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A new variant of Jensen inclusion and Hermite-Hadamard type inclusions for interval-valued functions

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Abstract. In this research, we give a new version of Jensen inclusion for interval-valued functions, which is called Jensen-Mercer inclusion. Moreover, we establish some new inclusions of the Hermite-Hadamard-Mercer type for interval-valued functions. Finally, we give some applications of newly established inequalities to make them more interesting for the readers.

1. Introduction

In literature, the well-known Jensen inequality [19] states that if Π is a convex function on $[\pi_1, \pi_2]$, then

$$\Pi\left(\sum_{j=1}^{n} \lambda_j \varkappa_j\right) \leq \sum_{j=1}^{n} \lambda_j \Pi\left(\varkappa_j\right) \tag{1}$$

where $\sum_{j=1}^{n} \lambda_j = 1$.

The Hermite-Hadamard (H-H) inequality, discovered by C. Hermite and J. Hadamard (see, also, [11], and [27, p.137]), is one of the most well-known inequalities in the theory of convex functions, with a geometrical interpretation and a wide range of applications. The H-H inequality is stated as:

$$\Pi\left(\frac{\pi_1 + \pi_2}{2}\right) \leq \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} \Pi(\varkappa)\, d\varkappa \leq \frac{\Pi(\pi_1) + \Pi(\pi_2)}{2} \tag{2}$$

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where $\Pi : I \to \mathbb{R}$ is a convex function over I and $\pi_1, \pi_2 \in I$ with $\pi_1 < \pi_2$. In the case of concave mappings, the above inequality satisfies in reverse order. We should point out that H-H inequality is a refinement of the concept of convexity, and it follows obviously from Jensen's inequality. In recent years, the H-H inequality for convex functions has gotten a lot of attention, and a lot of refinements and generalisations have been studied.

The following variant of Jensen inequality, known as the Jensen-Mercer, was demonstrated by Mercer [17]:

Theorem 1.1. *If Π is a convex function on $[\pi_1, \pi_2]$, then the following inequality is true:*

$$\Pi\left(\pi_1 + \pi_2 - \sum_{j=1}^{n} \lambda_j \varkappa_j\right) \leq \Pi(\pi_1) + \Pi(\pi_2) - \sum_{j=1}^{n} \lambda_j \Pi\left(\varkappa_j\right). \tag{3}$$

In [14], the idea of Jensen-Mercer inequality has been used by Kian and Moslehian, and the following H-H-Mercer inequality was demonstrated:

$$\begin{aligned} \Pi\left(\pi_1 + \pi_2 - \frac{\varkappa + \gamma}{2}\right) &\leq \Pi(\pi_1) + \Pi(\pi_2) - \frac{1}{\gamma - \varkappa} \int_{\varkappa}^{\gamma} \Pi(\tau)\, d\tau \\ &\leq \Pi(\pi_1) + \Pi(\pi_2) - \Pi\left(\frac{\varkappa + \gamma}{2}\right) \end{aligned} \tag{4}$$

$$\begin{aligned} \Pi\left(\pi_1 + \pi_2 - \frac{\varkappa + \gamma}{2}\right) &\leq \frac{1}{\gamma - \varkappa} \int_{\varkappa}^{\gamma} \Pi(\pi_1 + \pi_2 - \tau)\, d\tau \\ &\leq \frac{\Pi(\pi_1 + \pi_2 - \varkappa) + \Pi(\pi_1 + \pi_2 - \gamma)}{2} \\ &\leq \Pi(\pi_1) + \Pi(\pi_2) - \frac{\Pi(\varkappa) + \Pi(\gamma)}{2} \end{aligned} \tag{5}$$

where Π is convex function on $[\pi_1, \pi_2]$. For some recent studies linked to Jensen-Mercer inequality, one can consult [1, 2, 8, 22].

In contrast, interval analysis is a well-known example of set-valued analysis, which is the study of sets in the context of mathematical and general topology analysis. It was created as a solution to the interval instability of deterministic real-world phenomena that can be found in many mathematical or computer models. The technique of Archimede's, which is related to computing the diameter of a circle, is an old example of an interval enclosure. Moore, who is credited with being the first to use intervals in computational mathematics, published the first book on interval analysis in 1966 (see, [20]). Following the publication of his book, a number of scientists began to study the theory and applications of interval arithmetic. Nowadays, due to its applications, interval analysis is a valuable method in different fields that are intensely interested in ambiguous results. Computer graphics, experimental and computational physics, error analysis, robotics, and many other areas have applications.

In addition, several significant inequalities (H-H, Ostrowski, and others) for interval-valued functions have been studied in recent years. Chalco-Cano et al. obtained Ostrowski type inequalities for interval-valued functions in [6, 7] using the Hukuhara derivative for interval-valued functions. We refer readers to [5, 9, 10, 12, 13, 18, 23, 24, 28–30, 34, 35] for additional relevant results.

2. Interval Calculus and Inequalities

In this section, we provide notation and background information on interval analysis. The space of all closed intervals of $\mathbb{R}$ is denoted by I_c and Δ is a bounded element of I_c. We have the representation

$$\Delta = \left[\underline{\Theta_1}, \overline{\Theta_1}\right] = \left\{\tau \in \mathbb{R} : \underline{\Theta_1} \leq \tau \leq \overline{\Theta_1}\right\}$$

where $\underline{\Theta_1}, \overline{\Theta_1} \in \mathbb{R}$ and $\underline{\Theta_1} \leq \overline{\Theta_1}$. $L(\Delta) = \overline{\Theta_1} - \underline{\Theta_1}$ can be used to express the length of the interval $\Delta = \left[\underline{\Theta_1}, \overline{\Theta_1}\right]$. The left and right endpoints of interval Δ are denoted by the numbers $\underline{\Theta_1}$ and $\overline{\Theta_1}$, respectively. The interval Δ is said to be degenerate when $\underline{\Theta_1} = \overline{\Theta_1}$, and the form $\Delta = \Theta_1 = [\Theta_1, \Theta_1]$ is used. Also, if $\underline{\Theta_1} > 0$, we can say Δ is positive, and if $\overline{\Theta_1} < 0$, we can say Δ is negative. I_c^+ and I_c^- denote the sets of all closed positive intervals and closed negative intervals of $\mathbb{R}$, respectively. Between the intervals Δ and Λ, the Pompeiu-Hausdorff distance is defined by

$$d_H(\Delta, \Lambda) = d_H\left(\left[\underline{\Theta_1}, \overline{\Theta_1}\right], \left[\underline{\Theta_2}, \overline{\Theta_2}\right]\right) = \max\left\{\left|\underline{\Theta_1} - \underline{\Theta_2}\right|, \left|\overline{\Theta_1} - \overline{\Theta_2}\right|\right\}. \tag{6}$$

(I_c, d) is a complete metric space, as far as we know (see, [3]).
$|\Delta|$ denotes the absolute value of Δ, which is the maximum of the absolute values of its endpoints:

$$|\Delta| = \max\left\{\left|\underline{\Theta_1}\right|, \left|\overline{\Theta_1}\right|\right\}.$$

The following are the concepts for fundamental interval arithmetic operations for the intervals Δ and Λ:

$$\begin{aligned}
\Delta + \Lambda &= \left[\underline{\Theta_1} + \underline{\Theta_2}, \overline{\Theta_1} + \overline{\Theta_2}\right], \\
\Delta - \Lambda &= \left[\underline{\Theta_1} - \overline{\Theta_2}, \overline{\Theta_1} - \underline{\Theta_2}\right], \\
\Delta \cdot \Lambda &= [\min U, \max U] \text{ where } U = \left\{\underline{\Theta_1}\,\underline{\Theta_2}, \underline{\Theta_1}\,\overline{\Theta_2}, \overline{\Theta_1}\underline{\Theta_2}, \overline{\Theta_1}\,\overline{\Theta_2}\right\}, \\
\Delta / \Lambda &= [\min V, \max V] \text{ where } V = \left\{\underline{\Theta_1}/\underline{\Theta_2}, \underline{\Theta_1}/\overline{\Theta_2}, \overline{\Theta_1}/\underline{\Theta_2}, \overline{\Theta_1}/\overline{\Theta_2}\right\} \text{ and } 0 \notin \Lambda.
\end{aligned}$$

The interval Δ's scalar multiplication is defined by

$$\mu\Delta = \mu\left[\underline{\Theta_1}, \overline{\Theta_1}\right] = \begin{cases} \left[\mu\underline{\Theta_1}, \mu\overline{\Theta_1}\right], & \mu > 0; \\ \{0\}, & \mu = 0; \\ \left[\mu\overline{\Theta_1}, \mu\underline{\Theta_1}\right], & \mu < 0, \end{cases}$$

where $\mu \in \mathbb{R}$.
The opposite of the interval Δ is

$$-\Delta := (-1)\Delta = [-\overline{\Theta_1}, -\underline{\Theta_1}],$$

where $\mu = -1$.
In general, $-\Delta$ is not additive inverse for Δ, i.e. $\Delta - \Delta \neq 0$.

Definition 2.1. *[31] For some kind of the intervals $\Delta, \Lambda \in I_c$, we denote the the H-difference of Δ and Λ as the $\Omega \in I_c$, we have*

$$\Delta \ominus_g \Lambda = \Omega \Leftrightarrow \begin{cases} (i)\ \Delta = \Lambda + \Omega \\ \quad or \\ (ii)\ \Lambda = \Delta + (-\Omega). \end{cases}$$

It seems beyond contraversy that

$$\Delta \ominus_g \Lambda = \begin{cases} \left[\underline{\Theta_1} - \underline{\Theta_2}, \overline{\Theta_1} - \overline{\Theta_2}\right], \text{ if } L(\Delta) \geq L(\Lambda) \\ \left[\overline{\Theta_1} - \overline{\Theta_2}, \underline{\Theta_1} - \underline{\Theta_2}\right], \text{ if } L(\Delta) \leq L(\Lambda), \end{cases}$$

where $L(\Delta) = \overline{\Theta_1} - \underline{\Theta_1}$ and $L(\Lambda) = \overline{\Theta_2} - \underline{\Theta_2}$.

The definitions of operations generate a large number of algebraic properties, enabling I_c to be a quasilinear space (see, [16]). The following are some of these characteristics (see, [3, 15, 16, 20]):

(1) (Law of associative under +) $(\Delta + \Lambda) + C = \Delta + (\Lambda + C)$ for all $\Delta, \Lambda, C \in I_c$,
(2) (Additivity element) $\Delta + 0 = 0 + \Delta = \Delta$ for all $\Delta \in I_c$,
(3) (Law of commutative under +) $\Delta + \Lambda = \Lambda + \Delta$ for all $\Delta, \Lambda \in I_c$,
(4) (Law of cancellation under +) $\Delta + C = \Lambda + C \Longrightarrow \Delta = \Lambda$ for all $\Delta, \Lambda, C \in I_c$,
(5) (Law of associative under ×) $(\Delta \cdot \Lambda) \cdot C = \Delta \cdot (\Lambda \cdot C)$ for all $\Delta, \Lambda, C \in I_c$,
(6) (Law of commutative under ×) $\Delta \cdot \Lambda = \Lambda \cdot \Delta$ for all $\Delta, \Lambda \in I_c$,
(7) (Multiplicativity element) $\Delta \cdot 1 = 1 \cdot \Delta$ for all $\Delta \in I_c$,
(8) (The first law of distributivity) $\lambda(\Delta + \Lambda) = \lambda\Delta + \lambda\Lambda$ for all $\Delta, \Lambda \in I_c$ and all $\lambda \in \mathbb{R}$,
(9) (The second law of distributivity) $(\lambda + \mu)\Delta = \lambda\Delta + \mu\Delta$ for all $\Delta \in I_c$ and all $\lambda, \mu \in \mathbb{R}$.

Aside from any of these characteristics, the distributive law does not always apply to intervals. As an example, $\Delta = [1, 2], \Lambda = [2, 3]$ and $C = [-2, -1]$.

$$\Delta \cdot (\Lambda + C) = [0, 4],$$

whereas

$$\Delta \cdot \Lambda + \Delta \cdot C = [-2, 5].$$

Another distinct feature is the inclusion $\subseteq$, which is described by

$$\Delta \subseteq \Lambda \Longleftrightarrow \underline{\Theta_1} \geq \underline{\Theta_2} \text{ and } \overline{\Theta_1} \leq \overline{\Theta_2}.$$

In [20], Moore given the definition of the Riemann integral for functions of interval-valued. $IR_{([\pi_1,\pi_2])}$ and $R_{([\pi_1,\pi_2])}$ denote the set of all Riemann integrable interval-valued functions and real-valued functions on $[\pi_1, \pi_2]$, respectively. The following theorem defines a relationship between Riemann integrable (R-integrable) functions and (IR)-integrable functions (see, [21, pp. 131]):

Theorem 2.2. *For an interval-valued mapping $\Pi : [\pi_1, \pi_2] \to \mathbb{R}_I$ with $\Pi(\tau) = \left[\underline{\Pi}(\tau), \overline{\Pi}(\tau)\right]$. The mapping $\Pi \in IR_{([\pi_1,\pi_2])}$ if and only if $\underline{\Pi}(\tau), \overline{\Pi}(\tau) \in R_{([\pi_1,\pi_2])}$ and*

$$(IR)\int_{\pi_1}^{\pi_2} \Pi(\tau)d\tau = \left[(R)\int_{\pi_1}^{\pi_2} \underline{\Pi}(\tau)d\tau, (R)\int_{\pi_1}^{\pi_2} \overline{\Pi}(\tau)d\tau\right].$$

Zhao et al. defined the following convex interval-valued function in [32, 33]:

Definition 2.3. *For all $\varkappa, \gamma \in [\pi_1, \pi_2]$ and $\tau \in (0, 1)$, the h-convex mapping $\Pi : [\pi_1, \pi_2] \to \mathbb{R}_I^+$ is stated as:*

$$h(\tau)\Pi(\varkappa) + h(1 - \tau)\Pi(\gamma) \subseteq \Pi(\tau\varkappa + (1 - \tau)\gamma). \tag{7}$$

Where $h : [c, d] \to \mathbb{R}$ is a non-negative mapping, $h \neq 0$, $(0, 1) \subseteq [c, d]$. We'll show the set of all h-convex interval-valued functions with $SX(h, [\pi_1, \pi_2], \mathbb{R}_I^+)$.

The standard definition of a convex interval-valued function is (7) with $h(\tau) = \tau$ (see, [30]). In addition, if we take $h(\tau) = \tau^s$ into (7), then Definition 2.3 gives the definition of s-convex interval-valued function (see, [4]).

In [32], Zhao et al., used the $h-$convexity of interval-valued functions and obtained the following H-H inclusion:

Theorem 2.4. *If $\Pi \in SX(h, [\pi_1, \pi_2], \mathbb{R}_I^+)$ and $h\left(\frac{1}{2}\right) \neq 0$, then following inclusions are true:*

$$\frac{1}{2h\left(\frac{1}{2}\right)}\Pi\left(\frac{\pi_1+\pi_2}{2}\right) \supseteq \frac{1}{\pi_2-\pi_1}(IR)\int_{\pi_1}^{\pi_2}\Pi(\varkappa)d\varkappa \supseteq [\Pi(\pi_1)+\Pi(\pi_2)]\int_0^1 h(\tau)d\tau. \tag{8}$$

Remark 2.5. *(i) The Inclusions (8) becomes the following for $h(\tau) = \tau$:*

$$\Pi\left(\frac{\pi_1+\pi_2}{2}\right) \supseteq \frac{1}{\pi_2-\pi_1}(IR)\int_{\pi_1}^{\pi_2}\Pi(\varkappa)d\varkappa \supseteq \frac{\Pi(\pi_1)+\Pi(\pi_2)}{2}, \tag{9}$$

which Sadowska have discovered in [30].

(ii) The Inclusions (8) becomes the following for $h(\tau) = \tau^s$:

$$2^{s-1}\Pi\left(\frac{\pi_1+\pi_2}{2}\right) \supseteq \frac{1}{\pi_2-\pi_1}(IR)\int_{\pi_1}^{\pi_2}\Pi(\varkappa)d\varkappa \supseteq \frac{\Pi(\pi_1)+\Pi(\pi_2)}{s+1},$$

which Osuna-Gómez et al. have discovered in [25].

3. Main Results

We will study convex interval-valued functions and prove Jensen-Mercer inclusion for interval-valued functions in this section. We also use the newly proven Jensen-Mercer inclusion to prove H-H type inclusion for convex interval-valued function. In this section, we use $\Pi = \left[\underline{\Pi}, \overline{\Pi}\right]$ and $\mathcal{G} = \left[\underline{\mathcal{G}}, \overline{\mathcal{G}}\right]$ for brevity.

3.1. Convex interval-valued functions

Definition 3.1. *[30] A function $\Pi : [\pi_1, \pi_2] \to I_c^+$ is said to be a convex interval-valued, if for all $\varkappa, \gamma \in [\pi_1, \pi_2]$ and $\tau \in (0, 1)$, we have*

$$\tau\Pi(\varkappa) + (1-\tau)\Pi(\gamma) \subseteq \Pi(\tau\varkappa + (1-\tau)\gamma).$$

Lemma 3.2. *[30] A function $\Pi : [\pi_1, \pi_2] \to I_c^+$ is said to be a convex interval-valued if and only if $\underline{\Pi}$ is a convex function on $[\pi_1, \pi_2]$ and $\overline{\Pi}$ is a concave function on $[\pi_1, \pi_2]$.*

Theorem 3.3 (Jensen's Inclusion). *Let Π be a convex interval-valued function on $[\pi_1, \pi_2]$, then following inclusion is true:*

$$\Pi\left(\sum_{j=1}^{n}\lambda_j\varkappa_j\right) \supseteq \sum_{j=1}^{n}\lambda_j\Pi\left(\varkappa_j\right) \tag{10}$$

where $\sum_{j=1}^{n}\lambda_j = 1$.

Proof. Since $\Pi = \left[\underline{\Pi}, \overline{\Pi}\right]$ is convex interval-valued function, therefore $\underline{\Pi}$ and $\overline{\Pi}$ are convex and concave functions, respectively. Hence, from convexity of $\underline{\Pi}$, we have

$$\underline{\Pi}\left(\sum_{j=1}^{n}\lambda_j\varkappa_j\right) \leq \sum_{j=1}^{n}\lambda_j\underline{\Pi}\left(\varkappa_j\right) \tag{11}$$

and from concavity of $\overline{\Pi}$, we get

$$\overline{\Pi}\left(\sum_{j=1}^{n} \lambda_j \varkappa_j\right) \geq \sum_{j=1}^{n} \lambda_j \overline{\Pi}\left(\varkappa_j\right). \tag{12}$$

We get the resulting inclusion (10) by combining (11) and (12). □

Our goal is to show that there is a new variant of inclusion (10).

Theorem 3.4. *Let Π be a convex interval-valued function on $[\pi_1, \pi_2]$ such that $L(\Pi(\pi_2)) \geq L(\Pi(\pi_0))$ for all $\pi_0 \in [\pi_1, \pi_2]$, then following inclusion is true:*

$$\Pi\left(\pi_1 + \pi_2 - \sum_{j=1}^{n} \lambda_j \varkappa_j\right) \supseteq \Pi(\pi_1) + \Pi(\pi_2) \ominus_g \sum_{j=1}^{n} \lambda_j \Pi\left(\varkappa_j\right) \tag{13}$$

where $\sum_{j=1}^{n} \lambda_j = 1$.

Proof. Using the strategies used in the proof of Theorem 3.3 and inequality (3), one can easily prove the necessary inclusion (13). □

3.2. H-H-Mercer Inclusion

Theorem 3.5. *Let $\Pi : [\pi_1, \pi_2] \to I_c^+$ be a convex interval-valued function such that $L(\Pi(\pi_2)) \geq L(\Pi(\pi_0))$ for all $\pi_0 \in [\pi_1, \pi_2]$. Then*

$$\begin{aligned} \Pi\left(\pi_1 + \pi_2 - \frac{\varkappa + \gamma}{2}\right) &\supseteq \frac{1}{\gamma - \varkappa}(IR)\int_{\varkappa}^{\gamma} \Pi(\pi_1 + \pi_2 - \tau)\, d\tau \\ &\supseteq \frac{\Pi(\pi_1 + \pi_2 - \varkappa) + \Pi(\pi_1 + \pi_2 - \gamma)}{2} \\ &\supseteq \Pi(\pi_1) + \Pi(\pi_2) \ominus_g \frac{\Pi(\varkappa) + \Pi(\gamma)}{2}. \end{aligned} \tag{14}$$

Proof. From convexity of Π, we have

$$\Pi\left(\pi_1 + \pi_2 - \frac{\varkappa_1 + \gamma_1}{2}\right) \supseteq \frac{1}{2}\left[\Pi(\pi_1 + \pi_2 - \varkappa_1) + \Pi(\pi_1 + \pi_2 - \gamma_1)\right] \tag{15}$$

for all $\varkappa_1, \gamma_1 \in [\pi_1, \pi_2]$. By setting $\pi_1 + \pi_2 - \varkappa_1 = \tau(\pi_1 + \pi_2 - \varkappa) + (1 - \tau)(\pi_1 + \pi_2 - \gamma)$ and $\pi_1 + \pi_2 - \gamma_1 = (1 - \tau)(\pi_1 + \pi_2 - \varkappa) + \tau(\pi_1 + \pi_2 - \gamma)$, $\tau \in [0, 1]$ in (15), we get an inclusion

$$\begin{aligned} &\Pi\left(\pi_1 + \pi_2 - \frac{\varkappa + \gamma}{2}\right) \\ \supseteq\; &\frac{1}{2}\left[\Pi(\tau(\pi_1 + \pi_2 - \varkappa) + (1 - \tau)(\pi_1 + \pi_2 - \gamma)) + \Pi((1 - \tau)(\pi_1 + \pi_2 - \varkappa) + \tau(\pi_1 + \pi_2 - \gamma))\right]. \end{aligned} \tag{16}$$

We obtain the first inclusion in (14) by integrating the inclusion (16) with respect to τ over $[0, 1]$ and using the change of variables.

On the other hand, from convexity of Π, we have

$$\Pi(\tau(\pi_1 + \pi_2 - \varkappa) + (1 - \tau)(\pi_1 + \pi_2 - \gamma)) \supseteq \tau\Pi(\pi_1 + \pi_2 - \varkappa) + (1 - \tau)\Pi(\pi_1 + \pi_2 - \gamma) \tag{17}$$

and

$$\Pi((1 - \tau)(\pi_1 + \pi_2 - \varkappa) + \tau(\pi_1 + \pi_2 - \gamma)) \supseteq (1 - \tau)\Pi(\pi_1 + \pi_2 - \varkappa) + \tau\Pi(\pi_1 + \pi_2 - \gamma). \tag{18}$$

By adding the above inclusions and from inclusion (13), we have

$$\begin{aligned} &\Pi(\tau(\pi_1+\pi_2-\varkappa)+(1-\tau)(\pi_1+\pi_2-\gamma))+\Pi((1-\tau)(\pi_1+\pi_2-\varkappa)+\tau(\pi_1+\pi_2-\gamma)) \quad (19)\\ \supseteq\ &\Pi(\pi_1+\pi_2-\varkappa)+\Pi(\pi_1+\pi_2-\gamma)\\ \supseteq\ &2[\Pi(\pi_1)+\Pi(\pi_2)]\ominus_g[\Pi(\varkappa)+\Pi(\gamma)]. \end{aligned}$$

We obtain the second and third inclusions in (14) by integrating the inclusion (19) with respect to τ over $[0,1]$ and using the change of variables. □

Remark 3.6. *If we set $\underline{\Pi}=\overline{\Pi}$ in Theorem 3.8, then inclusions (14) reduces to the inequality (5).*

Remark 3.7. *If we use $\varkappa=\pi_1$ and $\gamma=\pi_2$ in Theorem 3.8, then inclusion (14) reduces to the inclusion (9).*

Theorem 3.8. *Let $\Pi,\mathcal{G}:[\pi_1,\pi_2]\to I_c^+$ be two convex interval-valued functions . Then*

$$\frac{1}{\gamma-\varkappa}(IR)\int_{\varkappa}^{\gamma}\Pi(\pi_1+\pi_2-\tau)\mathcal{G}(\pi_1+\pi_2-\tau)\,d\tau\supseteq\frac{1}{3}\mathcal{M}(\pi_1,\pi_2;\varkappa,\gamma)+\frac{1}{6}\mathcal{N}(\pi_1,\pi_2;\varkappa,\gamma) \quad (20)$$

where

$$\mathcal{M}(\pi_1,\pi_2;\varkappa,\gamma)=\Pi(\pi_1+\pi_2-\varkappa)\mathcal{G}(\pi_1+\pi_2-\varkappa)+\Pi(\pi_1+\pi_2-\gamma)\mathcal{G}(\pi_1+\pi_2-\gamma)$$

and

$$\mathcal{N}(\pi_1,\pi_2;\varkappa,\gamma)=\Pi(\pi_1+\pi_2-\varkappa)\mathcal{G}(\pi_1+\pi_2-\gamma)+\Pi(\pi_1+\pi_2-\gamma)\mathcal{G}(\pi_1+\pi_2-\varkappa).$$

Proof. From convexity of Π, we have

$$\Pi(\tau(\pi_1+\pi_2-\varkappa)+(1-\tau)(\pi_1+\pi_2-\gamma))\supseteq\tau\Pi(\pi_1+\pi_2-\varkappa)+(1-\tau)\Pi(\pi_1+\pi_2-\gamma) \quad (21)$$

and

$$\mathcal{G}(\tau(\pi_1+\pi_2-\varkappa)+(1-\tau)(\pi_1+\pi_2-\gamma))\supseteq\tau\mathcal{G}(\pi_1+\pi_2-\varkappa)+(1-\tau)\mathcal{G}(\pi_1+\pi_2-\gamma). \quad (22)$$

We get the following inclusion by multiplying (21) and (22)

$$\begin{aligned} &\Pi(\tau(\pi_1+\pi_2-\varkappa)+(1-\tau)(\pi_1+\pi_2-\gamma))\mathcal{G}(\tau(\pi_1+\pi_2-\varkappa)+(1-\tau)(\pi_1+\pi_2-\gamma)) \quad (23)\\ \supseteq\ &\tau^2\Pi(\pi_1+\pi_2-\varkappa)\mathcal{G}(\pi_1+\pi_2-\varkappa)+(1-\tau)^2\Pi(\pi_1+\pi_2-\gamma)\mathcal{G}(\pi_1+\pi_2-\gamma)\\ &+\tau(1-\tau)\Pi(\pi_1+\pi_2-\varkappa)\mathcal{G}(\pi_1+\pi_2-\gamma)+\tau(1-\tau)\Pi(\pi_1+\pi_2-\gamma)\mathcal{G}(\pi_1+\pi_2-\varkappa). \end{aligned}$$

Likewise, we have

$$\begin{aligned} &\Pi((1-\tau)(\pi_1+\pi_2-\varkappa)+\tau(\pi_1+\pi_2-\gamma))\mathcal{G}((1-\tau)(\pi_1+\pi_2-\varkappa)+\tau(\pi_1+\pi_2-\gamma)) \quad (24)\\ \supseteq\ &(1-\tau)^2\Pi(\pi_1+\pi_2-\varkappa)\mathcal{G}(\pi_1+\pi_2-\varkappa)+\tau^2\Pi(\pi_1+\pi_2-\gamma)\mathcal{G}(\pi_1+\pi_2-\gamma)\\ &+\tau(1-\tau)\Pi(\pi_1+\pi_2-\varkappa)\mathcal{G}(\pi_1+\pi_2-\gamma)+\tau(1-\tau)\Pi(\pi_1+\pi_2-\gamma)\mathcal{G}(\pi_1+\pi_2-\varkappa). \end{aligned}$$

We get the following inclusion by adding (23) and (24)

$$\Pi(\tau(\pi_1+\pi_2-\varkappa)+(1-\tau)(\pi_1+\pi_2-\gamma))\mathcal{G}(\tau(\pi_1+\pi_2-\varkappa)+(1-\tau)(\pi_1+\pi_2-\gamma)) \quad (25)$$

$$+\Pi\left(\left(1-\tau\right)\left(\pi_1+\pi_2-\varkappa\right)+\tau\left(\pi_1+\pi_2-\gamma\right)\right)\mathcal{G}\left(\left(1-\tau\right)\left(\pi_1+\pi_2-\varkappa\right)+\tau\left(\pi_1+\pi_2-\gamma\right)\right)$$

$$\supseteq \left[\tau^2+\left(1-\tau\right)^2\right]\mathcal{M}\left(\pi_1,\pi_2;\varkappa,\gamma\right)+2\tau\left(1-\tau\right)\mathcal{N}\left(\pi_1,\pi_2;\varkappa,\gamma\right).$$

We obtain the resulting inclusion (20) by integrating the inclusion (25) with respect to τ over $[0,1]$ and using the change of variables. □

Remark 3.9. *In Theorem 3.8 if we set $\varkappa=\pi_1$ and $\gamma=\pi_2$, then Theorem 3.8 becomes [32, Theorem 4.5 for $h(t)=t$].*

Corollary 3.10. *If we set $\underline{\Pi}=\overline{\Pi}$ in Theorem 3.8, then we have the following inequality*

$$\frac{1}{\gamma-\varkappa}\int_{\varkappa}^{\gamma}\Pi\left(\pi_1+\pi_2-\tau\right)\mathcal{G}\left(\pi_1+\pi_2-\tau\right)d\tau \leq \frac{1}{3}\mathcal{M}\left(\pi_1,\pi_2;\varkappa,\gamma\right)+\frac{1}{6}\mathcal{N}\left(\pi_1,\pi_2;\varkappa,\gamma\right). \tag{26}$$

Remark 3.11. *If we set $\varkappa=\pi_1$ and $\gamma=\pi_2$ in Corollary 3.10, then inequality (26) reduces to inequality (1) of [26, Theorem 1].*

Theorem 3.12. *Let $\Pi,\mathcal{G}:[\pi_1,\pi_2]\to I_c^+$ be two convex interval-valued functions such that $L\left(\Pi\left(\pi_2\right)\right)\geq L\left(\Pi\left(\pi_0\right)\right)$ for all $\pi_0\in[\pi_1,\pi_2]$. Then*

$$\begin{aligned}
&\frac{1}{\gamma-\varkappa}\left(IR\right)\int_{\varkappa}^{\gamma}\Pi\left(\pi_1+\pi_2-\tau\right)\mathcal{G}\left(\pi_1+\pi_2-\tau\right)d\tau \qquad (27)\\
\supseteq\ & \mathcal{M}\left(\pi_1,\pi_2\right)+\mathcal{N}\left(\pi_1,\pi_2\right)\ominus_g\frac{1}{2}\left[\mathcal{M}_1\left(\pi_1,\pi_2;\varkappa,\gamma\right)+\mathcal{N}_1\left(\pi_1,\pi_2;\varkappa,\gamma\right)\right.\\
&\left.+\mathcal{M}_2\left(\pi_1,\pi_2;\varkappa,\gamma\right)+\mathcal{N}_2\left(\pi_1,\pi_2;\varkappa,\gamma\right)\right]\\
&+\frac{1}{3}\mathcal{M}\left(\varkappa,\gamma\right)+\frac{1}{6}\mathcal{N}\left(\varkappa,\gamma\right)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{M}\left(u,v\right) &= \Pi\left(u\right)\mathcal{G}\left(u\right)+\Pi\left(v\right)\mathcal{G}\left(v\right),\\
\mathcal{N}\left(u,v\right) &= \Pi\left(u\right)\mathcal{G}\left(v\right)+\Pi\left(v\right)\mathcal{G}\left(u\right),\\
\mathcal{M}_1\left(\pi_1,\pi_2;\varkappa,\gamma\right) &= \Pi\left(\pi_1\right)\mathcal{G}\left(\varkappa\right)+\Pi\left(\pi_2\right)\left(\gamma\right),\\
\mathcal{N}_1\left(\pi_1,\pi_2;\varkappa,\gamma\right) &= \Pi\left(\pi_1\right)\mathcal{G}\left(\gamma\right)+\Pi\left(\pi_2\right)\mathcal{G}\left(\varkappa\right),\\
\mathcal{M}_2\left(\pi_1,\pi_2;\varkappa,\gamma\right) &= \Pi\left(\varkappa\right)\mathcal{G}\left(\pi_1\right)+\Pi\left(\gamma\right)\mathcal{G}\left(\pi_2\right)
\end{aligned}$$

and

$$\mathcal{N}_2\left(\pi_1,\pi_2;\varkappa,\gamma\right)=\Pi\left(\varkappa\right)\mathcal{G}\left(\pi_2\right)+\Pi\left(\gamma\right)\mathcal{G}\left(\pi_1\right).$$

Proof. From inclusion (13), we have

$$\Pi\left(\tau\left(\pi_1+\pi_2-\varkappa\right)+\left(1-\tau\right)\left(\pi_1+\pi_2-\gamma\right)\right)\supseteq\Pi\left(\pi_1\right)+\Pi\left(\pi_2\right)\ominus_g\left[\tau\Pi\left(\varkappa\right)+\left(1-\tau\right)\Pi\left(\gamma\right)\right] \tag{28}$$

and

$$\mathcal{G}\left(\tau\left(\pi_1+\pi_2-\varkappa\right)+\left(1-\tau\right)\left(\pi_1+\pi_2-\gamma\right)\right)\supseteq\mathcal{G}\left(\pi_1\right)+\mathcal{G}\left(\pi_2\right)\ominus_g\left[\tau\mathcal{G}\left(\varkappa\right)+\left(1-\tau\right)\mathcal{G}\left(\gamma\right)\right]. \tag{29}$$

We get the following inclusion by multiplying (28) and (29)

$$\begin{aligned}
&\Pi\left(\tau\left(\pi_1+\pi_2-\varkappa\right)+(1-\tau)\left(\pi_1+\pi_2-\gamma\right)\right)\mathcal{G}\left(\tau\left(\pi_1+\pi_2-\varkappa\right)+(1-\tau)\left(\pi_1+\pi_2-\gamma\right)\right) \qquad (30)\\
\supseteq\ &\left[\Pi\left(\pi_1\right)+\Pi\left(\pi_2\right)\right]\left[\mathcal{G}\left(\pi_1\right)+\mathcal{G}\left(\pi_2\right)\right]\ominus_g\left[\Pi\left(\pi_1\right)+\Pi\left(\pi_2\right)\right]\left[\tau\mathcal{G}(\varkappa)+(1-\tau)\mathcal{G}(\gamma)\right]\\
&\ominus_g\left[\mathcal{G}\left(\pi_1\right)+\mathcal{G}\left(\pi_2\right)\right]\left[\tau\Pi(\varkappa)+(1-\tau)\Pi(\gamma)\right]+\left[\tau\Pi(\varkappa)+(1-\tau)\Pi(\gamma)\right]\left[\tau\mathcal{G}(\varkappa)+(1-\tau)\mathcal{G}(\gamma)\right].
\end{aligned}$$

Likewise, we have

$$\begin{aligned}
&\Pi\left((1-\tau)\left(\pi_1+\pi_2-\varkappa\right)+\tau\left(\pi_1+\pi_2-\gamma\right)\right)\mathcal{G}\left((1-\tau)\left(\pi_1+\pi_2-\varkappa\right)+\tau\left(\pi_1+\pi_2-\gamma\right)\right) \qquad (31)\\
\supseteq\ &\left[\Pi\left(\pi_1\right)+\Pi\left(\pi_2\right)\right]\left[\mathcal{G}\left(\pi_1\right)+\mathcal{G}\left(\pi_2\right)\right]\ominus_g\left[\Pi\left(\pi_1\right)+\Pi\left(\pi_2\right)\right]\left[(1-\tau)\mathcal{G}(\varkappa)+\tau\mathcal{G}(\gamma)\right]\\
&\ominus_g\left[\mathcal{G}\left(\pi_1\right)+\mathcal{G}\left(\pi_2\right)\right]\left[(1-\tau)\Pi(\varkappa)+\tau\Pi(\gamma)\right]+\left[(1-\tau)\Pi(\varkappa)+\tau\Pi(\gamma)\right]\left[(1-\tau)\mathcal{G}(\varkappa)+\tau\mathcal{G}(\gamma)\right].
\end{aligned}$$

We get the following inclusion by adding (30) and (31)

$$\begin{aligned}
&\Pi\left(\tau\left(\pi_1+\pi_2-\varkappa\right)+(1-\tau)\left(\pi_1+\pi_2-\gamma\right)\right)\mathcal{G}\left(\tau\left(\pi_1+\pi_2-\varkappa\right)+(1-\tau)\left(\pi_1+\pi_2-\gamma\right)\right) \qquad (32)\\
&+\Pi\left((1-\tau)\left(\pi_1+\pi_2-\varkappa\right)+\tau\left(\pi_1+\pi_2-\gamma\right)\right)\mathcal{G}\left((1-\tau)\left(\pi_1+\pi_2-\varkappa\right)+\tau\left(\pi_1+\pi_2-\gamma\right)\right)\\
\supseteq\ &2\left[\mathcal{M}\left(\pi_1,\pi_2\right)+\mathcal{N}\left(\pi_1,\pi_2\right)\right]\\
&\ominus_g\left[\mathcal{M}_1\left(\pi_1,\pi_2;\varkappa,\gamma\right)+\mathcal{N}_1\left(\pi_1,\pi_2;\varkappa,\gamma\right)+\mathcal{M}_2\left(\pi_1,\pi_2;\varkappa,\gamma\right)+\mathcal{N}_2\left(\pi_1,\pi_2;\varkappa,\gamma\right)\right]\\
&+\left[\tau^2+(1-\tau)^2\right]\mathcal{M}(\varkappa,\gamma)+2\tau(1-\tau)\mathcal{N}(\varkappa,\gamma).
\end{aligned}$$

We obtain the resulting inclusion (27) by integrating the inclusion (32) with respect to τ over $[0,1]$ and using the change of variables. □

Remark 3.13. *In Theorem 3.12, if we assume $\varkappa=\pi_1$ and $\gamma=\pi_2$, then Theorem 3.12 becomes [32, Theorem 4.5 for $h(t)=t$].*

Corollary 3.14. *If we set $\underline{\Pi}=\overline{\Pi}$ in Theorem 3.12, then we have the following inequality*

$$\begin{aligned}
&\frac{1}{\gamma-\varkappa}\int_{\varkappa}^{\gamma}\Pi\left(\pi_1+\pi_2-\tau\right)\mathcal{G}\left(\pi_1+\pi_2-\tau\right)d\tau \qquad (33)\\
\leq\ &\mathcal{M}\left(\pi_1,\pi_2\right)+\mathcal{N}\left(\pi_1,\pi_2\right)-\frac{1}{2}\left[\mathcal{M}_1\left(\pi_1,\pi_2;\varkappa,\gamma\right)+\mathcal{N}_1\left(\pi_1,\pi_2;\varkappa,\gamma\right)\right.\\
&\left.+\mathcal{M}_2\left(\pi_1,\pi_2;\varkappa,\gamma\right)+\mathcal{N}_2\left(\pi_1,\pi_2;\varkappa,\gamma\right)\right]\\
&+\frac{1}{3}\mathcal{M}(\varkappa,\gamma)+\frac{1}{6}\mathcal{N}(\varkappa,\gamma).
\end{aligned}$$

Remark 3.15. *If we set $\varkappa=\pi_1$ and $\gamma=\pi_2$ in Corollary 3.14, then inequality (33) reduces to inequality (1) of [26, Theorem 1].*

Theorem 3.16. *Let $\Pi, \mathcal{G} : [\pi_1, \pi_2] \to I_c^+$ be two convex interval-valued functions. Then*

$$\begin{aligned} & 2\Pi\left(\pi_1+\pi_2-\frac{\varkappa+\gamma}{2}\right)\mathcal{G}\left(\pi_1+\pi_2-\frac{\varkappa+\gamma}{2}\right) \\ \supseteq\ & \frac{1}{\gamma-\varkappa}(IR)\int_{\varkappa}^{\gamma}\Pi(\pi_1+\pi_2-\tau)\mathcal{G}(\pi_1+\pi_2-\tau)\,d\tau+\frac{1}{6}\mathcal{M}(\pi_1,\pi_2;\varkappa,\gamma)+\frac{1}{3}\mathcal{N}(\pi_1,\pi_2;\varkappa,\gamma) \end{aligned} \tag{34}$$

where $\mathcal{M}(\pi_1, \pi_2; \varkappa, \gamma)$ and $\mathcal{N}(\pi_1, \pi_2; \varkappa, \gamma)$ are defined in Theorem 3.8.

Proof. From convexity of Π, we have

$$\begin{aligned} \Pi\left(\pi_1+\pi_2-\frac{\varkappa+\gamma}{2}\right) \supseteq\ & \frac{1}{2}\left[\Pi(\tau(\pi_1+\pi_2-\varkappa)+(1-\tau)(\pi_1+\pi_2-\gamma))\right. \\ & \left.+\Pi((1-\tau)(\pi_1+\pi_2-\varkappa)+\tau(\pi_1+\pi_2-\gamma))\right] \end{aligned} \tag{35}$$

and

$$\begin{aligned} \mathcal{G}\left(\pi_1+\pi_2-\frac{\varkappa+\gamma}{2}\right) \supseteq\ & \frac{1}{2}\left[\mathcal{G}(\tau(\pi_1+\pi_2-\varkappa)+(1-\tau)(\pi_1+\pi_2-\gamma))\right. \\ & \left.+\mathcal{G}((1-\tau)(\pi_1+\pi_2-\varkappa)+\tau(\pi_1+\pi_2-\gamma))\right]. \end{aligned} \tag{36}$$

We get the following inclusion by multiplying (35) and (36)

$$\begin{aligned} & \Pi\left(\pi_1+\pi_2-\frac{\varkappa+\gamma}{2}\right)\mathcal{G}\left(\pi_1+\pi_2-\frac{\varkappa+\gamma}{2}\right) \\ \supseteq\ & \frac{1}{4}\left[\Pi(\tau(\pi_1+\pi_2-\varkappa)+(1-\tau)(\pi_1+\pi_2-\gamma))\mathcal{G}(\tau(\pi_1+\pi_2-\varkappa)+(1-\tau)(\pi_1+\pi_2-\gamma))\right. \\ & +\Pi((1-\tau)(\pi_1+\pi_2-\varkappa)+\tau(\pi_1+\pi_2-\gamma))\mathcal{G}((1-\tau)(\pi_1+\pi_2-\varkappa)+\tau(\pi_1+\pi_2-\gamma)) \\ & +\Pi(\tau(\pi_1+\pi_2-\varkappa)+(1-\tau)(\pi_1+\pi_2-\gamma))\mathcal{G}((1-\tau)(\pi_1+\pi_2-\varkappa)+\tau(\pi_1+\pi_2-\gamma)) \\ & \left.+\Pi((1-\tau)(\pi_1+\pi_2-\varkappa)+\tau(\pi_1+\pi_2-\gamma))\mathcal{G}(\tau(\pi_1+\pi_2-\varkappa)+(1-\tau)(\pi_1+\pi_2-\gamma))\right] \\ \supseteq\ & \frac{1}{4}\left[\Pi(\tau(\pi_1+\pi_2-\varkappa)+(1-\tau)(\pi_1+\pi_2-\gamma))\mathcal{G}(\tau(\pi_1+\pi_2-\varkappa)+(1-\tau)(\pi_1+\pi_2-\gamma))\right. \\ & \left.+\Pi((1-\tau)(\pi_1+\pi_2-\varkappa)+\tau(\pi_1+\pi_2-\gamma))\mathcal{G}((1-\tau)(\pi_1+\pi_2-\varkappa)+\tau(\pi_1+\pi_2-\gamma))\right] \\ & +\frac{1}{4}\left[\{\tau\Pi(\pi_1+\pi_2-\varkappa)+(1-\tau)\Pi(\pi_1+\pi_2-\gamma)\}\right. \\ & \times\{(1-\tau)\mathcal{G}(\pi_1+\pi_2-\varkappa)+\tau\mathcal{G}(\pi_1+\pi_2-\gamma)\} \\ & +\{\tau\mathcal{G}(\pi_1+\pi_2-\varkappa)+(1-\tau)\mathcal{G}(\pi_1+\pi_2-\gamma)\} \\ & \left.\times\{(1-\tau)\Pi(\pi_1+\pi_2-\varkappa)+\tau\Pi(\pi_1+\pi_2-\gamma)\}\right] \\ =\ & \frac{1}{4}\left[\Pi(\tau(\pi_1+\pi_2-\varkappa)+(1-\tau)(\pi_1+\pi_2-\gamma))\mathcal{G}(\tau(\pi_1+\pi_2-\varkappa)+(1-\tau)(\pi_1+\pi_2-\gamma))\right. \end{aligned} \tag{37}$$

$$+\Pi\left(\left(1-\tau\right)\left(\pi_1+\pi_2-\varkappa\right)+\tau\left(\pi_1+\pi_2-\gamma\right)\right)\mathcal{G}\left(\left(1-\tau\right)\left(\pi_1+\pi_2-\varkappa\right)+\tau\left(\pi_1+\pi_2-\gamma\right)\right)\Big]$$

$$+\frac{1}{4}\left[2\tau\left(1-\tau\right)\mathcal{M}\left(\pi_1,\pi_2;\varkappa,\gamma\right)+\left\{\tau^2+\left(1-\tau\right)^2\right\}\mathcal{N}\left(\pi_1,\pi_2;\varkappa,\gamma\right)\right].$$

We obtain the resulting inclusion (34) by integrating the inclusion (37) with respect to τ over $[0,1]$ and using the change of variables. □

Remark 3.17. *If we assume $\varkappa = \pi_1$ and $\gamma = \pi_2$ in Theorem 3.16, then Theorem 3.16 becomes [32, Theorem 4.6 for $h(t) = t$].*

Corollary 3.18. *If we set $\underline{\Pi} = \overline{\Pi}$ in Theorem 3.16, then we have the following inequality*

$$\begin{aligned} & 2\Pi\left(\pi_1+\pi_2-\frac{\varkappa+\gamma}{2}\right)\mathcal{G}\left(\pi_1+\pi_2-\frac{\varkappa+\gamma}{2}\right) \\ \leq\ & \frac{1}{\gamma-\varkappa}\int_{\varkappa}^{\gamma}\Pi\left(\pi_1+\pi_2-\tau\right)\mathcal{G}\left(\pi_1+\pi_2-\tau\right)d\tau+\frac{1}{6}\mathcal{M}\left(\pi_1,\pi_2;\varkappa,\gamma\right)+\frac{1}{3}\mathcal{N}\left(\pi_1,\pi_2;\varkappa,\gamma\right) \end{aligned} \tag{38}$$

Remark 3.19. *If we set $\varkappa = \pi_1$ and $\gamma = \pi_2$ in Corollary 3.18, then inequality (38) reduces to inequality (2) of [26, Theorem 1].*

4. Application to Special Means

For arbitrary positive numbers π_1, π_2 $(\pi_1 \neq \pi_2)$, we consider the means as follows:

1. The arithmatic mean
$$\mathcal{A} = \mathcal{A}(\pi_1,\pi_2) = \frac{\pi_1+\pi_2}{2}.$$
2. The geometric mean
$$G{=}G\left(\pi_1,\pi_2\right) = \sqrt{\pi_1\pi_2}.$$
3. The logarithmic mean
$$\mathcal{L} = \mathcal{L}\left(\pi_1,\pi_2\right) = \frac{\pi_1-\pi_2}{\ln\pi_2-\ln\pi_1}.$$
4. The identric mean
$$\mathcal{I} = \mathcal{I}\left(\pi_1,\pi_2\right) = \begin{cases} \frac{1}{e}\left(\frac{\pi_2}{\pi_1}\right)^{\frac{1}{\pi_2-\pi_1}}, & \text{if } \pi_1 \neq \pi_2, \\ \pi_1, & \text{if } \pi_1 = \pi_2, \end{cases} \quad \pi_1,\pi_2 > 0.$$

Proposition 4.1. *For $\pi_1, \pi_2 \in (e, \infty)$, the following inclusion is true:*

$$\begin{aligned} & \left[\left(2\mathcal{A}\left(\pi_1,\pi_2\right)-\mathcal{A}\left(\varkappa,\gamma\right)\right)^{-1},\ln\left(2\mathcal{A}\left(\pi_1,\pi_2\right)-\mathcal{A}\left(\varkappa,\gamma\right)\right)\right] \\ \supseteq\ & \left[\mathcal{L}^{-1}\left(\pi_1+\pi_2-\varkappa,\pi_1+\pi_2-\gamma\right),\ln\mathcal{I}\left(\pi_1+\pi_2-\varkappa,\pi_1+\pi_2-\gamma\right)\right] \\ \supseteq\ & \left[\mathcal{A}\left(\left(2\mathcal{A}\left(\pi_1,\pi_2\right)-\varkappa\right)^{-1},\left(2\mathcal{A}\left(\pi_1,\pi_2\right)-\gamma\right)^{-1}\right),\ln G\left(2\mathcal{A}\left(\pi_1,\pi_2\right)-\varkappa,2\mathcal{A}\left(\pi_1,\pi_2\right)-\gamma\right)\right] \\ \supseteq\ & \left[2\mathcal{A}\left(\pi_1,\pi_2\right)-\mathcal{A}\left(\varkappa^{-1},\gamma^{-1}\right),\ln G^2\left(\pi_1,\pi_2\right)-\ln G\left(\varkappa,\gamma\right)\right]. \end{aligned} \tag{39}$$

Proof. We consider a convex interval-valued function $\Pi : [\pi_1, \pi_2] \subset (e, \infty) \to I_c^+$ which is defined as $\Pi = \left[\underline{\Pi}, \overline{\Pi}\right] = \left[\frac{1}{x}, \ln \varkappa\right]$ such that $\Pi(\pi_2) \geq \Pi(\pi_0)$ for all $\pi_0 \in [\pi_1, \pi_2]$, then we have

$$\begin{aligned}\Pi\left(\pi_1 + \pi_2 - \frac{\varkappa + \gamma}{2}\right) &= \left[\underline{\Pi}\left(\pi_1 + \pi_2 - \frac{\varkappa + \gamma}{2}\right), \overline{\Pi}\left(\pi_1 + \pi_2 - \frac{\varkappa + \gamma}{2}\right)\right] \\ &= \left[(2\mathcal{A}(\pi_1, \pi_2) - \mathcal{A}(\varkappa, \gamma))^{-1}, \ln(2\mathcal{A}(\pi_1, \pi_2) - \mathcal{A}(\varkappa, \gamma))\right],\end{aligned} \tag{40}$$

$$\begin{aligned}&\frac{1}{\gamma - \varkappa}(IR)\int_{\pi_1+\pi_2-\gamma}^{\pi_1+\pi_2-\varkappa} \Pi(\tau)\, d\tau \\ &= \left[\frac{1}{\gamma - \varkappa}(R)\int_{\pi_1+\pi_2-\gamma}^{\pi_1+\pi_2-\varkappa} \underline{\Pi}(\tau)\, d\tau, \frac{1}{\gamma - \varkappa}(R)\int_{\pi_1+\pi_2-\gamma}^{\pi_1+\pi_2-\varkappa} \overline{\Pi}(\tau)\, d\tau\right] \\ &= \left[\mathcal{L}^{-1}(\pi_1 + \pi_2 - \varkappa, \pi_1 + \pi_2 - \gamma), \ln \mathcal{I}(\pi_1 + \pi_2 - \varkappa, \pi_1 + \pi_2 - \gamma)\right],\end{aligned} \tag{41}$$

$$\begin{aligned}&\frac{\Pi(\pi_1 + \pi_2 - \varkappa) + \Pi(\pi_1 + \pi_2 - \gamma)}{2} \\ &= \left[\frac{\underline{\Pi}(\pi_1 + \pi_2 - \varkappa) + \underline{\Pi}(\pi_1 + \pi_2 - \gamma)}{2}, \frac{\overline{\Pi}(\pi_1 + \pi_2 - \varkappa) + \overline{\Pi}(\pi_1 + \pi_2 - \gamma)}{2}\right] \\ &= \left[\mathcal{A}\left((2\mathcal{A}(\pi_1, \pi_2) - \varkappa)^{-1}, (2\mathcal{A}(\pi_1, \pi_2) - \gamma)^{-1}\right), \ln G(2\mathcal{A}(\pi_1, \pi_2) - \varkappa, 2\mathcal{A}(\pi_1, \pi_2) - \gamma)\right]\end{aligned} \tag{42}$$

and

$$\begin{aligned}&\Pi(\pi_1) + \Pi(\pi_2) \ominus_g \frac{\Pi(\varkappa) + \Pi(\gamma)}{2} \\ &= \left[\underline{\Pi}(\pi_1) + \underline{\Pi}(\pi_2), \overline{\Pi}(\pi_1) + \overline{\Pi}(\pi_2)\right] - \left[\frac{\underline{\Pi}(\varkappa) + \underline{\Pi}(\gamma)}{2}, \frac{\overline{\Pi}(\varkappa) + \overline{\Pi}(\gamma)}{2}\right] \\ &= \left[\underline{\Pi}(\pi_1) + \underline{\Pi}(\pi_2) - \frac{\underline{\Pi}(\varkappa) + \underline{\Pi}(\gamma)}{2}, \overline{\Pi}(\pi_1) + \overline{\Pi}(\pi_2) - \frac{\overline{\Pi}(\varkappa) + \overline{\Pi}(\gamma)}{2}\right] \\ &= \left[2\mathcal{A}(\pi_1, \pi_2) - \mathcal{A}\left(\varkappa^{-1}, \gamma^{-1}\right), \ln G^2(\pi_1, \pi_2) - \ln G(\varkappa, \gamma)\right].\end{aligned} \tag{43}$$

We derive the necessary results from (40)-(43) and inclusions (14). □

5. Conclusion

In this work, we proposed Jensen-Mercer inclusion for interval-valued functions and developed H-H-Mercer type inclusion using the newly proposed Jensen-Mercer inclusion. We discussed the special cases of recently proven findings and found some recent and old results in the literature. It's a new problem that future researchers will be able to prove similar inclusions for different kinds fractional operators.

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