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On the partial boundary value condition basing on the diffusion coefficient

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Abstract. The paper follows with interest in a nonlinear parabolic equation coming from the electrorheological fluid

$$u_t = \mathrm{div}(a(x)|\nabla u|^{p(x)-2}\nabla u) + \sum_{i=1}^{N} \frac{\partial b_i(u,x,t)}{\partial x_i}$$

with $a(x)$ being positive in Ω. We study the well-posedness problem of the equation under the condition $b_i(\cdot,x,t) = 0$ on the partial boundary $\partial\Omega \setminus \Sigma_1$ for every $i = 1,2,\cdots,N$, where $\Sigma_1 = \{x \in \partial\Omega : a(x) > 0\}$. The stability of the weak solutions is obtained only basing on a partial boundary value condition $u(x,t) = 0, (x,t) \in \Sigma_1 \times (0,T)$.

1. Introduction

In recent years, the initial-boundary value problem of the electrorheological fluid equation [1, 21, 24]

$$u_t - \mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = 0,\ (x,t) \in Q_T = \Omega \times (0,T), \tag{1}$$

has been studied widely, one can refer to [2, 3, 26] and the references therein. Here $\Omega \subset \mathbb{R}^N$ is a bounded domain with suitably smooth boundary $\partial\Omega$, $1 < p(x) \in C^1(\overline{\Omega})$, and

$$p^+ = \max_{\bar{\Omega}} p(x),\ p^- = \min_{\bar{\Omega}} p(x).$$

Of course, if $p(x) \equiv p$, equation (1) becomes

$$u_t - \mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0,\ (x,t) \in Q_T, \tag{2}$$

which emerges in the non-Newtonian fluids mechanics theory and is called the evolutionary $p-$ Laplacian equation [2, 16, 22, 27, 28].

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In this paper, we generalize equation (1) to the following type

$$u_t = \mathrm{div}(a(x)|\nabla u|^{p(x)-2}\nabla u) + \sum_{i=1}^{N} \frac{\partial b_i(u,x,t)}{\partial x_i},\ (x,t) \in Q_T, \tag{3}$$

where the nonnegative function $a(x) \in C^1(\overline{\Omega})$ and $a(x) > 0$ in Ω, $b_i(s,x,t) \in C^1(\mathbb{R}\times\overline{Q}_T)$ is bounded when $|s|$ is bounded, $i = 1, 2, \cdots, N$. In order to study the well-posedness of weak solutions to equation (3), the initial value

$$u(x,0) = u_0(x),\ x \in \Omega \tag{4}$$

is always indispensable. Since $a(x)$ may be degenerate on the boundary $\partial\Omega$, the Dirichlet boundary value condition

$$u(x,t) = 0,\ (x,t) \in \partial\Omega \times (0,T) \tag{5}$$

may be overdetermined.

To see that, we can review some backgrounds. Firstly, we consider a linear degenerate equation

$$\frac{\partial u}{\partial t} - \mathrm{div}(a(x)\nabla u) - f_i(x)D_i u + c(x,t)u = g(x,t),\ (x,t) \in Q_T, \tag{6}$$

where $a(x), f_i(x), c(x,t)$ and $g(x,t)$ are smooth functions, $D_i = \frac{\partial}{\partial x_i}$, $a(x) \geq 0$. We can rewrite it as

$$\frac{\partial u}{\partial t} - a(x)\Delta u - (a_{x_i}(x) + f_i(x))D_i u + c(x,t)u = g(x,t),\ (x,t) \in Q_T. \tag{7}$$

According to Fichera-Oleinik theory [9, 23], besides the initial value condition (4), only a partial boundary value condition

$$u(x,t) = 0, (x,t) \in \Sigma_p \times (0,T) \tag{8}$$

matches up with equation (7), where

$$\Sigma_p = \{x \in \partial\Omega : f_i(x)n_i(x) < 0\} \cup \{x \in \partial\Omega : a(x) > 0\}$$

and $\overrightarrow{n} = \{n_i\}$ is the inner normal vector of Ω. In particular, if $f_i(x)n_i(x) \geq 0$ and $a(x) = 0$ for all $x \in \partial\Omega$, then

$$\Sigma_p = \emptyset.$$

This implies that, to obtain the well-posedness of the solutions to equation (7), the boundary value condition is dispensable in this case.

Secondly, we consider the equation

$$\frac{\partial u}{\partial t} = \mathrm{div}(d^\beta|\nabla u|^{p-2}\nabla u) + f(x,t,u),\quad (x,t) \in Q_T, \tag{9}$$

where $\beta > 0$, $d(x) = \mathrm{dist}(x, \partial\Omega)$ is the distance function from the boundary. If $f(x,t,u)$ is a Lipchitz function, then the stability of solutions to (9) was proved without any boundary value conditions [31]. Thus, the boundary value condition (5) may be replaced by the degeneracy of d^α. However, if $f(x,t,u)$ is not a Lipchitz function, the situation may change. In fact, Jiří Benedikt et.al [4, 5] had shown that the uniqueness of the solution to the following equation

$$u_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u) + q(x)|u|^{\alpha-1}u,\ (x,t) \in Q_T \tag{10}$$

is not true, where $0 < \alpha < 1$, $q(x) \geq 0$ and $q(x_0) > 0$ for some $x_0 \in \Omega$.

From the above brief reviews, we can say that how to give a suitable boundary condition matching a nonlinear parabolic equation is a difficult but very important problem, one can refer to [2, 7, 8, 10–14, 17, 19] and [29]-[35] et. al. for more information. In this paper, we will give the explicit formula Σ_p and obtain the stability of the solutions based on the partial boundary value condition (8), provided that $b_i(\cdot, x, t) = 0$ on the boundary.

2. The basic concepts and the main results

Let us introduce the basic functional spaces with variable exponents, for more details, see [9, 15, 36] et.al.

1. $L^{p(x)}(\Omega)$ space.

$$L^{p(x)}(\Omega)=\left\{u: u \quad is \quad a \quad measurable\ real-valued\ function, \int_\Omega |u(x)|^{p(x)}dx<\infty\right\},$$

it is equipped with the following Luxemburg's norm

$$\|u\|_{L^{p(x)}(\Omega)}=\inf\left\{\lambda>0: \int_\Omega \Big|\frac{u(x)}{\lambda}\Big|^{p(x)}dx\le 1\right\}.$$

The space $\left(L^{p(x)}(\Omega), \|.\|_{L^{p(x)}(\Omega)}\right)$ is a separable, uniformly convex Banach space.

2. $W^{1,p(x)}(\Omega)$ space.

$$W^{1,p(x)}(\Omega)=\left\{u\in L^{p(x)}(\Omega): |\nabla u|\in L^{p(x)}(\Omega)\right\},$$

it is endowed with the following norm

$$\|u\|_{W^{1,p(x)}}=\|u\|_{L^{p(x)}(\Omega)}+\|\nabla u\|_{L^{p(x)}(\Omega)}, \forall u\in W^{1,p(x)}(\Omega).$$

We use $W_0^{1,p(x)}(\Omega)$ to denote the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

Lemma 2.1. *(i) The space $\left(L^{p(x)}(\Omega), \|\cdot\|_{L^{p(x)}(\Omega)}\right)$, $\left(W^{1,p(x)}(\Omega), \|\cdot\|_{W^{1,p(x)}(\Omega)}\right)$ and $W_0^{1,p(x)}(\Omega)$ are reflexive Banach spaces.*

(ii) $p(x)$-Hölder's inequality. Let $q_1(x)$ and $q_2(x)$ be real functions with $\frac{1}{q_1(x)}+\frac{1}{q_2(x)}=1$ and $q_1(x)>1$. Then, the conjugate space of $L^{q_1(x)}(\Omega)$ is $L^{q_2(x)}(\Omega)$. If $u\in L^{q_1(x)}(\Omega)$ and $v\in L^{q_2(x)}(\Omega)$, then

$$\left|\int_\Omega uv dx\right|\le 2\|u\|_{L^{q_1(x)}(\Omega)}\|v\|_{L^{q_2(x)}(\Omega)}.$$

(iii)

$$If\ \|u\|_{L^{p(x)}(\Omega)}=1, then \int_\Omega |u|^{p(x)}dx=1.$$
$$If\ \|u\|_{L^{p(x)}(\Omega)}>1, then\ \|u\|^{p^-}_{L^{p(x)}(\Omega)}\le \int_\Omega |u|^{p(x)}dx\le \|u\|^{p^+}_{L^{p(x)}(\Omega)}.$$
$$If\ \|u\|_{L^{p(x)}(\Omega)}<1, then\ \|u\|^{p^+}_{L^{p(x)}(\Omega)}\le \int_\Omega |u|^{p(x)}dx\le \|u\|^{p^-}_{L^{p(x)}(\Omega)}.$$

In [36], Zhikov showed that

$$\begin{aligned}W_0^{1,p(x)}(\Omega)&\neq\{v\in W_0^{1,p(x)}(\Omega)\mid v|_{\partial\Omega}=0\}\\ &=\mathring{W}^{1,p(x)}(\Omega).\end{aligned}$$

Hence, the property of the space is different from the case when p is a constant. This fact implies that the general methods used in studying the well-posedness of weak solutions to the evolutionary $p-$Laplacian equation can not be used directly. However, if the exponent $p(x)$ satisfies the logarithmic Hölder continuity condition, i.e.

$$|p(x)-p(y)|\le\omega(|x-y|), \forall x,y\in\Omega, \quad |x-y|<\frac{1}{2}, \overline{\lim_{s\to 0^+}}\,\omega(s)\ln\left(\frac{1}{s}\right)=c<\infty,$$

then

$$W_0^{1,p(x)}(\Omega) = \mathring{W}^{1,p(x)}(\Omega).$$

Moreover, for any $u \in W^{1,p(x)}(\Omega)$, if u_ε is the mollified function of u, then by [19], we know that

$$\|\nabla u_\varepsilon\|_{L^{p(x)}(\Omega)} \leq c\|\nabla u\|_{L^{p(x)}(\Omega)}.$$

Now, we introduce the basic definitions and main results of this paper.

Definition 2.2. *A function $u(x,t)$ is said to be a weak solution of equation (3) with the initial value (4), if*

$$u \in L^\infty(Q_T),\ a(x)\,|\nabla u|^{p(x)} \in L^1(Q_T),\ u_t \in L^2(Q_T), \tag{11}$$

and for any function $\varphi \in L^\infty(0,T; W_0^{1,p(x)}(\Omega)) \bigcap L^2(Q_T)$,

$$\iint_{Q_T} \left(u_t\varphi + a(x)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi + \sum_{i=1}^N b_i(u,x,t)\cdot \varphi_{x_i} \right) dxdt = 0. \tag{12}$$

Moreover,

$$\lim_{t\to 0} \int_\Omega |\ u(x,t) - u_0(x)\ |\ dx = 0. \tag{13}$$

Definition 2.3. *A function $u(x,t)$ is said to be a weak solution of equation (3) with (4) and (5), if u satisfies Definition 2.2 and the partial boundary condition (8) in the sense of the trace.*

When $\int_\Omega a^{-\frac{1}{p(x)-1}}dx \leq c$, we can show that $\iint_{Q_T} |\nabla u| dxdt \leq c$. Then we can define the trace of u on the boundary $\partial\Omega$, so the partial boundary condition (8) is feasible.

The main results of the paper are the following stability theorems, in which the exponent $p(x)$ is required to satisfy the logarithmic Hölder continuity condition unexceptionally.

Theorem 2.4. *Let*

$$\Sigma_1 = \{x \in \partial\Omega : a(x) > 0\},$$

$u(x,t), v(x,t)$ be two solutions of equation (3) with the initial values $u_0(x), v_0(x)$ respectively, and with the same partial boundary value condition

$$u(x,t) = v(x,t) = 0,\ (x,t) \in \Sigma_1 \times (0,T). \tag{14}$$

If for large enough n,

$$n^{1-\frac{1}{p^+}} \left(\int_{\Omega\setminus\Omega_n} |\nabla a(x)|^{p(x)} dx \right)^{\frac{1}{p^+}} \leq c, \tag{15}$$

$$|b_i(u,x,t) - b_i(v,x,t)| \leq c a(x)^{\frac{1}{p(x)}} |u - v|, \tag{16}$$

then

$$\int_\Omega |u(x,t) - v(x,t)| dx \leq \int_\Omega |u(x,0) - v(x,0)| dx. \tag{17}$$

Here, $\Omega_n = \{x \in \Omega : a(x) > \frac{1}{n}\}$.

Theorem 2.5. *Let $u(x,t)$ and $v(x,t)$ be two weak solutions of equation (3) with the different initial values $u_0(x)$, $v_0(x)$ respectively, and with the same partial boundary value condition*

$$u(x,t) = v(x,t) = 0,\ (x,t) \in \Sigma_1 \times (0,T). \tag{18}$$

If

$$\nabla a(x) = 0,\ x \in \Sigma_2 = \partial\Omega \setminus \Sigma_1, \tag{19}$$

and for small $\lambda > 0$,

$$\int_{\Omega\setminus\Omega_\lambda} a(x)^{1-p(x)} |\nabla a|^{p(x)} dx \le c, \tag{20}$$

and $b_i(s,x,t)$ satisfies

$$|b_i(u,x,t) - b_i(v,x,t)| \le ca(x)^{\frac{1}{p(x)}} |u-v|, \tag{21}$$

then

$$\int_\Omega |u(x,t) - v(x,t)| dx \le \int_\Omega |u_0(x) - v_0(x)| dx. \tag{22}$$

Here $\Omega_\lambda = \{x \in \Omega : a(x) > \lambda\}$.

3. The existence

At the beginning of this section, we would like to point out that the conditions in the following Theorem 3.1 are not optimal, we only supply an existence result to assure the completeness of the paper.

Theorem 3.1. *If $p^- \ge 2$ and $\int_\Omega a^{-\frac{p(x)-2}{2}}(x)dx \le c$, there are constants β, c such that*

$$|b_i(s,x,t)| \leqslant c|s|^{1+\beta}, \quad |b_{is}(s,x,t)| \leqslant c|s|^\beta,\ \left|b_{ix_i}(s,x,t)\right| \leqslant c\,|s|^{1+\beta}, \tag{23}$$

and u_0 satisfies

$$u_0 \in L^\infty(\Omega), \quad a(x)\,|\nabla u_0|^{p^+} \in L^1(\Omega), \tag{24}$$

then there exists a solution of equation (3) with initial value condition (4), where $b_{is} = \frac{\partial b_i}{\partial s}, b_{ix_i} = \frac{\partial b_i}{\partial x_i}$, $i = 1, 2, \cdots, N$.

Proof. Let $u_{\varepsilon,0} \in C_0^\infty(\Omega)$ and $a(x)\left|\nabla u_{\varepsilon,0}\right|^{p^+} \in L^1(\Omega)$ be uniformly bounded, and $u_{\varepsilon,0}$ converges to u_0 in $W_0^{1,p^+}(\Omega)$. By considering the following approximate problem

$$u_{\varepsilon t} - \mathrm{div}((a(x)+\varepsilon)(|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{p(x)-2}{2}} \nabla u_\varepsilon) - \sum_{i=1}^N \frac{\partial b_i(u_\varepsilon, x, t)}{\partial x_i} = 0,\ (x,t) \in Q_T, \tag{25}$$

$$u_\varepsilon(x,t) = 0,\ (x,t) \in \partial\Omega \times (0,T), \tag{26}$$

$$u_\varepsilon(x,0) = u_{\varepsilon,0}(x),\ x \in \Omega. \tag{27}$$

It is well-known that the above problem has a unique weak solution ([20, 26])

$$u_\varepsilon \in L^\infty(Q_T) \bigcap L^1(0,T; W_0^{1,p(x)}(\Omega)),\ |u_\varepsilon| \le c. \tag{28}$$

Multiplying (23) by u_ε and integrating it over Q_T, it 's easy to prove that

$$\iint_{Q_T} (a(x)+\varepsilon)|\nabla u_\varepsilon|^{p(x)} dxdt \le c. \tag{29}$$

Multiplying (25) by $u_{\varepsilon t}$ and integrating it over Q_T, then

$$\begin{aligned}\iint_{Q_T} (u_{\varepsilon t})^2 dxdt \qquad &= \iint_{Q_T} \mathrm{div}((a(x)+\varepsilon)(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p(x)-2}{2}}\nabla u_\varepsilon)\cdot u_{\varepsilon t} dxdt \\ &+ \iint_{Q_T} u_{\varepsilon t}\frac{\partial b_i(u_\varepsilon,x,t)}{\partial x_i} dxdt.\end{aligned} \tag{30}$$

Noticing that

$$(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p(x)-2}{2}}\nabla u_\varepsilon\cdot\nabla u_{\varepsilon t} = \frac{1}{2}\frac{d}{dt}\int_0^{|\nabla u_\varepsilon(x,t)|^2+\varepsilon} s^{\frac{p(x)-2}{2}} ds.$$

Thus,

$$\begin{aligned}&\iint_{Q_T} \mathrm{div}((a(x)+\varepsilon)(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p(x)-2}{2}}\nabla u_\varepsilon) u_{\varepsilon t} dxdt \\ &= -\iint_{Q_T} (a(x)+\varepsilon)(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p(x)-2}{2}}\nabla u_\varepsilon\cdot\nabla u_{\varepsilon t} dxdt \\ &= -\frac{1}{2}\iint_{Q_T} (a(x)+\varepsilon)\frac{d}{dt}\int_0^{|\nabla u_\varepsilon(x,t)|^2+\varepsilon} s^{\frac{p(x)-2}{2}} dsdxdt.\end{aligned} \tag{31}$$

By (23) and (28),

$$\begin{aligned}&\iint_{Q_T} u_{\varepsilon t}\frac{\partial b_i(u_\varepsilon,x,t)}{\partial x_i} dxdt \\ &\leqslant \iint_{Q_T} \left|b_{iu_\varepsilon}(u_\varepsilon,x,t)\right|\left|u_{\varepsilon x_i}\right| |u_{\varepsilon t}| dxdt + \iint_{Q_T} \left|b_{ix_i}(u_\varepsilon,x,t)\right| |u_{\varepsilon t}| dxdt \\ &\leqslant \frac{1}{4}\iint_{Q_T} (u_{\varepsilon t})^2 dxdt + c\iint_{Q_T} |u_\varepsilon|^{2\beta}|\nabla u_\varepsilon|^2 dxdt \\ &+\frac{1}{4}\iint_{Q_T} (u_{\varepsilon t})^2 dxdt + c\iint_{Q_T} |u_\varepsilon|^{2(\beta+1)} dxdt \\ &\leqslant \frac{1}{4}\iint_{Q_T} (u_{\varepsilon t})^2 dxdt + c\iint_{Q_T} |u_\varepsilon|^{2\beta}|\nabla u_\varepsilon|^2 dxdt + \frac{1}{4}\iint_{Q_T} (u_{\varepsilon t})^2 dxdt + c.\end{aligned} \tag{32}$$

By Hölder's inequality, (28) and $\int_\Omega a^{-\frac{p(x)-2}{2}}(x)dx \le c$ yield

$$\begin{aligned}&\iint_{Q_T} |u_\varepsilon|^{2\beta}|\nabla u_\varepsilon|^2 dxdt \\ &\leqslant c\iint_{Q_T} |\nabla u_\varepsilon|^2 dxdt = c\iint_{Q_T} (a(x)+\varepsilon)^{-\frac{2}{p(x)}}\cdot(a(x)+\varepsilon)^{\frac{2}{p(x)}}|\nabla u_\varepsilon|^2 dxdt \\ &\le c\left(\iint_{Q_T} (a(x)+\varepsilon)^{-\frac{2}{p(x)-2}} dxdt\right)^m\cdot\left(\iint_{Q_T} (a(x)+\varepsilon)|\nabla u_\varepsilon|^{p(x)} dxdt\right)^{m_1} \\ &\le c.\end{aligned} \tag{33}$$

Here $m = \max_{x\in\overline{\Omega}} \frac{p(x)-2}{p(x)}$ or $\min_{x\in\overline{\Omega}} \frac{p(x)-2}{p(x)}$ according to (iii) of Lemma 2.1, $m_1 = \max_{x\in\overline{\Omega}} \frac{2}{p(x)}$ or $\min_{x\in\overline{\Omega}} \frac{2}{p(x)}$ has the same meaning.

Combining (30)-(33), we have

$$\iint_{Q_T} (u_{\varepsilon t})^2 dxdt + \iint_{Q_T} (a(x)+\varepsilon)\frac{d}{dt}\int_0^{|\nabla u_\varepsilon(x,t)|^2} s^{\frac{p(x)-2}{2}} dsdxdt \leqslant c,$$

by which implies that

$$\iint_{Q_T} (u_{\varepsilon t})^2 dxdt \leqslant c + c\int_\Omega (a(x)+\varepsilon)\left|\nabla u_{\varepsilon,0}\right|^{p(x)} dx \leqslant c. \tag{34}$$

By choosing a subsequence, letting $\varepsilon \to 0$, we may obtain $u_\varepsilon \to u$ a.e. in Q_T,where u satisfies (12). Meanwhile, we can show (13) in a similar way as that of the usual evolutionary $p-$Laplacian equation (see Ref. [22]). Then u is the solution of equation (3) with the initial value (4) in the sense of Definition 2.2. □

Lemma 3.2. *Assume that $\int_\Omega a^{-\frac{1}{p(x)-1}} dx \leq c$, let $u(x,t)$ be the solution of equation (3) with the initial value (4). Then*

$$\int_\Omega |\nabla u| dx \leq c. \tag{35}$$

Proof. Since $\int_\Omega a^{-\frac{1}{p(x)-1}} dx \leq c$, we have

$$\begin{aligned}
\int_\Omega |\nabla u|\, dx &= \int_{\{x\in\Omega; a^{\frac{1}{p(x)-1}}|\nabla u|\leqslant 1\}} |\nabla u|\, dx + \int_{\{x\in\Omega; a^{\frac{1}{p(x)-1}}|\nabla u|>1\}} |\nabla u|\, dx \\
&\leqslant \int_\Omega a^{-\frac{1}{p(x)-1}} dx + \int_\Omega a\,|\nabla u|^{p(x)}\, dx \\
&\leqslant c.
\end{aligned}$$

Lemma (3.2) is proved. □

By (35), the trace of $u(x,t)$ on the boundary $\partial\Omega$ can be defined in the traditional way.

Since $\int_\Omega a^{-\frac{p(x)-2}{2}}(x)dx \leq c$ implies $\int_\Omega a^{-\frac{1}{p(x)-1}} dx \leq c$, we have

Theorem 3.3. *If the conditions of Theorem 3.1 are true, then there is a solution of equation (3) with the usual initial-boundary conditions (4),(5)(or(8)).*

From the proof of Theorem 3.1, one can see that the condition $\int_\Omega a^{-\frac{p(x)-2}{2}}(x)dx \leq c$ and the assumption of (23) are only used to prove $u_t \in L^2(Q_T)$. If one relaxes the regularity of u_t, $p^- \geq 2$ can be generalized to $p^- > 1$. Also, one can relaxes the condition $\int_\Omega a^{-\frac{p(x)-2}{2}}(x)dx \leq c$, for instance, we have the following theorem.

Theorem 3.4. *If $p^- \geq 2$, $u_0(x)$ satisfies (24), and*

$$|b_{is}(s,x,t)| \leqslant ca^{\frac{1}{p(x)}},\ \left|b_{ix_i}(s,x,t)\right| \leqslant ca^{\frac{1}{p(x)}}, \tag{36}$$

then there is a solution of equation (3) with initial value (4).

4. The proof of Theorem 2.4

For any given positive integer n, let $g_n(s)$ be an odd function, and

$$g_n(s) = \begin{cases} 1, & |s| > \frac{1}{n}, \\ ns, & 0 \leq s \leqslant \frac{1}{n}. \end{cases}$$

Clearly,

$$\lim_{n\to\infty} g_n(s) = \mathrm{sgn}(s), s \in (-\infty, +\infty),$$

and

$$\lim_{n\to\infty} s g_n'(s) = 0. \tag{37}$$

Denoting that $\Sigma_1 = \{x \in \partial\Omega : a(x) > 0\}$ and $\Sigma_2 = \{x \in \partial\Omega : a(x) = 0\}$. Let $\phi(x)$ be a $C^1(\overline{\Omega})$ function satisfying

$$\phi(x)\mid_{x\in\Sigma_2} = 0, \ \ \phi(x)\mid_{x\in\overline{\Omega}\setminus\Sigma_2} > 0, \tag{38}$$

and

$$\Omega_n = \{x \in \Omega : \phi(x) > \frac{1}{n}\}. \tag{39}$$

Theorem 4.1. *Let $u(x,t), v(x,t)$ be two solutions of equation (3) with the initial values $u_0(x), v_0(x)$ respectively, and with the same partial boundary value condition*

$$u(x,t) = v(x,t) = 0, (x,t) \in \Sigma_1 \times (0,T). \tag{40}$$

If for sufficiently large n,

$$n\left(\int_{\Omega\setminus\Omega_n} a(x)|\nabla\phi(x)|^{p(x)}dx\right)^{\frac{1}{p^+}} \leq c, \tag{41}$$

and there exist functions $g_i(x,t)$ such that

$$|b_i(u,x,t) - b_i(v,x,t)| \leq c g_i(x,t)|u-v|, \int_\Omega g_i(x,t)^{q(x)} a(x)^{-\frac{1}{p(x)-1}} dx < \infty, \tag{42}$$

then

$$\int_\Omega |u(x,t) - v(x,t)|dx \leq \int_\Omega |u_0(x) - v_0(x)|dx.$$

Proof. Let $u(x,t)$ and $v(x,t)$ be two weak solutions of equation (1) with the initial values $u_0(x)$ and $v_0(x)$ respectively, and with the partial boundary value condition (40).

Let

$$\phi_n(x) = \begin{cases} 1, & \text{if } x \in \Omega_n, \\ n\phi(x), & \text{if } x \in \Omega \setminus \Omega_n, \end{cases} \tag{43}$$

and $\chi_{[\tau,s]}$ be the characteristic function of $[\tau, s) \subseteq [0,T)$. Then, since u and v satisfy the partial boundary value condition (40), one can choose $\chi_{[\tau,s]}\phi_n g_n(u-v)$ as the test function, and

$$\begin{aligned}&\int_\tau^s\int_\Omega \phi_n(x)g_n(u-v)\frac{\partial(u-v)}{\partial t}dxdt\\&+\int_\tau^s\int_\Omega a(x)(|\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v)\cdot\nabla(u-v)g_n'(u-v)\phi_n(x)dxdt\\&+\int_\tau^s\int_\Omega a(x)(|\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v)g_n(u-v)\nabla\phi_n dxdt\\&+\sum_{i=1}^N\int_\tau^s\int_\Omega[b_i(u,x,t)-b_i(v,x,t)][g_n'(u-v)(u-v)_{x_i}\phi_n(x)+g_n(u-v)\phi_{nx_i}]dxdt\\&=0.\end{aligned}\tag{44}$$

As usual,

$$\int_\tau^s\int_\Omega a(x)(|\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v)\cdot\nabla(u-v)g_n'(u-v)\phi_n(x)dxdt\geq 0.\tag{45}$$

Since $u_t\in L^2(Q_T)$, by Lebesgue's dominated convergence theorem, one has

$$\begin{aligned}&\lim_{n\to\infty}\int_\tau^s\int_\Omega \phi_n(x)g_n(u-v)\frac{\partial(u-v)}{\partial t}dxdt\\&=\int_\Omega|u(x,s)-v(x,s)|dx-\int_\Omega|u(x,\tau)-v(x,\tau)|dx.\end{aligned}\tag{46}$$

Obviously, for $i=1,2,\cdots,N$

$$\phi_{nx_i}(x)=\begin{cases}0, & \text{if } x\in\Omega_n,\\ n\phi_{x_i}(x), & \text{if } x\in\Omega\setminus\Omega_n.\end{cases}$$

By (40), one has

$$\begin{aligned}&\left|\int_\tau^s\int_\Omega a(x)(|\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v)g_n(u-v)\nabla\phi_n dxdt\right|\\&=\left|\int_\tau^s\int_{\Omega\setminus\Omega_n} a(x)(|\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v)g_n(u-v)\nabla\phi_n dx\right|\\&\leq\int_\tau^s n\int_{\Omega\setminus\Omega_n}a(x)(|\nabla u|^{p(x)-1}+|\nabla v|^{p(x)-1})\nabla\phi g_n(u-v)|dxdt\\&\leq c\int_\tau^s n\left(\int_{\Omega\setminus\Omega_n}a(x)(|\nabla u|^{p(x)}+|\nabla v|^{p(x)})dx\right)^{\frac{1}{q^+}}\left(\int_{\Omega\setminus\Omega_n}a(x)|\nabla\phi|^{p(x)}dx\right)^{\frac{1}{p^+}}dt\\&\leq c\int_\tau^s\left[\left(\int_{\Omega\setminus\Omega_n}a(x)|\nabla u|^{p(x)}dx\right)^{\frac{1}{q^+}}+\left(\int_{\Omega\setminus\Omega_n}a(x)|\nabla v|^{p(x)}dx\right)^{\frac{1}{q^+}}\right]\\&\qquad\cdot\left[n\left(\int_{\Omega\setminus\Omega_n}a(x)|\nabla\phi|^{p(x)}dx\right)^{\frac{1}{p^+}}\right]dt\\&\leq c\int_\tau^s\left(\int_{\Omega\setminus\Omega_n}a(x)|\nabla u|^{p(x)}dx\right)^{\frac{1}{q^+}}dt+c\int_\tau^s\left(\int_{\Omega\setminus\Omega_n}a(x)|\nabla v|^{p(x)}dx\right)^{\frac{1}{q^+}}dt,\end{aligned}\tag{47}$$

where $q(x)=\frac{p(x)}{p(x)-1}$, $q^+=\max_{x\in\overline{\Omega}}q(x)$. Thus,

$$\begin{aligned}
&\lim_{n\to\infty}\left|\int_\tau^s\int_\Omega a(x)(|\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v)(u-v)g_n(u-v)\nabla\phi_n dxdt\right|\\
&\le c\lim_{n\to\infty}\left[c\int_\tau^s\left(\int_{\Omega\setminus\Omega_n}a(x)|\nabla u|^{p(x)}dx\right)^{\frac{1}{q^+}}dt+c\int_\tau^s\left(\int_{\Omega\setminus\Omega_n}a(x)|\nabla v|^{p(x)}dx\right)^{\frac{1}{q^+}}dt\right]\\
&=0. \qquad (48)
\end{aligned}$$

Moreover, by (42), $\int_\Omega\sum_{i=1}^N g_i(x,t)^{q(x)}a(x)^{-\frac{1}{p(x)-1}}dx<\infty$, using Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}
&\lim_{n\to\infty}\left|\sum_{i=1}^N\int_\tau^s\int_\Omega[b_i(u,x,t)-b_i(v,x,t)]g_n'(u-v)(u-v)_{x_i}\phi_n(x)dxdt\right|\\
\le\ & c\lim_{n\to\infty}\int_\tau^s\int_\Omega\sum_{i=1}^N\left|g_i(x,t)a^{\frac{1}{p(x)}}a^{-\frac{1}{p(x)}}(u-v)_{x_i}\phi_n(x)(u-v)g_n'(u-v)\right|dxdt\\
\le\ & c\lim_{n\to\infty}\int_\tau^s\sum_{i=1}^N\left(\int_\Omega a(x)(|u_{x_i}|^{p(x)}+|v_{x_i}|^{p(x)})dx\right)^{\frac{1}{p^+}}\\
&\cdot\left(\int_\Omega g_i(x,t)^{q(x)}a(x)^{-\frac{1}{p(x)-1}}\left|(u-v)g_n'(u-v)\right|^{q(x)}dx\right)^{\frac{1}{q^+}}dt\\
\le\ & c\lim_{n\to\infty}\int_\tau^s\left(\int_\Omega a(x)(|\nabla u|^{p(x)}+|\nabla v|^{p(x)})dx\right)^{\frac{1}{p^+}}\\
&\cdot\left(\int_\Omega\sum_{i=1}^N g_i(x,t)^{q(x)}a(x)^{-\frac{1}{p(x)-1}}\left|(u-v)g_n'(u-v)\right|^{q(x)}dx\right)^{\frac{1}{q^+}}dt \qquad (49)
\end{aligned}$$

Once again, since $g_i(x,t)\ge 0$, $a(x)\ge 0$, by (42),

$$g_i(x,t)=0=a(x),x\in\Sigma_2,i=1,2,\cdots,N,\forall t\in[0,T], \qquad (50)$$

thus,

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{i=1}^N\left|\int_\tau^s\int_\Omega[b_i(u,x,t)-b_i(v,x,t)]\phi_{nx_i}g_n(u-v)dxdt\right|\\
&=\lim_{n\to\infty}\sum_{i=1}^N\left|\int_\tau^s\int_{\Omega\setminus\Omega_n}[b_i(u,x,t)-b_i(v,x,t)]\phi_{nx_i}g_n(u-v)dxdt\right|\\
&\le\lim_{n\to\infty}\sum_{i=1}^N n\int_\tau^s\int_{\Omega\setminus\Omega_n}g_i(x,t)|u-v||\phi_{x_i}(x)||g_n(u-v)|dxdt\\
&=\sum_{i=1}^N\int_\tau^s\left(\lim_{n\to\infty}n\int_{(\Omega\setminus\Omega_n)}g_i(x,t)|u-v||\phi_{x_i}(x)||g_n(u-v)|dx\right)dt\\
&=\sum_{i=1}^N\int_\tau^s\int_{\Sigma_1}g_i(x,t)|\phi_{x_i}(x)||u-v|d\Sigma dt\\
&=0. \qquad (51)
\end{aligned}$$

Let $n \to \infty$ in (44). Then

$$\int_\Omega |u(x,s) - v(x,s)|\, dx \leqslant \int_\Omega |u(x,\tau) - v(x,\tau)|\, dx. \tag{52}$$

By the arbitrary of τ, we have

$$\int_\Omega |u(x,s) - v(x,s)|\, dx \leqslant \int_\Omega |u_0(x) - v_0(x)|\, dx. \tag{53}$$

Theorem 4.1 is proved. □

Proof. [Proof of Theorem 2.4] We only need to choose

$$\phi(x) = a(x),$$

and

$$g_i(x,t) = a(x)^{\frac{1}{p(x)}}, i = 1, 2, \cdots, N$$

in Theorem 4.1, the conclusion is clear. □

Certainly, there are many choices of ϕ. For example, when x is close to the partial boundary Σ_2, $\phi(x) = d_{\Sigma_2}(x) = \text{dist}(x, \Sigma_2)$.

Instead of the condition (41), if the conditions (40) , (42) are still true, and

$$n \left(\int_{\Omega \setminus \Omega_n} a(x) dx \right)^{\frac{1}{p^+}} \le c, \tag{54}$$

then the same conclusion of Theorem 4.1 is true.

Only if we notice that

$$|\nabla \phi| = |\nabla d| = 1,$$

then the conclusion follows.

5. The proof of Theorem 2.5

Let $g_n(s)$ be defined as before and $\phi(x)$ satisfy (38) and

$$\nabla \phi(x) = 0, x \in \Sigma_2. \tag{55}$$

Theorem 5.1. *Let $u(x,t)$ and $v(x,t)$ be two weak solutions of equation (3) with the different initial values $u_0(x)$, $v_0(x)$ respectively,and with the same partial boundary value condition*

$$u(x,t) = v(x,t) = 0, (x,t) \in \Sigma_1 \times (0,T). \tag{56}$$

If for small $\lambda > 0$,

$$\int_{\Omega \setminus \Omega_\lambda} a(x) \left| \frac{\nabla \phi}{\phi} \right|^{p(x)} dx \le c, \tag{57}$$

$b_i(s,x,t)$ satisfies

$$|b_i(u,x,t) - b_i(v,x,t)| \le c g_i(x,t)|u - v|, \int_\Omega \left[a(x)^{-\frac{1}{p(x)}} g_i(x,t) \right]^{q(x)} dx < \infty, \tag{58}$$

then

$$\int_\Omega |u(x,t) - v(x,t)| dx \le \int_\Omega |u_0(x) - v_0(x)| dx.$$

Proof. For a small positive constant $\lambda > 0$, let

$$\phi_\lambda(x) = \begin{cases} 1, & \text{if } x \in \Omega_\lambda, \\ \frac{\phi(x)}{\lambda}, & \text{if } x \in \Omega \setminus \Omega_\lambda, \end{cases}$$

where ϕ satisfies (38), and $\Omega_\lambda = \{x \in \Omega : \phi(x) > \lambda\}$.

Since $u(x,t)$ and $v(x,t)$ satisfy the partial boundary value condition (55), by a process of limit, we can choose $g_n(\phi_\lambda(u-v))$ as the test function, then

$$\begin{aligned}
&\int_\Omega g_n(\phi_\lambda(u-v))\frac{\partial(u-v)}{\partial t}dx \\
&+\int_\Omega a(x)(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)\cdot \phi_\lambda \nabla(u-v) g_n'(\phi_\lambda(u-v))dx \\
&+\int_\Omega a(x)(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)\cdot \nabla\phi_\lambda (u-v) g_n'(\phi_\lambda(u-v))dx \\
&+\sum_{i=1}^N \int_\Omega (b_i(u,x,t)-b_i(v,x,t))\cdot (u-v)_{x_i} g_n'(\phi_\lambda(u-v))\phi_\lambda dx \\
&+\sum_{i=1}^N \int_\Omega (b_i(u,x,t)-b_i(v,x,t))\cdot \phi_{\lambda x_i}(u-v) g_n'(\phi_\lambda(u-v))dx \\
&= 0. \qquad (59)
\end{aligned}$$

Thus

$$\lim_{n\to\infty}\lim_{\lambda\to 0}\int_\Omega g_n(\phi_\lambda(u-v))\frac{\partial(u-v)}{\partial t}dx = \frac{d}{dt}\|u-v\|_{L^1(\Omega)}, \qquad (60)$$

$$\int_\Omega a(x)(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)\cdot \nabla(u-v) g_n'(\phi_\lambda(u-v))\phi_\lambda(x)dx \geq 0, \qquad (61)$$

Since $\nabla\phi_\lambda = \frac{\nabla\phi}{\lambda}$ when $x \in \Omega \setminus \Omega_\lambda$, $\nabla\phi_\lambda = 0$ when $x \in \Omega_\lambda$, we have

$$\begin{aligned}
&\left|\int_\Omega a(x)(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)\cdot \nabla\phi_\lambda (u-v) g_n'(\phi_\lambda(u-v))dx\right| \\
&= \left|\int_{\Omega\setminus\Omega_\lambda} a(x)(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)\cdot \frac{\nabla\phi}{\lambda}(u-v) g_n'(\phi_\lambda(u-v))dx\right| \\
&= \left|\int_{\Omega\setminus\Omega_\lambda} a(x)(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)\cdot \frac{1}{\lambda}\frac{\nabla\phi}{\phi_\lambda}\phi_\lambda(u-v) g_n'(\phi_\lambda(u-v))dx\right| \\
&\leq c\int_{\Omega\setminus\Omega_\lambda} a^{\frac{p(x)-1}{p(x)}}|(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)||a^{\frac{1}{p(x)}}\frac{\nabla\phi}{\phi}|dx \\
&\leq c\left\|a^{-\frac{p(x)-1}{p(x)}}(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)\right\|_{L^{\frac{p(x)}{p(x)-1}}(\Omega)}\|a(x)\frac{\nabla\phi}{\phi}\|_{L^{p(x)}(\Omega)} \\
&\leq c\left(\int_{\Omega\setminus\Omega_\lambda} a(x)(|\nabla u|^{p(x)} + |\nabla v|^{p(x)})dx\right)^{\frac{1}{q^+}}\left(\int_{\Omega\setminus\Omega_\lambda} a(x)\left|\frac{\nabla\phi}{\phi}\right|^{p(x)}dx\right)^{\frac{1}{p^+}} \\
&\leq c\left(\int_{\Omega\setminus\Omega_\lambda} a(x)\left|\frac{\nabla\phi}{\phi}\right|^{p(x)}dx\right)^{\frac{1}{p^+}}, \qquad (62)
\end{aligned}$$

where $q(x) = \frac{p(x)}{p(x)-1}$, $q^+ = \max_{x\in\overline{\Omega}} q(x)$. Then, it follows from (57) that

$$\lim_{\lambda\to 0}\left|\int_\Omega a(x)(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)\cdot\nabla\phi_\lambda(u-v)g_n'(\phi_\lambda(u-v))dx\right| = 0. \tag{63}$$

Since

$$|b_i(u,x,t) - b_i(v,x,t)| \le cg_i(x,t)|u-v|,$$

by (55) and the partial boundary value condition (56), there holds

$$\begin{aligned}
&\lim_{\lambda\to 0}\left|\int_\Omega (b_i(u,x,t) - b_i(v,x,t))g_n{}'(\phi_\lambda(u-v))(u-v)\phi_{\lambda x_i}(x)dx\right| \\
&\le \lim_{\lambda\to 0}\frac{1}{\lambda}\int_{\Omega\setminus\Omega_\lambda}|b_i(u,x,t) - b_i(v,x,t)|g_n{}'(\phi_\lambda(u-v))|\nabla\phi|dx \\
&\le c\int_{\partial\Omega} g_i(x,t)|u-v|g_n{}'(u-v)|\nabla\phi|d\Sigma \\
&= 0.
\end{aligned} \tag{64}$$

Moreover, by (58), we get

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{\lambda\to 0}\left|\int_\Omega (b_i(u,x,t) - b_i(v,x,t))(u-v)_{x_i}g_n{}'(\phi_\lambda(u-v))(u-v)\phi_\lambda(x)dx\right| \\
&\le \lim_{n\to\infty}\int_\Omega |b_i(u,x,t) - b_i(v,x,t)||(u-v)_{x_i}|g_n{}'(u-v)dx \\
&\le c\lim_{n\to\infty}\int_\Omega g_i(x,t)|(u-v)_{x_i}||u-v|g_n{}'(u-v)dx \\
&\le c\lim_{n\to\infty}\left(\int_\Omega \left[a(x)^{-\frac{1}{p(x)}}g_i(x,t)|(u-v)g_n{}'(u-v)|\right]^{\frac{p(x)}{p(x)-1}}dx\right)^{\frac{1}{q^+}} \\
&\quad\cdot\left(\int_\Omega a(x)(|\nabla u|^{p(x)} + |\nabla v|^{p(x)})dx\right)^{\frac{1}{p_1}} \\
&= 0.
\end{aligned} \tag{65}$$

Now, after letting $\lambda \to 0$, let $n \to \infty$ in (59). □

Proof. [Proof of Theorem 2.5] Only if we choose $\phi(x) = a(x)$ and $g_i(x,t) = a(x)^{\frac{1}{p(x)}}$, by Theorem 5.1, we know Theorem 2.5 is true. □

Competing interests

The author declares that he has no competing interests.

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