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# On symmetrized and Wright convexity

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**Abstract.** In this paper, we study symmetrized convex functions and their relations with other notions of convexity, namely Wright convexity; then, we introduce the notion of symmetrized log-convexity. Our results include equivalent statements and extensions of certain existing results in the literature. Applications that involve Hermite-Hadamard and means' inequalities will be presented.

## 1. Introduction

We recall that a function $f : J \to \mathbb{R}$ is said to be convex on the real interval $J$ if for each $a, b \in J$ and $0 \le \nu \le 1$, the inequality

$$f((1-\nu)a+\nu b) \le (1-\nu)f(a)+\nu f(b) \tag{1}$$

holds. Geometrically, a convex function lies above its chords in the interval of convexity. Several authors, including Marshall and Proschan [10], Wright [23], and Mercer [11], presented a large number of significant results for convex functions and related inequalities. See also [5, Chapter 5] for recent advances in convex functions and Jensen inequality. Convex functions are widely used in pure mathematics, functional analysis, optimization theory, and mathematical economics. Giant attention has been given to analyzing convex functions and their properties (e.g., [19], which contains an extensive bibliography).

Researchers have devoted a considerable amount of their time to find sharper inequalities than (1). In [1], it was shown that

$$f((1-\nu)a+\nu b)+2r\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right) \le (1-\nu)f(a)+\nu f(b) \tag{2}$$

and

$$f((1-\nu)a+\nu b)+2R\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right) \ge (1-\nu)f(a)+\nu f(b), \tag{3}$$

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where $r = \min\{\nu, 1-\nu\}$ and $R = \max\{\nu, 1-\nu\}$, as a refinement and a reverse of (1). These two inequalities have been used in the related literature to obtain many interesting results. We refer the reader to [12, 14, 18] as a sample of such work, where further investigations and applications have been made.

Among the interesting inequalities that govern convex functions is the Jensen-Mercer inequality, which states [11]

$$f\left(b+a-\sum_{i=1}^{n} w_i x_i\right) \le f(b) + f(a) - \sum_{i=1}^{n} w_i f(x_i),$$

where $f : [a,b] \to \mathbb{R}$ is convex, $a \le x_i \le b$, and $w_i \ge 0$ satisfying $\sum_{i=1}^{n} w_i = 1$.

The celebrated Hermite-Hadamard inequality furnishes estimates of the mean value of a convex function (see, e.g., [17, p. 50]). This inequality asserts that every convex function $f$ on an interval $[a,b]$ enjoys the inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{f(a)+f(b)}{2}. \tag{4}$$

Notice that every continuous function is convex if and only if it satisfies (4). We encourage interested readers to go more into the literature on convex functions and their inequalities, especially [13, 16, 20–22] and the references therein.

Inspired by the Mercer inequality and the even-odd decomposition, the following decomposition was first introduced in [7].

**Definition 1.1.** *Let $f : [a,b] \to \mathbb{R}$ be a given real-valued function. For $t \in [a,b]$, define*

$$\overset{\bullet}{f}(t) = \frac{f(t) + f(b+a-t)}{2} \text{ and } \underset{\bullet}{f}(t) = \frac{f(t) - f(b+a-t)}{2}.$$

*Then $\overset{\bullet}{f}$ is called the symmetrized transform of $f$, and $\underset{\bullet}{f}$ is the anti-symmetrized transform of $f$.*

Obviously, $f = \overset{\bullet}{f} + \underset{\bullet}{f}$. At this point, we notice that when $a = -b$, the above decomposition reduces to the even-odd decomposition.

We highlight that this symmetrized transform has been extensively studied in the context of convex functions, as seen in [6, 8, 9].

One can easily check that if $f$ is a convex function on $[a,b]$, then $\overset{\bullet}{f}$ is a convex function on $[a,b]$. However, the reverse assertion is not true in general [2].

In [2], Dragomir introduced the following definition of convexity (see also [3] for an equivalent definition):

**Definition 1.2.** *A function $f : [a,b] \to \mathbb{R}$ is said to be symmetrized convex (concave) on $[a,b]$ if $\overset{\bullet}{f}$ is convex (concave) on $[a,b]$.*

In the same paper, it has been shown that (4) holds whenever the symmetrized transform of $f$ is convex on $[a,b]$.

Another notion related to convexity is the so-called Wright convexity [23]. We say that $f : J \to \mathbb{R}$ is a *Wright-convex* function on $J \subseteq \mathbb{R}$, if for any $0 \le \nu \le 1$

$$f((1-\nu)a + \nu b) + f(\nu a + (1-\nu)b) \le f(a) + f(b) \tag{5}$$

for all $a, b \in J$.

It was shown in [19, p. 233] that (5) is equivalent to

$$f(x+\delta)+f(y)\le f(y+\delta)+f(x),$$

for all $x, y+\delta \in [a,b]$ with $x<y$ and $\delta \ge 0$.

The direct connection between the notions of convexity and Wright-convexity was established in [15, Corollary 5], as follows.

**Lemma 1.3.** *Let $f : J \to \mathbb{R}$ be a function. Then $f$ is Wright-convex if and only if there exists a convex function $g : J \to \mathbb{R}$ and an additive function $h : \mathbb{R} \to \mathbb{R}$ such that*

$$f(x) = g(x) + h(x); \ \ (x \in J). \tag{6}$$

One of the main motivations of this paper is the equivalence of the following four assertions:

(i) The symmetrized transform of $f$ is convex on $[a,b]$.

(ii) $f$ is Wright-convex on $[a,b]$.

(iii) For $x, y, (1+\nu)y-\nu x \in [a,b]$ and $0 \le \nu \le 1$,

$$f((1-\nu)x+\nu y)+f(y)\le f((1+\nu)y-\nu x)+f(x). \tag{7}$$

(iv) For all $x, y+\delta \in [a,b]$ with $x<y$ and $\delta \ge 0$,

$$f(x+\delta)+f(y)\le f(y+\delta)+f(x). \tag{8}$$

Many other results will be shown for symmetrized convex functions and the related notions. Then, similar to symmetrized convexity, we introduce the notion of symmetrized log-convexity and present many results that extend the existing ones.

## 2. Symmetrized convex functions

We begin by showing the following equivalence.

**Theorem 2.1.** *Let $f : [a,b] \to \mathbb{R}$ be a given real-valued function. The following are equivalent:*

*(i) The symmetrized transform of $f$ is convex on $[a,b]$.*

*(ii) $f$ is Wright-convex on $[a,b]$.*

*(iii) For $x, y, (1+\nu)y-\nu x \in [a,b]$ and $0 \le \nu \le 1$,*

$$f((1-\nu)x+\nu y)+f(y)\le f((1+\nu)y-\nu x)+f(x). \tag{9}$$

*(iv) For all $x, y+\delta \in [a,b]$ with $x<y$ and $\delta \ge 0$,*

$$f(x+\delta)+f(y)\le f(y+\delta)+f(x). \tag{10}$$

*Proof.* $(i) \Leftrightarrow (ii)$ From the definition of $\overset{\bullet}{f}$, we have for $0 \le \nu \le 1$,

$$\begin{aligned}&\overset{\bullet}{f}\left((1-\nu)a+\nu b\right)\le (1-\nu)\overset{\bullet}{f}(a)+\nu\overset{\bullet}{f}(b)\\ \Longleftrightarrow\ & f((1-\nu)a+\nu b)+f(\nu a+(1-\nu)b)\le f(a)+f(b).\end{aligned}$$

$(i) \Rightarrow (iii)$ If the symmetrized transform of $f$ is convex, we conclude that [4]

$$(1+\nu)\overset{\bullet}{f}(a) - \nu\overset{\bullet}{f}(b) \le \overset{\bullet}{f}((1+\nu)a - \nu b); \ (\nu \ge 0 \text{ or } \nu \le -1), \tag{11}$$

which is analogous to stating that

$$f(a) + f(b) \le f((1+\nu)a - \nu b) + f((1+\nu)b - \nu a).$$

This inequality is of interest in itself. We consider the case $\nu \ge 0$. Putting $a = (1-t)x + ty$, $b = y$, and $\nu = \frac{t}{1-t}$ (in this case $\frac{1}{1+\nu} = 1 - t$ and $\frac{\nu}{1+\nu} = t$). Consequently,

$$(1+\nu)a - \nu b = \frac{(1-t)x + ty - ty}{1-t} = x,$$

and

$$(1+\nu)b - \nu a = \frac{y - t((1-t)x + ty)}{1-t} = (1+t)y - tx.$$

Accordingly, for $0 \le t \le 1$

$$f((1-t)x + ty) - f((1+t)y - tx) \le f(x) - f(y)$$

as expected.

$(iii) \Rightarrow (i)$ We assume the inequality (9) holds. The same argument above shows that (11) is valid for $\nu \ge 0$. From this inequality, we can show the convexity of $\overset{\bullet}{f}$. We present it for convenience to the readers, although its method is standard (see, e.g., [4]). Putting for $\nu \ge 0$,

$$a := (1-s)\alpha + s\beta, \ b := \beta, \ \nu := \frac{s}{1-s}.$$

Then we have

$$\frac{1}{1+\nu} = 1 - s, \ \frac{\nu}{1+\nu} = s, \ (1+\nu)a - \nu b = \alpha.$$

Therefore,

$$\overset{\bullet}{f}((1-s)\alpha + s\beta) = \overset{\bullet}{f}(a) \le \frac{1}{1+\nu}\overset{\bullet}{f}((1+\nu)a - \nu b) + \frac{1}{1+\nu}\overset{\bullet}{f}(b) = (1-s)\overset{\bullet}{f}(\alpha) + \overset{\bullet}{f}(\beta).$$

$(iv) \Rightarrow (iii)$ If we take $\delta := \nu(y-x) \ge 0$ with $0 \le \nu \le 1$ in (10), then $x, y + \delta = (1+\nu)y - \nu x \in [a,b]$. In addition, $y \in [a,b]$ since $x \le y \le y + \delta$. Thus (10) implies (9).

$(iii) \Rightarrow (iv)$ Assume that (9) holds. For $0 \le \nu \le 1$ and $x, y, (1+\nu)y - \nu x \in [a,b]$ such that $x < y$, take $\nu(y-x) =: \delta$. Then $x, y + \delta \in [a,b]$ and $\delta \ge 0$. Thus (9) implies (10). □

We can state the following decomposition of functions by combining Theorem 2.1 and Lemma 1.3.

**Corollary 2.2.** *Let the symmetrized transform of $f$ be convex on $[a,b]$. Then a convex function $g : [a,b] \to \mathbb{R}$ and an additive function $h : [a,b] \to \mathbb{R}$ exist, such that*

$$F(t) = 2(g(t) + h(t)) - f(t) + f(a+b-t)$$

*is a convex function.*

We noticed that (2) and (3) hold for convex functions $f$. In the following, we show analogous inequalities knowing that the symmetrized transform is convex.

**Theorem 2.3.** *Let $f : [a, b] \to \mathbb{R}$ be such that the symmetrized transform of $f$ is convex. Then for any $0 \le v \le 1$,*

$$\begin{aligned}
&\frac{f(a)+f(b)}{2} - 2R\left(\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right)\\
&\le \frac{1}{2}\left(f\left((1-v)a+vb\right) + f\left(va+(1-v)b\right)\right)\\
&\le \frac{f(a)+f(b)}{2} - 2r\left(\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right),
\end{aligned}$$

*where $r = \min\{v, 1-v\}$ and $R = \max\{v, 1-v\}$.*

*Proof.* Since the symmetrized transform of $f$ is convex on $[a, b]$, the inequalities (2) and (3) imply

$$\begin{aligned}
&(1-v)\overset{\bullet}{f}(a) + v\overset{\bullet}{f}(b) - 2R\left(\frac{\overset{\bullet}{f}(a)+\overset{\bullet}{f}(b)}{2} - \overset{\bullet}{f}\left(\frac{a+b}{2}\right)\right)\\
&\le \overset{\bullet}{f}\left((1-v)a+vb\right)\\
&\le (1-v)\overset{\bullet}{f}(a) + v\overset{\bullet}{f}(b) - 2r\left(\frac{\overset{\bullet}{f}(a)+\overset{\bullet}{f}(b)}{2} - \overset{\bullet}{f}\left(\frac{a+b}{2}\right)\right).
\end{aligned}$$

Since

$$\begin{aligned}
&\overset{\bullet}{f}\left((1-v)a+vb\right) = \frac{1}{2}\left(f\left((1-v)a+vb\right) + f\left(va+(1-v)b\right)\right),\\
&\overset{\bullet}{f}\left(\frac{a+b}{2}\right) = f\left(\frac{a+b}{2}\right),
\end{aligned}$$

and

$$\overset{\bullet}{f}(a) = \overset{\bullet}{f}(b) = \frac{f(a)+f(b)}{2},$$

the desired result follows. □

**Remark 2.4.** *As we can see in the above, if $\overset{\bullet}{f}$ is convex on $[a, b]$, then we find that the function $f$ is mid-convex by taking $v = 0$ or $1$ in Theorem 2.3. Notice that mid-point convexity is not equivalent to convexity. In fact, every mid-point convex function is convex under some mild conditions, such as continuity.*

**Remark 2.5.** *If the anti-symmetrized transform of $f$ is convex on $[a, b]$, by the similar way to the proof of Theorem 2.3, we then have*

$$f((1-v)a+vb) - f((1-v)b+va) = (1-2v)\left(f(a) - f(b)\right)$$

*which is true for special cases $v = 0, 1$ and $1/2$, clearly.*

The following shows Hermite-Hadamard-type inequalities under certain assumptions on the symmetrized transform.

**Corollary 2.6.** *Let $f : [a, b] \to \mathbb{R}$ be such that the symmetrized transform of $f$ is convex on $[a, b]$. Then*

$$\begin{aligned}
3f\left(\frac{a+b}{2}\right) - \frac{f(a)+f(b)}{2} &\le 2\int_0^1 f\left((1-v)a+vb\right)dv\\
&\le \frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right).
\end{aligned}$$

*Proof.* Since $\min\{a,b\} = (a+b-|a-b|)/2$ and $\max\{a,b\} = (a+b+|a-b|)/2$, for $a,b \geq 0$, we have by Theorem 2.3

$$\begin{aligned}
&\frac{f(a)+f(b)}{2} - (1+|2\nu-1|)\left(\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right) \\
&\leq \frac{1}{2}\left(f((1-\nu)a+\nu b) + f(\nu a+(1-\nu)b)\right) \qquad (12)\\
&\leq \frac{f(a)+f(b)}{2} - (1-|2\nu-1|)\left(\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right).
\end{aligned}$$

Taking integral over $\nu \in [0,1]$, and using the fact that

$$\int_0^1 f((1-\nu)a+\nu b)\,d\nu = \int_0^1 f(\nu a+(1-\nu)b)\,d\nu,$$

we get

$$\begin{aligned}
3f\left(\frac{a+b}{2}\right) - \frac{f(a)+f(b)}{2} &\leq 2\int_0^1 f((1-\nu)a+\nu b)\,d\nu \\
&\leq \frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right),
\end{aligned}$$

as desired. □

**Remark 2.7.** *Corollary 2.6 is equivalent to*

$$\begin{aligned}
&\frac{f(a)+f(b)}{2} - \frac{3}{2}\left(\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right) \\
&\leq \int_0^1 f((1-\nu)a+\nu b)\,d\nu \\
&\leq \frac{f(a)+f(b)}{2} - \frac{1}{2}\left(\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right).
\end{aligned}$$

*Indeed, the first inequality provides a counterpart for the second inequality in* (4), *and the second inequality improves the second inequality in* (4).

The following is usually referred to as a Fejér-type inequality.

**Theorem 2.8.** *Let $f : [a,b] \to \mathbb{R}$ be such that the symmetrized transform of $f$ is convex on $[a,b]$, and let $p : [0,1] \to (0,\infty)$. Then*

$$\begin{aligned}
&\left(\int_0^1 p(\nu)\,d\nu\right)\frac{f(a)+f(b)}{2} - \int_0^1 p(\nu)(1+|2\nu-1|)\,d\nu\left(\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right) \\
&\leq \frac{1}{2}\left(\int_0^1 p(\nu) f((1-\nu)a+\nu b)\,d\nu + \int_0^1 p(\nu) f(\nu a+(1-\nu)b)\,d\nu\right) \\
&\leq \left(\int_0^1 p(\nu)\,d\nu\right)\frac{f(a)+f(b)}{2} - \int_0^1 p(\nu)(1-|2\nu-1|)\,d\nu\left(\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right).
\end{aligned}$$

*Proof.* We get the desired result by multiplying the inequality (12) by $p(\nu)$ and then integrating this inequality over $0 \le \nu \le 1$. $\square$

The forthcoming theorem provides the Jensen inequality for Wright-convex functions.

**Proposition 2.9.** *Let $f : J \to \mathbb{R}$ be a Wright-convex function, let $x_1, x_2, \dots, x_n \in J$, and let $w_1, w_2, \dots, w_n \ge 0$ such that $\sum_{i=1}^{n} w_i = 1$. Then*

$$f\left(\sum_{i=1}^{n} w_i x_i\right) + \sum_{i=1}^{n} w_i h(x_i) - h\left(\sum_{i=1}^{n} w_i x_i\right) \le \sum_{i=1}^{n} w_i f(x_i)$$

*where $h : \mathbb{R} \to \mathbb{R}$ is an additive function.*

*Proof.* If we replace $x$ by $\sum_{i=1}^{n} w_i x_i$, in the decomposition (6), we get

$$f\left(\sum_{i=1}^{n} w_i x_i\right) = g\left(\sum_{i=1}^{n} w_i x_i\right) + h\left(\sum_{i=1}^{n} w_i x_i\right). \tag{13}$$

On the other hand, if we multiply (6) by $w_i$, sum over $i$ from 1 to $n$, we obtain

$$\sum_{i=1}^{n} w_i f(x_i) = \sum_{i=1}^{n} w_i g(x_i) + \sum_{i=1}^{n} w_i h(x_i). \tag{14}$$

Combining two equalities (13) and (14), together, one can write

$$\begin{aligned}
\sum_{i=1}^{n} w_i f(x_i) - f\left(\sum_{i=1}^{n} w_i x_i\right) &= \sum_{i=1}^{n} w_i h(x_i) - h\left(\sum_{i=1}^{n} w_i x_i\right) \\
&\quad + \sum_{i=1}^{n} w_i g(x_i) - g\left(\sum_{i=1}^{n} w_i x_i\right) \\
&\ge \sum_{i=1}^{n} w_i h(x_i) - h\left(\sum_{i=1}^{n} w_i x_i\right).
\end{aligned}$$

The above inequality is obtained from the Jensen inequality for the convex function $g$. $\square$

So far, we have studied possible relations assuming that the symmetrized transform is convex. Next, we discuss the convexity of the anti-symmetrized transform of $f$. The following is a Jensen-type inequality.

**Theorem 2.10.** *Let $f : [a,b] \to \mathbb{R}$ be such that the anti-symmetrized transform of $f$ is convex on $[a,b]$, and let $x_1, x_2, \dots, x_n \in [a,b]$. If $w_1, w_2, \dots, w_n$ are positive numbers such that $\sum_{i=1}^{n} w_i = 1$, then*

$$f\left(\sum_{i=1}^{n} w_i x_i\right) \le \sum_{i=1}^{n} w_i f(x_i) + f\left(a + b - \sum_{i=1}^{n} w_i x_i\right) - \sum_{i=1}^{n} w_i f(a + b - x_i).$$

*Proof.* It follows from Definition 1.1 that

$$\overset{\bullet}{f}\left(\sum_{i=1}^{n} w_i x_i\right) = \frac{1}{2}\left(f\left(\sum_{i=1}^{n} w_i x_i\right) + f\left(a + b - \sum_{i=1}^{n} w_i x_i\right)\right), \tag{15}$$

$$\sum_{i=1}^{n} w_i \underset{\bullet}{f}(x_i) = \frac{1}{2}\left(\sum_{i=1}^{n} w_i f(x_i) - \sum_{i=1}^{n} w_i f(a + b - x_i)\right).$$

Now, if $\underset{\bullet}{f}$ is convex, we have, by the Jensen inequality,

$$\underset{\bullet}{f}\left(\sum_{i=1}^{n} w_i x_i\right) \le \frac{1}{2}\left(\sum_{i=1}^{n} w_i f(x_i) - \sum_{i=1}^{n} w_i f(a+b-x_i)\right). \tag{16}$$

Using (15) and (16), and the fact that $f = \overset{\bullet}{f} + \underset{\bullet}{f}$, we get

$$\begin{aligned} f\left(\sum_{i=1}^{n} w_i x_i\right) &= \overset{\bullet}{f}\left(\sum_{i=1}^{n} w_i x_i\right) + \underset{\bullet}{f}\left(\sum_{i=1}^{n} w_i x_i\right) \\ &\le \frac{1}{2}\left(f\left(\sum_{i=1}^{n} w_i x_i\right) + f\left(a+b-\sum_{i=1}^{n} w_i x_i\right)\right) \\ &\quad + \frac{1}{2}\left(\sum_{i=1}^{n} w_i f(x_i) - \sum_{i=1}^{n} w_i f(a+b-x_i)\right). \end{aligned}$$

Thus,

$$f\left(\sum_{i=1}^{n} w_i x_i\right) \le \sum_{i=1}^{n} w_i f(x_i) + f\left(a+b-\sum_{i=1}^{n} w_i x_i\right) - \sum_{i=1}^{n} w_i f(a+b-x_i).$$

This completes the proof. $\square$

**Remark 2.11.** *If in Theorem 2.10, $f$ is convex, we obtain a refinement of the Jensen inequality, thanks to the Jensen-Mercer inequality [11, Theorem 1.2]. To show that such a function exists, consider the function $f(t) = t^3$ on the interval $0 < a < b$. Since $\underset{\bullet}{f}''(t) = 3(2t-(a+b))$, for $t \ge (a+b)/2$, $\underset{\bullet}{f}(t)$ is convex.*

## 3. Symmetrized log-convexity

This section is partly motivated by the results presented in [2, 3]. The function $f : J \to (0,\infty)$ is called log-convex (concave) if $\log f(x)$ is convex (concave). Therefore we have

$$f : \log\text{–convex} \Leftrightarrow f((1-\nu)a+\nu b) \le f(a)^{1-\nu} f(b)^{\nu}, \quad (a, b \in J,\ 0 \le \nu \le 1).$$

Motivated by Definition 1.1, if $f : [a,b] \to (0,\infty)$, we define the symmetrized log-transform of $f$ as

$$\overset{\circ}{f}(t) = \sqrt{f(t)\, f(a+b-t)},$$

and the anti-symmetrized log-transform as

$$\underset{\circ}{f}(t) = \sqrt{\frac{f(t)}{f(a+b-t)}}.$$

According to this definition, we see that $f = \overset{\circ}{f}\,\underset{\circ}{f}$. Notice that if $f$ is log-convex, then $\overset{\circ}{f}$ is log-convex too. To show this,

$$\begin{aligned} \overset{\circ}{f}((1-\nu)a+\nu b) &= \sqrt{f((1-\nu)a+\nu b)\, f(a+b-((1-\nu)a+\nu b))} \\ &= \sqrt{f((1-\nu)a+\nu b)\, f((1-\nu)b+\nu a)} \\ &\le \sqrt{f(a)^{1-\nu} f(b)^{\nu} f(b)^{1-\nu} f(a)^{\nu}} \\ &= \sqrt{\left(f(a)^{1-\nu} f(b)^{1-\nu}\right)\left(f(b)^{\nu} f(a)^{\nu}\right)} \\ &= \overset{\circ}{f}(a)^{1-\nu}\, \overset{\circ}{f}(b)^{\nu}. \end{aligned}$$

A function $f : [a, b] \to (0, \infty)$ is said to be symmetrized log-convex (concave) on $[a, b]$ if symmetrized transform $\mathring{f}$ is log-convex (concave) on $[a, b]$.

It is easy to show that if $\mathring{f}$ and $\underset{\circ}{f}$ are log-convex functions, then $f$ is log-convex too. Indeed, we have

$$f((1-\nu)a+\nu b) = \mathring{f}((1-\nu)a+\nu b)\underset{\circ}{f}((1-\nu)a+\nu b) \le \mathring{f}(a)^{1-\nu}\mathring{f}(b)^{\nu}\underset{\circ}{f}(a)^{1-\nu}\underset{\circ}{f}(b)^{\nu} = f(a)^{1-\nu}f(b)^{\nu}.$$

Our first result in this context is the following Hermite-Hadamard-type inequality.

**Theorem 3.1.** *Let $f : [a, b] \to (0, \infty)$ be such that $\mathring{f}$ is log-convex. Then*

$$f\left(\frac{a+b}{2}\right) \le \exp\left(\frac{1}{2}\int_0^1 \log\left(f\left((1-\nu)a+\nu b\right) f\left((1-\nu)b+\nu a\right)\right) d\nu\right) \le \sqrt{f(a)\, f(b)}.$$

*Proof.* Since the symmetrized transform of $f$ is log-convex, $\log \mathring{f}$ is convex. So, it satisfies the Hermite-Hadamard inequality. Namely, we can write

$$\log \mathring{f}\left(\frac{a+b}{2}\right) \le \int_0^1 \log \mathring{f}\left((1-\nu)a+\nu b\right) d\nu \le \frac{\log \mathring{f}(a) + \log \mathring{f}(b)}{2} = \log \sqrt{\mathring{f}(a)\, \mathring{f}(b)}.$$

Consequently, we have

$$f\left(\frac{a+b}{2}\right) \le \exp\left(\frac{1}{2}\int_0^1 \log f\left((1-\nu)a+\nu b\right) f\left((1-\nu)b+\nu a\right) d\nu\right) \le \sqrt{f(a)\, f(b)},$$

since

$$\mathring{f}\left((1-\nu)a+\nu b\right) = \sqrt{f\left((1-\nu)a+\nu b\right) f\left((1-\nu)b+\nu a\right)}$$

and

$$\mathring{f}(a) = \mathring{f}(b) = \sqrt{f(a)\, f(b)}, \quad \mathring{f}\left(\frac{a+b}{2}\right) = f\left(\frac{a+b}{2}\right).$$

This completes the proof. □

In the following example, we present applications to numerical means.

**Example 3.2.**

(i) *If we take $f(x) := 1/x$ for $x > 0$, then*

$$\frac{d^2\left(\log \mathring{f}(x)\right)}{dx^2} = \frac{1}{2}\left(\frac{1}{x^2} + \frac{1}{(a+b-x)^2}\right) \ge 0.$$

*Hence we have*

$$\frac{\log a + \log b}{2} \le \frac{b - a + a\log a - b\log b}{a-b} \le \log\left(\frac{a+b}{2}\right)$$

*which implies*

$$\mathcal{G}(a, b) \le \mathcal{I}(a, b) \le \mathcal{A}(a, b),$$

*where $\mathcal{G}(a, b) := \sqrt{ab}$, $\mathcal{I}(a, b) := \frac{1}{e}\left(\frac{a^a}{b^b}\right)^{\frac{1}{a-b}}$ and $\mathcal{A}(a, b) := \frac{a+b}{2}$.*

*(ii) If we take* $f(x) := \exp(1/x)$ *for* $x > 0$*, then*

$$\frac{d^2\left(\log \mathring{f}(x)\right)}{dx^2} = \frac{1}{x^3} + \frac{1}{(a+b-x)^3} \geq 0,$$

*since* $x \in [a, b] \subset (0, \infty)$*. Thus we have*

$$\exp\left(\frac{2}{a+b}\right) \leq \exp\left(\frac{\log a - \log b}{a-b}\right) \leq \sqrt{\exp(1/a)\exp(1/b)}$$

*which indicates*

$$\mathcal{H}(a,b) \leq \mathcal{L}(a,b) \leq \mathcal{A}(a,b)$$

*where* $\mathcal{H}(a,b) := \mathcal{A}^{-1}(a^{-1}, b^{-1}) = \dfrac{2ab}{a+b}$ *and* $\mathcal{L}(a,b) := \dfrac{a-b}{\log a - \log b}$ *with* $\mathcal{L}(a,a) := a$*.*

On the other hand, we can state the following Fejé-type inequality.

**Theorem 3.3.** *Let* $f : [a,b] \to (0,\infty)$ *be such that* $\mathring{f}$ *is log-convex, and let* $p : [0,1] \to (0,\infty)$*. Then*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_0^1 p(v)\,dv &\leq \int_0^1 p(v)\ \sqrt{f((1-v)a+vb)\,f((1-v)b+va)}dv \\ &\leq \sqrt{f(a)\,f(b)}\int_0^1 p(v)\,dv. \end{aligned} \tag{17}$$

*Proof.* Thanks to the assumption on $f$, it follows that

$$\begin{aligned} \mathring{f}\left(\frac{a+b}{2}\right) &= \mathring{f}\left(\frac{((1-v)a+vb)+((1-v)b+va)}{2}\right) \\ &\leq \sqrt{\mathring{f}\ ((1-v)a+vb)\,\mathring{f}\ ((1-v)b+va)} \\ &\leq \sqrt{\mathring{f}\ (a)\,\mathring{f}\ (b)} \quad (\text{by } \log\text{–convexity of } \mathring{f}). \end{aligned}$$

Multiplying the above inequalities by $p(v)$ and then integrating this inequality over $0 \leq v \leq 1$, we obtain

$$\begin{aligned} \int_0^1 p(v)\,dv\mathring{f}\left(\frac{a+b}{2}\right) &\leq \int_0^1 p(v)\ \sqrt{\mathring{f}\ ((1-v)a+vb)\,\mathring{f}\ ((1-v)b+va)}dv \\ &\leq \int_0^1 p(v)\,dv\sqrt{\mathring{f}\ (a)\,\mathring{f}\ (b)}, \end{aligned}$$

which implies (17). □

Another refinement for log-convex inequalities can be stated via the transformations mentioned above.

**Theorem 3.4.** *Let* $f : [a,b] \to (0,\infty)$ *be such that* $\mathring{f}$ *is log-convex. Then*

$$f\left(\frac{a+b}{2}\right) \leq \sqrt{f(t)\,f(a+b-t)} \leq \sqrt{f(a)\,f(b)}.$$

*Proof.* Since $\overset{\circ}{f}$ is log-convex, we infer that

$$\begin{aligned}\overset{\circ}{f}\left(\frac{a+b}{2}\right) &= \overset{\circ}{f}\left(\frac{a+b-t+t}{2}\right)\\ &\leq \sqrt{\overset{\circ}{f}(t)\overset{\circ}{f}(a+b-t)}.\end{aligned}$$

On the other hand, since $\log\overset{\circ}{f}$ is convex, we can use Jensen-Mercer inequality so that we have

$$\overset{\circ}{f}(a+b-t) \leq \frac{\overset{\circ}{f}(a)\overset{\circ}{f}(b)}{\overset{\circ}{f}(t)}.$$

Thus,

$$\overset{\circ}{f}\left(\frac{a+b}{2}\right) \leq \sqrt{\overset{\circ}{f}(t)\overset{\circ}{f}(a+b-t)} \leq \sqrt{\overset{\circ}{f}(a)\overset{\circ}{f}(b)}.$$

Since

$$\overset{\circ}{f}\left(\frac{a+b}{2}\right) = f\left(\frac{a+b}{2}\right), \overset{\circ}{f}(a) = f(a), \overset{\circ}{f}(b) = f(b)$$

and

$$\overset{\circ}{f}(t)\overset{\circ}{f}(a+b-t) = f(t)f(a+b-t)$$

we get

$$f\left(\frac{a+b}{2}\right) \leq \sqrt{f(t)f(a+b-t)} \leq \sqrt{f(a)f(b)}$$

as desired. □

On the other hand, multiplicative versions of (2) and (3) can be stated as follows.

**Theorem 3.5.** *Let* $f:[a,b]\to(0,\infty)$ *be such that* $\overset{\circ}{f}$ *is log-convex. If* $0\leq\nu\leq1$, *then*

$$\begin{aligned}&\left(\frac{f\left(\frac{a+b}{2}\right)}{\sqrt{f(a)f(b)}}\right)^{2R}\sqrt{f(a)f(b)}\\ &\leq \sqrt{f((1-\nu)a+\nu b)f((1-\nu)b+\nu a)}\\ &\leq \left(\frac{f\left(\frac{a+b}{2}\right)}{\sqrt{f(a)f(b)}}\right)^{2r}\sqrt{f(a)f(b)}\end{aligned}$$

*where* $r=\min\{\nu,1-\nu\}$ *and* $R=\max\{\nu,1-\nu\}$.

*Proof.* Under the given assumptions, we have

$$\begin{aligned}&2r\left(\frac{\log\overset{\circ}{f}(a)+\log\overset{\circ}{f}(b)}{2}-\log\overset{\circ}{f}\left(\frac{a+b}{2}\right)\right)\\ &\leq (1-\nu)\log\overset{\circ}{f}(a)+\nu\log\overset{\circ}{f}(b)-\log\overset{\circ}{f}((1-\nu)a+\nu b)\\ &\leq 2R\left(\frac{\log\overset{\circ}{f}(a)+\log\overset{\circ}{f}(b)}{2}-\log\overset{\circ}{f}\left(\frac{a+b}{2}\right)\right).\end{aligned}$$

This means that

$$\left(\frac{\sqrt{\overset{\circ}{f}(a)\overset{\circ}{f}(b)}}{\overset{\circ}{f}\left(\frac{a+b}{2}\right)}\right)^{2r}\overset{\circ}{f}((1-\nu)a+\nu b)\le\overset{\circ}{f}(a)^{1-\nu}\overset{\circ}{f}(b)^{\nu}\le\left(\frac{\sqrt{\overset{\circ}{f}(a)\overset{\circ}{f}(b)}}{\overset{\circ}{f}\left(\frac{a+b}{2}\right)}\right)^{2R}\overset{\circ}{f}((1-\nu)a+\nu b).$$

Using the definition of $\overset{\circ}{f}$, we get the desired result. □

**Remark 3.6.** *If the anti-symmetrized transform of $f$ is log-convex on $[a, b]$, by the similar way to the proof of Theorem 3.5, we then have*

$$\frac{f((1-\nu)a+\nu b)}{f((1-\nu)b+\nu a)}=\left(\frac{f(a)}{f(b)}\right)^{1-2\nu},$$

*which is true for special cases $\nu = 0, 1$ and $1/2$, clearly.*

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