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Formulae for the Drazin inverse of elements in a ring

Xiaolan Qin[a], Linzhang Lu[a,b,*]

[a] *School of Mathematical Science, Guizhou Normal University, P. R. China*
[b] *School of Mathematical Science, Xiamen University, P. R. China*

Abstract. This paper studies additive properties of the Drazin inverse in a ring $\mathcal{R}$. Some necessary and sufficient conditions for the Drazin invertible are given. Furthermore, we derive additive formulae under conditions weaker than those used in some resent papers on the subject. These extend the main results of Wei and Deng (J. Linear Multilinear Algebra, 59(12) (2011) 1319-1329) and Wang et al. (Filomat, 30(2016), 1185-1193)

1. Introduction

Throughout this paper, $\mathcal{R}$ is an associative ring with an identity. $\mathcal{R}^{-1}$ denotes the set of all invertible elements in $\mathcal{R}$. λ, λ', μ and μ' always stand for nonzero complex numbers. The commutant of an element $a \in \mathcal{R}$ is defined as $\mathrm{comm}(a) = \{x \in \mathcal{R} : ax = xa\}$. Let us recall that the Drazin inverse of $a \in \mathcal{R}$ is the element $b \in \mathcal{R}$ (denoted by a^D) which satisfies the following equations [6]:

$$bab = b, \qquad ab = ba, \qquad a^k = a^{k+1}b,$$

for some positive integer k. The smallest integer k is called the Drazin index of a, denoted by $\mathrm{ind}(a)$. If $\mathrm{ind}(a) = 1$, then b is called the group inverse of a and is denoted by $a^\#$. The subset of $\mathcal{R}$ composed of Drazin invertible elements will be denote by $\mathcal{R}^D$. The conditions in the definition of Drazin inverse are equivalent to:

$$bab = b, \qquad ab = ba, \qquad a - a^2b \text{ is nilpotent.}$$

The notation a^π means $1 - aa^D$ for any Drazin invertible element $a \in \mathcal{R}$. Observe that by the definition of the Drazin inverse, $aa^\pi = a^\pi a$ is nilpotent.

The Drazin inverse has applications in a number of areas such as singular linear systems [23], the theory of finite Markov chains [11, 15, 16], numerical analysis [8, 10, 14, 20, 22] and so on [1, 2]. Drazin first studied the Drazin inverse of the sum of two Drazin invertible elements in a ring in his ccelebrated paper in [6]. In this paper, Drazin was able to deduce a formula for the Drazin inverse of $a + b$ when $ab = ba = 0$. The general question of how to express the Drazin inverse of $a + b$ as a function of a, b and the Drazin inverses

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* Corresponding author: Linzhang Lu
Email addresses: qinxiaolan18@163.com (Xiaolan Qin), llz@gznu.edu.cn, lzlu@xmu.edu.cn (Linzhang Lu)

of a and b without side conditions, is very difficult and remains open. Hartwig et al. [9], expressed $(A+B)^D$ under the one-side condition $AB = 0$, where A, B are complex square matrices. This result was extended to bounded linear operators on an arbitrary complex Banach space by Djordjević and Wei in [7]. Later, it was extended for morphisms on arbitrary additive categories by Chen at al. in [5]. More related results can be found in [3, 12, 13, 17, 18, 24].

The motivation for this article was the result in Wang et al. [19], Wei and Deng [21] and Zhuang et al. [25]. In [19], the authors proved that $a-b \in \mathcal{R}^D$ if and only if $aa^D(a-b)bb^D \in \mathcal{R}^D$, if $ab = \lambda ba$ for some nonzero complex λ. In [21], the authors considered the relations between the Drazin inverse of $A+B$ and $1+A^DB$ for two commutative complex matrices A and B. In [25], Zhuang et al. extended the result in [21] to a ring $\mathcal{R}$, and it was proved that $a+b \in \mathcal{R}^D$ if and only if $1+a^Db \in \mathcal{R}^D$. In this paper, our main contributions are to generalize the results of [19] and [21, 25] for the Drazin inverse $(ab)^D$ and $(a \pm b)^D$ under the weaker conditions.

The paper is organized as follows. In section 2, we will deduce some lemmas. In section 3, we investigate Drazin invertibility of the product of $a, b \in \mathcal{R}^D$ which will be repeatedly used in the sequel. Then we characterize the relations of $a-b$, $aa^D(a-b)$, $(a-b)bb^D$ and $aa^D(a-b)bb^D$. In section 4, we introduce some new conditions and give the explicit expressions for $(a \pm b)^D$.

2. Preliminaries

The following lemmas are required in what follows.

Lemma 2.1. *Let $a, b \in \mathcal{R}^D$. If $a^2b = \lambda ba^2$, then $aa^Db = baa^D$.*

Proof. Assume k=ind(a). Let $p = aa^D$. By hypothesis, we have

$$\begin{aligned} pb - pbp &= (a^D)^{2k}a^{2k}b(1-aa^D) = (a^D)^{2k}a^{2(k-1)}a^2b(1-aa^D) \\ &= \lambda(a^D)^{2k}a^{2(k-1)}ba^2(1-aa^D) = \cdots = \lambda^k(a^D)^{2k}ba^{2k}(1-aa^D) \\ &= \lambda^k(a^D)^{2k}ba^{2k}(1-aa^D)^2 = \lambda^k(a^D)^{2k}b(a^k - a^{k+1}a^D)^2. \end{aligned}$$

From the definition of the Drazin inverse, we obtain $a^k - a^{k+1}a^D = 0$, and so $pb - pbp = 0$. Hence $pb = pbp$. Likewise, $bp = pbp$. Accordingly, $aa^Db = pb = bp = baa^D$, as desired. □

Analogously to Lemma 2.1, we have the following result.

Lemma 2.2. *Let $a, b \in \mathcal{R}^D$. If $b^2a = \mu ab^2$, then $bb^Da = abb^D$.*

Lemma 2.3. *Let $a, b \in \mathcal{R}$ with $aba = \lambda a^2b$. Then for any positive integer i, the following hold:*

$$(1)\ a^{i+1}b = \lambda^{-1}a^iba = \lambda^{-i}aba^i. \tag{2.1}$$

$$(2)\ (ab)^i = \lambda^{\frac{(i-1)i}{2}}a^ib^i. \tag{2.2}$$

Proof. (1) From $aba = \lambda a^2b$, we have

$$a^{i+1}b = a^{i-1}a^2b = \lambda^{-1}a^{i-1}aba = \lambda^{-1}a^iba.$$

Also, we have that

$$\begin{aligned} a^{i+1}b &= \lambda^{-1}a^iba = \lambda^{-1}a^{i-2}a^2ba \\ &= \lambda^{-2}a^{i-2}aba^2 = \lambda^{-2}a^{i-3}a^2ba^2 \\ &= \lambda^{-3}a^{i-3}aba^3 = \cdots = \lambda^{-i}aba^i. \end{aligned}$$

(2) Under the assumption of this lemma, we have

$$(ab)^i = (ab)^{i-2}abab = \lambda(ab)^{i-2}a^2b^2 = \lambda^{1+2}(ab)^{i-3}a^3b^3$$
$$= \cdots = \lambda^{\sum_{k=0}^{k=i-1} k}a^ib^i = \lambda^{\frac{(i-1)i}{2}}a^ib^i.$$

□

Lemma 2.4. *Let $a, b \in \mathcal{R}^D$. If $aba = \lambda a^2b = \lambda' ba^2$ and $bab = \mu b^2a = \mu' ab^2$, then*

(1) $aba^D = \lambda^{-1}a^Dab$. (2.3)

(2) $bab^D = \mu^{-1}b^Dba$.

Proof. Assume $k = max\{ind(a), ind(b)\}$.

(1) By hypotheses, we get

$$\begin{aligned} a^D(a^{k+1}b) &\overset{(2.1)}{=} a^D(\lambda^{-k}aba^k) = \lambda^{-k}a^D(aba^{k+1})a^D \\ &\overset{(2.1)}{=} \lambda^{-k}a^D(\lambda^{k+1}a^{k+1}ab)a^D = \lambda(a^Da^{k+1})aba^D \\ &= \lambda a^kaba^D = \lambda a^{k+1}ba^D. \end{aligned}$$

It follows that

$$\begin{aligned} a^Dab &= (a^D)^{k+1}a^kab = (a^D)^k(a^Da^{k+1}b) = \lambda(a^D)^ka^{k+1}ba^D \\ &= \lambda(a^D)^{k-1}(a^Da^{k+1}b)a^D = \lambda^2(a^D)^{k-1}a^{k+1}b(a^D)^2 \\ &= \cdots = \lambda^{k+1}(a^{k+1}b)(a^D)^{k+1} \\ &\overset{(2.1)}{=} \lambda^{k+1}(\lambda^{-k}aba^k)(a^D)^{k+1} \\ &= \lambda aba^D. \end{aligned}$$

(2) The proof is similar to (1). □

Lemma 2.5. *Let $a, b \in \mathcal{R}^D$. If $aba = \lambda a^2b = \lambda' ba^2$ and $bab = \mu b^2a = \mu' ab^2$, then*

(1) $ba^Db = \lambda b^2a^D$.

(2) $ab^Da = \mu a^2b^D$.

Proof. It is enough to prove (1) since we can obtain (2) by the symmetry of a and b.

(1) By hypotheses, we obtain $ab(a^D)^2 = (aba^D)a^D \overset{(2.3)}{=} \lambda^{-1}a^D(aba^D) \overset{(2.3)}{=} \lambda^{-2}(a^D)^2ab$. Since $\lambda a^2b = \lambda' ba^2$ implies that $aa^Db = baa^D$ by Lemma 2.1, it follows that

$$\begin{aligned} ba^Db &= b(a^D)^2ab = \lambda^2bab(a^D)^2 = \lambda^2b(aba^D)a^D \\ &\overset{(2.3)}{=} \lambda^2b(\lambda^{-1}a^Dab)a^D = \lambda bba^Daa^D = \lambda b^2a^D. \end{aligned}$$

□

3. Main result 1

Under the conditions $aba = \lambda a^2b = \lambda' ba^2$ and $bab = \mu b^2a = \mu' ab^2$, Chen and Sheibani [4] considered the relations of $a + b$, $(a + b)bb^D$, $aa^D(a + b)$ and $aa^D(a + b)bb^D$ in a Banach algebra, but they did not deduce the formulae of $(a + b)^D$, $\left[aa^D(a + b)\right]^D$, $\left[(a + b)bb^D\right]^D$ and $\left[aa^D(a + b)bb^D\right]^D$. In this section, we extend the results in [4] to a ring case. Moreover, we give explicit representations of $(a - b)^D$, $\left[aa^D(a - b)\right]^D$, $\left[(a - b)bb^D\right]^D$ and $\left[aa^D(a - b)bb^D\right]^D$.

First, we start with a theorem that is an extension of [19, Lemma 2.2 (3)].

Theorem 3.1. *Let $a, b \in \mathcal{R}^D$. If $aba = \lambda a^2 b = \lambda' ba^2$ and $bab = \mu b^2 a = \mu' ab^2$, then $ab \in \mathcal{R}^D$ and*

$$(ab)^D = b^D a^D = \mu^{-1} a^D b^D.$$

Proof. Let $x = b^D a^D$. We will prove that $ab \in \mathcal{R}^D$ and the Drazin inverse of ab is x, i.e., $(ab)x = x(ab)$, $x(ab)x = x$, and $(ab)^k = (ab)^{k+1}x$ for some positive integer k.

First we give some useful equalities

$$\begin{aligned}(ab)^i a^D &= (ab)^{i-1} aba^D \overset{(2.3)}{=} \lambda^{-1}(ab)^{i-1} a^D ab = \lambda^{-1}(ab)^{i-2} aba^D ab \\ &\overset{(2.3)}{=} \lambda^{-2}(ab)^{i-2} a^D (ab)^2 = \cdots = \lambda^{-i} a^D (ab)^i,\end{aligned} \tag{3.1}$$

so we have

$$a^i b^i a^D \overset{(2.2)}{=} \lambda^{-\frac{(i-1)i}{2}} \left[(ab)^i a^D\right] \overset{(3.1)}{=} \lambda^{-\frac{(i-1)i}{2}} \left[\lambda^{-i} a^D (ab)^i\right] = \lambda^{-\frac{i(i+1)}{2}} a^D (ab)^i. \tag{3.2}$$

Note that $\lambda a^2 b = \lambda' ba^2$ and $\mu b^2 a = \mu' ab^2$ imply that $aa^D b = baa^D$ and $bb^D a = abb^D$ by Lemma 2.1 and Lemma 2.2.

(1) It can easily be verified that $(ab)x = (abb^D)a^D = b^D(baa^D) = b^D a^D ab = x(ab)$.
(2) We easily find that $x(ab)x = b^D(a^D ab)b^D a^D = bb^D(a^D ab^D)a^D = b^D bb^D a^D aa^D = b^D a^D = x$.
(3) Take $k = max\{ind(a), ind(b)\}$. From the definition of the Drazin inverse, we have that

$$\begin{aligned}(ab)^{k+1}x &= (ab)^{k+1} b^D a^D \overset{(2.2)}{=} \lambda^{\frac{k(k+1)}{2}} a^{k+1}(b^{k+1} b^D) a^D = \lambda^{\frac{k(k+1)}{2}} a^{k+1} b^k a^D \\ &= \lambda^{\frac{k(k+1)}{2}} a(a^k b^k a^D) \overset{(3.2)}{=} \lambda^{\frac{k(k+1)}{2}} a\left[\lambda^{-\frac{k(k+1)}{2}} a^D (ab)^k\right] \\ &= aa^D (ab)^k \overset{(2.2)}{=} \lambda^{\frac{(k-1)k}{2}} a^D a^{k+1} b^k = \lambda^{\frac{(k-1)k}{2}} a^k b^k \\ &\overset{(2.2)}{=} \lambda^{\frac{(k-1)k}{2}} \left[\lambda^{\frac{-(k-1)k}{2}} (ab)^k\right] = (ab)^k.\end{aligned}$$

Hence, $(ab)^D = b^D a^D$. Similarly, we can check that $(ab)^D = \mu^{-1} a^D b^D$. □

Remark 3.2. *Note that the conditions given in Theorem 3.1 are symmetric.*

Corollary 3.3. [19, Lemma 2.2(3)] *Let $a, b \in \mathcal{R}^D$. If $ab = \lambda ba$, then $ab \in \mathcal{R}^D$ and $(ab)^D = b^D a^D = \lambda^{-1} a^D b^D$.*

Proof. From $ab = \lambda ba$, we have $aba = \lambda^{-1} a^2 b = \lambda ba^2$ and $bab = \lambda b^2 a = \lambda^{-1} ab^2$. This completes the proof by Theorem 3.1. □

In 2017, Zhu and Chen [24] assumed the following two conditions in $a, b \in \mathcal{R}^D$,

$$a^2 b = aba,\ b^2 a = bab. \tag{3.3}$$

We observe that the conditions (3.3) and $aba = \lambda a^2 b = \lambda' ba^2$, $bab = \mu b^2 a = \mu' ab^2$ are independent. Precisely, in the first example, the conditions (3.3) hold, but the conditions $aba = \lambda a^2 b = \lambda' ba^2$, $bab = \mu b^2 a = \mu' ab^2$ are not applicable.

Example 3.4. *Let $\mathcal{R} = M_2(R)$, and take*

$$a = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},\ b = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{R}^D.$$

Since $aba \neq \lambda' ba^2$ and $bab \neq \mu' ab^2$, we know that the conditions of Theorem 3.1 do not hold. On the other hand, we can observe that $a^2b = aba$ and $b^2a = bab$. By [24, Theorem 3.1], $(ab)^D = a^D b^D$.

From computing, we get $a^D = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $b^D = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 0 \end{bmatrix}$, $a^D b^D = \begin{bmatrix} \frac{1}{3} & 0 \\ \frac{1}{3} & 0 \end{bmatrix}$ and $(ab)^D = \begin{bmatrix} \frac{1}{3} & 0 \\ \frac{1}{3} & 0 \end{bmatrix} = a^D b^D$. But $b^D a^D = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 0 \end{bmatrix} \neq (ab)^D$.

Also, we construct matrices a and b such that $aba = \lambda a^2 b = \lambda' ba^2$, $bab = \mu b^2 a = \mu' ab^2$ are met, but (3.3) are not satisfied in the next example.

Example 3.5. *Let $\mathcal{R} = M_2(\mathbb{C})$, and take*

$$a = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix},\ b = \begin{bmatrix} 0 & 2i \\ 2i & 0 \end{bmatrix} \in \mathcal{R}^D.$$

After calculating, we get that $aba = -a^2b = -ba^2$, $bab = -b^2a = -ab^2$. Hence, the conditions of Theorem 3.1 hold. However, $aba \neq a^2b$, $bab \neq b^2a$.

By elementary computations, we obtain $a^D b^D = \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}$, $b^D a^D = \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$.

By Theorem 3.1, $(ab)^D = b^D a^D = \mu^{-1} a^D b^D$. Here $(ab)^D = \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} = b^D a^D = -a^D b^D$.

Now we start the first of our main theorems, which extends the result under the condition $ab = \lambda ba$ in [19, Theorem 2.3].

Theorem 3.6. *Let $a, b \in \mathcal{R}^D$ be such that $aba = \lambda a^2 b = \lambda' ba^2$ and $bab = \mu b^2 a = \mu' ab^2$. Then the following conditions are equivalent:*

(1) $a - b \in \mathcal{R}^D$.
(2) $c = aa^D(a - b) \in \mathcal{R}^D$.
(3) $e = (a - b)bb^D \in \mathcal{R}^D$.
(4) $w = aa^D(a - b)bb^D \in \mathcal{R}^D$.

In this case,

$$(a - b)^D = c^D - a^\pi(1 - b^D aa^\pi)^{-1} b^D, \tag{3.4}$$

$$(a - b)^D = e^D + a^D(1 - bb^\pi a^D)^{-1} b^\pi, \tag{3.5}$$

$$(a - b)^D = w^D + a^D(1 - bb^\pi a^D)^{-1} b^\pi - a^\pi(1 - b^D aa^\pi)^{-1} b^D, \tag{3.6}$$

where $c^D = aa^D(a - b)^D$, $e^D = (a - b)^D bb^D$, $w^D = aa^D(a - b)^D bb^D$.

Proof. Recall that aa^π and bb^π are nilpotent and their indexes of nilpotency are the Drazin indexes of a and b, respectively. Let s=index(a) and t=index(b).

(1) ⇒ (2) Since $\lambda a^2 b = \lambda' ba^2$, by Lemma 2.1, we get $aa^D b = baa^D$ which implies $aa^D(a - b) = (a - b)aa^D$. Applying Corollary 3.3, we obtain $aa^D(a - b) \in \mathcal{R}^D$ and $\left[aa^D(a - b)\right]^D = aa^D(a - b)^D$.

(2) ⇒ (1) Assume that c is Drazin invertible and let us define $x = c^D - a^\pi(1 - b^D aa^\pi)^{-1} b^D$.

By Lemma 2.5(2) and $a^\pi a^s = 0$, we get $(b^D aa^\pi)^s = \mu^{\frac{(s-1)s}{2}} b^D a^\pi a^s (b^D)^{s-1} = 0$. Thus, according to [12, Lemma 1.1], we get $1 - b^D aa^\pi$ is invertible and

$$(1 - b^D aa^\pi)^{-1} = 1 + b^D aa^\pi + (b^D aa^\pi)^2 + \cdots + (b^D aa^\pi)^{s-1}.$$

Since $aa^D b = baa^D$ it follows that $a^\pi b = ba^\pi$ and $c(a - b) = (a - b)c$. From these, by [6, Theorem 1], we get $a^\pi b^D = b^D a^\pi$ and $c^D(a - b) = (a - b)c^D$.

Since $(1-b^Daa^\pi)b^D = b^D(1-aa^\pi b^D)$, combining $a^\pi b^D = b^Da^\pi$, we derive

$$\begin{aligned}
&(a-b)a^\pi(1-b^Daa^\pi)^{-1}b^D\\
&=(a-b)a^\pi b^D(1-aa^\pi b^D)^{-1}\\
&=(a-b)b^Da^\pi(1-aa^\pi b^D)^{-1}\\
&=-bb^D(1-ab^D)a^\pi(1-aa^\pi b^D)^{-1}\\
&=-bb^D(1-ab^Da^\pi-ab^Daa^D)a^\pi(1-aa^\pi b^D)^{-1}\\
&=-bb^D(1-aa^\pi b^D)a^\pi(1-aa^\pi b^D)^{-1}\\
&=-bb^Da^\pi.
\end{aligned}$$

So, we get

$$(a-b)x=(a-b)\left[c^D-a^\pi(1-b^Daa^\pi)^{-1}b^D\right]=(a-b)c^D+bb^Da^\pi.$$

Similar to the above way, we also have $\left[a^\pi(1-b^Daa^\pi)^{-1}b^D\right](a-b)=-bb^Da^\pi$. So, it follows

$$x(a-b)=\left[c^D-a^\pi(1-b^Daa^\pi)^{-1}b^D\right](a-b)=c^D(a-b)+bb^Da^\pi \tag{3.7}$$

and $x(a-b)=(a-b)x$.

Next, we give the proof of $x(a-b)x=x$. Let $x(a-b)=x'+x''$, where $x'=c^D(a-b)$ and $x''=b^Dba^\pi$.

Observe that $c+a^\pi(a-b)=aa^D(a-b)+(1-aa^D)(a-b)=a-b$. From $a^\pi c=ca^\pi=(a-b)aa^Da^\pi=0$, we get $a^\pi c^D=c^Da^\pi=(c^D)^2ca^\pi=0$. In view of the relations above and taking into account the following identities

$$\begin{aligned}
xx'&=\left[c^D-a^\pi(1-b^Daa^\pi)^{-1}b^D\right]c^D(a-b)\\
&=(c^D)^2(a-b)=(c^D)^2\left[c+a^\pi(a-b)\right]\\
&=c^D
\end{aligned}$$

and

$$\begin{aligned}
xx''&=\left[c^D-a^\pi(1-b^Daa^\pi)^{-1}b^D\right]b^Dba^\pi\\
&=\left[-a^\pi(1-b^Daa^\pi)^{-1}b^D\right]b^Dba^\pi\\
&=-(1-b^Daa^\pi)^{-1}b^Da^\pi\\
&=x-c^D,
\end{aligned}$$

we conclude that $x(a-b)x=x(x'+x'')=x$.

Finally, we will prove that $(a-b)-(a-b)^2x$ is nilpotent. Since $a-b=c+a^\pi(a-b)$, $ca^\pi=0$ and $a^\pi c^D=0$, we have

$$(a-b)^2c^D=\left[c+a^\pi(a-b)\right]^2c^D=\left[c^2+2ca^\pi(a-b)+a^\pi(a-b)^2\right]c^D=c^2c^D=c-cc^\pi.$$

Also we have $(a-b)b^Dba^\pi=(a-b)(1-b^\pi)a^\pi=aa^\pi-ba^\pi-aa^\pi b^\pi+bb^\pi a^\pi$.

From (3.7) and the above two equalities, we get

$$\begin{aligned}
&(a-b)-(a-b)^2x\\
&=(a-b)-(a-b)\left[c^D(a-b)+bb^Da^\pi\right]\\
&=(a-b)-(c-cc^\pi+aa^\pi-ba^\pi-aa^\pi b^\pi+bb^\pi a^\pi)\\
&=(a-b)-\left[(a-b)aa^D+(a-b)a^\pi-aa^\pi b^\pi+bb^\pi a^\pi-cc^\pi\right]\\
&=(a-b)-\left[(a-b)-aa^\pi b^\pi+bb^\pi a^\pi-cc^\pi\right]\\
&=aa^\pi b^\pi-bb^\pi a^\pi+cc^\pi.
\end{aligned}$$

Note that $(aa^\pi b^\pi - bb^\pi a^\pi)^k = (a-b)^k a^\pi b^\pi$ and $(a-b)^k = \sum_{i+j=k} \lambda_{i,j} a^j b^i$. Let $k \geq 2\ max\{s,t\}$. Then we have $(aa^\pi b^\pi - bb^\pi a^\pi)^k = 0$. Hence $aa^\pi b^\pi - bb^\pi a^\pi$ is nilpotent.

Since cc^π and $aa^\pi b^\pi - bb^\pi a^\pi$ are nilpotent and $(aa^\pi b^\pi - bb^\pi a^\pi)cc^\pi = cc^\pi(aa^\pi b^\pi - bb^\pi a^\pi) = 0$, it follows from [25, Lemma 1(2)] that $aa^\pi b^\pi - bb^\pi a^\pi + cc^\pi$ is nilpotent. Therefore, we have proved $a-b \in \mathcal{R}^D$ and $(a-b)^D = x$, i.e., the expression (3.4).

(1) ⇒ (3) This is similar to (1) ⇒ (2).

(3) ⇒ (1) Assume that e is Drazin invertible and let $y = e^D + a^D(1-bb^\pi a^D)^{-1}b^\pi$. Since $\mu b^2 a = \mu' ab^2$, by Lemma 2.2, we have $bb^D a = abb^D$ and $b^\pi a = ab^\pi$. From $bb^D a = abb^D$, it is easy to obtain that $e(a-b) = (a-b)e$ and

$$e^D(a-b) = (a-b)e^D. \tag{3.8}$$

By Lemma 2.5(1) and $b^t b^\pi = 0$, we have $(bb^\pi a^D)^t = \lambda^{\frac{(t-1)t}{2}} b^t b^\pi (a^D)^t = 0$. It follows from [12, Lemma 1.1] that $1 - bb^\pi a^D$ is invertible $(1-bb^\pi a^D)^{-1} = 1 + bb^\pi a^D + (bb^\pi a^D)^2 + \cdots + (bb^\pi a^D)^{t-1}$.

Since $b^\pi a = ab^\pi$, we have that $b^\pi a^D = a^D b^\pi$ by [6, Theorem 1]. From this, we obtain

$$\begin{aligned}
&(a-b)a^D(1-bb^\pi a^D)^{-1}b^\pi \\
&= aa^D(1-ba^D)b^\pi(1-bb^\pi a^D)^{-1} \\
&= aa^D(1-bb^\pi a^D - bbb^D a^D)b^\pi(1-bb^\pi a^D)^{-1} \\
&= aa^D(1-bb^\pi a^D)b^\pi(1-bb^\pi a^D)^{-1} \\
&= aa^D b^\pi.
\end{aligned} \tag{3.9}$$

Hence, we have

$$(a-b)y = (a-b)\left[e^D + a^D(1-bb^\pi a^D)^{-1}b^\pi\right] = (a-b)e^D + aa^D b^\pi. \tag{3.10}$$

Since $a^D(1-bb^\pi a^D) = (1-a^D bb^\pi)a^D$, then using the same way as in the proof of (3.9), we also obtain $\left[a^D(1-bb^\pi a^D)^{-1}b^\pi\right](a-b) = (1-a^D bb^\pi)^{-1}a^D b^\pi(a-b) = aa^D b^\pi$. So, it follows

$$y(a-b) = \left[e^D + a^D(1-bb^\pi a^D)^{-1}b^\pi\right](a-b) = e^D(a-b) + aa^D b^\pi. \tag{3.11}$$

Combining (3.8), (3.10) and (3.11), we get $y(a-b) = (a-b)y$.

We now prove that $y(a-b)y = y$. Let $y(a-b) = y' + y''$, where $y' = e^D(a-b)$ and $y'' = aa^D b^\pi$.

The following equality will be useful:

$$e + (a-b)b^\pi = (a-b)bb^D + (a-b)(1-bb^D) = a-b. \tag{3.12}$$

From $b^\pi e = eb^\pi = (a-b)bb^D b^\pi = 0$, we get $b^\pi e^D = e^D b^\pi = (e^D)^2 eb^\pi = 0$. Thus, we obtain

$$y'e^D = (e^D)^2\left[e + (a-b)b^\pi\right] = (e^D)^2 e + (e^D)^2(a-b)b^\pi = e^D$$

and $y''e^D = aa^D b^\pi e^D = 0$. Similarly, it is easy to get $y'a^D(1-bb^\pi a^D)^{-1}b^\pi = 0$.

So, we get

$$\begin{aligned}
y(a-b)y &= (y'+y'')\left[e^D + a^D(1-bb^\pi a^D)^{-1}b^\pi\right] \\
&= y'e^D + y'a^D(1-bb^\pi a^D)^{-1}b^\pi \\
&+ y''e^D + y''a^D(1-bb^\pi a^D)^{-1}b^\pi \\
&= e^D + aa^D b^\pi a^D(1-bb^\pi a^D)^{-1}b^\pi \\
&= e^D + a^D(1-bb^\pi a^D)^{-1}b^\pi \\
&= y.
\end{aligned}$$

Finally, let us prove that $(a-b)-(a-b)^2y$ is nilpotent. From $eb^\pi = b^\pi e = 0$ and $b^\pi e^D = 0$, which together with equality (3.12), we have

$$(a-b)^2e^D = [e+(a-b)b^\pi]^2\, e^D = \left[e^2 + 2(a-b)b^\pi e + (a-b)^2 b^\pi\right] e^D = e^2e^D = e - ee^\pi \tag{3.13}$$

and

$$(a-b)aa^Db^\pi = (a-b)(1-a^\pi)b^\pi = ab^\pi - bb^\pi - aa^\pi b^\pi + bb^\pi a^\pi. \tag{3.14}$$

Then by equalities (3.10), (3.12), (3.13) and (3.14), we obtain

$$\begin{aligned}
&(a-b)-(a-b)^2y\\
&= (a-b)-(a-b)\left[(a-b)e^D + aa^Db^\pi\right]\\
&= (a-b)-(a-b)^2e^D-(a-b)aa^Db^\pi\\
&= e+(a-b)b^\pi - e + ee^\pi - ab^\pi + bb^\pi + aa^\pi b^\pi - bb^\pi a^\pi\\
&= aa^\pi b^\pi - bb^\pi a^\pi + ee^\pi.
\end{aligned}$$

The rest of the proof follows in much the same way as the proof of (2) ⇒ (1). Hence, $a-b \in \mathcal{R}^D$ and $(a-b)^D = e^D + a^D(1-bb^\pi a^D)^{-1}b^\pi$.

(3) ⇒ (4) Since aa^D commutes with a, b and b^D, we get $aa^D(a-b)bb^D = (a-b)bb^Daa^D$. Thus, from Corollary 3.3 it follows that w^D exists and $w^D = \left[aa^D(a-b)bb^D\right]^D = (aa^D)^D\left[(a-b)bb^D\right]^D = aa^De^D$.

(4) ⇒ (3) To check that $e \in \mathcal{R}^D$, let $p_1 = abb^D$, $p_2 = b^2b^D$. In view of Lemma 2.2, $abb^D = bb^Da$, and then we may apply Corollary 3.3 to give $p_1, p_2 \in \mathcal{R}^D$ and $p_1^D = a^D(bb^D)^D = a^Dbb^D$.

It is easy to verify that $p_1p_2p_1 = \lambda p_1^2p_2 = \lambda' p_2p_1^2$, $p_2p_1p_2 = \mu p_2^2p_1 = \mu' p_1p_2^2$, therefore p_1 and p_2 satisfy the conditions of Theorem 3.6.

In addition, $p_1p_1^D(p_1-p_2) = abb^Da^Dbb^D(abb^D - b^2b^D) = aa^D(a-b)bb^D \in \mathcal{R}^D$. Applying (2) ⇒ (1) to p_1 and p_2, we conclude that $(a-b)bb^D = p_1 - p_2 \in \mathcal{R}^D$.

Further, the equality $(a-b)^D = w^D + a^D(1-bb^\pi a^D)^{-1}b^\pi - a^\pi(1-b^Daa^\pi)^{-1}b^D$ appearing in (3.6) follows from (2) ⇒ (1) and (3) ⇒ (1). □

Remark 3.7. *(1) Let us observe that the expression for $(a-b)^D$ in* [19, Theorem 2.3] *(in this paper the expression (3.6)). If we assume that $c = aa^D(a-b)$ (or $e = (a-b)bb^D$) instead of $w = aa^D(a-b)bb^D$, we get a much simpler expression for $(a-b)^D$, i.e., the expression (3.4) (or the expression (3.5)).*

(2) In Theorem 3.6, the conditions $aba = \lambda a^2b = \lambda' ba^2$ and $bab = \mu b^2a = \mu' ab^2$ are weaker than $ab = \lambda ba$ of [19, Theorem 2.3]. *Since $ab = \lambda ba$, by the proof of Corollary 3.3 we get $aba = \lambda a^2b = \lambda' ba^2$ and $bab = \mu b^2a = \mu' ab^2$. However, in general, the converse is false. The following example can illustrate this fact.*

Example 3.8. *Let $\mathcal{R} = M_3(R)$, and take*

$$a = \begin{bmatrix} 0 & 0 & 1\\ 0 & 0 & 1\\ 0 & 0 & 0 \end{bmatrix},\ b = \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 1 & -1 & 0 \end{bmatrix} \in \mathcal{R}^D.$$

Then we have $aba = \lambda a^2b = \lambda' ba^2$ and $bab = \mu b^2a = \mu' ab^2$. However,

$$ab = \begin{bmatrix} 1 & -1 & 0\\ 1 & -1 & 0\\ 0 & 0 & 0 \end{bmatrix} \neq \lambda ba = \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{bmatrix}.$$

4. Main result 2

The problem of finding $(a+b)^D$ was studied in [25], where the authors gave some representations for $(a+b)^D$ under the assumption $ab = ba$. In this section, we derive some new representations for $(a \pm b)^D$ under the conditions $a^2b = aba$ and $b^2a = bab = ab^2$, which weaker than $ab = ba$. Now, we begin with the following lemma.

Lemma 4.1. [24, Lemma 2.4] *Let $a, b \in \mathcal{R}^D$ with $a^2b = aba$ and $b^2a = bab$. Then*

$$(1)\ \{ab, a^Db, ab^D, a^Db^D\} \subseteq comm(a). \tag{4.1}$$

$$(2)\ \{ba, b^Da, ba^D, b^Da^D\} \subseteq comm(b). \tag{4.2}$$

Lemma 4.2. *Let $a, b \in \mathcal{R}^D$ with $a^2b = aba$ and $b^2a = bab = ab^2$. Then for any positive integer i, the following hold:*

$$(1)\ ab^D = b^Da. \tag{4.3}$$

$$(2)\ ba^Db = a^Db^2 = b^2a^D. \tag{4.4}$$

$$(3)\ a^Dbba^D = ba^Da^Db = (a^Db)^2 = (a^D)^2b^2. \tag{4.5}$$

$$(4)\ (a^Db)^{i+1} = a^Db(ba^D)^i = (a^D)^{i+1}b^{i+1}. \tag{4.6}$$

$$(5)\ (ba^D)(a^Db)^{i+1} = (a^Db)^{i+2}. \tag{4.7}$$

Proof. (1) Since $b^2a = ab^2$, applying Lemma 2.2, we have $abb^D = bb^Da$. Then $b^Da = b^Dbb^Da \overset{(4.2)}{=} b^Dbab^D = ab^Dbb^D = ab^D$.

(2) From $b^2a = ab^2$, by [6, Theorem 1], $b^2a^D = a^Db^2$. Hence, we have $ba^Db \overset{(4.2)}{=} b^2a^D = a^Db^2$.

(3) It is easy to check that $a^Dbba^D \overset{(4.4)}{=} ba^Dba^D \overset{(4.1)}{=} ba^Da^Db$ and $a^Dbba^D \overset{(4.2)}{=} a^Dba^Db \overset{(4.1)}{=} (a^D)^2b^2$.

(4) We will prove this result using mathematical induction on $k+1$. It is just (3) for $i = 1$. Now, we will assume that it holds for k, i.e., $(a^Db)^{k+1} = a^Db(ba^D)^k = (a^D)^{k+1}b^{k+1}$. For $i = k+1$,

$$(a^Db)^{k+2} = a^Db(a^Db)^{k+1} = a^Dba^Db(ba^D)^k \overset{(4.5)}{=} a^Dbba^D(ba^D)^k = a^Db(ba^D)^{k+1}$$

and

$$(a^Db)^{k+2} = a^Db(a^Db)^{k+1} = a^Db(a^D)^{k+1}b^{k+1} \overset{(4.1)}{=} (a^D)^{k+1}a^Dbb^{k+1} = (a^D)^{k+2}b^{k+2}.$$

(5) Under the assumption of this lemma, we have $ba^D(a^Db)^{i+1} = ba^Da^Db(a^Db)^i \overset{(4.5)}{=} (a^Db)^2(a^Db)^i = (a^Db)^{i+2}$. □

Lemma 4.3. *Let $a, b \in \mathcal{R}^D$ with $a^2b = aba$ and $b^2a = bab = ab^2$. If $a_1 = aa^\pi b^\pi$ and $a_2 = a^\pi bb^\pi$, then $a_1 - a_2$ is nilpotent.*

Proof. Firstly, we prove that $a_1 = aa^\pi b^\pi$ is nilpotent. In view of Lemma 2.2, bb^D commutes with a and a^D it follows that $b^\pi a = ab^\pi$ and $b^\pi a^D = a^Db^\pi$ and, thus

$$b^\pi a^\pi = b^\pi(1 - aa^D) = b^\pi - b^\pi aa^D = b^\pi - aa^Db^\pi = a^\pi b^\pi. \tag{4.8}$$

In the rest of the proof, we will use frequently that $\{bb^D, b^\pi, a, a^D, a^\pi\}$ is a commutative family. Hence, we get $aa^\pi b^\pi = b^\pi aa^\pi$. Since aa^π is nilpotent, $aa^\pi b^\pi = a_1$ is nilpotent by [25, Lemma 1(1)].

Secondly, we will show that $a_2 = a^\pi bb^\pi$ is nilpotent. As

$$\begin{aligned}(a^\pi bb^\pi)^2 &= a^\pi b^\pi ba^\pi bb^\pi = a^\pi b^\pi b(1 - aa^D)bb^\pi \\ &= a^\pi b^\pi(b^2 - baa^Db)b^\pi \overset{(4.1)}{=} a^\pi b^\pi(b^2 - baba^D)b^\pi \\ &= a^\pi b^\pi(b^2 - ab^2a^D)b^\pi \overset{(4.4)}{=} a^\pi b^\pi(b^2 - aa^Db^2)b^\pi \\ &= a^\pi b^\pi a^\pi b^2b^\pi \overset{(4.8)}{=} a^\pi a^\pi b^\pi b^2b^\pi = a^\pi(bb^\pi)^2.\end{aligned}$$

By induction, $(a^\pi bb^\pi)^n = a^\pi (bb^\pi)^n$ for every integer $n \geq 1$. Since bb^π is nilpotent, $a^\pi bb^\pi = a_2$ is nilpotent.

Finally, we shall prove that $a_1 - a_2$ is nilpotent. Since $\{bb^D, b^\pi, a, a^D, a^\pi\}$ is a commutative family, we derive

$$\begin{aligned} a_1a_2a_1 &= aa^\pi b^\pi a^\pi bb^\pi aa^\pi b^\pi = a^\pi b^\pi a^\pi (ab)aa^\pi b^\pi b^\pi \\ &\overset{(4.1)}{=} a^\pi b^\pi a^\pi aaa^\pi (bb^\pi)b^\pi = a^\pi b^\pi a^\pi aa(a^\pi b^\pi)bb^\pi \\ &= a^\pi (b^\pi a^\pi a)ab^\pi a^\pi bb^\pi = aa^\pi b^\pi aa^\pi b^\pi a^\pi bb^\pi \\ &= a_1^2a_2 \end{aligned}$$

and

$$\begin{aligned} a_2a_1a_2 &= a^\pi bb^\pi aa^\pi b^\pi a^\pi bb^\pi = a^\pi b^\pi baa^\pi a^\pi bb^\pi b^\pi \\ &= a^\pi b^\pi ba(1-aa^D)bb^\pi b^\pi = a^\pi b^\pi \left[bab - baa(a^Db)\right] b^\pi b^\pi \\ &\overset{(4.1)}{=} a^\pi b^\pi \left[(ba)b - (ba^D)baa\right] b^\pi b^\pi \overset{(4.2)}{=} a^\pi b^\pi bbaa^\pi b^\pi b^\pi \\ &= a^\pi b^\pi b(ba^\pi)aa^\pi b^\pi b^\pi \overset{(4.2)}{=} a^\pi b^\pi ba^\pi b(aa^\pi b^\pi)b^\pi \\ &= a^\pi bb^\pi a^\pi bb^\pi aa^\pi b^\pi \\ &= a_2^2a_1. \end{aligned}$$

Therefore, we can prove that $a_1^2a_2 = a_1a_2a_1$ and $a_2a_1a_2 = a_2^2a_1$.

As a_1 and a_2 are nilpotent, $aa^\pi b^\pi - a^\pi bb^\pi = a_1 - a_2$ is nilpotent by [24, Lemma 2.2 (2)]. □

Lemma 4.4. *Let $a, b \in \mathcal{R}^D$ with $a^2b = aba$ and $b^2a = bab = ab^2$ and $w = aa^D(a-b)bb^D \in \mathcal{R}^D$. Suppose $b_1 = ww^\pi$ and $b_2 = aa^\pi b^\pi - a^\pi bb^\pi$. Then $b_1 + b_2$ is nilpotent.*

Proof. First, we will give some useful equalities. By the proof of Lemma 4.3, we have that $\{bb^D, b^\pi, a, a^D, a^\pi\}$ is a commutative family. This means that $a^\pi w = wa^\pi = aa^D(a-b)bb^Da^\pi \overset{(4.1)}{=} a(a-b)a^Dbb^Da^\pi = a(a-b)bb^Da^Da^\pi = 0$ and $wb^\pi = b^\pi w = b^\pi aa^D(a-b)bb^D = aa^D(a-b)bb^\pi b^D = 0$.

Next, we will prove that $b_1 + b_2$ is nilpotent. Using the previous equations, we obtain that

$$b_1b_2 = ww^\pi(aa^\pi b^\pi - a^\pi bb^\pi) = w^\pi wa^\pi(ab^\pi - bb^\pi) = 0$$

and $b_2b_1 = (aa^\pi b^\pi - a^\pi bb^\pi)ww^\pi = (aa^\pi - a^\pi b)b^\pi ww^\pi = 0$. Hence, $b_1b_2 = b_2b_1 = 0$.

By Lemma 4.3, $a_1 - a_2 = b_2$ is nilpotent. Since b_1 and b_2 are nilpotent, and b_1 commutes with b_2, then by using [25, Lemma 1(2)], it follows that $b_1 + b_2$ is nilpotent. □

We are now ready to prove the other of our main results.

Theorem 4.5. *Let $a, b \in \mathcal{R}^D$ be such that $a^2b = aba$, $b^2a = bab = ab^2$ and $ind(a) = s$, $ind(b) = t$. Then the following conditions are equivalent:*

(1) $a - b \in \mathcal{R}^D$.

(2) $\xi = 1 - a^Db \in \mathcal{R}^D$.

(3) $w = aa^D(a-b)bb^D \in \mathcal{R}^D$.

In this case,

$$(a-b)^D = a^D\xi^D - a^\pi b(a^D\xi^D)^2 - \sum_{i=0}^{s-1}(b^D)^{i+1}a^ia^\pi, \tag{4.9}$$

$$(a-b)^D = w^D - \sum_{i=0}^{s-1}(b^D)^{i+1}a^ia^\pi + \sum_{i=0}^{t-1}(a^D)^{i+1}b^ib^\pi - a^\pi b(a^D)^2, \tag{4.10}$$

where $\xi^D = a^\pi + a^2a^D(a-b)^D$, $w^D = aa^D(a-b)^Dbb^D$.

Proof. (1) ⇒ (3) Assume that $a-b \in \mathcal{R}^D$, let $w = cl$, $y = aa^D(a-b)^D bb^D$, where $c = aa^D(a-b)$, $l = bb^D$. We will prove that $w = aa^D(a-b)bb^D \in \mathcal{R}^D$ and $w^D = y$. By Lemma 4.1, we have $aa^D aa^D(a-b) = aa^D(a-b)aa^D$ and

$$\begin{aligned}(a-b)aa^D(a-b) &= a^3a^D - aa^D(ab) - ba^2a^D + ba(a^Db)\\ &\overset{(4.1)}{=} a^3a^D - abaa^D - ba^2a^D + (ba^D)ba\\ &\overset{(4.2)}{=} a^3a^D - abaa^D - ba^2a^D + b^2aa^D\\ &= (a^2 - ab - ba + b^2)aa^D\\ &= (a-b)^2aa^D.\end{aligned}$$

By [24, Theorem 3.1], it follows that $c \in \mathcal{R}^D$ and $c^D = \left[aa^D(a-b)\right]^D = aa^D(a-b)^D$.

By the proof of Lemma 4.3, we have that $\{bb^D, b^\pi, a, a^D, a^\pi\}$ is a commutative family. Then we get

$$cl = aa^D(a-b)bb^D = aa^Dbb^D(a-b) = bb^Daa^D(a-b) = lc.$$

Thus, utilizing Corollary 3.3, we obtain $w \in \mathcal{R}^D$ and

$$w^D = \left[aa^D(a-b)bb^D\right]^D = \left[aa^D(a-b)\right]^D (bb^D)^D = aa^D(a-b)^D bb^D = y.$$

(3) ⇒ (1) Assume that w is Drazin invertible and let us define

$$x = w^D - \sum_{i=0}^{s-1}(b^D)^{i+1}a^ia^\pi + \sum_{i=0}^{t-1}(a^D)^{i+1}b^ib^\pi - a^\pi b(a^D)^2 = x_1 - x_2 + x_3,$$

where $x_1 = w^D$, $x_2 = \sum_{i=0}^{s-1}(b^D)^{i+1}a^ia^\pi$, $x_3 = \sum_{i=0}^{t-1}(a^D)^{i+1}b^ib^\pi - a^\pi b(a^D)^2$.

Next, we will prove that x is the Drazin inverse of $a-b$, i.e., we will prove that $x(a-b) = (a-b)x$, $x(a-b)x = x$ and $(a-b)-(a-b)^2x$ is nilpotent.

Step 1 First we prove that $x(a-b) = (a-b)x$. In light of Lemma 4.1, we have

$$\begin{aligned}(a-b)a^\pi b(a^D)^2 &= (a-b)(1-aa^D)b(a^D)^2\\ &= ab(a^D)^2 - bb(a^D)^2 - (a-b)aa^Db(a^D)^2\\ &\overset{(4.1)}{=} aa^Dba^D - bb(a^D)^2 - (a-b)aa^Da^Dba^D\\ &\overset{(4.2)}{=} aa^Dba^D - ba^Dba^D - (a-b)a^Dba^D\\ &= (a-b)a^Dba^D - (a-b)a^Dba^D\\ &= 0.\end{aligned}$$

Hence

$$\begin{aligned}(a-b)x &= (a-b)\left[w^D - \sum_{i=0}^{s-1}(b^D)^{i+1}a^ia^\pi + \sum_{i=0}^{t-1}(a^D)^{i+1}b^ib^\pi - a^\pi b(a^D)^2\right]\\ &= (a-b)\left[w^D - \sum_{i=0}^{s-1}(b^D)^{i+1}a^ia^\pi + \sum_{i=0}^{t-1}(a^D)^{i+1}b^ib^\pi\right]\\ &= z_1 - z_2 + z_3,\end{aligned} \tag{4.11}$$

where $z_1 = (a-b)w^D$, $z_2 = (a-b)\sum_{i=0}^{s-1}(b^D)^{i+1}a^ia^\pi$, $z_3 = (a-b)\sum_{i=0}^{t-1}(a^D)^{i+1}b^ib^\pi$.

Second we show $x_1(a-b) = z_1$, $x_2(a-b) = z_2$ and $x_3(a-b) = z_3$. Since $w = aa^D(a-b)bb^D$, we have $w = (1-a^\pi)(a-b)(1-b^\pi)$ and

$$a - b = w + (a-b)b^\pi + a^\pi(a-b) - a^\pi(a-b)b^\pi. \tag{4.12}$$

From Lemma 4.2, we have $ab^D = b^D a$, which together with $bb^D a = abb^D$, and then we have

$$w(a-b) = aa^D(a-b)bb^D(a-b) = (a-b)aa^D(a-b)bb^D = (a-b)w.$$

Accordingly, by [6, Theorem 1], we get $x_1(a-b) = w^D(a-b) = (a-b)w^D = z_1$.

By elementary computations, we obtain

$$b^D a^\pi b = bb^D - b^D a(a^D b) \overset{(4.1)}{=} bb^D - (b^D a^D)ba \overset{(4.2)}{=} bb^D - bb^D aa^D = bb^D a^\pi. \tag{4.13}$$

So we have

$$\begin{aligned}
x_2(a-b) - z_2 &= \sum_{i=0}^{s-1}(b^D)^{i+1}a^i a^\pi (a-b) - (a-b)\sum_{i=0}^{s-1}(b^D)^{i+1}a^i a^\pi \\
&= \sum_{i=0}^{s-1}(b^D)^{i+1}a^{i+1}a^\pi - \sum_{i=0}^{s-1}(b^D)^{i+1}a^i a^\pi b - a\sum_{i=0}^{s-1}(b^D)^{i+1}a^i a^\pi + b\sum_{i=0}^{s-1}(b^D)^{i+1}a^i a^\pi \\
&\overset{(4.3)}{=} \sum_{i=0}^{s-1}(b^D a)^{i+1}a^\pi - b^D\sum_{i=0}^{s-1}(b^D a)^i a^\pi b - \sum_{i=0}^{s-1}(b^D a)^{i+1}a^\pi + bb^D\sum_{i=0}^{s-1}(b^D a)^i a^\pi \\
&\overset{(4.2)}{=} -\sum_{i=0}^{s-1}(b^D a)^i b^D a^\pi b + \sum_{i=0}^{s-1}(b^D a)^i bb^D a^\pi \\
&= 0.
\end{aligned}$$

Hence, $x_2(a-b) = z_2$.

Next, we will prove that $x_3(a-b) = z_3$. Also, the following equalities will be useful: taking into account the following identities

$$\begin{aligned}
ba^D b^\pi - a^D bb^\pi &= ba^D(1-bb^D) - a^D b(1-bb^D) \\
&= ba^D - (ba^D b)b^D - a^D b + a^D b^2 b^D \\
&\overset{(4.4)}{=} ba^D - a^D b^2 b^D - a^D b + a^D b^2 b^D \\
&= ba^D - a^D b
\end{aligned}$$

and

$$\begin{aligned}
a^\pi b(a^D)^2(a-b) &\overset{(4.1)}{=} a^\pi ba^D(a-b)a^D = a^\pi(ba^D a - ba^D b)a^D \\
&\overset{(4.4)}{=} a^\pi(ba^D a - a^D b^2)a^D = a^\pi ba^D aa^D - a^\pi a^D b^2 a^D \\
&= (1-aa^D)ba^D = ba^D - a(a^D b)a^D \\
&\overset{(4.1)}{=} ba^D - a^D aa^D b = ba^D - a^D b,
\end{aligned}$$

we have $(ba^D b^\pi - a^D bb^\pi) - a^\pi b(a^D)^2(a-b) = 0$. Using the above relations and $ab^\pi = b^\pi a$, we conclude

$$\begin{aligned}
x_3(a-b) - z_3 &= \left[\sum_{i=0}^{t-1}(a^D)^{i+1}b^i b^\pi - a^\pi b(a^D)^2\right](a-b) - (a-b)\sum_{i=0}^{t-1}(a^D)^{i+1}b^i b^\pi \\
&= -a^\pi b(a^D)^2(a-b) - \sum_{i=0}^{t-1}(a^D)^{i+1}b^i b^\pi b + \sum_{i=0}^{t-1}(a^D)^{i+1}b^i b^\pi a - a\sum_{i=0}^{t-1}(a^D)^{i+1}b^i b^\pi + b\sum_{i=0}^{t-1}(a^D)^{i+1}b^i b^\pi \\
&\overset{(4.6)}{=} -a^\pi b(a^D)^2(a-b) - \sum_{i=0}^{t-1}(a^D b)^{i+1}b^\pi + a^D\sum_{i=0}^{t-1}(a^D b)^i b^\pi a - aa^D\sum_{i=0}^{t-1}(a^D b)^i b^\pi + ba^D\sum_{i=0}^{t-1}(a^D b)^i b^\pi
\end{aligned}$$

$$\stackrel{(4.1)}{=} -a^\pi b(a^D)^2(a-b) - \sum_{i=0}^{t-1}(a^Db)^{i+1}b^\pi + \sum_{i=0}^{t-1}(a^Db)^i aa^Db^\pi - \sum_{i=0}^{t-1}(a^Db)^i aa^Db^\pi + ba^D\sum_{i=0}^{t-1}(a^Db)^ib^\pi$$
$$= ba^D\sum_{i=0}^{t-1}(a^Db)^ib^\pi - \sum_{i=0}^{t-1}(a^Db)^{i+1}b^\pi - a^\pi b(a^D)^2(a-b)$$
$$\stackrel{(4.7)}{=} \left[ba^Db^\pi + \sum_{i=1}^{t-1}(a^Db)^{i+1}b^\pi\right] - \left[a^Dbb^\pi + \sum_{i=1}^{t-1}(a^Db)^{i+1}b^\pi\right] - a^\pi b(a^D)^2(a-b)$$
$$= (ba^Db^\pi - a^Dbb^\pi) - a^\pi b(a^D)^2(a-b) = 0.$$

Consequently, $x_3(a-b) = z_3$. It follows that $x(a-b) = (a-b)x$.

Step 2 We give the proof of $x(a-b)x = x$. By the proof of Lemma 4.4, we have $a^\pi w = wa^\pi = 0$ and $wb^\pi = b^\pi w = 0$. Hence, from equality (4.11), $w^Da^\pi = a^\pi w^D = a^\pi w(w^D)^2 = 0$ and $w^Db^\pi = b^\pi w^D = b^\pi w(w^D)^2 = 0$, we obtain

$$x(a-b)x = x(a-b)\left[w^D - \sum_{i=0}^{s-1}(b^D)^{i+1}a^ia^\pi + \sum_{i=0}^{t-1}(a^D)^{i+1}b^ib^\pi\right]$$
$$= (a-b)\left[w^D - \sum_{i=0}^{s-1}(b^D)^{i+1}a^ia^\pi + \sum_{i=0}^{t-1}(a^D)^{i+1}b^ib^\pi\right]^2$$
$$= (a-b)(w^D)^2 - (a-b)w^D\sum_{i=0}^{s-1}(b^D)^{i+1}a^ia^\pi + (a-b)w^D\sum_{i=0}^{t-1}(a^D)^{i+1}b^ib^\pi$$
$$+ (a-b)\sum_{i=0}^{s-1}(b^D)^{i+1}a^ia^\pi\sum_{i=0}^{s-1}(b^D)^{i+1}a^ia^\pi + (a-b)\sum_{i=0}^{t-1}(a^D)^{i+1}b^ib^\pi\sum_{i=0}^{t-1}(a^D)^{i+1}b^ib^\pi$$
$$= m_1 + m_2 + m_3 + m_4 + m_5,$$

where

$$m_1 = (a-b)(w^D)^2,\ m_2 = -(a-b)w^D\sum_{i=0}^{s-1}(b^D)^{i+1}a^ia^\pi,$$
$$m_3 = (a-b)w^D\sum_{i=0}^{t-1}(a^D)^{i+1}b^ib^\pi,\ m_4 = (a-b)\sum_{i=0}^{s-1}(b^D)^{i+1}a^ia^\pi\sum_{i=0}^{s-1}(b^D)^{i+1}a^ia^\pi,$$
$$m_5 = (a-b)\sum_{i=0}^{t-1}(a^D)^{i+1}b^ib^\pi\sum_{i=0}^{t-1}(a^D)^{i+1}b^ib^\pi.$$

Now we prove $m_1 + m_2 + m_3 + m_4 + m_5 = x$.

Since $ab^D = b^Da$, we may apply [6, Theorem 1] to give $a^Db^D = b^Da^D$. So, we have

$$b^Da^\pi = b^D - b^Da^Da = b^D - (a^Db^D)a \stackrel{(4.1)}{=} b^D - aa^Db^D = a^\pi b^D. \tag{4.14}$$

From the equality (4.12), $a^\pi w^D = 0$ and $b^\pi w^D = 0$, we derive

$$\begin{aligned}(a-b)w^D &= [w + (a-b)b^\pi + a^\pi(a-b) - a^\pi(a-b)b^\pi]\,w^D \\ &= ww^D + a^\pi(a-b)w^D = ww^D + a^\pi w^D(a-b) = ww^D.\end{aligned} \tag{4.15}$$

Thus $m_1 = (a-b)(w^D)^2 = \left[(a-b)w^D\right]w^D = ww^Dw^D = w^D$. From the equalities (4.14) and $w^Da^\pi = 0$, we deduce that

$$m_2 = -(a-b)w^D\sum_{i=0}^{s-1}(b^D)^{i+1}a^ia^\pi = -(a-b)w^Da^\pi\sum_{i=0}^{s-1}(b^D)^{i+1}a^i = 0.$$

By the proof of Lemma 4.3, we have $b^\pi a^D = a^D b^\pi$. Combining $b^\pi a^D = a^D b^\pi$ and $w^D b^\pi = 0$, we get

$$m_3 = (a-b)w^D \sum_{i=0}^{t-1} (a^D)^{i+1} b^i b^\pi = (a-b)w^D b^\pi \sum_{i=0}^{t-1} (a^D)^{i+1} b^i = 0.$$

Secondly, we have to prove $m_4 = -\sum_{i=0}^{s-1} (b^D)^{i+1} a^i a^\pi$ and $m_5 = \sum_{i=0}^{t-1} (a^D)^{i+1} b^i b^\pi - a^\pi b (a^D)^2$, respectively. In view of Lemma 4.1 and Lemma 4.2, we have the following equality:

$$(a^D b - b a^D) a^D b^\pi = -a^\pi b (a^D)^2. \tag{4.16}$$

Using the above relations and $a^s a^\pi = 0$, we get

$$\begin{aligned}
m_4 &= (a-b) \sum_{i=0}^{s-1} (b^D)^{i+1} a^i a^\pi \sum_{i=0}^{s-1} (b^D)^{i+1} a^i a^\pi \\
&= -b \sum_{i=0}^{s-1} (b^D)^{i+1} a^i a^\pi \sum_{i=0}^{s-1} (b^D)^{i+1} a^i a^\pi + a \sum_{i=0}^{s-1} (b^D)^{i+1} a^i a^\pi \sum_{i=0}^{s-1} (b^D)^{i+1} a^i a^\pi \\
&= -\left[b b^D a^\pi + \sum_{i=1}^{s-1} (b^D a)^i a^\pi \right] \sum_{i=0}^{s-1} (b^D)^{i+1} a^i a^\pi + a \sum_{i=0}^{s-1} (b^D)^{i+1} a^i a^\pi \sum_{i=0}^{s-1} (b^D)^{i+1} a^i a^\pi \\
&= -b b^D a^\pi \sum_{i=0}^{s-1} (b^D)^{i+1} a^i a^\pi - \sum_{i=1}^{s-1} (b^D a)^i a^\pi \sum_{i=0}^{s-1} (b^D)^{i+1} a^i a^\pi + a \sum_{i=0}^{s-1} (b^D)^{i+1} a^i a^\pi \sum_{i=0}^{s-1} (b^D)^{i+1} a^i a^\pi \\
&\overset{(4.14)}{=} -\sum_{i=0}^{s-1} (b^D)^{i+1} a^i a^\pi - \sum_{i=1}^{s-1} (b^D a)^i a^\pi \sum_{i=0}^{s-1} (b^D)^{i+1} a^i a^\pi + a \sum_{i=0}^{s-1} (b^D)^{i+1} a^i a^\pi \sum_{i=0}^{s-1} (b^D)^{i+1} a^i a^\pi \\
&\overset{(4.3)}{=} -\sum_{i=0}^{s-1} (b^D)^{i+1} a^i a^\pi - \sum_{i=1}^{s-1} (ab^D)^i a^\pi \sum_{i=0}^{s-1} (b^D)^{i+1} a^i a^\pi + \sum_{i=0}^{s-1} (ab^D)^{i+1} a^\pi \sum_{i=0}^{s-1} (b^D)^{i+1} a^i a^\pi \\
&= -\sum_{i=0}^{s-1} (b^D)^{i+1} a^i a^\pi - \sum_{i=1}^{s-1} (ab^D)^i a^\pi \sum_{i=0}^{s-1} (b^D)^{i+1} a^i a^\pi + \left[\sum_{i=1}^{s-1} (ab^D)^i a^\pi + (ab^D)^s a^\pi \right] \sum_{i=0}^{s-1} (b^D)^{i+1} a^i a^\pi \\
&\overset{(4.3)}{=} -\sum_{i=0}^{s-1} (b^D)^{i+1} a^i a^\pi - \sum_{i=1}^{s-1} (ab^D)^i a^\pi \sum_{i=0}^{s-1} (b^D)^{i+1} a^i a^\pi + \sum_{i=1}^{s-1} (ab^D)^i a^\pi \sum_{i=0}^{s-1} (b^D)^{i+1} a^i a^\pi + (b^D)^s a^s a^\pi \sum_{i=0}^{s-1} (b^D)^{i+1} a^i a^\pi \\
&= -\sum_{i=0}^{s-1} (b^D)^{i+1} a^i a^\pi.
\end{aligned}$$

Similarly, one can show that

$$\begin{aligned}
m_5 &= (a-b) \sum_{i=0}^{t-1} (a^D)^{i+1} b^i b^\pi \sum_{i=0}^{t-1} (a^D)^{i+1} b^i b^\pi = \sum_{i=0}^{t-1} (a^D)^{i+1} b^i b^\pi + (a^D b - b a^D) a^D b^\pi \\
&\overset{(4.16)}{=} \sum_{i=0}^{t-1} (a^D)^{i+1} b^i b^\pi - a^\pi b (a^D)^2.
\end{aligned}$$

So, we get $x(a-b)x = x$.

Step 3 Now we will prove that $(a-b)-(a-b)^2x$ is nilpotent. According to the equality (4.11), we have

$$\begin{aligned}(a-b)^2x &= \left[w^D - \sum_{i=0}^{s-1}(b^D)^{i+1}a^ia^\pi + \sum_{i=0}^{t-1}(a^D)^{i+1}b^ib^\pi\right](a-b)^2 \\ &= w^D(a-b)^2 - \sum_{i=0}^{s-1}(b^D)^{i+1}a^ia^\pi(a-b)^2 + \sum_{i=0}^{t-1}(a^D)^{i+1}b^ib^\pi(a-b)^2 = I_1+I_2+I_3,\end{aligned} \tag{4.17}$$

where $I_1 = w^D(a-b)^2$, $I_2 = -\sum_{i=0}^{s-1}(b^D)^{i+1}a^ia^\pi(a-b)^2$ and $I_3 = \sum_{i=0}^{t-1}(a^D)^{i+1}b^ib^\pi(a-b)^2$. Using the expression (4.15), we get

$$I_1 = w(w^D)^2(a-b)^2 = w\left[(a-b)w^D\right]^2 = w(ww^D)^2 = w - ww^\pi. \tag{4.18}$$

By Lemma 4.1 and Lemma 4.2, it is sufficient to prove

$$\begin{aligned}I_2 &\overset{(4.3)}{=} -\sum_{i=0}^{s-1}(b^Da)^{i+1}aa^\pi + \sum_{i=0}^{s-1}(b^Da)^{i+1}a^\pi b + b^D\sum_{i=0}^{s-1}(b^Da)^ia^\pi ba - \sum_{i=0}^{s-1} b^D(b^Da)^ia^\pi b^2 \\ &\overset{(4.2)}{=} -\sum_{i=0}^{s-1}(b^Da)^{i+1}aa^\pi + \sum_{i=0}^{s-1}(b^Da)^{i+1}a^\pi b + \sum_{i=0}^{s-1}(b^Da)^ib^Da^\pi ba - \sum_{i=0}^{s-1}(b^Da)^ib^Da^\pi b^2 \\ &\overset{(4.13)}{=} -\sum_{i=0}^{s-1}(b^Da)^{i+1}aa^\pi + \sum_{i=0}^{s-1}(b^Da)^{i+1}a^\pi b + \sum_{i=0}^{s-1}(b^Da)^ibb^Daa^\pi - \sum_{i=0}^{s-1}(b^Da)^ibb^Da^\pi b \\ &= bb^Daa^\pi + \sum_{i=0}^{s-1}(b^Da)^{i+1}a^\pi b - \sum_{i=0}^{s-1}(b^Da)^ibb^Da^\pi b \\ &\overset{(4.2)}{=} bb^Daa^\pi + \sum_{i=0}^{s-1}(b^Da)^{i+1}a^\pi b - \sum_{i=0}^{s-1} bb^D(b^Da)^ia^\pi b \\ &= bb^Daa^\pi - bb^Da^\pi b.\end{aligned} \tag{4.19}$$

Similarly,

$$I_3 = -aa^Dbb^\pi + aa^Db^\pi a. \tag{4.20}$$

Combining (4.12), (4.17), (4.18), (4.19) and (4.20) gives

$$\begin{aligned}&(a-b)-(a-b)^2x \\ &= [w + (a-b)b^\pi + a^\pi(a-b) - a^\pi(a-b)b^\pi] - (w - ww^\pi) \\ &- (bb^Daa^\pi - bb^Da^\pi b - aa^Dbb^\pi + aa^Db^\pi a) \\ &= ww^\pi + aa^\pi b^\pi - a^\pi bb^\pi \\ &= b_1 + b_2.\end{aligned}$$

It follows from Lemma 4.4, $(a-b)-(a-b)^2x = b_1+b_2$ is nilpotent. Therefore, we have proved $a-b \in \mathcal{R}^D$ and $(a-b)^D = x$, i.e., the expression (4.10).

(1) ⇒ (2) To check that $\xi \in \mathcal{R}^D$, we write ξ as $\xi = 1 - a^Db = h_1 + h_2$, where $h_1 = a^\pi$, $h_2 = a^D(a-b)$.

It follows from Lemma 4.1 that $(a^D)^2(a-b) = a^D(a-b)a^D$ and $(a-b)^2a^D = (a-b)a^D(a-b)$. Utilizing [24, Theorem 3.1] gets $a^D(a-b) = h_2 \in \mathcal{R}^D$ and $h_2^D = \left[a^D(a-b)\right]^D = (a^D)^D(a-b)^D = a^2a^D(a-b)^D$.

By Lemma 4.1, we have that a^Db commutes with aa^D. Then $a^D(a-b) \in comm(a^\pi)$ and $h_1h_2 = h_2h_1 = 0$. It follows from [6, corollary 1] that $\xi^D = a^\pi + a^2a^D(a-b)^D$.

(2) ⇒ (1) Assume that ξ is Drazin invertible, similarly as in the proof of (3) ⇒ (1), we have that $a-b \in \mathcal{R}^D$ and $(a-b)^D$ is represented as in (4.9). □

Using Theorem 4.5, we can verify the following corollary, which generalizes [25, Theorem 3].

Corollary 4.6. *Let $a, b \in \mathcal{R}^D$ be such that $a^2b = aba$, $b^2a = bab = ab^2$ and $ind(a) = s$, $ind(b) = t$. Then the following conditions are equivalent:*
(1) $a + b \in \mathcal{R}^D$.
(2) $\xi' = 1 + a^D b \in \mathcal{R}^D$.
(3) $w' = aa^D(a + b)bb^D \in \mathcal{R}^D$.
In this case,

$$(a+b)^D = a^D\xi'^D + a^\pi b(a^D\xi'^D)^2 + \sum_{i=0}^{s-1}(b^D)^{i+1}(-a)^i a^\pi, \tag{4.21}$$

$$(a+b)^D = w'^D + \sum_{i=0}^{s-1}(b^D)^{i+1}(-a)^i a^\pi + \sum_{i=0}^{t-1}(a^D)^{i+1}(-b)^i b^\pi + a^\pi b(a^D)^2,$$

where $\xi'^D = a^\pi + a^2a^D(a + b)^D$, $w'^D = aa^D(a + b)^D bb^D$.

Proof. By virtue of [3, Theorem 2.2], $-b \in \mathcal{R}^D$. Applying Theorem 4.5 to a and $-b$, we complete the proof. □

Remark 4.7. *(1) Given $a, b \in \mathcal{R}$, the equality $ab = ba$ of* [25, Theorem 3] *implies that $a^2b = aba$, $b^2a = bab = ab^2$ of Corollary 4.6. In addition, the expressions of $(a + b)^D$ from* [25, Theorem 3] *can be derived from (4.21).*
(2) Let $\mathcal{R} = M_n(\mathbb{C})$, then Corollary 4.6 covers [21, Theorem 2].

Finally, to show that our conditions are strictly weaker than the assumption $ab = ba$, we construct matrices a, b satisfying the conditions of Corollary 4.6, but $ab = ba$ does not hold.

Example 4.8. *Let $\mathcal{R} = M_2(R)$, and take* $a = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathcal{R}^D$.

Since $ab \neq ba$, the representations for $(a + b)^D$ fail to apply in [25, Theorem 3]. On the other hand, we can observe that $a^2b = aba$ and $b^2a = bab = ab^2$. Therefore, according to the formulae in Corollary 4.6, we get

$$(a+b)^D = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}.$$

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