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Multiplicative Laplace transform in $q-$ calculus

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Abstract. In this study, we introduce q^*- (or $q-$multiplicative) Laplace transform by means of q^*-integral. Some properties of q^*-Laplace transform are presented. Also, q^*-Laplace transform can be utilized for solving q^*-linear differential equations.

1. Introduction

Non-Newtonian calculus provided scientists to a different perspective for problems encountered in science and engineering. Grossman and Katz examined Non-Newtonian calculus composing of branches of geometric, anageometric and biogeometric calculus [20]. Afterwards, geometric calculus is named as multiplicative calculus by Stanley [25]. Thereafter, some works in respect to multiplicative calculus are provided by Campell [13]. Bashirov et al. [8] introduced notions of multiplicative calculus at length and some applications to the features of derivative and integral operators of this calculus. Recently, some authors [6, 9, 17–19, 24, 29–33] have demonstrated that multiplicative calculus can be utilized efficiently in the solution of problems in some science and engineering areas. In [30], Yalcin et al. applied the Laplace transform to multiplicative calculus. Here, let's talk about quantum calculus on which we will build the Laplace transform in multiplicative calculus.

Quantum calculus is new area for study of calculus. The quantum calculus or $q-$calculus was initiated by Jackson [21, 22] in 1909 and 1910. But, this type of calculus had already been invented by Euler and Jacobi. Adams [2], Carmichael [14] provided the theory of linear $q-$difference equations. Analytic theory of linear $q-$difference equations was studied by Trjitzinsky [26] in 1933.

Today, it attracted interest as a result of increasing demand of mathematics that models quantum computing. $q-$calculus serve as a relation between mathematics and physics. There are numerous applications of $q-$calculus in various mathematical fields such as number theory, combinatorics, orthogonal polynomials, hypergeometric functions and quantum theory, mechanics, and the theory of relativity. In a great number of fundamental features of quantum calculus are explained in a book written by Kac and Cheung [23]. In 2004, variational $q-$calculus was examined by Bangerezako [7]. Bohner and Hudson [11] examined Euler-type boundary value problems in quantum calculus in 2007. Ahmad proved existence of solutions for

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Boundary-value problems for nonlinear third-order $q-$difference equations in 2011 [3]. The $q-$exponential and $q-$trigonometric functions were developed by Cieslinski [16]. Yu and Wang [36] solved nonlinear second-order $q-$difference equations with first order $q-$derivatives. Alp and Sarıkaya [5] describe features of quantum integral called $q-$integral. Apart from this, $q-$calculus has many applications in different subjects [4].

There are many basic aspects of $q-$calculus. It has been shown that $q-$calculus is subarea of more extensive mathematical area of time scale calculus. Time scales afford a framework for investigating dynamic equations on both discrete and continuous domains [10, 12, 35]. Evaluation of derivatives and integrals on time scale and hence the solution of differential equations is quite complicated. Thus, the results attained by studying the $q-$calculus, which has numerous applications in many areas, are exceedingly significant.

Very significant work has been performed in conventional case and $q-$calculus on the Laplace transform. Solutions of initial value problems are obtained by utilizing the Laplace transform [1, 5, 15]. For instance, $h-$ and $q-$Laplace transformations are defined by Bohner and Guseinov [10]. Ucar and Albayrak [28] examined $q-$Laplace integral operators and applications in 2012.

There are studies that combine $q-$calculus and multiplicative calculus. Yener and Emiroglu [34] introduced $q-$analogue of some fundamental ideas of multiplicative calculus and they named it as q^*-calculus. They presented q^*-calculus and some essential theorems about derivatives, integrals and infinite products which are verified within this calculus.

In this study, we present q^*-Laplace transform by the help of q^*-integral. Some features of q^*-Laplace transform are given. Additionally, we successfully can apply the q^*-Laplace transform to q^*-linear differential equations.

2. Preliminaries

In this section, we provide some essential ideas of multiplicative calculus, $q-$calculus, $q-$analogue of multiplicative calculus and some related concepts as conventional Laplace transform, multiplicative Laplace transform and $q-$Laplace transform. Let us now explain the definitions and theorems necessary to understand the topic.

2.1. Multiplicative calculus and its fundamental properties

Definition 2.1. *[8, 9, 13, 25] Let $f : \mathbb{R} \longrightarrow \mathbb{R}^+$ be a function. The multiplicative derivative of f is defined as follow:*

$$\frac{d^* f}{dt}(t) = f^*(t) = \lim_{h \to 0} \left(\frac{f(t+h)}{f(t)} \right)^{\frac{1}{h}}.$$

Utilizing features of usual derivative, multiplicative derivative is given by

$$\frac{d^* f}{dt}(t) = f^*(t) = e^{\frac{f'(t)}{f(t)}} = e^{(\ln \circ f)'(t)}$$

where $(\ln \circ f)(t) = \ln f(t)$. By n times iterated multiplicative differentiation operation, a positive f function possesses an $n-th$ order multiplicative derivative at the point t and described as

$$f^{*(n)}(t) = e^{(\ln \circ f)^{(n)}(t)}.$$

Theorem 2.2. *[8, 9, 13, 25] If a positive function f has multiplicative derivative at point t, then it has classical derivative and the relation between them can be demonstrated as*

$$f'(t) = f(t) \ln f^*(t).$$

Theorem 2.3. *[8, 9, 13, 25] Let f and g be multiplicative differentiable. If c is arbitrary constant, then cf, fg, $f+g$, f/g, f^g have multiplicative derivative and their multiplicative derivatives can be demonstrated as*

(1) $(cf)^*(t) = f^*(t)$,
(2) $(fg)^*(t) = f^*(t)\, g^*(t)$,
(3) $(f+g)^*(t) = f^*(t)^{\frac{f(t)}{f(t)+g(t)}}\, g^*$,
(4) $\left(\dfrac{f}{g}\right)^* = \dfrac{f^*(t)}{g^*(t)}$,
(5) $(f^g)^*(t) = f^*(t)^{g(t)}\, f(t)^{g'(t)}$.

Theorem 2.4. *[8, 9, 13, 25] $f^*(t) = 1$ for every $t \in (a,b)$ necessary and sufficient condition $f(t) = c > 0$ is constant function on (a,b).*

Theorem 2.5. *[8, 9, 13, 25] Let g be multiplicative differentiable, f have derivative in classical sense. If*

$$f(t) = (g \circ h)(t),$$

then, we can write the following

$$f^*(t) = [g^*(h(t))]^{h'(t)}.$$

Theorem 2.6. *[8, 9, 13, 25] If f is positive function, then $f^*(t) = 1$ if and only if $f'(t) = 0$.*

Definition 2.7. *[8, 9, 13, 25] Let f be positive, bounded and Riemann integrable function on $[a,b]$, then multiplicative integral is defined by*

$$\int_a^b f(t)^{dt} = e^{\int_a^b (\ln f(t))dt}.$$

This multiplicative integral possesses the following features:

(1) $\displaystyle\int_a^b \left(f(t)^k\right)^{dt} = \left(\int_a^b (f(t))^{dt}\right)^k, k \in \mathbb{R}$,
(2) $\displaystyle\int_a^b (f(t)\, g(t))^{dt} = \int_a^b (f(t))^{dt} \int_a^b (g(t))^{dt}$,
(3) $\displaystyle\int_a^b \left(\frac{f(t)}{g(t)}\right)^{dt} = \frac{\int_a^b (f(t))^{dt}}{\int_a^b (g(t))^{dt}}$,
(4) $\displaystyle\int_a^b f(t)^{dt} = \int_a^c f(t)^{dt} \int_c^b f(t)^{dt}, a \le c \le b$

where f and g are integrable on $[a,b]$ in meaning of multiplicative integral.

Definition 2.8. *[30] Assume that f is a positive function on $[0,\infty)$. Then multiplicative Laplace transform of $f(t)$ is represented by*

$$\mathcal{L}_m\{f(t)\} = F_m(s) = \int_0^\infty f(t)^{e^{-st}dt} = e^{\int_0^\infty e^{-st}\ln f(t)dt} = e^{\mathcal{L}\{\ln f(t)\}},$$

where $\mathcal{L}$ is classical Laplace transform.

2.2. $q-$calculus and its basic properties

Definition 2.9. *[21–23] Let $q \in (0,1)$. $q-$natural number $[n]_q$ is defined by,*

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}.$$

Definition 2.10. *[21–23] Factorial of a q-number $[n]_q$ is represented by*

$$[n]_q! = \begin{cases} 1, & n = 0 \\ [n]_q[n-1]_q[n-2]_q \cdots [1]_q, & n = 1, 2, 3, \cdots \end{cases}$$

Definition 2.11. *[21, 22] $q-$analogue of $(x-a)_q^n$ is denoted by*

$$(x-a)_q^n = \begin{cases} 1, & n = 0 \\ [x-a]_q[x-qa]_q[x-q^2a]_q \cdots [x-q^{n-1}a]_q, & n = 1, 2, 3, \cdots \end{cases}$$

Definition 2.12. *[21–23] $q-$binomial coefficient is defined as*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{k![n-k]!},$$

where $n, k \in \mathbb{N}$.

Definition 2.13. *[21–23] $q-$differential of f is represented by*

$$\begin{aligned} d_q f(x) &= f(qx) - f(x) \\ d_q x &= (q-1)x, \end{aligned}$$

where $q \in (0,1)$ and f is arbitrary function.

Definition 2.14. *[21–23] The $q-$derivative of f on a subset of $\mathbb{R}$ is denoted by*

$$\begin{aligned} D_q f(x) &= \frac{d_q f(x)}{d_q x}, \ x \neq 0, \\ D_q f(0) &= \lim_{x \to 0} D_q f(x), \end{aligned} \tag{1}$$

where qx and x should be in the domain of f and D_q is $q-$difference operator.

It is obvious that if $f(x)$ is differential, then we can acquire the following limit

$$\lim_{q \to 1} D_q f(x) = \frac{df(x)}{dx} = f'(x).$$

Higher order $q-$derivatives are represented by

$$D_q^n f(t) = D_q D_q^{n-1} f(t),$$

where n is element of $\mathbb{N}$.

Theorem 2.15. *[23] If f, g are $q-$differentiable functions, then D_q is a linear operator. That is to say, for any constants α and β*

$$D_q\{\alpha f(t) + \beta g(t)\} = \alpha D_q\{f(t)\} + \beta D_q\{g(t)\}.$$

Remark 2.16. *[23] If f, $g \neq 0$ are $q-$differentiable functions, then the following statements are hold.*

$$\begin{aligned} D_q\{f(t)\,g(t)\} &= f(qt)\,D_q g(t) + g(t)\,D_q f(t), \\ D_q\left\{\frac{f(t)}{g(t)}\right\} &= \frac{g(t)\,D_q f(t) - f(t)\,D_q g(t)}{g(qt)\,g(t)}. \end{aligned}$$

Definition 2.17. *[15] $q-$analogues of classical exponential function e^x is denoted by*

$$\begin{aligned} e_q^x &= \sum_{j=0}^{\infty} \frac{x^j}{[j]!}, \\ E_q^x &= \sum_{j=0}^{\infty} q^{\frac{j(j-1)}{2}} \frac{x^j}{[j]!}. \end{aligned}$$

$q-$exponential functions hold relations as follows:

$$e_q^x E_q^{-x} = E_q^x e_q^{-x} = 1,\ E_q^x = e_{\frac{1}{q}}^x,\ E_q^0 = 1.$$

Moreover, for $E_q^{-x} = \frac{1}{e_q^x}$ we obtain

$$\lim_{x\to\infty} E_q^{-x} = \lim_{x\to\infty} \frac{1}{e_q^x} = 0.$$

Definition 2.18. *[15] The $q-$analogues of traditional trigonometric functions are defined by*

$$\begin{aligned} sin_q x &= \tfrac{e_q^{ix} - e_q^{-ix}}{2i}, \quad Sin_q x = \tfrac{E_q^{ix} - E_q^{-ix}}{2i}, \\ cos_q x &= \tfrac{e_q^{ix} + e_q^{-ix}}{2}, \quad Cos_q x = \tfrac{E_q^{ix} + E_q^{-ix}}{2}. \end{aligned}$$

$q-$cosine and $q-$sine hyperbolic functions are

$$cosh_q ax = \frac{e_q^{ax} + e_q^{-ax}}{2} \text{ and } sinh_q ax = \frac{e_q^{ax} - e_q^{-ax}}{2}.$$

Definition 2.19. *[21–23] Jackson integral or $q-$integral of $f(x)$ is represented by*

$$\int f(x)\ d_q x = (1-q)\,x \sum_{j=0}^{\infty} q^j f\left(q^j x\right).$$

Remark 2.20. *[21–23] From the definition above, we can easily get a more comprehensive formula as*

$$\begin{aligned} \int f(x)\,D_q g(x)\,d_q x &= (1-q)\,x \sum_{j=0}^{\infty} q^j f\left(q^j x\right) D_q g\left(q^j x\right) \\ &= (1-q)\,x \sum_{j=0}^{\infty} q^j f\left(q^f x\right) \frac{g\left(q^j x\right) - g\left(q^{j+1} x\right)}{(1-q)\,q^j x}, \end{aligned}$$

or

$$\int f(x)\,d_q g(x) = \sum_{j=0}^{\infty} f\left(q^j x\right)\left(g\left(q^j x\right) - g\left(q^{j+1} x\right)\right).$$

Definition 2.21. *[21–23] Let $0 < a < b$. Definite $q-$integral can be defined as follows*

$$\int_0^b f(x)\, d_q x = (1-q)\, b \sum_{j=0}^{\infty} q^j f\left(q^j b\right)$$

and

$$\int_a^b f(x)\, d_q x = \int_0^b f(x)\, d_q x - \int_0^a f(x)\, d_q x. \tag{2}$$

By using (1) *and* (2)*, we obtain*

$$\int_0^x D_q f(t)\, d_q t = f(x) - f(0).$$

Definition 2.22. *[15] $q-$Laplace transform of $f(t)$ is denoted by*

$$\mathcal{L}_q\{f(t)\} = F(s) = \int_0^{\infty} E_q(-qst)\, f(t)\, d_q t,\ s > 0.$$

Then,

$$\mathcal{L}_q\{\alpha f(t) + \beta g(t)\} = \alpha \mathcal{L}_q\{f(t)\} + \beta \mathcal{L}_q\{g(t)\}$$

where α and β are constants.

2.3. q^*-calculus and its properties

Definition 2.23. *[34] Let $0 < q < 1$ and f be a positive function. q^*- derivative of f is defined by*

$$D_q^*\{f(x)\} = \left(\frac{f(qx)}{f(x)}\right)^{\frac{1}{(q-1)x}}.$$

It can be easliy observed that the following limit is hold:

$$\lim_{q \to 1} D_q^*\{f(x)\} = \frac{d^*}{dx} f(x).$$

$n-th$ order q^*-derivative or q^*-derivative is represented by

$$f_q^{*(n)}(x) = D_q^{*(n)}\{f(x)\} = D_q^*\left\{D_q^{*(n-1)} f(x)\right\}$$

where n is element of $\mathbb{N}$.

Theorem 2.24. *[34] Assume that $0 < q < 1$ and f is a $q-$differentiable positive function. Then, q^*-derivative of f can be given by*

$$D_q^*\{f(x)\} = e_q^{D_q\{\ln f(x)\}}. \tag{3}$$

Corollary 2.25. *[34] Suppose that $0 < q < 1$ and f is positive function. $n-th$ order q^*-derivative of f is denoted by*

$$D_q^{*(n)}\{f(x)\} = e_q^{D_q^{(n)}\{\ln f(x)\}}. \tag{4}$$

Theorem 2.26. *[34] Let f, g be q^*-differentiable and α, β be positive constants. Then, the following rules can be easily obtained:*

(1) $D_q^*\{\alpha f(x)\} = D_q^*\{f(x)\}$,

(2) $D_q^*\{f(x)g(x)\} = D_q^*\{f(x)\}D_q^*\{g(x)\}$,

(3) $D_q^*\left\{\dfrac{f(x)}{g(x)}\right\} = D_q^*\{f(x)\}/D_q^*\{g(x)\}$,

(4) $D_q^*\{(f\circ g)(x)\} = D_{q,g}\{f(g(x))\}^{g_q(x)}$,

(5) $D_q^*\left\{f(x)^{g(x)}\right\} = \left(D_q^*\{f(x)\}\right)^{g(qx)} f(x)^{D_q\{g(x)\}}$.

Definition 2.27. *[34] q^*-integral can be given by*

$$\begin{aligned}\int f(x)^{d_qx} &= e_q^{\int \ln f(x)d_qx} \\ &= e_q^{(1-q)x\sum_{i=0}^{\infty} q^i \ln f(q^ix)},\end{aligned} \tag{5}$$

where q is element of $(0,1)$.

Note that

$$\lim_{q\to 1}\int_a^b f(x)^{d_qx} = \int_a^b f(x)^{d_qx}.$$

Definition 2.28. *[34] Assume that f is a positive function on (a,b). Then, q^*-integral is denoted by*

$$\int_b^a f(x)^{d_qx} = e_q^{\int_0^b \ln f(x)d_qx - \int_0^a \ln f(x)d_qx}. \tag{6}$$

Remark 2.29. *Since the properties $e_q^{\ln x} = x$ and $\ln e_q^x = x$ are not provided, e should be taken instead of e_q in formulas* (3)−(6).

3. Main Results

In the present section, the properties of this new transform will be given by establishing q-multiplicative Laplace transform with aid of multiplicative Laplace transform in multiplicative calculus [30].

Definition 3.1. *Let f be a positive function given on $[0,\infty)$. Therefore, q-multiplicative Laplace transform or q^*-Laplace transform of $f(t)$ is given by*

$$\mathcal{L}_{q^*}\{f(t)\} = F_{q^*}(s) = \int_0^\infty f(t)^{E_q^{-qst}d_qt} = e^{\int_0^\infty E_q^{-qst}\ln f(t)d_qt} = e^{\mathcal{L}_q\{\ln f(t)\}} \tag{7}$$

By the way of definition, q^*-Laplace transform of some fundamental functions can be written as follows.

$$\mathcal{L}_{q^*}\{1\} = F_{q^*}(s) = e^{\int_0^\infty E_q^{-qst}\ln 1 d_qt} = e^0 = 1, \tag{8}$$

$$\mathcal{L}_{q^*}\{e^t\} = F_{q^*}(s) = e^{\int_0^\infty E_q^{-qst}\ln e^t d_qt} = e^{\int_0^\infty E_q^{-qst}t d_qt} = e^{\mathcal{L}_q\{t\}} = e^{\frac{1}{s^2}}, \tag{9}$$

$$\mathcal{L}_{q^*}\{e^{e_q^{at}}\} = F_{q^*}(s) = e^{\mathcal{L}_q\{e_q^{at}\}} = e^{\frac{1}{s-a}}, \tag{10}$$

$$\mathcal{L}_{q^*}\{e^{E_q^{at}}\} = F_{q^*}(s) = e^{\mathcal{L}_q\{E_q^{at}\}} = e^{\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}}\frac{a^n}{s^{n+1}}}, \tag{11}$$

$$\mathcal{L}_{q^*}\{e^{sin_q at}\} = F_{q^*}(s) = e^{\mathcal{L}_q\{sin_q at\}} = e^{\frac{a}{s^2+a^2}}, \tag{12}$$

$$\mathcal{L}_{q^*}\{e^{cos_q at}\} = F_{q^*}(s) = e^{\mathcal{L}_q\{cos_q at\}} = e^{\frac{s}{s^2+a^2}}, \tag{13}$$

$$\mathcal{L}_{q^*}\{e^{sinh_q at}\} = F_{q^*}(s) = e^{\mathcal{L}_q\{sinh_q at\}} = e^{\frac{a}{s^2-a^2}}, \tag{14}$$

$$\mathcal{L}_{q^*}\{e^{cosh_q at}\} = F_{q^*}(s) = e^{\mathcal{L}_q\{cosh_q at\}} = e^{\frac{s}{s^2-a^2}}. \tag{15}$$

Theorem 3.2. *q^*-Laplace transform has linearity property in the meaning of multiplicative. That is to say, let c_1, c_2 be arbitrary exponents and f_1, f_2 be two given positive functions which have q^*-Laplace transform. Therefore,*

$$\mathcal{L}_{q^*}\{f_1^{c_1} f_2^{c_2}\} = \mathcal{L}_{q^*}\{f_1\}^{c_1} \mathcal{L}_{q^*}\{f_2\}^{c_2}.$$

Proof. Using the definition of q^*-Laplace transform, we have

$$\begin{aligned}\mathcal{L}_{q^*}\{f_1^{c_1} f_2^{c_2}\} &= e^{\int_0^\infty E_q^{-qst} \ln\left(f_1^{c_1} f_2^{c_2}\right) d_q t} \\ &= e^{\int_0^\infty E_q^{-qst}\left(c_1 \ln f_1 + c_2 \ln f_2\right) d_q t} \\ &= \left(e^{\int_0^\infty E_q^{-qst} \ln f_1}\right)^{c_1} \left(e^{\int_0^\infty E_q^{-qst} \ln f_2}\right)^{c_2} \\ &= \mathcal{L}_{q^*}\{f_1(t)\}^{c_1} \mathcal{L}_{q^*}\{f_2(t)\}^{c_2}.\end{aligned}$$

Hence, it appears that

$$\mathcal{L}_{q^*}\{f_1^{c_1} f_2^{c_2}\} = \mathcal{L}_{q^*}\{f_1\}^{c_1} \mathcal{L}_{q^*}\{f_2\}^{c_2}.$$

□

Theorem 3.3. *If $\mathcal{L}_{q^*}\{f(t)\} = F_{q^*}(s)$, then we get*

$$\mathcal{L}_{q^*}\{f(at)\} = F_{q^*}\left(\frac{s}{a}\right)^{\frac{1}{a}}.$$

Proof. Using the definition, we get

$$\mathcal{L}_{q^*}\{f(at)\} = e^{\int_0^\infty \ln f(at) E_q^{-qst} d_q t}.$$

If we apply substitution in the integral above, we have

$$\begin{aligned}\mathcal{L}_{q^*}\{f(at)\} &= e^{\frac{1}{a}\int_0^\infty \ln f(u) E_q^{\frac{-qsu}{a}} d_q u} \\ &= \left(e^{\int_0^\infty \ln f(u) E_q^{\frac{-qsu}{a}} d_q u}\right)^{\frac{1}{a}}.\end{aligned}$$

Hence, we get

$$\mathcal{L}_{q^*}\{f(at)\} = F_{q^*}\left(\frac{s}{a}\right)^{\frac{1}{a}}.$$

□

Definition 3.4. *Let f be a positive function given on $[0, \infty)$, and $\ln f(t)$ be a function on $[0, \infty)$. If there exist positive constants t_0, K and α such that*

$$|f(t)| \leq K e^{e_q^{\alpha t}}$$

for $t > t_0$, then we can say that f function is α-double exponential order.

Theorem 3.5. *If f is a positive function, that satisfy property of $\alpha-$double exponential order for $t > t_0$ on $[0, \infty)$, and $\ln f(t)$ is a piecewise continuous function on $[0, \infty)$, then for $s > \alpha$, $\mathcal{L}_{q^*}\{f(t)\}$ exists.*

Proof. We are going to prove that for $s > \alpha$, integral $\int_0^\infty E_q^{-qst} \ln f(t) d_q t$ is convergent. Firstly, we write the integral as follows,

$$\int_0^\infty E_q^{-qst} \ln f(t) d_q t = \int_0^{t_0} E_q^{-qst} \ln f(t) d_q t + \int_{t_0}^\infty E_q^{-qst} \ln f(t) d_q t. \tag{16}$$

The first expression in (16) is convergent for arbitrary s due to $\ln f(t)$ and therefore $E_q^{-qst} \ln f(t)$ is piecewise continuous given on $[0, t_0]$. However, as $f(t)$ satisfy property of $\alpha-$double exponential order,

$$|f(t)| \leq K e^{e_q^{\alpha t}}$$

for $t > t_0$. Consequently,

$$|E_q^{-qst} \ln f(t)| = E_q^{-qst} |\ln f(t)| \leq \left(\ln K + e_q^{\alpha t}\right) E_q^{-qst}$$

for all $t > t_0$, and

$$\frac{\ln K}{s} E_q^{-st_0} + \frac{1}{s-\alpha} < \infty$$

for $s > \alpha$. That is to say, for $t > t_0$, second expression in the right hand side of (16) is convergent. It shows that both of integrals in right side of (16) exist, therefore for $s > \alpha$ $\mathcal{L}_{q^*} \{f(t)\}$ exists. □

Theorem 3.6. *If $f(t)$ is a positive function that holds $\alpha-$double exponenetial order for $t > t_0$ on $(0, \infty]$, and $\ln f(t)$ is piecewise function that satisfies property of continuity given on $(0, \infty]$, then*

$$\lim_{s\to\infty} \mathcal{L}_{q^*} \{f(t)\} = 1.$$

Proof. By the way of (3.5), we have

$$\begin{aligned} \mathcal{L}_{q^*} \{f(t)\} &\leq e^{\int_0^\infty \ln\left(K e^{e_q^{\alpha t}}\right) E_q^{-qst} d_q t} \\ &\leq e^{\int_0^\infty (\ln K) E_q^{-qst} d_q t + \int_0^\infty e^{\alpha t} E_q^{-qst} d_q t} \\ &\leq e^{(\ln K)\left(\frac{E_q^{-st_0}}{s}\right)} e^{\frac{1}{s-\alpha}}. \end{aligned}$$

for all $s > \alpha$. Lastly, taking limit from two sides as $s \to \infty$, we obtain

$$\lim_{s\to\infty} \mathcal{L}_{q^*} \{f(t)\} \leq \lim_{s\to\infty} e^{(\ln K)\left(\frac{E_q^{-st_0}}{s}\right)} \lim_{s\to\infty} e^{\frac{1}{s-\alpha}} = 1.$$

□

Theorem 3.7. *If $f(t)$ is a function satisfying $\alpha-$double exponential order given on $[0, B]$ and $f_q^*(t)$ (or $D_q^*[f(t)]$) is a piecewise function that holds property of continuity on $[0, B]$, then q^*-Laplace transform of q^*-derivative is*

$$\mathcal{L}_{q^*} \left\{f_q^*(t)\right\} = \frac{1}{f(0)} F_{q^*}(s)^s$$

for $s > \alpha$.

Proof. By the way of the definition, we get

$$\begin{aligned}\mathcal{L}_{q^*}\left\{f_q^*(t)\right\} = \mathcal{L}_{q^*}\left\{D_q^*[f(t)]\right\} = \int_0^\infty D_q^*[f(t)]^{E_q^{-qst}d_qt} &= e^{\int_0^\infty E_q^{-qst}\ln D_q^*\left[\ln f(t)\right]d_qt}\\ &= e^{\int_0^\infty E_q^{-qst}\ln e_q^{D_q\left[\ln f(t)\right]d_qt}}\\ &= e^{\int_0^\infty E_q^{-qst}D_q\left[\ln f(t)\right]d_qt}\\ &= e^{\lim_{B\to\infty}\int_0^B E_q^{-qst}D_q\left[\ln f(t)\right]d_qt}\end{aligned}$$

Here, $t_0, t_1, ..., t_n$ indicate the points of discontinuities of function $D_q^*[f(t)]$. By means of these points, the integral can be written as follows

$$\int_0^B E_q^{-qst}D_q[\ln f(t)]\,d_qt = \int_0^{t_1} E_q^{-qst}D_q[\ln f(t)]\,d_qt + \int_{t_1}^{t_2} E_q^{-qst}D_q[\ln f(t)]\,d_qt + ... + \int_{t_n}^{B} E_q^{-qst}D_q[\ln f(t)]\,d_qt.$$

Through the integration by parts method in quantum calculus, we have

$$\begin{aligned}\int_0^B E_q^{-qst}D_q[\ln f(t)] &= E_q^{-qst}\ln f(t)|_0^{t_1} + E_q^{-qst}\ln f(t)|_{t_1}^{t_2} + ... + E_q^{-qst}\ln f(t)|_{t_n}^{B}\\ &\quad + s\left(\int_0^{t_1}\ln f(t)E_q^{-qst}d_qt + \int_{t_1}^{t_2}\ln f(t)E_q^{-qst}d_qt + ... + \int_{t_n}^{B}\ln f(t)E_q^{-qst}d_qt\right).\end{aligned}$$

Since $f(t)$ is continuous, we can combine domains of integration of the above expression in one domain. Therefore, we get

$$e^{\int_0^B E_q^{-qst}D_q\left[f(t)\right]d_qt} = e^{E_q^{-qsB}\ln f(B) - \ln f(0) + s\left(\int_0^B \ln f(t)E_q^{-qst}d_qt\right)}.$$

Since $B\to\infty$, $E_q^{-qsB}\ln f(B)\to 0$ and $e^{E_q^{-qsB}\ln f(B)}\to 1$ for $s>\alpha$. Therefore, we have

$$\mathcal{L}_{q^*}\left\{f_q^*(t)\right\} = \mathcal{L}_{q^*}\left\{D_q^*[f(t)]\right\} = e^{-\ln f(0)}\left(e^{\int_0^\infty \ln f(t)E_q^{-qst}d_qt}\right)^s = \frac{1}{f(0)}F_{q^*}(s)^s.$$

By modifying the method in [30], we proved the theorem. □

If we interchange $f(t)$ and $f_q^*(t)$ in this theorem by $f_q^*(t)$ and $f_q^{**}(t)$, respectively, q^*-Laplace transformation of $f_q^{**}(t)$ is as the following

$$\mathcal{L}_{q^*}\left\{f_q^{**}(t)\right\} = \frac{1}{f(0)^s f_q^*(0)}F_{q^*}(s)^{s^2},$$

for $s>\alpha$. If we apply method of induction to this theorem, then q^*-Laplace transformation of function $f_q^{*(n)}(t)$ can be generalised as in the following result.

Corollary 3.8. *If f, $f_q^*(t)$, $f_q^{**}(t)$, ..., $f_q^{*(n-1)}(t)$ are functions which satisfies property of continuity, $f_q^{*(n)}(t)$ is piecewise function that holds property of continuity defined on the interval $0\le t\le A$. Furthermore assume that there have positive real numbers K, α and t_0 such that*

$$|f(t)|\le Ke_q^{e^{\alpha t}}, |f_q^*(t)|\le Ke_q^{e^{\alpha t}}, ..., |f_q^{*(n-1)}(t)|\le Ke_q^{e^{\alpha t}}$$

for $t\ge t_0$, then for $s>\alpha$ there exist q^*-Laplace transform $\mathcal{L}_{q^*}\left\{f_q^{*(n)}(t)\right\}$ and we can calculate it by means of the formula in the following

$$\mathcal{L}_{q^*}\left\{f_q^{*(n)}(t)\right\} = \frac{1}{f(0)^{s^{n-1}} f_q^*(0)^{s^{n-2}} f_q^{**}(0)^{s^{n-3}} ... f_q^{*(n-1)}(0)}F_{q^*}(s)^{s^n}. \tag{17}$$

Theorem 3.9. *If f_1 and f_2 are positive function definite functions satisfying continuity, then $f_1 = f_2$ if and only if $\mathcal{L}_{q^*}\{f_1\} = \mathcal{L}_{q^*}\{f_2\}$.*

Proof. $f_1 = f_2$ if and only if $\ln f_1 = \ln f_2$. Using $q-$Laplace transform of quantum calculus, we obtain

$$\mathcal{L}_q\{\ln f_1\} = \mathcal{L}_q\{\ln f_2\} \Leftrightarrow e^{\mathcal{L}_q\{\ln f_1\}} = e^{\mathcal{L}_q\{\ln f_2\}} \Leftrightarrow \mathcal{L}_{q^*}\{f_1\} = \mathcal{L}_{q^*}\{f_2\}.$$

□

Definition 3.10. *If $F_{q^*}(s)$ is the q^*-Laplace transform of a positive, continuous function f, that is*

$$\mathcal{L}_{q^*}\{f\} = F,$$

then $\mathcal{L}_{q^*}^{-1}\{F\}$ is named as the inverse q^*-Laplace transform of F.

Theorem 3.11. *Let $\mathcal{L}_{q^*}^{-1}\{F(s)\} = f(t)$ and $\mathcal{L}_{q^*}^{-1}\{G(s)\} = g(t)$, then we have*

$$\mathcal{L}_{q^*}^{-1}\{F_{q^*}(s)^{G(s)}\} = \int_0^t f(x)^{g(t-qx)^{d_qx}}.$$

Proof. If we apply q^*-Laplace transform to integral $\int_0^t f(x)^{g(t-qx)^{d_qx}}$, then we get

$$\begin{aligned}\mathcal{L}_{q^*}\left\{\int_0^t f(x)^{g(t-qx)^{d_qx}}\right\} &= \mathcal{L}_{q^*}\left\{e^{\int_0^t g(t-qx)\ln f(x)d_qx}\right\} \\ &= e^{\mathcal{L}_q\left\{\ln\left(e^{\int_0^t g(t-qx)\ln f(x)d_qx}\right)\right\}} \\ &= e^{\mathcal{L}_q\left\{\int_0^t g(t-qx)\ln f(x)d_qx\right\}}.\end{aligned}$$

By the way of convolution property of quantum calculus [15], we have

$$\begin{aligned}\mathcal{L}_{q^*}\left\{\int_0^t f(x)^{g(t-qx)^{d_qx}}\right\} &= e^{\mathcal{L}_q\{\ln f(x)\}\mathcal{L}_q\{g(x)\}} \\ &= \left(e^{\mathcal{L}_q\{\ln f(x)\}}\right)^{\mathcal{L}_q\{g(x)\}} \\ &= \left(\mathcal{L}_{q^*}\{f(x)\}\right)^{G(s)}.\end{aligned}$$

□

Theorem 3.12. *Inverse q^*-Laplace transform satisfies linearity in the meaning of multiplicative. That is to say, if c_1, c_2 are arbitrary exponents and f_1, f_2 are two given functions, which have q^*-Laplace transform F_1, F_2 respectively. Therefore,*

$$\mathcal{L}_{q^*}^{-1}\left\{F_1^{c_1}F_2^{c_2}\right\} = \mathcal{L}_{q^*}^{-1}\{F_1\}^{c_1}\mathcal{L}_{q^*}^{-1}\{F_2\}^{c_2}.$$

Proof. Assume that f_1 and f_2 are functions which satisfy property of continuity such that $F_1 = \mathcal{L}_{q^*}\{f_1\}$ and $F_2 = \mathcal{L}_{q^*}\{f_2\}$. On the basis of the definition, it is known that $\mathcal{L}_{q^*}\{F_1\} = f_1$ and $\mathcal{L}_{q^*}\{F_2\} = f_2$. Using property in Theorem 3.2, we get

$$\mathcal{L}_{q^*}\left\{f_1^{c_1}f_2^{c_2}\right\} = \mathcal{L}_{q^*}\{f_1\}^{c_1}\mathcal{L}_{q^*}\{f_2\}^{c_2} = F_1^{c_1}F_2^{c_2}.$$

Based on the definition of inverse q^*-Laplace transform we have

$$\mathcal{L}_{q^*}^{-1}\left\{F_1^{c_1}F_2^{c_2}\right\} = f_1^{c_1}f_2^{c_2} = \mathcal{L}_{q^*}^{-1}\{F_1\}^{c_1}\mathcal{L}_{q^*}^{-1}\{F_2\}^{c_2}.$$

□

4. Applications to q^*-linear differential equations

Formula for q^*-Laplace transforms of q^*-derivatives is given by (17). This formula contains q^*-Laplace transform of f and f, $f_q^*(t)$, $f_q^{**}(t)$, $f_q^{*(3)}(t)$, ..., $f_q^{*(n-1)}(t)$ functions and the values of these functions at $x = 0$. Hence, we can use it to get solutions of the initial value problems, especially q^*-linear differential equations with constant exponentials. We obtain solutions by applying q^*-Laplace transform to both sides of equations in these problems.

For instance, take into account the second order q^*-initial value problem

$$\left(y_q^{**}\right)^{a_0}\left(y_q^*\right)^{a_1}y^{a_2} = 1, \tag{18}$$

$$y(0) = b_0, \quad y_q^*(0) = b_1. \tag{19}$$

Here, a_0, a_1, a_2 and $b_0, b_1 > 0$ are defined as constants. If we apply q^*-Laplace transform (7) to both sides of (18) and through the q^*-linearity property of q^*-Laplace transform, then we obtain

$$\mathcal{L}_{q^*}\{y_q^{**}\}^{a_0}\mathcal{L}_{q^*}\{y_q^*\}^{a_1}\mathcal{L}_{q^*}\{y\}^{a_2} = \mathcal{L}_{q^*}\{1\}.$$

Now, using formula (17) and the initial conditions (19), we get

$$\left\{\frac{Y_{q^*}^{s^2}}{y(0)^s y_q^*(0)}\right\}^{a_0}\left\{\frac{Y_{q^*}(s)^s}{y(0)}\right\}^{a_1}\left\{Y_{q^*}(s)\right\}^{a_2} = 1.$$

If we rearrange the equation, then we get

$$\frac{Y_{q^*}(s)^{a_0s^2+a_1s+a_2}}{\left\{b_0^s b_1\right\}^{a_0}\{b_0\}^{a_1}} = 1.$$

In that case, we can write

$$\begin{aligned}
Y_{q^*}(s)^{a_0s^2+a_1s+a_2} &= b_0^{a_0s+a_1}b_1^{a_0}\\
Y_{q^*}(s) &= \left(b_0^{a_0s+a_1}b_1^{a_0}\right)^{\frac{1}{a_0s^2+a_1s+a_2}}\\
Y_{q^*}(s) &= (b_0)^{\frac{a_0s+a_1}{a_0s^2+a_1s+a_2}}(b_1)^{\frac{a_0}{a_0s^2+a_1s+a_2}}.
\end{aligned}$$

Thus, we apply inverse q^*-Laplace transform, we get

$$y(t) = \mathcal{L}_{q^*}^{-1}\left\{Y_{q^*}(s)\right\},$$

then the solution is found by considering the equalites (8)-(15) according to the values of the constants a_0, a_1, a_2.

We can apply this method to any order of linear differential equation with constant exponential.

Example 4.1. *Take into account the following q^*-differential equation*

$$y_q^* = y$$

with initial condition

$$y(0) = e.$$

Applying q^*-Laplace transform (7) *of two sides of the q^*- differential equation and through conditions which are given, we obtain*

$$
\begin{aligned}
&\mathcal{L}_{q^*}\left\{y_q^*\right\} = \mathcal{L}_{q^*}\left\{y\right\} \\
&\frac{Y_{q^*}(s)^s}{y(0)} = Y_{q^*}(s) \\
&Y_{q^*}(s)^{s-1} = e \\
&Y_{q^*}(s) = e^{\frac{1}{s-1}}
\end{aligned}
$$

Then, we take the inverse q^*-Laplace transform, we get

$$
\begin{aligned}
&\mathcal{L}_{q^*}^{-1}\left\{Y_{q^*}(s)\right\} = \mathcal{L}_{q^*}^{-1}\left\{e^{\frac{1}{s-1}}\right\} \\
&y\left(t\right) = e^{e_q^t}. \qquad (20)
\end{aligned}
$$

By taking limit of (20) *as $q \to 1$, we obtain $y\left(t\right) = e^{e^t}$ in [30].*

Example 4.2. *Take into consideration following q^*-differential equation*

$$y_q^{**} y_q^* = y$$

with initial conditions

$$y\left(0\right) = e, \quad y_q^*\left(0\right) = 1.$$

Applying q^*-Laplace transform (7) *of two sides of the q^*-differential equation and through conditions which are given, we get*

$$
\begin{aligned}
&\mathcal{L}_{q^*}\left\{y_q^{**} y_q^*\right\} = \mathcal{L}_{q^*}\left\{y\right\} \\
&\mathcal{L}_{q^*}\left\{y_q^{**}\right\} \mathcal{L}_{q^*}\left\{y_q^*\right\} = \mathcal{L}_{q^*}\left\{y\right\} \\
&\frac{Y_{q^*}\left(s\right)^{s^2+s-1}}{y\left(0\right)^{s+1} y_{q^*}\left(0\right)} = 1
\end{aligned}
$$

If we rearrange last equation, then we obtain

$$
\begin{aligned}
&Y_{q^*}\left(s\right)^{s^2+s-1} = e^{s+1} \\
&Y_{q^*}\left(s\right) = e^{\frac{s+1}{s^2+s-1}}
\end{aligned}
$$

Lastly, applying inverse q^*-Laplace transform, we have

$$
\begin{aligned}
&\mathcal{L}_{q^*}^{-1}\left\{Y_{q^*}\left(s\right)\right\} = \mathcal{L}_{q^*}^{-1}\left\{e^{\frac{s+1}{s^2+s-1}}\right\} \\
&y\left(t\right) = e^{\frac{5-\sqrt{5}}{10} e_q^{\left(-\frac{\sqrt{5}+1}{2}\right)t} + \frac{5+\sqrt{5}}{10} e_q^{\left(\frac{\sqrt{5}-1}{2}\right)t}}.
\end{aligned}
$$

5. Conclusions

The Laplace transform is one of the most useful and effective transforms in mathematical physics. It is effectively used in solving initial value problems in the classical case. It also has applications in different fields such as cryptology. Because of this efficiency and importance of this transformation, we defined the Laplace transform, which was previously defined in multiplicative calculus, in $q-$calculus in this study. We applied this transformation we defined to some problems in q^*-calculus and obtained solutions. Our results are a generalization of the present results.

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