Filomat 37:17 (2023), 5581–5589
https://doi.org/10.2298/FIL2317581L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Soft Carathéodory extension theorem and soft outer measure

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Abstract. Molodsov defined the concept of a soft set, which is widely used when drawing conclusionsif we have unclear, incomplete and imprecise information. Soft set theory became an interest of many researchers who defined new concepts such as soft measure, soft σ-algebra, soft premeasure, soft outer measure, soft content, etc. In this paper, it was shown that with a certain extension, we can construct a soft measure on the obtained soft σ-algebra starting from a soft premeasure on a soft semiring.

1. Introduction

During the study and analysis of vaque, inaccurate, uncertain and incomplete informations from the real life, in addition to probability theory, it is desirable and useful to use some other mathematical tools. One such tool is Molodtsov soft set theory (see [9] or [10]). The soft set theory was presented to the public in 1999, by Molodtsov [9], as a new mathematical tool for presenting data in which appear various types of uncertainty and ambiguity. Since 1999, the study and development of this mathematical tool has become more frequent and is advancing very rapidly.

After defining soft sets as a mathematical term, it is logical that certain operations are defined among such sets. In a large number of papers, many operations with soft sets are defined and their applications are listed. The most famous operations with soft sets, as well as their properties, were defined and proved by Maji et al. [6]. Ali et al. [1] and Yang [19] also pointed out some omissions and shortcomings in defining these operations. In order to fill the gaps in defining operations on soft sets, the papers [1], [2], [11], [15], [16], [18] also contribute to the theory of soft sets in this context. Maji et al. [5], [6] used soft set theory in decision-making problems and defined some more operations. Chen et al. [3] studied some basic problems in which soft set theory would be successfully applied.

In the paper [8], Samanta and Majumdar presented the idea of soft mappings and studied some of their properties. In the paper [12], Riaz and Naim studied measurable soft mappings and some applications of soft set theory. Soft σ-algebras and their basic properties were also studied in [12] and [4]. The subject of study of Riaz et al. [13] are soft measure and soft outer measure, and Stojanović and Boričić Joksimović defined soft content and soft premeasure in the paper [17]. Using the defined terms and the obtained results, we decide to continue studying the structures on which the mentioned mappings are defined.

2020 *Mathematics Subject Classification*. Primary 06D72.
Keywords. Soft σ-algebra; Soft semiring; Soft measure; Soft premeasure; Soft outer measure; Carathéodory theorem.
Received: 04 July 2022; Accepted: 27 January 2023
Communicated by Hari M. Srivastava
This work was supported by the Serbian Ministry of Education, Science and Technological Development (Agreement No. 451-03-68/2022-14/ 200122).
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The question is, can a soft measure on a soft σ-algebra, by a certain extension, be obtained by upgrading a mapping that is not a soft measure on a structure that is not a soft σ-algebra? The answer to the question is yes, and in this paper the main result is that, starting from the soft premeasure on a soft semiring, a soft measure is constructed on the obtained soft σ-algebra from the initial soft semiring.

2. Preliminaries

This section provides basic definitions and basic properties in soft set theory, as well as statements that are necessary in this paper, and have been proven by many authors.

Let X be an initial universe set and E_X be the set of all possible parameters under consideration with respect to X. The power set of X is denoted by $\mathcal{P}(X)$ and A is a subset of E. Usually, parameters are attributes, characteristics, or properties of objects in X. In what follows, E_X (simply denoted by E) always means the universe set of parameters with respect to X, unless otherwise specified.

Definition 2.1. *[9] A pair (F, A) is called a soft set over X where $A \subseteq E$ and $F : A \to \mathcal{P}(X)$ is a set valued mapping. In other words, a soft set over X is a parameterized family of subsets of the universe X. For $\forall e \in A$, $F(e)$ may be considered as the set of e-approximate elements of the soft set (F, A). It is worth noting that $F(e)$ may be arbitrary. Some of them may be empty, and some may have nonempty intersection.*

Definition 2.2. *[7] A soft set F_A on the universe X is defined by the set of ordered pairs $F_A = \{(e, f_A(e)) \mid e \in E, f_A(e) \in \mathcal{P}(X)\}$, where $f_A : E \to \mathcal{P}(X)$, such that $f_A(e) \neq \emptyset$ if $e \in A \subseteq E$ and $f_A(e) = \emptyset$, if $e \notin A$. Here, f_A is called an approximate function of the soft set F_A. The value of $f_A(e)$ may be arbitrary.*

Note that the set of all soft sets over X will be denoted by $\mathcal{S}(X, E)$.

Definition 2.3. *[2] Let $F_A \in \mathcal{S}(X, E)$. If $f_A(e) = \emptyset$ for all $e \in E$, then F_A is called an empty soft set, denoted by F_Φ or Φ. $f_A(e) = \emptyset$ means that there is no element in X related to the parameter $e \in E$. Therefore, we do not display such elements in the soft sets, as it is meaningless to consider such parameters.*

Definition 2.4. *[2] Let $F_A \in \mathcal{S}(X, E)$. If $f_A(e) = X$ for all $e \in A$, then F_A is called an A-universal soft set, denoted by $F_{\widetilde{A}} = \widetilde{A}$. If $A = E$, then the A-universal soft set is called a universal soft set, denoted by $F_{\widetilde{E}} = \widetilde{E}$.*

Definition 2.5. *[14] Let Y be a nonempty subset of X, then $\widetilde{Y}$ denotes the soft set Y_E over X for which $Y(e) = Y$, for all $e \in E$. In particular, X_E will be denoted by $\widetilde{X}$.*

Definition 2.6. *[2] Let $F_A, G_B \in \mathcal{S}(X, E)$. Then F_A is a soft subset of G_B, denoted by $F_A \sqsubseteq G_B$ if $f_A(e) \subseteq g_B(e)$, for all $e \in E$.*

Definition 2.7. *[2] Let $F_A, G_B \in \mathcal{S}(X, E)$. Then F_A and G_B are soft equal, denoted by $F_A = G_B$, if and only if $f_A(e) = g_B(e)$, for all $e \in E$.*

Definition 2.8. *[2] Let $F_A, G_B \in \mathcal{S}(X, E)$. Then, the soft union $F_A \sqcup G_B$ of F_A and G_B is defined by the approximate functions $h_{A\cup B}(e) = f_A(e) \cup g_B(e)$, for all $e \in E$.*

Definition 2.9. *[2] Let $F_A, G_B \in \mathcal{S}(X, E)$. Then, the soft intersection $F_A \sqcap G_B$ of F_A and G_B is defined by the approximate functions $h_{A\cap B}(e) = f_A(e) \cap g_B(e)$, for all $e \in E$.*

Definition 2.10. *[2] Let $F_A, G_B \in \mathcal{S}(X, E)$. Then, the soft difference $F_A \backslash G_B$ of F_A and G_B is defined by the approximate functions $h_{A\backslash B}(e) = f_A(e) \backslash g_B(e)$, for all $e \in E$.*

Definition 2.11. *[2] The soft complement F_A^c of $F_A \in \mathcal{S}(X, E)$ is defined by the approximate function $f_{A^c}(e) = f_A^c(e)$, where $f_A^c(e)$ is the complement of the set $f_A(e)$; that is, $f_A^c(e) = X \backslash f_A(e)$, for all $e \in E$.*

Definition 2.12. *[20] Let I be an arbitrary index set and $\{(F_A)_i\}_{i\in I}$ be a subfamily of $\mathcal{S}(X, E)$.*

- *The union of these soft sets is the soft set* G_C*, where* $g_C(e) = \cup_{i\in I}(F_A)_i(e)$ *for each* $e \in E$*. We write* $G_C = \sqcup_{i\in I}(F_A)_i$*.*
- *The intersection of these soft sets is the soft set* H_D*, where* $h_D(e) = \cap_{i\in I}(F_A)_i(e)$ *for each* $e \in E$*. We write* $H_D = \sqcap_{i\in I}(F_A)_i$*.*

As in the classical set theory, so in the soft set theory, a significant place takes the study of special collections of sets, ie. a collection of sets with specific properties. For studing soft measure it is necessary to define collections of soft sets like soft semiring, soft ring, soft σ-algebra, etc.

Definition 2.13. *[4] A collection* $\widetilde{\mathcal{A}}$ *of soft subsets of* $\widetilde{X}$ *is called a soft* σ*-algebra on* $\widetilde{X}$ *if, and only if, it satisfies the following conditions:*

- $\Phi \in \widetilde{\mathcal{A}}$*,*
- *if* $F_A \in \widetilde{\mathcal{A}}$*, then* $F_A^c = \widetilde{X}\backslash F_A \in \widetilde{\mathcal{A}}$*,*
- *if* $(F_A)_1, (F_A)_2, (F_A)_3 \ldots$ *is a countable collection of soft sets in* $\widetilde{\mathcal{A}}$*, then* $\bigsqcup_{i=1}^{\infty}(F_A)_i \in \widetilde{\mathcal{A}}$*.*

  The pair $(\widetilde{X}, \widetilde{\mathcal{A}})$ *is called a soft measurable space and* $(F_A)_i \in \widetilde{\mathcal{A}}$ *is called a measurable soft set.*

Definition 2.14. *[13] The smallest soft* σ*-algebra* $\widetilde{\mathcal{H}}$ *containing some soft collection* $\widetilde{\mathcal{G}}$ *of soft subsets of* $\widetilde{X}$ *is called the soft* σ*-algebra generated by* $\widetilde{\mathcal{G}}$*.*

Most significant soft mappings for this paper are those mappings that meet certain requirements on soft set collections.

Definition 2.15. *[13] Let* $\widetilde{\mathcal{A}}$ *be a soft* σ*-algebra of soft subsets over* $\widetilde{X}$ *and* $\widetilde{\mu}$ *be a soft real-valued mapping on* $\widetilde{\mathcal{A}}$*. Let* $((F_A)_i)_{i\in\mathbb{N}}$ *be a sequence of soft sets in* $\widetilde{\mathcal{A}}$*. The soft mapping* $\widetilde{\mu}$ *is called:*

- *finitely soft sub-additive, if*
$$\widetilde{\mu}\left(\bigsqcup_{i=1}^{n}(F_A)_i\right) \leqslant \sum_{i=1}^{n}\widetilde{\mu}((F_A)_i),$$
- *countably soft sub-additive, if*
$$\widetilde{\mu}\left(\bigsqcup_{i=1}^{\infty}(F_A)_i\right) \leqslant \sum_{i=1}^{\infty}\widetilde{\mu}((F_A)_i),$$
- *finitely soft additive, if*
$$\widetilde{\mu}\left(\bigsqcup_{i=1}^{n}(F_A)_i\right) = \sum_{i=1}^{n}\widetilde{\mu}((F_A)_i),$$
  where $(F_A)_i$*'s are pairwise soft disjoint,*
- *countably soft additive or soft* σ*-additive, if*
$$\widetilde{\mu}\left(\bigsqcup_{i=1}^{\infty}(F_A)_i\right) = \sum_{i=1}^{\infty}\widetilde{\mu}((F_A)_i),$$
  where $(F_A)_i$*'s are pairwise soft disjoint,*
- *soft monotone, if* $F_A \sqsubseteq G_B$ *then* $\widetilde{\mu}(F_A) \leqslant \widetilde{\mu}(G_B)$*, for all* $F_A, G_B \in \widetilde{\mathcal{A}}$*.*

As well as the classical set theory, the soft set theory defines the notion of measure [13], called soft measure.

Definition 2.16. *[13] Let $\widetilde{\mathcal{A}}$ be a soft σ-algebra of soft subsets of a set $\widetilde{X}$ and $\widetilde{\mu}$ be an extended soft real-valued mapping on $\widetilde{\mathcal{A}}$. Then $\widetilde{\mu}$ is called a soft measure on $\widetilde{\mathcal{A}}$, if*

- $\widetilde{\mu}(\Phi) = 0$,
- $\widetilde{\mu}(F_A) \geqslant 0$ *for each* $F_A \in \widetilde{\mathcal{A}}$,
- $\widetilde{\mu}$ *is countably soft additive, i.e.,*

$$\widetilde{\mu}\left(\bigsqcup_{i=1}^{\infty}(F_A)_i\right) = \sum_{i=1}^{\infty}\widetilde{\mu}((F_A)_i),$$

$(F_A)_i$'s being pairwise soft disjoint.

If $\widetilde{\mu}$ is a soft measure on a soft σ-algebra $\widetilde{\mathcal{A}}$, then the triplet $(\widetilde{X}, \widetilde{\mathcal{A}}, \widetilde{\mu})$ is called a soft measure space.

Based on all the above, the question arises whether it is always necessary to have a soft σ-algebra to define a soft measure? The paper [17] discusses the idea of first defining a suitable $[0, \infty]$-valued function on some set $\mathcal{E}$ that is not a soft σ-algebra and then extending this function to a soft measure on the soft σ-algebra generated by $\mathcal{E}$. Useful structures arising in the soft set theory, during the consideration of this problem, are soft rings and soft semirings.

Definition 2.17. *A collection $\widetilde{\mathcal{S}}$ of soft subsets of $\widetilde{X}$ is called a soft semiring on $\widetilde{X}$ if, and only if, it satisfies the following conditions:*

- $\Phi \in \widetilde{\mathcal{S}}$,
- *if* $F_A, G_B \in \widetilde{\mathcal{S}}$, *then* $F_A \sqcap G_B \in \widetilde{\mathcal{S}}$,
- *if* $F_A, G_B \in \widetilde{\mathcal{S}}$, *then there exist soft disjoint* $(C_H)_1, \ldots, (C_H)_n \in \widetilde{\mathcal{S}}$, $n \in \mathbb{N}$, *such that*

$$F_A \backslash G_B = \bigsqcup_{i=1}^{n}(C_H)_i.$$

Definition 2.18. *A collection $\widetilde{\mathcal{R}}$ of soft subsets of $\widetilde{X}$ is called a soft ring on $\widetilde{X}$ if, and only if, it satisfies the following conditions:*

- $\Phi \in \widetilde{\mathcal{R}}$,
- *if* $F_A, G_B \in \widetilde{\mathcal{R}}$, *then* $F_A \sqcup G_B \in \widetilde{\mathcal{R}}$,
- *if* $F_A, G_B \in \widetilde{\mathcal{R}}$, *then* $F_A \backslash G_B \in \widetilde{\mathcal{R}}$.

Definition 2.19. *The smallest soft ring $\rho(\widetilde{\mathcal{R}})$ containing some soft collection $\widetilde{\mathcal{E}}$ of soft subsets of $\widetilde{X}$ is called the soft ring generated by $\widetilde{\mathcal{E}}$.*

Theorem 2.20. *If $\widetilde{\mathcal{S}}$ is a soft semiring on $\widetilde{X}$ and $\rho(\widetilde{\mathcal{S}}) = \{\bigsqcup_{i=1}^{n}(F_A)_i \mid n \in \mathbb{N}, (F_A)_1, \ldots, (F_A)_n \in \widetilde{\mathcal{S}}$ soft disjoint$\}$, then $\rho(\widetilde{\mathcal{S}})$ is a soft ring.*

In the paper [17], besides considering the terms of soft measure and soft outer measure, the terms of soft content and soft premeasure are defined. Also, some claims that are necessary for us to realize our idea have been proven.

Definition 2.21. *[17] Let $\widetilde{\mathcal{S}}$ be a soft semiring of soft subsets of a set $\widetilde{X}$ and $\widetilde{\nu}$ be an extended soft real-valued mapping on $\widetilde{\mathcal{S}}$. Then $\widetilde{\nu}$ is called a soft content on $\widetilde{\mathcal{S}}$, if*

- $\widetilde{\nu}(\Phi) = 0$,
- $\widetilde{\nu}$ *is finitely soft additive, i.e., if* $n \in \mathbb{N}$ *and* $(F_A)_1, (F_A)_2, \dots, (F_A)_n \in \widetilde{\mathcal{S}}$ *are soft disjoint soft sets such that* $\bigsqcup_{i=1}^{n} (F_A)_i \in \widetilde{\mathcal{S}}$, *then*

$$\widetilde{\nu}\left(\bigsqcup_{i=1}^{n} (F_A)_i\right) = \sum_{i=1}^{n} \widetilde{\nu}((F_A)_i).$$

The soft content $\widetilde{\nu}$ *is called soft finite or soft bounded if, and only if,* $\widetilde{\nu}(F_A) < \infty$ *for each* $F_A \in \widetilde{\mathcal{S}}$; *it is called soft* σ*-finite if, and only if, there exists a sequence* $((F_A)_i)_{i\in\mathbb{N}}$ *in* $\widetilde{\mathcal{S}}$ *such that* $\widetilde{X} = \bigsqcup_{i=1}^{\infty} (F_A)_i$ *and* $\widetilde{\nu}((F_A)_i) < \infty$ *for each* $i \in \mathbb{N}$.

Definition 2.22. *[17] A soft content* $\widetilde{\nu}$ *is called a soft premeasure if, and only if, it is countably soft additive (also called soft* σ*-additive), i.e., if* $((F_A)_i)_{i\in\mathbb{N}}$ *is a sequence in* $\widetilde{\mathcal{S}}$ *consisting of soft disjoint soft sets such that* $\bigsqcup_{i=1}^{\infty} (F_A)_i \in \widetilde{\mathcal{S}}$, *then*

$$\widetilde{\nu}\left(\bigsqcup_{i=1}^{\infty} (F_A)_i\right) = \sum_{i=1}^{\infty} \widetilde{\nu}((F_A)_i).$$

Theorem 2.23. *[17] Let* $\widetilde{\mathcal{S}}$ *be a soft semiring of soft subsets of a set* $\widetilde{X}$ *and let* $\widetilde{\mu} : \widetilde{\mathcal{S}} \to [0, \infty]$ *be a soft content. Furthermore, let* $\widetilde{\mathcal{R}} = \rho(\widetilde{\mathcal{S}})$. *Then there is a unique extension* $\widetilde{\nu} : \widetilde{\mathcal{R}} \to [0, \infty]$ *of* $\widetilde{\mu}$ *such that* $\widetilde{\nu}$ *is a soft content. Moreover* $\widetilde{\nu}$ *is a soft premeasure if, and only if,* $\widetilde{\mu}$ *is a soft premeasure.*

3. Soft outer measures and Carathéodory extension Theorem

The main idea of this paper is to construct a soft measure by extending a soft premeasure. An important tool for this extension is the mapping introduced in [13], called soft outer measure. The paper [17] shows in what way, starting from a soft outer measure, we can get a soft measure on the corresponding soft σ-algebra.

Definition 3.1. *[13] A non-negative soft extended real-valued set function* $\widetilde{\mu}^*$ *defined on* $\mathcal{P}(\widetilde{X})$ *is called a soft outer measure, if* $\widetilde{\mu}^*$ *satisfies the following conditions (1) - (3):*

1. $\widetilde{\mu}^*(\Phi) = 0$.
2. $\widetilde{\mu}^*$ *is soft monotone, i.e.,* $F_A \sqsubseteq G_B \Rightarrow \widetilde{\mu}^*(F_A) \leqslant \widetilde{\mu}^*(G_B)$.
3. $\widetilde{\mu}^*$ *is countably soft sub-additive (also called* σ*-subadditive), i.e.,*

$$\widetilde{\mu}^*\left(\bigsqcup_{i=1}^{\infty} (F_A)_i\right) \leqslant \sum_{i=1}^{\infty} \widetilde{\mu}^*((F_A)_i).$$

Definition 3.2. *We call* $F_A \in \mathcal{P}(\widetilde{X})$ *soft* $\widetilde{\mu}^*$*-measurable if, and only if,*

$$(\forall Q_P \in \mathcal{P}(\widetilde{X}))\widetilde{\mu}^*(Q_P) \geqslant \widetilde{\mu}^*(Q_P \sqcap F_A) + \widetilde{\mu}^*(Q_P \sqcap F_A^c). \tag{1}$$

Theorem 3.3. *[17] Let* $\widetilde{\mu}^*$ *be a soft outer measure. Define*

$$\mathcal{A}_{\widetilde{\mu}^*} = \{F_A \sqsubseteq \widetilde{X} \mid F_A \text{ is soft } \widetilde{\mu}^*\text{-measurable}\}.$$

Then $\mathcal{A}_{\widetilde{\mu}^*}$ *is a soft* σ*-algebra and* $\widetilde{\mu}^* \upharpoonright_{\mathcal{A}_{\widetilde{\mu}^*}}$ *is a soft measure.*

The following theorem gives us a usefull metod for constructing a soft outer measure. We will use this method in the construction of soft outer measure from a given soft premeasure on a given soft semiring. The obtained soft outher measure can be easily extended to the measure using the Theorem 3.3.

Theorem 3.4. *Let $\widetilde{X}$ be a soft set, $\widetilde{\mathcal{S}} \sqsubseteq \mathcal{P}(\widetilde{X})$ with $\Phi \in \widetilde{\mathcal{S}}$, and $\widetilde{\mu} : \widetilde{\mathcal{S}} \to [0,\infty]$ with $\widetilde{\mu}(\Phi) = 0$. Then $\widetilde{\mu}^* : \mathcal{P}(\widetilde{X}) \to [0,\infty]$,*

$$\widetilde{\mu}^*(F_A) = \inf\left\{\sum_{i=1}^{\infty} \widetilde{\mu}((F_A)_i) \mid ((F_A)_i)_{i\in\mathbb{N}} \text{ sequence in } \widetilde{\mathcal{S}}, F_A \sqsubseteq \bigsqcup_{i=1}^{\infty}(F_A)_i\right\}, \ \inf\emptyset = \infty \tag{2}$$

defines a soft outer measure on $\widetilde{X}$. This soft outer measure is called the soft outer measure induced by $\widetilde{\mu}$.

Proof. Setting $(F_A)_i = \Phi$ for each $i \in \mathbb{N}$ in (2) shows $\widetilde{\mu}^*(\Phi) = 0$.

If $F_A \sqsubseteq G_B \sqsubseteq \widetilde{X}$ and $((F_A)_i)_{i\in\mathbb{N}}$ is a sequence in $\widetilde{\mathcal{S}}$ with $G_B \sqsubseteq \bigsqcup_{i=1}^{\infty}(F_A)_i$, then $F_A \sqsubseteq \bigsqcup_{i=1}^{\infty}(F_A)_i$, showing $\widetilde{\mu}^*(F_A) \leqslant \widetilde{\mu}^*(G_B)$.

Only the soft σ-subadditivity of $\widetilde{\mu}^*$ requires slightly more effort: Let $((F_A)_i)_{i\in\mathbb{N}}$ be a sequence in $\mathcal{P}(\widetilde{X})$ and $F_A = \bigsqcup_{i=1}^{\infty}(F_A)_i$. We need to show

$$\widetilde{\mu}^*(F_A) \leqslant \sum_{i=1}^{\infty} \widetilde{\mu}^*((F_A)_i). \tag{3}$$

Since (3) trivially holds if, for at least one $i \in \mathbb{N}$, $\widetilde{\mu}^*((F_A)_i) = \infty$, we may assume $\widetilde{\mu}^*((F_A)_i) < \infty$ for each $i \in \mathbb{N}$. Let $\varepsilon \in \mathbb{R}^+$. Then, for each $i \in \mathbb{N}$, there exists a sequence $((G_B)_{ik})_{k\in\mathbb{N}}$ in $\widetilde{\mathcal{S}}$ such that

$$(F_A)_i \sqsubseteq \bigsqcup_{k=1}^{\infty}(G_B)_{ik} \quad \text{and} \quad \sum_{k=1}^{\infty} \widetilde{\mu}((G_B)_{ik}) < \widetilde{\mu}^*((F_A)_i) + \varepsilon \cdot 2^{-i}.$$

Since $((G_B)_{ik})_{(i,k)\in\mathbb{N}^2}$ is a countable family (i.e. a sequence) in $\widetilde{\mathcal{S}}$ with $F_A \sqsubseteq \bigsqcup_{(i,k)\in\mathbb{N}^2}(G_B)_{ik}$,

$$\widetilde{\mu}^*(F_A) \leqslant \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} \widetilde{\mu}((G_B)_{ik}) \leqslant \sum_{i=1}^{\infty}(\widetilde{\mu}^*((F_A)_i) + \varepsilon \cdot 2^{-i}) = \varepsilon + \sum_{i=1}^{\infty} \widetilde{\mu}^*((F_A)_i)$$

implies (3) and establishes the case. □

The following Lemma describes some properties of soft content and soft premeasure on a given soft ring.

Lemma 3.5. *Let $\widetilde{\mathcal{R}}$ be a soft ring of soft subsets over $\widetilde{X}$ and let $\widetilde{\mu} : \widetilde{\mathcal{R}} \to [0,\infty]$ be a soft content. Then the following rules hold, where we assume $F_A \in \widetilde{\mathcal{R}}$ as well as $(F_A)_i \in \widetilde{\mathcal{R}}$ for each $i \in \mathbb{N}$.*

(a) If the $(F_A)_i$ are soft disjoint and $\bigsqcup_{i=1}^{\infty}(F_A)_i \sqsubseteq F_A$, then

$$\sum_{i=1}^{\infty} \widetilde{\mu}((F_A)_i) \leqslant \widetilde{\mu}(F_A);$$

(b) Soft subadditivity: For each $n \in \mathbb{N}$, one has

$$\widetilde{\mu}\left(\bigsqcup_{i=1}^{n}(F_A)_i\right) \leqslant \sum_{i=1}^{n} \widetilde{\mu}((F_A)_i);$$

(c) If $\widetilde{\mu}$ is a soft premeasure, then one also has soft σ-subadditivity: If $F_A \sqsubseteq \bigsqcup_{i=1}^{\infty}(F_A)_i$, then

$$\widetilde{\mu}(F_A) \leqslant \sum_{i=1}^{\infty} \widetilde{\mu}((F_A)_i).$$

Proof. (a) This part follows from the property of finite soft additivity and soft monotonicity. As $\bigsqcup_{i=1}^{n}(F_A)_i \sqsubseteq F_A$, for all $n \in \mathbb{N}$. then we have that

$$\widetilde{\mu}(F_A) = \widetilde{\mu}\left(\bigsqcup_{i=1}^{n}(F_A)_i\right) + \widetilde{\mu}\left(F_A \setminus \left(\bigsqcup_{i=1}^{n}(F_A)_i\right)\right) \geqslant \widetilde{\mu}\left(\bigsqcup_{i=1}^{n}(F_A)_i\right).$$

As $(F_A)_i$ are soft disjoint, then it is

$$\widetilde{\mu}(F_A) \geqslant \sum_{i=1}^{n} \widetilde{\mu}((F_A)_i), \text{ for all } n \in \mathbb{N}.$$

From the previously obtained, we have that $\widetilde{\mu}(F_A) \geqslant \sum_{i=1}^{\infty} \widetilde{\mu}((F_A)_i)$.

(b) Define $(G_A)_i := (F_A)_i \setminus \bigsqcup_{j=1}^{i-1}(G_A)_j$, for all $i = 1, \ldots, n$. Then, $(G_A)_i$ are soft disjoint, $\bigsqcup_{j=1}^{n}(F_A)_j = \bigsqcup_{j=1}^{n}(G_A)_j$ and $(G_A)_i \sqsubseteq (F_A)_i$, for all $i = 1, \ldots, n$. Hence, using soft monotonicity and definition of soft content we have that

$$\widetilde{\mu}\left(\bigsqcup_{i=1}^{n}(F_A)_i\right) = \widetilde{\mu}\left(\bigsqcup_{i=1}^{n}(G_A)_i\right) = \sum_{i=1}^{n} \widetilde{\mu}((G_A)_i) \leqslant \sum_{i=1}^{n} \widetilde{\mu}((F_A)_i).$$

(c) It follows from Definiton 2.22. applied to the construction of soft sets described in part (b) and soft monotonicity, i.e.,

$$\widetilde{\mu}(F_A) \leqslant \widetilde{\mu}\left(\bigsqcup_{i=1}^{\infty}(F_A)_i\right) \leqslant \sum_{i=1}^{\infty} \widetilde{\mu}((F_A)_i). \quad \square$$

The main result of this paper is to extend soft premeasure to soft measure.

Theorem 3.6. *(Soft Carathéodory extension Theorem) Let $\widetilde{\mathcal{S}}$ be a soft semiring on the soft set $\widetilde{X}$ and let $\widetilde{\mu} : \widetilde{\mathcal{S}} \to [0, \infty]$ be a soft content. If $\widetilde{\mu}^*$ is defined by* (2)*, then the following holds:*

- *(a) $\widetilde{\mu}^*$ is a soft outer measure and each $F_A \in \widetilde{\mathcal{S}}$ is soft $\widetilde{\mu}^*$-measurable (i.e. $\widetilde{\mathcal{S}} \sqsubseteq \sigma(\widetilde{\mathcal{S}}) \sqsubseteq \mathcal{A}_{\widetilde{\mu}^*}$);*
- *(b) If $\widetilde{\mu}$ is a soft premeasure, then $\widetilde{\mu}^* \restriction_{\widetilde{\mathcal{S}}} = \widetilde{\mu}$ (thus, in this case, $\widetilde{\mu}^* \restriction_{\mathcal{A}_{\widetilde{\mu}^*}}$ is a soft measure that extends the soft premeasure $\widetilde{\mu}$ to a soft σ-algebra containing $\sigma(\widetilde{\mathcal{S}})$);*
- *(c) If $\widetilde{\mu}$ is not a soft premeasure, then $(\exists F_A \in \widetilde{\mathcal{S}})\widetilde{\mu}^*(F_A) < \widetilde{\mu}(F_A)$.*

Proof. (a) According to Theorem 3.4., $\widetilde{\mu}^*$ is a soft outer measure. It remains to show that each $F_A \in \widetilde{\mathcal{S}}$ is soft $\widetilde{\mu}^*$-measurable, i.e. we have to verify (1), given some $F_A \in \widetilde{\mathcal{S}}$. It will be convenient to extend the soft content $\widetilde{\mu}$ to a soft content on $\widetilde{\mathcal{R}} = \rho(\widetilde{\mathcal{S}})$ via Theorem 2.23. (we still call the soft extension $\widetilde{\mu}$). Then (2) and Theorem 2.20. imply

$$(\forall Q_P \in \mathcal{P}(\widetilde{X}))\ \widetilde{\mu}^*(Q_P) = \inf\left\{\sum_{i=1}^{\infty} \widetilde{\mu}((F_A)_i) \mid ((F_A)_i)_{i\in\mathbb{N}} \text{ sequence in } \widetilde{\mathcal{R}}, Q_P \sqsubseteq \bigsqcup_{i=1}^{\infty}(F_A)_i\right\}. \tag{4}$$

To verify (1), let $Q_P \in \mathcal{P}(\widetilde{X})$. As (1) always holds if $\widetilde{\mu}^*(Q_P) = \infty$, we assume $\widetilde{\mu}^*(Q_P) < \infty$. For each sequence $((F_A)_i)_{i\in\mathbb{N}}$ in $\widetilde{\mathcal{S}}$ with $Q_P \sqsubseteq \bigsqcup_{i=1}^{\infty}(F_A)_i$ (where $\widetilde{\mu}^*(Q_P) < \infty$ guarantees such a sequence exists), the following must hold (using that $\widetilde{\mu}$ is a soft content on $\widetilde{\mathcal{R}}$):

$$\sum_{i=1}^{\infty} \widetilde{\mu}((F_A)_i) = \sum_{i=1}^{\infty} \widetilde{\mu}((F_A)_i \sqcap F_A) + \sum_{i=1}^{\infty} \widetilde{\mu}((F_A)_i \setminus F_A) \geqslant \widetilde{\mu}^*(Q_P \sqcap F_A) + \widetilde{\mu}^*(Q_P \sqcap F_A^c). \tag{5}$$

Now taking the infimum in (5) over all admissible sequences $((F_A)_i)_{i\in\mathbb{N}}$ in $\widetilde{\mathcal{S}}$ according to (2), we obtain $\widetilde{\mu}^*(Q_P) \geqslant \widetilde{\mu}^*(Q_P \sqcap F_A) + \widetilde{\mu}^*(Q_P \sqcap F_A^c)$, proving $\widetilde{\mathcal{S}} \sqsubseteq \mathcal{A}_{\widetilde{\mu}^*}$ as claimed.

(b) The inequality $\widetilde{\mu}^* \upharpoonright_{\widetilde{\mathcal{S}}} \leqslant \widetilde{\mu}$ is immediate from (2). It remains to show $\widetilde{\mu}^* \upharpoonright_{\widetilde{\mathcal{S}}} \geqslant \widetilde{\mu}$. We know from Theorem 2.23. that, if $\widetilde{\mu}$ is a soft premeasure on $\widetilde{\mathcal{S}}$, then the extended $\widetilde{\mu}$ is a soft premeasure on $\widetilde{\mathcal{R}}$. Thus, if $((F_A)_i)_{i\in\mathbb{N}}$ is a sequence in $\widetilde{\mathcal{R}}$ such that $Q_P \sqsubseteq \bigsqcup_{i=1}^{\infty}(F_A)_i$, $Q_P \in \widetilde{\mathcal{S}}$, then, by Lemma 3.5.(c), $\widetilde{\mu}(Q_P) \leqslant \sum_{i=1}^{\infty} \widetilde{\mu}((F_A)_i)$. Taking the infimum over all admissible sequences according to (4) yields $\widetilde{\mu}(Q_P) \leqslant \widetilde{\mu}^*(Q_P)$ and $\widetilde{\mu} \leqslant \widetilde{\mu}^* \upharpoonright_{\widetilde{\mathcal{S}}}$ as needed.

(c) If $\widetilde{\mu}$ is not a soft premeasure on $\widetilde{\mathcal{S}}$, then there exists $F_A \in \widetilde{\mathcal{S}}$ and a sequence $((F_A)_i)_{i\in\mathbb{N}}$ of disjoint soft sets in $\widetilde{\mathcal{S}}$ such that $F_A = \bigsqcup_{i=1}^{\infty}(F_A)_i$ and $\widetilde{\mu}(F_A) \neq \sum_{i=1}^{\infty} \widetilde{\mu}((F_A)_i)$. Due to Lemma 3.5.(a) (applied to $\widetilde{\mu}$ on $\widetilde{\mathcal{R}}$), $\sum_{i=1}^{\infty} \widetilde{\mu}((F_A)_i) \leqslant \widetilde{\mu}(F_A)$. Thus,

$$\widetilde{\mu}^*(F_A) \leqslant \sum_{i=1}^{\infty} \widetilde{\mu}((F_A)_i) < \widetilde{\mu}(F_A)$$

which establishes the case. □

Example 3.7. *Let $X = \{h_1, h_2, h_3\}$ and $E = \{e_1, e_2\}$. Collection $\widetilde{\mathcal{S}} = \{(F_A)_1, (F_A)_2, (F_A)_3, (F_A)_4\}$, where*
$(F_A)_1 = \Phi$,
$(F_A)_2 = \{(e_1, \{h_1\}), (e_2, \{h_2\})\}$,
$(F_A)_3 = \{(e_1, \{h_2\}), (e_2, \{h_1\})\}$ *i*
$(F_A)_4 = \{(e_1, \{h_3\}), (e_2, \{h_3\})\}$,
is one soft semipring on $\widetilde{X}$.
Notice the mapping $\widetilde{\mu} : \widetilde{\mathcal{S}} \to [0, \infty]$ be defined as

$$\widetilde{\mu}(F_A) = \begin{cases} 0, & F_A = \Phi, \\ 1, & F_A \neq \Phi. \end{cases}$$

It is clear that the mapping $\widetilde{\mu}$ is a soft content, and also a soft premeasure on soft semiring $\widetilde{\mathcal{S}}$.

Now, the soft semirnig $\widetilde{\mathcal{S}}$ can be extanded to a minimal soft σ-algebra

$$\{(F_A)_1, (F_A)_2, (F_A)_3, (F_A)_4, (F_A)_5, (F_A)_6, (F_A)_7, (F_A)_8\},$$

where
$(F_A)_1 = \Phi$,
$(F_A)_2 = \{(e_1, \{h_1\}), (e_2, \{h_2\})\}$,
$(F_A)_3 = \{(e_1, \{h_2\}), (e_2, \{h_1\})\}$,
$(F_A)_4 = \{(e_1, \{h_3\}), (e_2, \{h_3\})\}$,
$(F_A)_5 = \{(e_1, \{h_2, h_3\}), (e_2, \{h_1, h_3\})\}$,
$(F_A)_6 = \{(e_1, \{h_1, h_3\}), (e_2, \{h_2, h_3\})\}$,
$(F_A)_7 = \{(e_1, \{h_1, h_2\}), (e_2, \{h_1, h_2\})\}$ *and*
$(F_A)_8 = \widetilde{X}$.
Let $\widetilde{\mu}^*$ be a mapping defined as in (2). It is clear, that it is
$\widetilde{\mu}^*((F_A)_1) = 0$, $\widetilde{\mu}^*((F_A)_2) = 1$, $\widetilde{\mu}^*((F_A)_3) = 1$, $\widetilde{\mu}^*((F_A)_4) = 1$,
$\widetilde{\mu}^*((F_A)_5) = 2$, $\widetilde{\mu}^*((F_A)_6) = 2$, $\widetilde{\mu}^*((F_A)_7) = 2$ *i* $\widetilde{\mu}^*((F_A)_8) = 3$.
Hence, $\widetilde{\mu}^* \upharpoonright_{\mathcal{A}_{\widetilde{\mu}^*}}$ is a soft measure that extends the soft premeasure $\widetilde{\mu}$ to a soft σ-algebra containing $\sigma(\widetilde{\mathcal{S}})$.

4. Conclusion

In his research, Molodsov presented several possible applications of the theory of soft sets [9]. In addition to many applications, it is important to highlight the applications in which soft measure is used. This work represents a continuation of research in the field of soft measure theory. Also, this paper represents an important basis for an even better application of the theory of soft sets. For future research, it would be interesting to examine whether the soft measure obtained by the extension of the soft premeasure on a certain collection of soft sets will be unique.

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