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Determining the positive definiteness of even-order weakly symmetric tensors via Brauer-type Z-eigenvalue inclusion sets

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Abstract. This article presents sufficient conditions for the positive definiteness of even-order weakly symmetric tensors, based on some new Brauer-type Z-eigenvalue inclusion sets. In fact, these inclusion sets are obtained using the partitions of the index set, which improves some of the existing results.

1. Introduction

The positive definiteness of a homogeneous polynomial

$$f_{\mathcal{A}}(x) = \mathcal{A}x^m = x^T(\mathcal{A}x^{m-1}) = \sum_{i_1,i_2,\dots,i_m=1}^{n} a_{i_1i_2\dots i_m}x_{i_1}x_{i_2}\dots x_{i_m}, \tag{1}$$

where $\mathcal{A} = (a_{i_1i_2\dots i_m}) \in \mathbb{R}^{[m,n]}$ is an m-order n-dimensional real tensor with $i_j \in [n] := \{1,2,\dots,n\}$ for $j \in [m]$, and $\mathcal{A}x^{m-1}$ is an n-vector in $\mathbb{R}^n$, whose i-th component is

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2,i_3,\dots,i_m=1}^{n} a_{ii_2\dots i_m}x_{i_2}\dots x_{i_m},$$

is widely used in spectral hypergraph theory [9], automatical control [7] and etc. For higher order tensors, the following concept of Z-eigenvalues have been introduced in [8].

Definition 1.1. *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$. If there exists a nonzero real vector x and a real number λ such that*

$$\mathcal{A}x^{m-1} = \lambda x \quad and \quad x^Tx = 1, \tag{2}$$

then λ is called a Z-eigenvalue of $\mathcal{A}$ and x a Z-eigenvector of $\mathcal{A}$ associated with λ.

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A Z-identity tensor was defined in [4, 8] to propose a shifted power method for calculating Z-eigenpairs and investigate an extension of the characteristic polynomial for symmetric even-order tensors, respectively (for details, see [5, 10, 11]). In this article, we establish some Z-eigenvalue inclusion sets with parameters by Z-identity tensors.

Definition 1.2. *A tensor $\mathcal{I}_Z = (e_{i_1 i_2 \ldots i_m}) \in \mathbb{R}^{[m,n]}$, with m being even is called a Z-identity tensor if*

$$\mathcal{I}_Z x^{m-1} = x, \tag{3}$$

for any vector $x \in \mathbb{R}^n$ with $x^T x = 1$.

Note that the even-order n dimension Z-identity tensor is not unique in general. For instance, each of the following tensors is a Z-identity tensor:
Case I. Let $\mathcal{I}_1 = (e_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$, where

$$e_{i_1 i_2 \cdots i_m} = \begin{cases} 1 & i_1 = i_2,\ i_3 = i_4, \ldots, i_{m-1} = i_m \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Additionally, define $\prod_m$ is the set of all permutations of $(1, \ldots, m)$ and δ is the standard Kronecker delta, i.e., $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$.
Case II. Let $\mathcal{I}_2 = (e_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$, where

$$e_{i_1 i_2 \cdots i_m} = \frac{1}{m!} \sum_{\pi \in \prod_m} \delta_{i_{\pi(1)}\, i_{\pi(2)}}\, \delta_{i_{\pi(3)}\, i_{\pi(4)}} \ldots \delta_{i_{\pi(m-1)}\, i_{\pi(m)}}. \tag{5}$$

Recently, many people have focused on the Z-eigenvalue localization sets of higher order tensors (see for instance [1, 2, 12]). Unfortunately, the inclusion sets always include zero and could not be used to determine the positive definiteness of higher order tensors. In order to overcome this defect, Li et al. [5] presented a Z-eigenvalue inclusion interval for even-order tensors as follows:

Theorem 1.3. *[5, Theorem 2] Let $\mathcal{A} = (a_{i_1 i_2 \ldots i_m}) \in \mathbb{R}^{[m,n]}$ and $\mathcal{I} = (e_{i_1 i_2 \ldots i_m}) \in \mathbb{R}^{[m,n]}$ be a Z-identity tensor with m being even. Then for any real vector $\mu = (\mu_1, \ldots, \mu_n)^T \in \mathbb{R}^n$*

$$\sigma_Z(\mathcal{A}) \subseteq \mathcal{G}(\mathcal{A}, \mu) = \bigcup_{i \in [n]} \left(\mathcal{G}_i(\mathcal{A}, \mu_i) := \left\{ z \in \mathbb{R} \ :\ \left| z - \mu_i \right| \leq r_i(\mathcal{A}, \mu_i) \right\} \right), \tag{6}$$

where $r_i(\mathcal{A}, \mu_i) = \sum_{i_2, \ldots i_m \in [n]} \left| a_{i i_2 \ldots i_m} - \mu_i e_{i i_2 \ldots i_m} \right|$. Further, $\sigma_Z(\mathcal{A}) \subseteq \bigcap_{\mu \in \mathbb{R}^n} \mathcal{G}(\mathcal{A}, \mu)$.

As pointed out in [8] that an m-degree homogeneous polynomial $f(x)$ defined by (1) is positive definite, i.e., $f(x) > 0$ for any $x \in \mathbb{R}^n \setminus \{0\}$, if and only if the real symmetric tensor $\mathcal{A}$ is positive definite, and that an even-order real symmetric tensor is positive definite if and only if all of its Z-eigenvalues are positive. Here, a tensor $\mathcal{A}$ is said to be symmetric [8] if its entries $a_{i_1 i_2 \ldots i_m}$ are invariant under any permutation of m indices $(a_{i_1 i_2 \ldots i_m})$, and weakly symmetric [1] if the associated homogeneous polynomial (1) satisfied $\nabla \mathcal{A} x^m = m \mathcal{A} x^{m-1}$. It should be noted for $m = 2$, symmetric tensors and weakly symmetric tensors are the same. It's worth noting that a symmetric tensor must be a weakly symmetric tensor, but not vice versa. Therefore, some conclusions that are valid for symmetric tensors maybe not be applicable for weakly symmetric tensors.

Recently, several significant results have arisen to solve the problem of deciding positive-definiteness of an even-order symmetric tensor based on their special structure [3, 5, 6, 10]. For even-order real weakly symmetric tensors, Shen et al. [11] proposed two Brauer-type inclusion sets for identified the positive definiteness. In this paper, by improving the existing the Brauer-type inclusion sets, we will propose some sufficient conditions for the positive definiteness of even-order weakly symmetric tensors.

The rest of this paper is organized as follows: In Section 2, we establish some new Brauer-type Z-eigenvalue inclusion sets of even-order tensors. Moreover, by an example we show that the inclusion sets are more precise than existing results. In Section 3 based on the inclusion sets, we obtain some sufficient conditions to identify the positive definiteness of even-order weakly symmetric tensors. Finally, the numerical example shows the validity of our results.

2. Some new Brauer-type Z-eigenvalue inclusion intervals for even-order tensors

In this section, we present some new Brauer-type Z-eigenvalue inclusion sets by categorizing the elements of tensors, and show that this inclusion sets are sharper than existing results.

By partitioning the index set, we shall use the following notations and conventions.

$$\Lambda_i := \left\{(i_2,\ \ldots, i_m)\ :\ (\mathcal{I}_1)_{ii_2\ldots i_m} = 1,\quad i_2,\ \ldots, i_m \in [n]\right\},\quad i \in [n],$$
$$\overline{\Lambda}_i := \left\{(i_2,\ \ldots, i_m)\ :\ (\mathcal{I}_1)_{ii_2\ldots i_m} = 0,\quad i_2,\ \ldots, i_m \in [n]\right\},\quad i \in [n].$$

$$\Delta\ := \{(i_2, \ldots, i_m) : i_2 \neq \ldots \neq i_m,\ \textit{or only two of}\ i_2, \ldots, i_m \in [n]\ \textit{are the same}\},$$
$$\overline{\Delta}\ := \{(i_2, \ldots, i_m) : (i_2, \ldots, i_m) \notin \Delta,\ i_2, \ldots, i_m \in [n]\}.$$

$$\Omega_j\ := \{(i_2, \ldots, i_m) : i_k = j\ \textit{for some}\ k \in \{2, \ldots, m\},\ \textit{where}\ j, i_2, \ldots, i_m \in [n]\},$$
$$\overline{\Omega}_j\ := \{(i_2, \ldots, i_m) : i_k \neq j\quad \textit{for any}\ k \in \{2, \ldots, m\},\ \textit{where}\ j, i_2, \ldots, i_m \in [n]\}.$$

For $\mathcal{A} = (a_{i_1 i_2 \ldots i_m}) \in \mathbb{R}^{[m,n]}$, $i \neq j$ and $\Re \in \left\{\Lambda_i, \Delta, \Omega_j\right\}$ the following notations are used repeatedly in our proofs.

$$r_i(\mathcal{A}) = \sum_{i_2,\ldots,i_m \in [n]} \left|a_{ii_2\ldots i_m}\right|,\qquad r_i^j(\mathcal{A}) = r_i(\mathcal{A}) - \left|a_{ijj\ldots j}\right|,$$
$$r_i^{\Re}(\mathcal{A}) = \sum_{i_2,\ldots,i_m \in \Re} \left|a_{ii_2,\ldots,i_m}\right|,\qquad r_i^{\overline{\Re}}(\mathcal{A}) = \sum_{i_2,\ldots,i_m \in \overline{\Re}} \left|a_{ii_2,\ldots,i_m}\right|,$$
$$r_i^{\Re}(\mathcal{A}, \mu_i) = \sum_{i_2,\ldots,i_m \in \Re} \left|a_{ii_2,\ldots,i_m} - \mu_i e_{ii_2\ldots i_m}\right|,$$
$$r_i^{\overline{\Re}}(\mathcal{A}, \mu_i) = \sum_{i_2,\ldots,i_m \in \overline{\Re}} \left|a_{ii_2,\ldots,i_m} - \mu_i e_{ii_2\ldots i_m}\right|,$$
$$\beta_i = \max_{i_2,\ldots,i_m \in \Lambda_i} \left\{\left|a_{ii_2\ldots i_m} - \mu_i e_{ii_2\ldots i_m}\right|\right\},$$
$$M_i(\mathcal{A}, \mu_i) = \beta_i + \frac{1}{(m-2)^{\frac{m-2}{2}}} r_i^{\overline{\Lambda}_i \cap \Delta}(\mathcal{A}) + r_i^{\overline{\Lambda}_i \cap \overline{\Delta}}(\mathcal{A}),$$
$$M_i^{\Omega_i}(\mathcal{A}, \mu_i) = \beta_i + \frac{1}{(m-2)^{\frac{m-2}{2}}} r_i^{\overline{\Lambda}_i \cap \Delta \cap \Omega_i}(\mathcal{A}) + r_i^{\overline{\Lambda}_i \cap \overline{\Delta} \cap \Omega_i}(\mathcal{A}),$$
$$M_i^j(\mathcal{A}, \mu_i) = \beta_i + \frac{1}{(m-2)^{\frac{m-2}{2}}} r_i^{j\,\overline{\Lambda}_i \cap \Delta}(\mathcal{A}) + r_i^{j\,\overline{\Lambda}_i \cap \overline{\Delta}}(\mathcal{A}).$$

Obviously, for any $i \in [n]$, we have $r_i(\mathcal{A}) = r_i^{\Re}(\mathcal{A}) + r_i^{\overline{\Re}}(\mathcal{A})$ and $r_i(\mathcal{A}, \mu_i) = r_i^{\Re}(\mathcal{A}, \mu_i) + r_i^{\Re}(\mathcal{A}, \mu_i)$.

To begin with, we need the following lemma.

Lemma 2.1. *[10, Lemma 2.2] Let $x_1^2 + \cdots + x_n^2 = 1$, where $x_i \in \mathbb{R}$, $i \in [n]$. If $y_1, \ldots, y_k$ are arbitrary k entries of $x_1, \ldots, x_n$ then*

$$|y_1||y_2| \ldots |y_k| \leq \frac{1}{k^{\frac{k}{2}}}.$$

By modifying the Theorem 1.3, we have the following theorem.

Theorem 2.2. *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$, with m being even. Then for any real vector $\mu = (\mu_1, \ldots, \mu_n)^T \in \mathbb{R}^n$*

$$\sigma_Z(\mathcal{A}, \mu) \subseteq \Phi(\mathcal{A}, \mu) = \bigcup_{i \in [n]} \left(\Phi_i(\mathcal{A}, \mu_i) := \{\lambda \in \mathbb{R}\ :\ |\lambda - \mu_i| \leq M_i(\mathcal{A}, \mu_i)\}\right).$$

Proof. Let $\lambda \in \sigma_Z(\mathcal{A})$ with a corresponding Z-eigenvalue x, then (2) holds. Let $|x_t| = \max_{i\in[n]} |x_i|$, and μ_t be an arbitrary real number. By the th-equality of (2), one can obtain that

$$\begin{aligned}(\lambda - \mu_t)\, x_t &= \sum_{i_2,\dots,i_m\in\Lambda_t} (a_{ti_2,\dots,i_m} - \mu_t e_{ti_2,\dots,i_m})\, x_{i_2} \dots x_{im} \\ &+ \sum_{i_2,\dots,i_m\in\overline{\Lambda_t}\cap\Delta} a_{ti_2,\dots,i_m}\, x_{i_2} \dots x_{i_m} + \sum_{i_2,\dots,i_m\in\overline{\Lambda_t}\cap\overline{\Delta}} a_{ti_2,\dots,i_m}\, x_{i_2} \dots x_{i_m}.\end{aligned} \tag{7}$$

Taking modulus and using the triangle inequality for (7) give

$$|\lambda - \mu_t|\,|x_t| \le \beta_t \sum_{i_2,\dots,i_m\in\Lambda_t} x_{i_2} \dots x_{i_m} + \sum_{i_2,\dots,i_m\in\overline{\Lambda_t}\cap\Delta} \left|a_{ti_2,\dots,i_m}\right| \left|y_1\right| \dots \left|y_{m-2}\right| |x_t| + \sum_{i_2,\dots,i_m\in\overline{\Lambda_t}\cap\overline{\Delta}} \left|a_{ti_2,\dots,i_m}\right| |x_t|^{m-1},$$

where $\left|y_1\right|, \dots, \left|y_{m-2}\right|$ are taken by the following methods:

Case I. If $i_2 \neq \dots \neq i_m$, then we can enlarge any one of $\left|x_{i_2}\right|, \dots, \left|x_{i_m}\right|$ to $|x_t|$ and keep the others (can be taken as $\left|y_1\right|, \dots, \left|y_{m-2}\right|$) unchanged;

Case II. If only two of $i_2, \dots, i_m$ are the same, then we can enlarge one of the two same elements to $|x_t|$ and keep the others (can be taken as $\left|y_1\right|, \dots, \left|y_{m-2}\right|$) unchanged.

Using Lemmas 2.1 and Eq. (3), we have

$$|\lambda - \mu_t||x_t| \le |x_t|\left(\beta_t + \frac{1}{(m-2)^{\frac{m-2}{2}}} r_t^{\overline{\Lambda_t}\cap\Delta}(\mathcal{A}) + r_t^{\overline{\Lambda_t}\cap\overline{\Delta}}(\mathcal{A})\right), \tag{8}$$

which implies that $\lambda \in \Phi_t(\mathcal{A}, \mu) \subseteq \Phi(\mathcal{A}, \mu)$. Thus, we complete the proof. □

In the next, we establish some Brauer-type Z-eigenvalue inclusion sets for even-order tensors.

Theorem 2.3. *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ with m being even. Then for any real vector $\mu = (\mu_1, \dots, \mu_n)^T \in \mathbb{R}^n$*

a) $\sigma_Z(\mathcal{A}) \subseteq \mathcal{P}(\mathcal{A}, \mu) = \bigcup_{i\in[n]} \bigcap_{j\in[n], i\neq j} \mathcal{P}_{i,j}(\mathcal{A}, \mu)$,

b) $\sigma_Z(\mathcal{A}) \subseteq \mathcal{D}(\mathcal{A}, \mu) = \left(\bigcup_{i\in[n]} \bigcap_{j\in[n], i\neq j} \mathcal{X}_{i,j}(\mathcal{A}, \mu)\right) \bigcup \left(\bigcup_{i\in[n]} \bigcap_{j\in[n], i\neq j} \mathcal{Y}_{i,j}(\mathcal{A}, \mu)\right)$,

where

$$\mathcal{P}_{i,j}(\mathcal{A}, \mu) = \left\{\lambda \in \mathbb{R} : \left(\left|\lambda - \mu_i\right| - M_i^{\Omega_i}(\mathcal{A}, \mu_i)\right) \left|\lambda - \mu_j\right| \le r_i^{\overline{\Omega_i}}(\mathcal{A}, \mu_i)\, M_j\left(\mathcal{A}, \mu_j\right)\right\},$$

$$\mathcal{X}_{i,j}(\mathcal{A}, \mu) = \left\{\lambda \in \mathbb{R} : \left(\left|\lambda - \mu_i\right| - M_i^{\Omega_i}(\mathcal{A}, \mu_i)\right) \left(\left|\lambda - \mu_j\right| - M_j^i\left(\mathcal{A}, \mu_j\right)\right) \le r_i^{\overline{\Omega_i}}(\mathcal{A}, \mu_i)\, |a_{ji\dots i}|\right\},$$

$$\mathcal{Y}_{i,j}(\mathcal{A}, \mu) = \left\{\lambda \in \mathbb{R} : \left(\left|\lambda - \mu_i\right| - M_i^{\Omega_i}(\mathcal{A}, \mu_i)\right) < 0, \qquad \left(\left|\lambda - \mu_j\right| - M_j^i\left(\mathcal{A}, \mu_j\right)\right) < 0\right\}.$$

Proof. Let $\lambda \in \sigma_Z(\mathcal{A})$ with a corresponding Z-eigenvalue x. Let $|x_t| \ge |x_s| \ge \max_{\substack{k\in[n]\\ k\neq s, k\neq t}} |x_k|$, and μ_t be an arbitrary real number.

a) By the th-equality of (2), one can obtain that

$$(\lambda - \mu_t)\, x_t = \sum_{i_2,\dots,i_m\in\Omega_t} (a_{ti_2,\dots,i_m} - \mu_t e_{ti_2,\dots,i_m})\, x_{i_2} \dots x_{im} + \sum_{i_2,\dots,i_m\in\overline{\Omega_t}} (a_{ti_2,\dots,i_m} - \mu_t e_{ti_2,\dots,i_m})\, x_{i_2} \dots x_{i_m}.$$

Similar to the proof of Theorem 2.2, we get

$$|\lambda - \mu_t||x_t| \leq \left(\beta_t + \frac{1}{(m-2)^{\frac{m-2}{2}}} r_t^{\overline{\Lambda_t}\cap\Delta\cap\Omega_t}(\mathcal{A}) + r_t^{\overline{\Lambda_t}\cap\overline{\Delta}\cap\Omega_t}(\mathcal{A})\right)|x_t| + r_t^{\overline{\Omega_t}}(\mathcal{A},\mu_t)\,|x_s|^{m-1},$$

which implies that

$$\left(\left|\lambda - \mu_t\right| - M_t^{\Omega_t}(\mathcal{A},\mu_t)\right)|x_t| \leq r_t^{\overline{\Omega_t}}(\mathcal{A})\,|x_s|. \tag{9}$$

If $|x_s| = 0$, by (9), we deduce $\left(|\lambda| - M_t^{\Omega_t}(\mathcal{A},\mu_t)\right) \leq 0$. Thus $\lambda \in \mathcal{P}_{t,s}(\mathcal{A},\mu) \subseteq \mathcal{P}(\mathcal{A},\mu)$.

Otherwise, $|x_s| > 0$. Using (8), we have

$$\left|\lambda - \mu_s\right||x_s| \leq M_s(\mathcal{A},\mu_s)\,|x_t|. \tag{10}$$

Multiplying inequalities (9) and (10) yields, $\left(\left|z - \mu_t\right| - M_t^{\Omega_t}(\mathcal{A},\mu_t)\right)\left|\lambda - \mu_s\right| \leq r_t^{\overline{\Omega_t}}(\mathcal{A})\,M_s(\mathcal{A},\mu_s)$, which implies $\lambda \in \mathcal{P}_{t,s}(\mathcal{A},\mu) \subseteq \mathcal{P}(\mathcal{A},\mu)$. Hence, the conclusion follows.

b) By characterization of (9), for any $t, s \in [n]$, $s \neq t$, we have

$$\left(\left|\lambda - \mu_t\right| - M_t^{\Omega_t}(\mathcal{A},\mu_t)\right)|x_t| \leq r_t^{\overline{\Omega_t}}(\mathcal{A})\,|x_s|. \tag{11}$$

If $|x_s| = 0$, by (11), we deduce $\left(\left|\lambda - \mu_t\right| - M_t^{\Omega_t}(\mathcal{A},\mu_t)\right) \leq 0$. When $\left(\left|\lambda - \mu_s\right| - M_s^t(\mathcal{A},\mu_s)\right) \geq 0$, we have

$$\left(\left|\lambda - \mu_t\right| - M_t^{\Omega_t}(\mathcal{A},\mu_t)\right)\left(\left|\lambda - \mu_s\right| - M_s^t(\mathcal{A},\mu_s)\right) \leq 0 \leq r_t^{\overline{\Omega_t}}(\mathcal{A},\mu_t)\,|a_{st\ldots t}|,$$

which implies $\lambda \in \bigcap_{s\in[n],t\neq s} \mathcal{X}_{t,s}(\mathcal{A},\mu) \subseteq \mathcal{D}(\mathcal{A},\mu)$ from the arbitrariness of s. When $\left(\left|\lambda - \mu_s\right| - M_s^t(\mathcal{A},\mu_s)\right) < 0$, from the arbitrariness of s, we have $\lambda \in \bigcap_{s\in[n],t\neq s} \mathcal{Y}_{t,s}(\mathcal{A},\mu) \subseteq \mathcal{D}(\mathcal{A},\mu)$.

Otherwise, $|x_s| > 0$. Moreover, using (2), we have

$$\begin{aligned}
\left|\lambda - \mu_s\right||x_s| &\leq |a_{st\ldots t}|\,|x_t^{m-1}| + \sum_{\substack{i_2,\ldots,i_m\in[n]\\ \delta_{ti_2\ldots i_m}=0}} \left|a_{si_2,\ldots,i_m} - \mu_s e_{si_2,\ldots,i_m}\right| \left|x_{i_2}\right| \ldots \left|x_{i_m}\right| \\
&\leq |a_{st\ldots t}|\,|x_t| + \left(\beta_s + \frac{1}{(m-2)^{\frac{m-2}{2}}} r_s^{t\,\overline{\Lambda_s}\cap\Delta}(\mathcal{A}) + r_s^{t\,\overline{\Lambda_s}\cap\overline{\Delta}}(\mathcal{A})\right)|x_s|.
\end{aligned} \tag{12}$$

When $\left(\left|\lambda - \mu_t\right| - M_t^{\Omega_t}(\mathcal{A},\mu_t)\right) \geq 0$ or $\left(\left|\lambda - \mu_s\right| - M_s^t(\mathcal{A},\mu_s)\right) \geq 0$ holds, multiplying (11) and (12) yields,

$$\left(\left|\lambda - \mu_t\right| - M_t^{\Omega_t}(\mathcal{A},\mu_t)\right)\left(\left|\lambda - \mu_s\right| - M_s^t(\mathcal{A},\mu_s)\right) \leq r_t^{\overline{\Omega_t}}(\mathcal{A},\mu_t)\,|a_{st\ldots t}|,$$

which implies $\lambda \in \bigcap_{s\in[n],t\neq s} \mathcal{X}_{t,s}(\mathcal{A}) \subseteq \mathcal{D}(\mathcal{A})$ from the arbitrariness of s.

When $\left(\left|\lambda - \mu_t\right| - M_t^{\Omega_t}(\mathcal{A},\mu_t)\right) < 0$ and $\left(\left|\lambda - \mu_s\right| - M_s^t(\mathcal{A},\mu_s)\right) < 0$ hold, from the arbitrariness of s, we have $\lambda \in \bigcap_{s\in[n],t\neq s} \mathcal{Y}_{t,s}(\mathcal{A}) \subseteq \mathcal{D}(\mathcal{A})$. Hence, the conclusion follows. □

In the following example, we show the efficiency of our results.

Example 2.4. *([11, Example 1]) Consider the tensor $\mathcal{A} = \left(a_{ijkl}\right) \in \mathbb{R}^{[4,2]}$, with entries defined as follows:*

$$a_{1111} = 10,\quad a_{1122} = 9,\quad a_{1121} = a_{1222} = -1,\quad a_{2222} = 5,\quad a_{2211} = 6,\quad a_{2122} = a_{2111} = -1,$$

and other $a_{ijkl} = 0$. By computations, we get that $\sigma_Z(\mathcal{A}) = \{5, 10\}$. Taking Z-identity tensor $\mathcal{I}_1$. In the following, setting $\mu_1 = (10, 7)^T$, $\mu_2 = (9, 5)^T$ and $\mu_3 = (9, 5.5)^T$, we compute Table 1 to show the comparisons different methods with our results.

Table 1: The effect of parameters on the inclusion set

	Inclusion set with $\mu = (10, 7)^T$	Inclusion set with $\mu = (9, 5)^T$	Inclusion set with $\mu = (9, 5.5)^T$
Theorem 2 of [5]	$[2, 13]$	$[2, 12]$	$[2.5, 12]$
Theorem 1 of [11]	$[2.595, 12.851]$	$[2.522, 11.462]$	$[3, 11.5]$
Theorem 2 of [11]	$[2.595, 12.791]$	$[2.618, 11.462]$	$[3.541, 11.5]$
Theorem 2.3 part (a)	$[4.078, 12.172]$	$[3.078, 10.922]$	$[4, 10.872]$
Theorem 2.3 part (b)	$[4.264, 11.914]$	$[3.264, 10.736]$	$[4.197, 10.736]$

3. Z-eigenvalues-based sufficient conditions for the positive definiteness of even-order tensors

In this section, as an application, some sufficient conditions for testing the positive (semi-)definiteness of even-order weakly symmetric tensors are given.

Based on variational property of weakly symmetric tensors given in [11], the following result obtained.

Lemma 3.1. *[11, Lemma 1] $\mathcal{A} = (a_{i_1 i_2 \ldots i_m}) \in \mathbb{R}^{[m,n]}$ be a weakly symmetric tensor. Then, $f_{\mathcal{A}}(x) = \mathcal{A}x^m$ is positive definite if and only if its Z-eigenvalues are positive.*

Li et al. [5] proposed the following theorem to test the positive definiteness of polynomial systems via Gershgorin-type Z-eigenvalue inclusion sets.

Theorem 3.2. *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be a symmetric tensor with $m \geq 4$ being even. If there exists a positive real vector $\mu = (\mu_1, \ldots, \mu_n)^T \in \mathbb{R}^n$ such that $\mu_i > r_i(\mathcal{A}, \mu_i)$ for all $i \in [n]$, then $\mathcal{A}$ is positive definite.*

Based on Theorems 2.2 and 2.3, the following Z-eigenvalues based sufficient conditions can be obtained.

Theorem 3.3. *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be a weakly symmetric tensor with $m \geq 4$ being even. Then $\mathcal{A}$ is positive (semi-)definite, if there exists a positive real vector $\mu = (\mu_1, \ldots, \mu_n)^T \in \mathbb{R}^n$, such that at least one of the following conditions holds:*

a) $\mu_i > (\geq) M_i(\mathcal{A}, \mu_i) \quad \forall i \in [n].$

b) $\left(\mu_i - M_i^{\Omega_i}(\mathcal{A}, \mu_i)\right)\mu_j > (\geq) r_i^{\overline{\Omega_i}}(\mathcal{A}, \mu_i) M_j\left(\mathcal{A}, \mu_j\right) \quad \forall i, j \in [n],\ i \neq j.$

c) $\left(\mu_i - M_i^{\Omega_i}(\mathcal{A}, \mu_i)\right)\left(\mu_j - M_j^i\left(\mathcal{A}, \mu_j\right)\right) > (\geq) r_i^{\overline{\Omega_i}}(\mathcal{A}, \mu_i)\, |a_{ji\ldots i}|, \quad \forall i, j \in [n],\ i \neq j,$

and

$$\mu_i > (\geq) M_i^{\Omega_i}(\mathcal{A}, \mu_i) \qquad \text{and} \qquad \mu_j > (\geq) M_j^i\left(\mathcal{A}, \mu_j\right), \quad \forall i, j \in [n],\ i \neq j.$$

Proof. We prove that $\mathcal{A}$ is positive definite, and by a similar way one can prove that $\mathcal{A}$ is positive semi-definite. Let λ be a Z-eigenvalue of $\mathcal{A}$.
a) Suppose that $\lambda < 0$. From Theorem 2.2, we have $\lambda \in \Phi(\mathcal{A})$, hence, there is an $i_0 \in [n]$ such that

$$\left|\lambda - \mu_{i_0}\right| \leq M_i(\mathcal{A}, \mu_{i_0}).$$

On the other hand, for this index i_0, by $\mu_{i_0} > 0$ and $\lambda < 0$, we have

$$\left|\lambda - \mu_{i_0}\right| \geq \mu_{i_0} \geq M_i(\mathcal{A}, \mu_{i_0}).$$

This is a contradiction, and hence $\lambda > 0$. When $\mathcal{A}$ is a weakly symmetric tensor and all Z-eigenvalues are positive, we obtain that $\mathcal{A}$ is positive definite (by Lemma 3.1).
b) Suppose that $\lambda < 0$. From Theorem 2.3, we have $\lambda \in \mathcal{P}(\mathcal{A}, \mu)$. Thus, there exists $i_0 \in [n]$ such that

$$\left(\left|\lambda - \mu_{i_0}\right| - M_{i_0}^{\Omega_{i_0}}(\mathcal{A}, \mu_{i_0})\right)\left|\lambda - \mu_{j_0}\right| \leq r_{i_0}^{\overline{\Omega}_{i_0}}(\mathcal{A}, \mu_i) M_{j_0}\left(\mathcal{A}, \mu_{j_0}\right) \qquad \forall j_0 \neq i_0.$$

On the other hand, it follows from $\mu_{i_0} > 0$ and $\lambda < 0$ that

$$\left(\left|\lambda - \mu_{i_0}\right| - M_{i_0}{}^{\Omega_{i_0}}(\mathcal{A}, \mu_{i_0})\right)\left|\lambda - \mu_{j_0}\right| \geq \left(\mu_{i_0} - M_{i_0}{}^{\Omega_{i_0}}(\mathcal{A}, \mu_{i_0})\right)\mu_{j_0} \geq r_{i_0}^{\overline{\Omega}_{i_0}}(\mathcal{A}, \mu_i) M_{j_0}\left(\mathcal{A}, \mu_{j_0}\right) \quad \forall j_0 \neq i_0.$$

This is a contradiction. Therefore, $\mathcal{A}$ is positive definite.
c) The proof is obtained similar to the proof of part (b) and using Theorem 2.3. □

Compared with Theorems 3 and 4 of [11], our conclusions can more accurately determine the positive definiteness for even-order weakly symmetric tensors, as we show in the next example.

Example 3.4. *Let $\mathcal{A} = \left(a_{ijkl}\right) \in \mathbb{R}^{[4,2]}$, be a weakly symmetric tensor with entries defined as follows:*

$$a_{1111} = 6, \quad a_{1211} = 3, \quad a_{1221} = a_{1212} = 1, \quad a_{1121} = a_{1112} = 0, \quad a_{1122} = 4, \quad a_{1222} = \frac{2}{3};$$
$$a_{2111} = a_{2112} = a_{2121} = 1, \quad a_{2211} = 4, \quad a_{2221} = a_{2122} = 0, \quad a_{2212} = 2, \quad a_{2222} = 6.$$

By computations, we obtain that the minimum Z-eigenvalue is 4.9479. Hence, $\mathcal{A}$ is positive definite. Taking the Z-identity tensor $\mathcal{I}_Z$ as Case I or Case II, we cannot find positive real number μ_1 such that

$$\mu_1 > r_1(\mathcal{A}, \mu_1) \quad \textit{and} \quad \mu_1 > r_1^2(\mathcal{A}, \mu_1),$$

which shows that Theorem 3.2 of [5] and Theorems 3,4 of [11] fails to check the positive definiteness of weakly symmetric tensor $\mathcal{A}$. Setting $\mu = (6, 6)^T$, from part (a) of Theorem 3.3, we verify

$$\mu_1 = 6 > 5.1667 = M_1(\mathcal{A}, \mu_1) \quad \textit{and} \quad \mu_2 = 6 > 5 = M_2(\mathcal{A}, \mu_2),$$

which implies that $\mathcal{A}$ is positive definite. The verification method of other parts are similar to part (a).

4. Conclusions

In this paper, we firstly presented a new Z-eigenvalue localization set $\Phi(\mathcal{A}, \mu)$ for even-order tensors, which is a generalization of the set $\mathcal{G}(\mathcal{A}, \mu)$. Then, by classifying the index set, we obtained some optimal sets $\mathcal{P}(\mathcal{A}, \mu)$ and $\mathcal{D}(\mathcal{A}, \mu)$. By Example 2.4, we showed that these are tighter than existing results. Based on these sets, we attained some sufficient conditions for the positive (semi-)definiteness of even-order real weakly symmetric tensors. Finally in Example 3.4, we indicated the efficiency of our results.

Declaration of competing interest

The authors declare that they have no conflict of interest.

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