}{2}$ and $\mathcal{D}(x) \geq 2$.*

The main result of this paper is as following.

Theorem 1.4. *Let G be a connected graph on n vertices and F be the E-subgraph of G. If there are two vertices $x, y \in V(F)$ and an edge $xy \in E(F)$ such that $d_F(v) \leq 3$ for every vertex $v \in V(F)\backslash\{x, y\}$, then G has a path-decomposition $\mathcal{D}_1$ such that $|\mathcal{D}_1| \leq \frac{n+1}{2}$ and $\mathcal{D}_1(x) \geq 2$, and a path-decomposition $\mathcal{D}_2$ such that $|\mathcal{D}_2| \leq \frac{n+1}{2}$ and $\mathcal{D}_2(y) \geq 2$.*

2. Technical Lemmas

In a graph G, the set of neighbors of a vertex x is denoted by $N_G(x)$, the set of the edges incident with x is denoted by $E_G(x)$ and its degree by $d_G(x) = |E_G(x)|$. For a subgraph H of G and a vertex $x \in V(G)$, $N_H(x)$ is the set of the neighbors of x in H, $E_H(x)$ is the set of the edges incident with x in H, and $d_H(x) = |E_H(x)|$ is the degree of x in H. For $B \subseteq E(G)$, $G\backslash B$ is the graph obtained from G by deleting all the edges of B. For $X \in V(G)$, $G - X$ is the graph obtained from G by deleting all the vertices of X together with all the edges with at least one end in X. (When $X = \{x\}$, we simplify the notation to $G - x$.) The following easy observation will be used throughout the paper.

Observation 2.1. *Suppose that $\mathcal{D}$ is a path-decomposition of a graph G. Then $\mathcal{D}(v) \geq 1$ if $d_G(v)$ is odd.*

Definition 2.2. *Let w be a vertex in a graph G and B be a set of edges incident to w. Let $H = G\backslash B$ and $\mathcal{D}$ be a path-decomposition of H. For a subset $A \subseteq B$, say $A = \{wx_i : 1 \leq i \leq k\}$, we say that A is addible at w with respect to $\mathcal{D}$ if $H \cup A$ has a path-decomposition $\mathcal{D}^*$ such that*
(i) $|\mathcal{D}^*| = |\mathcal{D}|$;
(ii) $\mathcal{D}^*(w) = \mathcal{D}(w) + |A|$ and $\mathcal{D}^*(x_i) = \mathcal{D}(x_i) - 1$, $1 \leq i \leq k$;
(iii) $\mathcal{D}^*(v) = \mathcal{D}(v)$ for each $v \in V(G)\backslash\{w, x_1, ..., x_k\}$.

We say that $\mathcal{D}^*$ a transformation of $\mathcal{D}$ by adding A at w. The next lemma is from [3].

Lemma 2.3. *(Lemma 3.6, [3]) Let w be a vertex in a graph G and $x_1, x_2, ..., x_s$ be neighbors of w in G. Let $H = G\backslash\{wx_1, wx_2, ..., wx_s\}$. If H has a path-decomposition $\mathcal{D}$ such that $\mathcal{D}(v) \geq 1$ for every vertex $v \in N_G(w)$, then for any vertex $x \in \{x_1, x_2, ..., x_s\}$, there is an edge set $B \subseteq \{wx_1, wx_2, ..., wx_s\}$ such that $wx \in B$, $|B| \geq \lceil \frac{s}{2} \rceil$, and B is addible at w with respect to $\mathcal{D}$.*

The next lemma is from [4].

Lemma 2.4. *(Lemma 5, [4]) Suppose that w is a vertex in a graph G and $x_1, x_2, ..., x_k$ are neighbors of w in G. Let $H = G\backslash\{wx_1, wx_2, ..., wx_k\}$. If H has a path-decomposition $\mathcal{D}$ such that for some integer l, $|\{v \in N_H(x_i) : \mathcal{D}(v) = 0\}| \leq l$ for each i, $1 \leq i \leq k$, and $\mathcal{D}(w) \geq l + k$, then G has a path-decomposition $\mathcal{D}^*$ such that*

(i) $|\mathcal{D}^*| = |\mathcal{D}|$;

(ii) $\mathcal{D}^*(w) \geq l$ and $\mathcal{D}^*(x_i) = \mathcal{D}(x_i) + 1$, $1 \leq i \leq k$;

(iii) $\mathcal{D}^*(v) = \mathcal{D}(v)$ for each vertex $v \in V(G)\backslash\{w, x_1, ..., x_k\}$.

3. Proof of Main Theorem

Proof of Theorem 1.4.

By the hypothesis of G, $S_{2,2}$ is the graph that has the fewest edges.The two center vertices of $S_{2,2}$ are denoted by x and y, respectively. The two leaf vertices of $S_{2,2}$ are denoted by v_1 and v_2, respectively (see Figure 1).

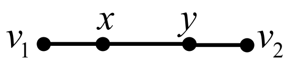

Figure 1: $S_{2,2}$.

Let $\mathcal{D}_1 = \{v_1x, xyv_2\}$ and $\mathcal{D}_2 = \{v_1xy, yv_2\}$. Because $|\mathcal{D}_1| = |\mathcal{D}_2| = 2 < \frac{4+1}{2}$ and $\mathcal{D}_1(x) \geq 2$, $\mathcal{D}_2(y) \geq 2$, the theorem holds. If the theorem is not true, choose G to be a counterexample with $|E(G)|$ minimum. Then $|E(G)| \geq 4$.

Claim 1. For any $z \in V(F)$, $G - z$ is connected.

If the claim is not true, then there are two connected nontrivial subgraphs G_1 and G_2 such that $V(G_1) \cap V(G_2) = \{z\}$, $E(G_1) \cup E(G_2) = E(G)$ and $z \in V(F)$. Let F_i be the E-subgraph of G_i, $i = 1, 2$. Obviously, F_i is a subgraph of F, $i = 1, 2$. Since $d_G(z)$ is even, we have that $d_{G_1}(z) \equiv d_{G_2}(z) \pmod 2$.

Because $xy \in E(G)$ and $xy \in E(F)$, x and y are both in either G_1 or G_2.

Case 1. $z \neq x, y$.

Assuming that $x, y \in V(G_2)$.

Subcase 1.1. Both $d_{G_1}(z)$ and $d_{G_2}(z)$ are even.

In the current case, $|V(F_1)| \geq 1$. According to Theorem 1.3, G has a path decomposition $\mathcal{P}_1$ such that $|\mathcal{P}_1| \leq \frac{|V(G_1)|+1}{2}$ and $\mathcal{P}_1(z) \geq 2$. Let P_1 and P_2 be two paths in $\mathcal{P}_1$ having z as an endvertex.

Because $x, y \in V(G_2)$ and $d_{G_2}(z)$ is even, $|V(F_2)| \geq 3$. By the minimality of G, G_2 has a path-decomposition $\mathcal{P}_2$ such that $\mathcal{P}_2(x) \geq 2$ and a path-decomposition $\mathcal{P}'_2$ such that $\mathcal{P}'_2(y) \geq 2$. $d_{G_2}(z)$ is even. If z is not the end vertex of any path in $\mathcal{P}_2$, let $Q \in \mathcal{P}_2$ and $z \in V(Q)$. The two segments of Q divided by z are denoted by Q_1 and Q_2. If z is the end vertex of some paths in $\mathcal{P}_2$, there are at least two such paths. Choose two paths from $\mathcal{P}_2$ with z as the end vertex, denoted by Q_1 and Q_2, respectively.

Let $\mathcal{D}_1 = (\mathcal{P}_1\backslash\{P_1, P_2\}) \cup (\mathcal{P}_2\backslash\{Q_1 \cup Q_2\}) \cup \{P_1 \cup Q_1, P_2 \cup Q_2\}$, then $|\mathcal{D}_1| \leq \frac{|V(G_1)|+1}{2} - 2 + \frac{|V(G_2)|+1}{2} - 1 + 2 = \frac{|V(G)|+1}{2} = \frac{n+1}{2}$ and $\mathcal{D}_1(x) \geq 2$. Similarly, we can use $\mathcal{P}_1$ and $\mathcal{P}'_2$ to find a path-decomposition $\mathcal{D}_2$ of G such that $|\mathcal{D}_2| \leq \frac{n+1}{2}$ and $\mathcal{D}_2(y) \geq 2$, contradicting that G is a counterexample.

Subcase 1.2. Both $d_{G_1}(z)$ and $d_{G_2}(z)$ are odd.

If the degree of every vertex of G_1 is odd, then there is a path-decomposition $\mathcal{P}_1$ of G_1 such that $|\mathcal{P}_1| \leq \frac{|V(G_1)|+1}{2}$, $\mathcal{P}_1(z) \geq 1$, by Theorem 1.1 and Observation 2.1. If the number of even degree vertices in G_1 is greater than or equal to 1, then there is a path-decomposition $\mathcal{P}_1$ of G_1 such that $|\mathcal{P}_1| \leq \frac{|V(G_1)|+1}{2}$, $\mathcal{P}_1(z) \geq 1$, by Theorem 1.3 and Observation 2.1. So, in either case, G_1 always has a path-decomposition $\mathcal{P}_1$, such that $|\mathcal{P}_1| \leq \frac{|V(G_1)|+1}{2}$, $\mathcal{P}_1(z) \geq 1$. Let P_1 be a path in $\mathcal{P}_1$ that ends at z. By the minimality of G, G_2 has a path-decomposition $\mathcal{P}_2$ such that $|\mathcal{P}_2| \leq \frac{|V(G_2)|+1}{2}$, $\mathcal{P}_2(x) \geq 2$ and a path-decomposition $\mathcal{P}'_2$ such that $|\mathcal{P}'_2| \leq \frac{|V(G_2)|+1}{2}$, $\mathcal{P}'_2(y) \geq 2$. For path-decomposition $\mathcal{P}_2$ or $\mathcal{P}'_2$, z is the end vertex of at least one path, by Observation 2.1. Let Q_1 and Q'_1 be a path in $\mathcal{P}_2$ and $\mathcal{P}'_2$ that ends at z, respectively. Let $\mathcal{D}_1 = (\mathcal{P}_1\backslash\{P_1\}) \cup (\mathcal{P}_2\backslash\{Q_1\}) \cup \{P_1 \cup Q_1\}$

and $\mathcal{D}_2 = (\mathcal{P}_1\backslash\{P_1\}) \cup (\mathcal{P}_2\backslash\{Q_1'\}) \cup \{P_1 \cup Q_1'\}$. Then $|\mathcal{D}_1| \leq \frac{|V(G_1)|+1}{2} - 1 + \frac{|V(G_2)|+1}{2} - 1 + 1 = \frac{|V(G)|+1}{2} = \frac{n+1}{2}$, $|\mathcal{D}_2| \leq \frac{|V(G)|+1}{2} = \frac{n+1}{2}$ and $\mathcal{D}_1(x) \geq 2$, $\mathcal{D}_2(y) \geq 2$, contradicting that G is a counterexample.

Case 2. $z = x$ or y.

Without loss of generality, we assume that $z = x$ and $y \in V(G_1)$. Because $d_G(y)$ is even and $y \in V(G_1)$, we can choose G_1 such that $G_1 - x$ is connected and $|E(G_1)| \geq 2$.

Subcase 2.1. Both $d_{G_1}(x)$ and $d_{G_2}(x)$ are even.

In the current case, $x, y \in V(F_1)$. By the minimality of G, G_1 has a path-decomposition $\mathcal{P}_1$ such that $|\mathcal{P}_1| \leq \frac{|V(G_1)|+1}{2}$, $\mathcal{P}_1(x) \geq 2$ and a path-decomposition $\mathcal{P}_1'$ such that $|\mathcal{P}_1'| \leq \frac{|V(G_1)|+1}{2}$, $\mathcal{P}_1'(y) \geq 2$. Because $x \in V(F_2)$, there are at least one vertex of even degree in G_2. By Theorem 1.3, G_2 has a path-decomposition $\mathcal{P}_2$ such that $|\mathcal{P}_2| \leq \frac{|V(G_2)|+1}{2}$, $\mathcal{P}_2(x) \geq 2$. In $\mathcal{P}_2$, we choose two paths with x as the end vertex, denoted by Q_1 and Q_2, respectively. In $\mathcal{P}_1$, we choose two paths with x as the end vertex, denoted by P_1 and P_2, respectively. Let $\mathcal{D}_1 = (\mathcal{P}_1\backslash\{P_1\}) \cup (\mathcal{P}_2\backslash\{Q_1\}) \cup \{P_1 \cup Q_1\}$, then $|\mathcal{D}_1| \leq \frac{|V(G_1)|+1}{2} - 1 + \frac{|V(G_2)|+1}{2} - 1 + 1 = \frac{|V(G)|+1}{2} = \frac{n+1}{2}$ and $\mathcal{D}_1(x) \geq 2$. In $\mathcal{P}_1'$, $\mathcal{P}_1'(x) = 0$ or $\mathcal{P}_1'(x) \geq 2$. If $\mathcal{P}_1'(x) = 0$, we choose a path from $\mathcal{P}_1'$ containing x, denoted by P. We divide P from x into two segments, denoted by P_1 and P_2, respectively. If $\mathcal{P}_1'(x) \geq 2$, we choose two paths with x as the end vertex, denoted by P_1 and P_2, respectively. Let $\mathcal{D}_2 = (\mathcal{P}_1'\backslash\{P_1 \cup P_2\}) \cup (\mathcal{P}_2\backslash\{Q_1, Q_2\}) \cup \{P_1 \cup Q_1, P_2 \cup Q_2\}$. Then $|\mathcal{D}_2| \leq \frac{|V(G_1)|+1}{2} - 1 + \frac{|V(G_2)|+1}{2} - 2 + 2 = \frac{|V(G)|+1}{2} = \frac{n+1}{2}$ and $\mathcal{D}_2(y) \geq 2$, contradicting that G is a counterexample.

Subcase 2.2. Both $d_{G_1}(x)$ and $d_{G_2}(x)$ are odd.

(i) $|E(G_2)| \geq 2$.

Let H_i be the connected graph obtained from G_i by adding a new edge xw, where w is a new vertex, $i = 1, 2$. The E-subgraph of H_i is denoted by F_i', $i = 1, 2$. Then $xy \in E(F_1')$, $x \in F_i'$ and $|E(H_i)| \leq |E(G)|$, $i = 1, 2$. By the minimality of G, H_1 has a path-decomposition $\mathcal{P}_1$ such that $|\mathcal{P}_1| \leq \frac{|V(H_1)|+1}{2}$, $\mathcal{P}_1(x) \geq 2$ and a path-decomposition $\mathcal{P}_1'$ such that $|\mathcal{P}_1'| \leq \frac{|V(H_1)|+1}{2}$, $\mathcal{P}_1'(y) \geq 2$. Because $d_{H_2}(x)$ is even, the number of even degree vertices of H_2 is greater than or equal to 1. By Theorem 1.3, H_2 has a path-decomposition $\mathcal{P}_2$ such that $|\mathcal{P}_2| \leq \frac{|V(H_2)|+1}{2}$, $\mathcal{P}_2(x) \geq 2$. Next, we construct the path-decomposition $\mathcal{D}_1$ of G such that $|\mathcal{D}_1| \leq \frac{n+1}{2}$, $\mathcal{D}_1(x) \geq 2$.

In $\mathcal{P}_1$, we choose the path which contains the edge xw, denoted by P_1. In $\mathcal{P}_1\backslash\{P_1\}$, we choose one path with x as the end vertex, denoted by P_2. In $\mathcal{P}_2$, we choose the path which contains the edge xw, denoted by Q_1. In $\mathcal{P}_2\backslash\{Q_1\}$, we choose one path with x as the end vertex, denoted by Q_2.

Let $P = (P_1\backslash xw) \cup (Q_1\backslash xw)$ and $Q = P_2 \cup Q_2$. If neither Q_1 nor P_1 is the single edge xw, let $\mathcal{D}_1 = (\mathcal{P}_1\backslash\{P_1, P_2\}) \cup (\mathcal{P}_2\backslash\{Q_1, Q_2\}) \cup \{P, Q\}$. Then $|\mathcal{D}_1| \leq \frac{|V(H_1)|+1}{2} - 2 + \frac{|V(H_2)|+1}{2} - 2 + 2 = \frac{|V(G)|+1}{2} = \frac{n+1}{2}$ and $\mathcal{D}_1(x) \geq 2$. If both $Q_1 = xw$ and $P_1 = xw$, let $\mathcal{D}_1 = (\mathcal{P}_1\backslash\{Q_1\}) \cup (\mathcal{P}_2\backslash\{Q_2\})$. Then $|\mathcal{D}_1| \leq \frac{|V(H_1)|+1}{2} - 1 + \frac{|V(H_2)|+1}{2} - 1 = \frac{|V(G)|+1}{2} = \frac{n+1}{2}$ and $\mathcal{D}_1(x) \geq 2$. If exactly one of Q_1 and P_1 is the single edge xw, say $P_1 = xw$, $Q_1 \neq xw$. Let $\mathcal{D}_1 = (\mathcal{P}_1\backslash\{P_1, P_2\}) \cup (\mathcal{P}_2\backslash\{Q_1, Q_2\}) \cup \{Q, Q_1\backslash xy\}$, then $|\mathcal{D}_1| \leq \frac{|V(G)|+1}{2}$ and $\mathcal{D}_1(x) \geq 2$.

In the following, we construct the path-decomposition $\mathcal{D}_2$ of G such that $|\mathcal{D}_2| \leq \frac{n+1}{2}$, $\mathcal{D}_2(y) \geq 2$. In G_1, the number of even degree vertices is greater than or equal to 1, and the degree of every vertex except y of F_1 less than or equal to three. By Theorem 1.3, G_1 has a path-decomposition $\mathcal{P}_1$ such that $|\mathcal{P}_1| \leq \frac{|V(G_1)|+1}{2} = \frac{n+1}{2}$, $\mathcal{P}_1(y) \geq 2$. Because $d_{G_1}(x)$ is odd, $\mathcal{P}_1 \geq 1$, by Observation 2.1. In $\mathcal{P}_1$, we choose one path with x as the end vertex, denoted by P_1. By Theorem 1.1 or 1.2, G_2 has a path-decomposition $\mathcal{P}_2$ such that $|\mathcal{P}_2| \leq \frac{|V(G_2)|+1}{2} = \frac{n+1}{2}$. By Observation 2.1, $\mathcal{P}_2(x) \geq 1$. In $\mathcal{P}_2$, we choose one path with x as the end vertex, denoted by P_2. Let $\mathcal{D}_2 = (\mathcal{P}_1\backslash\{P_1\}) \cup (\mathcal{P}_2\backslash\{P_2\}) \cup \{P_1, P_2\}$. Then $|\mathcal{D}_2| \leq \frac{|V(G_1)|+1}{2} - 1 + \frac{|V(G_2)|+1}{2} - 1 + 1 = \frac{|V(G)|+1}{2} = \frac{n+1}{2}$ and $\mathcal{D}_2(y) \geq 2$, contradicting that G is a counterexample.

(ii) $|E(G_2)| = 1$.

G_2 is a single edge, say $G_2 = xw_1$. Let $R = G_1 - x$. By the choice of G_1, R is connected. Let $E_F(x) = \{xx_1, xx_2, ..., xx_m\}$, $m = d_F(x)$. Let $H = G\backslash E_F(x)$ and F_H be the E-subgraph of H.

In the following, we construct the path-decomposition $\mathcal{D}_1$ of G such that $|\mathcal{D}_1| \leq \frac{n+1}{2}$, $\mathcal{D}_1(x) \geq 2$.

(1) $m < d_{G_1}(x)$.

Because $R = G_1 - x$ is connected and $m < d_{G_1}(x)$, H is connected.

If m is even, then $d_H(x)$ is even, and $y \in \{x_1, x_2, ..., x_m\}$. So, $d_H(y)$ is odd. By Theorem 1.3, there is a path-decomposition $\mathcal{P}$ of H such that $|\mathcal{P}| \le \frac{|V(H)|+1}{2} \le \frac{n+1}{2}$ and $\mathcal{P}(x) \ge 2$. By Lemma 2.3, there is an edge set $B \subseteq E_F(x)$ such that $|B| \ge \lceil \frac{m}{2} \rceil$, $xy \in B$ and B is addible at x with respect to $\mathcal{P}$.

If m is odd, then x is odd degree in H, and H has a path-decomposition $\mathcal{P}$ such that $|\mathcal{P}| \le \frac{n+1}{2}$, by Theorem 1.1 or 1.2. By Observation 2.1, $\mathcal{P}(x) \ge 1$. By Lemma 2.3, there is an edge set $B \subseteq E_F(x)$ and $xy \in B$ such that $|B| \ge \lceil \frac{m}{2} \rceil$ and B is addible at x with respect to $\mathcal{P}$.

In either case, $H \cup B$ has a path-decomposition $\mathcal{P}'$, a transformation of $\mathcal{P}$ by adding B at x, such that $|\mathcal{P}'| \le \frac{n+1}{2}$ and $\mathcal{P}'(x) \ge m - \lceil \frac{m}{2} \rceil + 2$. Since $d_F(v) \le 3$ for every vertex $V(F)\backslash\{x, y\}$. So, every vertex $v \in E_F(x)\backslash B$, $d_F(v) \le 3$ and $d_{F_H}(v) \le 2$.

By Lemma 2.4, with $l = 2$ and $k = m - \lceil \frac{m}{2} \rceil$, G has a path-decomposition $\mathcal{P}^*$ such that $|\mathcal{P}^*| = |\mathcal{P}'| \le \frac{n+1}{2}$ and $\mathcal{P}^*(x) \ge 2$.

(2) $m = d_{G_1}(x)$.

Because $d_G(x)$ is even and $d_{G_2}(x) = 1$, m is odd, say $m = 2k + 1$. There are no new even vertices in $R = G_1 - x$. The degree of x and all vertices adjacent to x are odd. By Theorem 1.1 or 1.2, there is a path-decomposition $\mathcal{R}$ of R such that $|\mathcal{R}| \le \frac{|V(R)|+1}{2}$ and $\mathcal{R}(x_i) \ge 1$ for all i, $1 \le i \le m$. By Lemma 2.3, there is an edge set $B \subseteq E_F(x)$, $xy \in B$, such that $|B| \ge k + 1$ and B is addible at x with respect to $\mathcal{R}$. Let $\mathcal{R}'$ be a transformation of $\mathcal{R}$ by adding B at x. Then $\mathcal{R}'$ is a path-decomposition of $R \cup B$ such that $|\mathcal{R}'| \le \frac{|R|+1}{2}$ and $\mathcal{R}'(x) \ge |B| \ge k + 1$. Let $\mathcal{P}' = \mathcal{R}' \cup \{xw_1\}$, which is a path-decomposition of $R \cup B \cup \{xw_1\}$. Note that $|V(R)| = |V(G)| - 2$. So, $|\mathcal{P}'| \le \frac{|V(R)|+1}{2} + 1 = \frac{n+1}{2}$ and $\mathcal{P}'(x) \ge |B| + 1 \ge k + 2$. By Lemma 2.4, with $l = 2$, we obtain a path-decomposition $\mathcal{D}_1$ of G such that $|\mathcal{D}_1| \le \frac{n+1}{2}$ and $\mathcal{D}_1(x) \ge 2$.

Next, we will find a path-decomposition $\mathcal{D}_2$ of G, such that $|\mathcal{D}_2| \le \frac{n+1}{2}$ and $\mathcal{D}_2(y) \ge 2$. Let $I = G\backslash\{xw_1\}$ and F_I be the E-subgraph of I. Because the number of even vertices in I is greater than or equal to one, and only $d_{F_I}(y)$ may be greater than three, I has a path-decomposition $\mathcal{P}$ such that $|\mathcal{P}| \le \frac{|V(I)|+1}{2}$ and $\mathcal{P}(y) \ge 2$, by Theorem 1.3. Because $d_I(x)$ is odd, $\mathcal{P}(x) \ge 1$, by Observation 2.1. In $\mathcal{P}$, we choose one path with x as the end vertex, denoted by P. Let $Q = P \cup \{xw_1\}$ and $\mathcal{D}_2 = (\mathcal{P}\backslash\{P\}) \cup \{Q\}$. Then $|\mathcal{D}_2| \le \frac{|V(I)|+1}{2} - 1 + 1 < \frac{|V(G)|+1}{2} = \frac{n+1}{2}$ and $\mathcal{D}_2(y) \ge 2$, contradicting that G is a counterexample. This proves Claim 1.

Claim 2. At least one of $d_F(x)$ and $d_F(y)$ is even.

Suppose, to the contrary, that $d_F(x)$ and $d_F(y)$ are odd. Let $E_F(x) = \{xw_1, xw_2, ..., xw_m\}$, where $m = d_F(x)$ and $w_m = y$. Let $H = G\backslash E_F(x)$. By Claim 1, H is connected. Note that the degree of x and y are odd in H. By Theorem 1.1 or 1.2, H has a path-decomposition $\mathcal{P}_1$ such that $|\mathcal{P}_1| \le \frac{n+1}{2}$. By Observation 2.1, $\mathcal{P}_1(x) \ge 1$, $\mathcal{P}_1(y) \ge 1$. By Lemma 2.3, to add a set $B \subseteq E_F(x)$ at x with $|B| \ge \lceil \frac{m}{2} \rceil$ and $xy \in B$, we can get a path-decomposition $\mathcal{P}_2$ of $H \cup B$ from $\mathcal{P}_1$. Since $|B| \ge \lceil \frac{m}{2} \rceil$ and m is odd, $|B| \ge \frac{m+1}{2}$, $\mathcal{P}_2(x) \ge \frac{m+1}{2} + 1 = \frac{m+3}{2}$ and $|\mathcal{P}_2| \le \frac{n+1}{2}$. By applying Lemma 2.4, with $l = 2$, we obtain a path-decomposition $\mathcal{D}_1$ of G such that $|\mathcal{D}_1| \le \frac{n+1}{2}$ and $\mathcal{D}_1(x) \ge 2$. Because $d_F(y)$ is odd, we can obtain the path-decomposition $\mathcal{D}_2$ in the same way as above such that $|\mathcal{D}_2| \le \frac{n+1}{2}$ and $\mathcal{D}_2(y) \ge 2$, contradicting that G is a counterexample. This proves Claim 2.

Because $xy \in E_F(x)$ and $xy \in E_F(y)$, $d_F(x) \ne 0$ and $d_F(y) \ne 0$. By Claim 2, at least one of $d_F(x)$ and $d_F(y)$ is even. Without loss of generality, suppose $d_F(x)$ is even. So, $d_F(x) \ge 2$.

In the following, we will find a path-decomposition $\mathcal{D}$ of G, such that $|\mathcal{D}| \le \frac{n+1}{2}$, $\mathcal{D}(x) \ge 2$ and $\mathcal{D}(y) \ge 2$.

Let $E_F(x) = \{xx_1, xx_2, ..., xx_m\}$, $m = d_F(x) \ge 2$ is even. Let $xx_m = xy$, $m = 2k$ and $k \ge 1$. Let $S = E_F(x)\backslash\{xx_m\}$. Thus $|S| = 2k - 1$. Suppose $H = G\backslash S$. By Claim 1, H is connected. $d_H(x)$ is odd and $d_H(y)$ is even. By Theorem 1.3, there is a path-decomposition $\mathcal{P}$ of H such that $|\mathcal{P}| \le \frac{n+1}{2}$ and $\mathcal{P}(y) \ge 2$. By Observation 2.1, $\mathcal{P}(x)$ and $\mathcal{P}(v) \ge 1$, $v \in N_G(x)$. By Lemma 2.3, there is an edge set $B \subseteq S$, such that $|B| \ge k$ and B is addible at x with respect to $\mathcal{P}$. Let $\mathcal{P}'$ be a transformation of $\mathcal{P}$ by adding B at x. Then $\mathcal{P}'$ is a path-decomposition of $H \cup B$ such that $|\mathcal{P}'| \le \frac{n+1}{2}$ and $\mathcal{P}'(x) \ge k + 1$. Note that $|S\backslash B| \le k - 1$. By Lemma 2.4, with $l = 2$, G has a path-decomposition $\mathcal{D}$ of G, such that $|\mathcal{D}| \le \frac{n+1}{2}$, $\mathcal{D}_2(x) \ge 2$ and $\mathcal{D}_2(y) \ge 2$, contradicting that G is a counterexample. ∎

Date availability statement

Because no new data were created or analyzed in this study, data sharing is not applicable to this article.

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